Time Consistent Portfolio Management

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Motivation

- There are at least two examples in portfolio management that are time inconsistent:
  - Maximizing utility of intertemporal consumption, life insurance and final wealth assuming a non-exponential discount rate.
  - Mean-variance utility.
- It means that the agents may have an incentive to deviate from their decisions that were optimal in the past.
- It is not often the case that management decisions are irreversible; there will usually be many opportunities to reverse a decision which, as times goes by, seems ill-advised.

Introduction

- we consider the optimal life insurance purchase, consumption and portfolio management strategies for a wage earner subject to mortality risk.
- the wage earner receives income continuously at some rate, but this is terminated by the wage earner's death or retirement whichever happens first.
- the wage earner needs to buy life insurance to protect his/her family from his/her premature death.
- aside from consumption and life insurance purchase, the wage earner has also the opportunity to invest in the capital market.

Literature Review (Selected)

- Yaari (1965) provided the key idea for research on life insurance, consumption and/or portfolio decisions under an uncertain lifetime.
- Richard (1975) used dynamic programming approach for a life-cycle life insurance and consumption/investment problem with uncertain life time.
- Pliska and Ye (2007) studied optimal life insurance and consumption for an income earner whose lifetime is random and unbounded.
- Ye (2007) considered optimal life insurance, consumption and portfolio choice problem under uncertain lifetime.
- Huang et al. (2008) investigated optimal life insurance, consumption and portfolio choice problem under uncertain lifetime with stochastic income process.

Maximizing utility of intertemporal consumption, life insurance and final wealth

- We work in a Black and Scholes world, with riskless rate r :

$$dS(t) = S(t)\left[\alpha\, dt + \sigma\, dW(t)\right], \quad 0 \leq t \leq \infty,$$

- Consider a self-financing portfolio. The total wealth value is X, the amount invested in the stock is ζ, the life insurance is p and the consumption rate is c. Then with $\mu = \alpha - r$:

$$dX^{\zeta,c,p}(t) = rX^{\zeta,c,p}(t)dt - c(t)dt - p(t)dt + i(t)dt + \zeta(t)(\alpha\, dt + \sigma dW(t)).$$

- T is a benchmark deterministic horizon (e.g. retirement time).
- the stopping time τ is lifetime; it has density $g(t)$, CDF $G(t)$ and

hazard function $\lambda(t) = \frac{g(t)}{1-G(t)}$.

- The investor at time $t \in [0, T]$ uses the criterion:

$$\begin{aligned} J(t,x,\zeta,c,p) &\triangleq \mathbb{E}\Big[\int_t^{T\wedge\tau} h(s-t)U_\gamma(c(s))\,ds + nh(T-t)U_\gamma(X^{\zeta,c,p}(T))1_{\{\tau>T|\tau>t\}} \\ &+ m(\tau-t)\hat{h}(\tau-t)U_\gamma(Z^{\zeta,c,p}(\tau))1_{\{\tau\leq T|\tau>t\}}\Big| X^{\zeta,c,p}(t)=x\Big], \end{aligned}$$

  - $U_\gamma(c) = -\frac{1}{\gamma}c^\gamma$, $\gamma < 1$
  - $n > 0$ is a constant.
  - $m(t) > 0$ is a continuous function.
  - h and $\hat{h}$ are discount functions. .
  - $Z(t) = \eta(t)X(t) + l(t)p(t)$, where $\eta(t)$ and $l(t)$ are prescribed deterministic and continuous functions.

- The risk criterion functional J can be written

$$
\begin{aligned}
J(t,x,\zeta,c,p) &= \mathbb{E}\Big[\int_t^T Q(s,t)U_\gamma(c(s))\,ds \\
&+ \int_t^T q(s,t)U_\gamma(Z^{t,x}(s))\,ds + nQ(T,t)U_\gamma(X^{t,x}(T))\Big],
\end{aligned}
$$

where

$$
\begin{aligned}
q(s,t) &\triangleq \bar{h}(s-t)\lambda(s)\exp\{-\int_t^s \lambda(z)\,dz\}, \quad \bar{h}(t) \triangleq m(t)\hat{h}(t) \\
Q(s,t) &\triangleq h(s-t)\exp\{-\int_t^s \lambda(z)\,dz\}
\end{aligned}
$$

Examples

$$\begin{aligned} J(t,x,\zeta,c,p) &= \mathbb{E}\Big[\int_t^T Q(s,t)U_\gamma(c(s))\,ds \\ &+ \int_t^T q(s,t)U_\gamma(Z^{t,x}(s))\,ds + nQ(T,t)U_\gamma(X^{t,x}(T))\Big], \end{aligned}$$

- $h(t) = \hat{h}(t) = \exp(-\rho t), \quad m(t) = n = 1.$
- $h(t) = \exp(-\rho t)$ and $\hat{h}(t) \neq h(t)$.
- $h(t) = (1+at)^{-\frac{b}{a}}$ (hyperbolic discounting).
- $h(t) = \hat{h}(t) = \exp(-\rho t), \qquad m(t) =$ nonconstant.

The last three cases have strong empirical support, but they do not fall within the classical framework. Indeed, they give rise to time inconsistency on the part of the investor, so that there is no implementable optimal portfolio.

Time-inconsitency for dummies

Consider an individual who wants to start running:

- If he starts today, he will suffer -1 today (pain), but gain +2 tomorrow (health).
- He has a non-constant discount rate: a stream u_t is valued today $(t = 0)$ at

$$u_0 + \frac{1}{2}\sum_{t=1}^{\infty} \rho^t u_t \quad \text{for some } \rho \in (\frac{1}{2}, 1)$$

- Starting today yields a utility of $-1 + \rho < 0$.
- Starting tomorrow yields a utility of $\frac{\rho(-1+2\rho)}{2} > 0$.
- So he decides today to start tomorrow. Unfortunately, when tomorrow comes, it becomes today, and he decides again to start the next day.

Time-inconsistency for mathematicians, financiers and economists

Assume for simplicity $\eta(t) = \lambda(t) = m(t) = n = 1,\ l(t) = l$; then

$$J(t,x,\zeta,c,p) = \mathbb{E}\Big[\int_t^T h(s-t)U_\gamma(c(s))\,ds + \int_t^T h(s-t)U_\gamma(Z^{t,x}(s))\,ds + h(T-t)U_\gamma(X^{t,x}(T))\Big].$$

- The HJB equation written for the investor at time t is

$$\frac{\partial V}{\partial s}(t,s,x) + \sup_{\zeta,c}\Big[(r+\mu\zeta - c)x\frac{\partial V}{\partial x}(t,s,x) + \frac{1}{2}\sigma^2\zeta^2x^2\frac{\partial^2 V}{\partial x^2}(t,s,x)$$

$$+\frac{h'(s-t)}{h(s-t)}V(t,s,x) + U_\gamma(xc) + U_\gamma(x+lp)\Big] = 0, \quad V(t,T,x) = U_\gamma(x)$$

which obviously depends on t (so every $t-$day the investor changes his criterion of optimality).

Time-inconsitency Literature Review (Selected)

- Dynamic inconsistent behavior was first formalized analytically by Stortz.
- Further work by Pollak, Peleg and Yaari, Goldmann on this issue advocates that the policies to be followed should be the output of an intra-personal game.
- This idea is implemented by Laibson who considers a discrete time consumption-investment economy without uncertainty.
- Krusell and Smith consider a discrete non-stochastic Ramsey paradigm with hyperbolic discounting.
- Ekeland and Lazrak looked at the Ramsey problem of economic growth with non-exponential discounting in continuous time.

Markov Strategies

- A Markov strategy is a pair $(F_1(t,x), F_2(t,x), F_3(t,x))$ of smooth functions.
- Investment, consumption and insurance are given by:

$$\zeta(t) = F_1(t, X(t)), \quad c(t) = F_2(t, X(t)), \quad p(t) = F_3(t, X(t)).$$

and the wealth is given by (SDE):

$$dX(s) = [i(s)+rX(s)+\mu F_1(s, X(s))-F_2(s, X(s))-F_3(s, X(s))]ds+\sigma F_1(s, X(s))dW(s).$$

- Moreover

$$J(t, x, F_1, F_2, F_3) = J(t, x, \zeta, c, p).$$

Equilibrium Strategies

- We say that (F_1, F_2, F_3) is an equilibrium strategy if at any time t, the investor finds that he has no incentive to change it during the infinitesimal period $[t, t+\epsilon]$.

- Definition: (F_1, F_2, F_3) is an equilibrium strategy if at any time t, for every ζ and c :

$$\lim_{\epsilon \downarrow 0} \frac{J(t, x, F_1, F_2, F_3) - J(t, x, \zeta_\epsilon, c_\epsilon, p_\epsilon)}{\epsilon} \geq 0,$$

where the process $\{\zeta_\epsilon(s), c_\epsilon(s), p_\epsilon\}_{s \in [0,T]}$ is defined by:

$$[\zeta_\epsilon(s), c_\epsilon(s), p_\epsilon(s)] = \begin{cases} [F_1(s, X(s)), F_2(s, X(s)), F_3(s, X(s)))] & 0 \leq s \leq t \\ [\zeta(s), c(s), p(s)] & t \leq s \leq t + \epsilon \\ [F_1(s, X(s)), F_2(s, X(s)), F_3(s, X(s)))] & t + \epsilon \leq s \leq T \end{cases}$$

and the equilibrium wealth process is given by the SDE:

$$dX(s) = [i(s) + rX(s) + \mu F_1(s, X(s)) - F_2(s, X(s)) - F_3(s, X(s))]ds + \sigma F_1(s, X(s))dW(s).$$

Main Result

- Theorem An equilibrium strategy is given by

$$F_1(t,x) = -\frac{\mu \frac{\partial v}{\partial x}(t,x)}{\sigma^2 \frac{\partial^2 v}{\partial x^2}(t,x)}, \;\; F_2(t,x) = \left(\frac{\partial v}{\partial x}(t,x)\right)^{\frac{1}{\gamma-1}},$$

$$F_3(t,x) = \frac{1}{l(t)}\left[\left(\frac{1}{m}\frac{\partial v}{\partial x}(t,x)\right)^{\frac{1}{\gamma-1}} - \eta(t)x\right],$$

where $m \triangleq m(0)$, and v is the value function and it satisfies the integral equation

$$v(t,x) = J(t,x,F_1,F_2,F_3).$$

- Sketch of The Proof

– In a first step the integral equation $v(t,x) = J(t,x,F_1,F_2,F_3)$ is transformed into a PDE with non-local term

$$\frac{\partial v}{\partial t}(t,x) + \left(rx + \mu F_1(t,x) - F_2(t,x) - F_3(t,x) + i(t)\right)\frac{\partial v}{\partial x}(t,x) +$$

$$\frac{\sigma^2 F_1^2(t,x)}{2}\frac{\partial^2 v}{\partial x^2}(t,x) + U_\gamma(F_2(t,x)) + mU_\gamma(x + l(t)F_3(t,x)) =$$

$$\mathbb{E}\Big[\int_t^T \frac{\partial Q}{\partial t}(s,t)U_\gamma(F_2(s,\bar{X}^{t,x}(s)))\,ds +$$

$$\int_t^T \frac{\partial q}{\partial t}(s,t)U_\gamma(\bar{Z}^{t,x}(s))\,ds + n\frac{\partial Q}{\partial t}(T,t)U_\gamma(\bar{X}^{t,x}(T))\Big]$$

– In a second step (assuming concavity of $v(t,x)$ in x) the PDE with nonlocal term can be expressed as

$$\frac{\partial v}{\partial t}(t,x) + \sup_{\zeta,c,p}\Bigg[\bigg(r+\mu\zeta - c - p + i(t)\bigg)\frac{\partial v}{\partial x}(t,x) +$$

$$\frac{1}{2}\sigma^2\zeta^2\frac{\partial^2 v}{\partial x^2}(t,x) + U_\gamma(c) + mU_\gamma(\eta(t)x + l(t)p)\Bigg] =$$

$$\mathbb{E}\Bigg[\int_t^T \frac{\partial Q}{\partial t}(s,t)U_\gamma(F_2(s,\bar{X}^{t,x}(s)))\,ds +$$

$$\int_t^T \frac{\partial q}{\partial t}(s,t)U_\gamma(\bar{Z}^{t,x}(s))\,ds + n\frac{\partial Q}{\partial t}(T,t)U_\gamma(\bar{X}^{t,x}(T))\Bigg]$$

– In a third step we show that

$$\lim_{\epsilon\downarrow 0}\frac{J(t,x,\bar{\zeta},\bar{c},\bar{p}) - J(t,x,\zeta_\epsilon,c_\epsilon,p_\epsilon)}{\epsilon} =$$

$$\Bigg[\frac{\partial v}{\partial t}(t,x) + \bigg(rx + \mu F_1(t,x) - F_2(t,x) - F_3(t,x) + i(t)\bigg)\frac{\partial v}{\partial x}(t,x) +$$

$$\frac{\sigma^2 F_1^2(t,x)}{2}\frac{\partial^2 v}{\partial x^2}(t,x) + U_\gamma(F_2(t,x))) + mU_\gamma(x + l(t)F_3(t,x))\Big] -$$

$$\left[\frac{\partial v}{\partial t}(t,x) + \Big(r + \mu\zeta(t) - c(t) - p(t) + i(t))\Big)\frac{\partial v}{\partial x}(t,x) +\right.$$

$$\left.\frac{1}{2}\sigma^2\zeta^2(t)\frac{\partial^2 v}{\partial x^2}(t,x) + U(c(t)) + mU_\gamma(x + l(t)p(t))\right] =$$

$$\frac{\partial v}{\partial t}(t,x) + \sup_{\zeta,c,p}\left[\Big(r + \mu\zeta - c - p + i(t)\Big)\frac{\partial v}{\partial x}(t,x) +\right.$$

$$\left.\frac{1}{2}\sigma^2\zeta^2\frac{\partial^2 v}{\partial x^2}(t,x) + U_\gamma(c) + mU_\gamma(\eta(t)x + l(t)p)\right] -$$

$$\frac{\partial v}{\partial t}(t,x) + \left[\Big(r + \mu\zeta(t) - c(t) - p(t) + i(t))\Big)\frac{\partial v}{\partial x}(t,x) +\right.$$

$$\left.\frac{1}{2}\sigma^2\zeta^2(t)\frac{\partial^2 v}{\partial x^2}(t,x) + U(c(t)) + mU_\gamma(x + l(t)p(t))\right] \geq 0.$$

Ansatz

- Search for v of the form

$$v(t,x) = a(t)U_\gamma(x + b(t)),$$

where the functions $a(t)$, $b(t)$ are to be found. Then

$$\begin{aligned} F_1(t,x) &= \frac{\mu(x+b(t))}{\sigma^2(1-\gamma)}, \; F_2(t,x) = [a(t)]^{\frac{1}{\gamma-1}}(x+b(t)), \\ F_3(t,x) &= \frac{1}{l(t)}\left[([\frac{a(t)}{m}]^{\frac{1}{\gamma-1}} - \eta(t))x + [\frac{a(t)}{m}]^{\frac{1}{\gamma-1}} b(t))\right]. \end{aligned}$$

- the integral equation $v(t,x) = J(t,x,F_1,F_2,F_3)$ leads to

$$b(s) = \int_s^T i(u) e^{-\int_u^s \left(r + \frac{\eta(u)}{l(u)}\right) du} ds,$$

- in the literature $b(t)$ is known as time-t value of human capital.

$$a(t)=\int_t^T [Q(s,t)+q(s,t)](a(s))^{\frac{\gamma}{\gamma-1}} e^{K(s-t)+\left(\int_t^s \frac{\gamma\eta(z)}{l(z)}-\gamma(a(z))^{\frac{1}{\gamma-1}}\left(1+\frac{1}{m^{\frac{1}{\gamma-1}}l(z)}\right)dz\right)} ds$$

$$+nQ(T,t)e^{K(T-t)+\left(\int_t^T \frac{\gamma\eta(z)}{l(z)}-\gamma(a(z))^{\frac{1}{\gamma-1}}\left(1+\frac{1}{m^{\frac{1}{\gamma-1}}l(z)}\right)dz\right)}, \qquad a(T)=n.$$

- This ODE (with nonlocal term) is shown to have a unique solution. Moreover we solve it numerically.

Logarithmic utility

- Take $U_\gamma(x) = \ln x$ (corresponding to $\gamma = 0$). In this case we get closed form solution

$$a(t) = \int_t^T [Q(s,t) + q(s,t)]\, ds + nQ(T,t),$$

$$\begin{aligned} F_1(t,x) &= \frac{\mu(x+b(t))}{\sigma^2},\ F_2(t,x) = [a(t)]^{-1}(x+b(t)), \\ F_3(t,x) &= \frac{1}{l(t)}\left[([\frac{a(t)}{m}]^{-1} - \eta(t))x + [\frac{a(t)}{m}]^{-1} b(t))\right]. \end{aligned}$$

The case of exponential discounting

- In this case the integral equation $v(t,x) = J(t,x,F_1,F_2,F_3)$ becomes the classical HJB

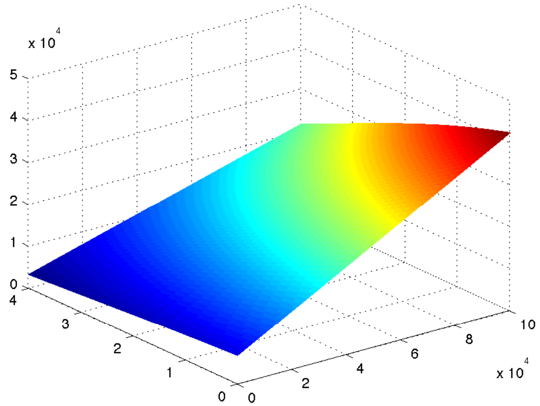
$$-(\lambda(t)+\rho)v(t,x)+\frac{\partial v}{\partial t}(t,x)+\left(rx+\mu F_1(t,x)-F_2(t,x)-F_3(t,x)+i(t)\right)\frac{\partial v}{\partial x}(t,x)+$$

$$\frac{\sigma^2 F_1^2(t,x)}{2}\frac{\partial^2 v}{\partial x^2}(t,x)+U_\gamma(F_2(t,x))+mU_\gamma(\eta(t)x+l(t)F_3(t,x)) = 0.$$

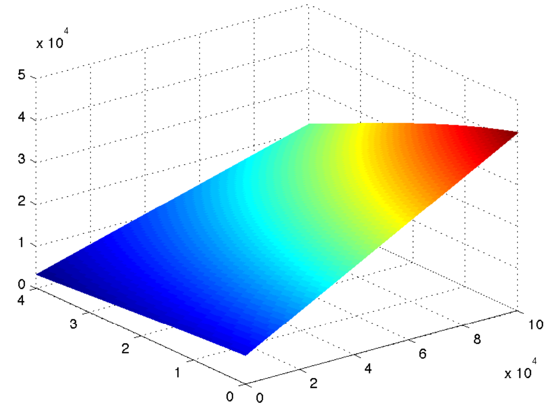
- Therefore in this case equilibrium strategy coincides with optimal strategy.

Time varying Pareto weight versus constant Pareto weight

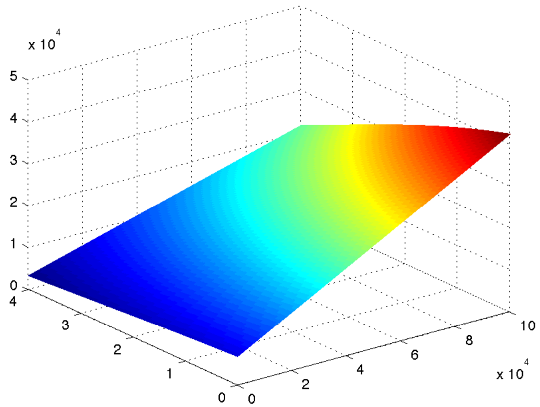
- We want to study the effect of a time varying Pareto weight on the equilibrium strategies
- Let $T = 4$, $r = 0.05$, $\mu = 0.07$, $\sigma = 0.2$, $p = -1$, $N = 1000$, $\rho = 0.8$, $\lambda(t) = \frac{1}{200} + \frac{9}{8000}t$, $l(t) = \frac{1}{\lambda(t)}$, $\eta(t) = 1$.
- The discount function is exponential $h(t) = \hat{h}(t) = \exp(-\rho t)$ with $\rho = 0.8$. The Pareto weight is $m(t) = \log(\frac{T+\epsilon-t}{\epsilon})$ with $\epsilon = 10^{-15}$.



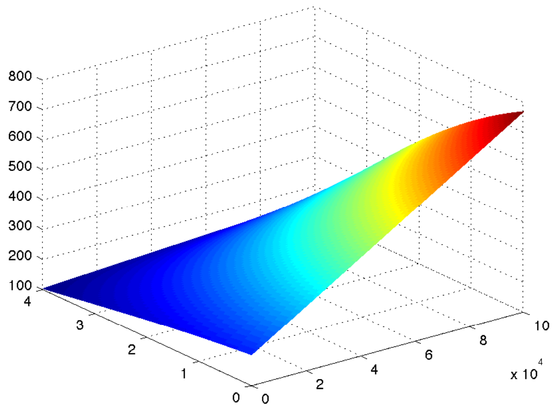
(a) $F_2(t,x)$ with m constant



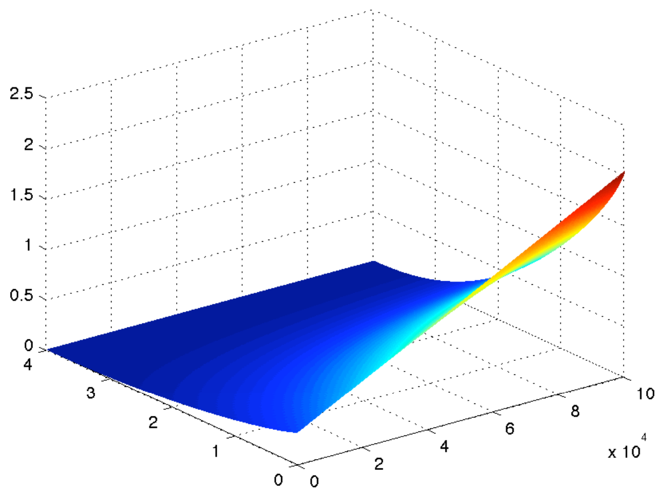
(b) $F_2(t,x)$ with m variable



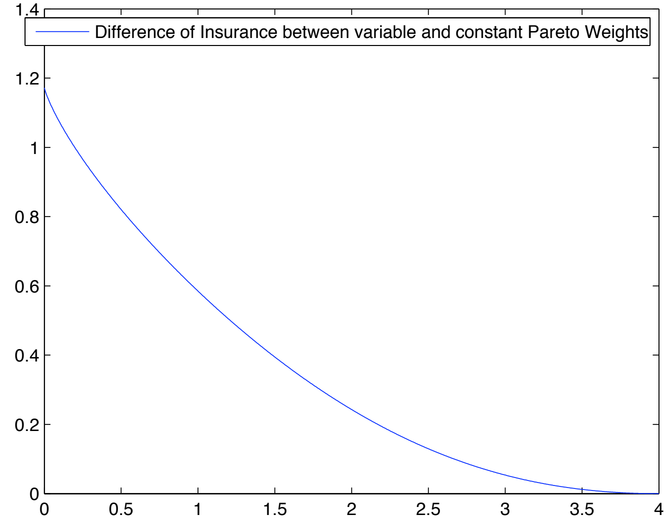
(c) $F_3(t,x)$ with m constant



(d) $F_3(t,x)$ with m variable



(e) difference in $F_3(t, x)$ for constant and variable m



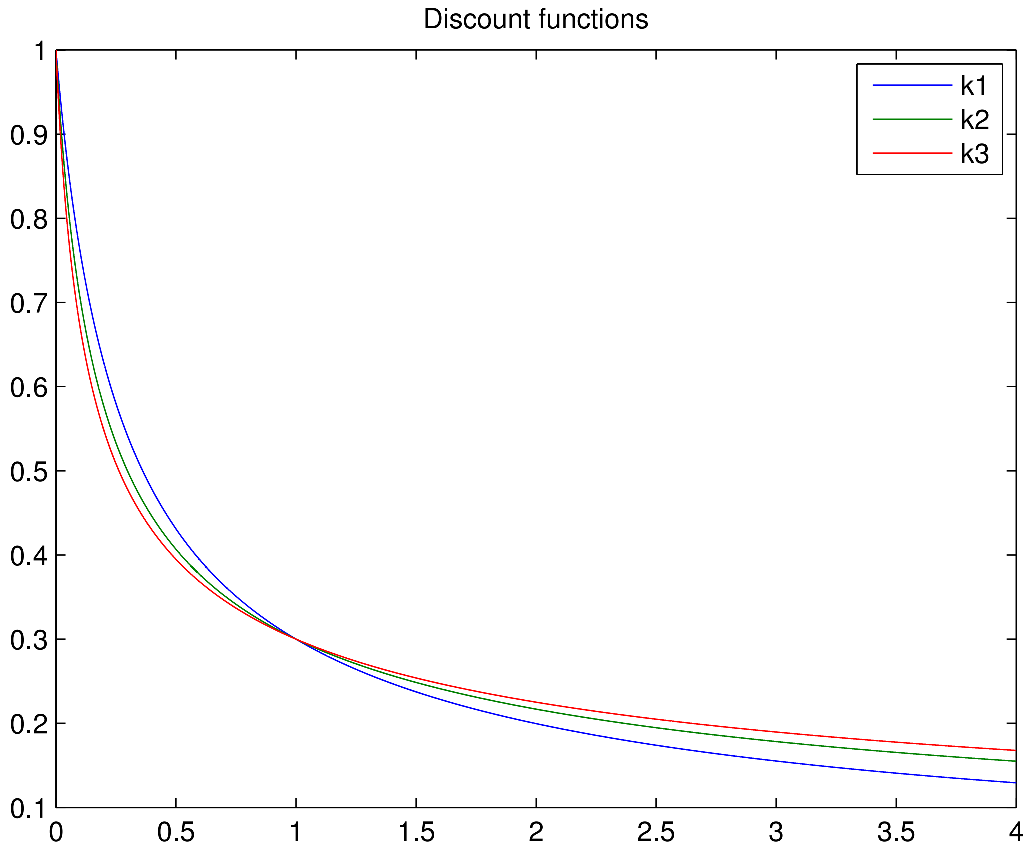
(f) section of (e) with x constant

Merton Problem with hyperbolic discounting

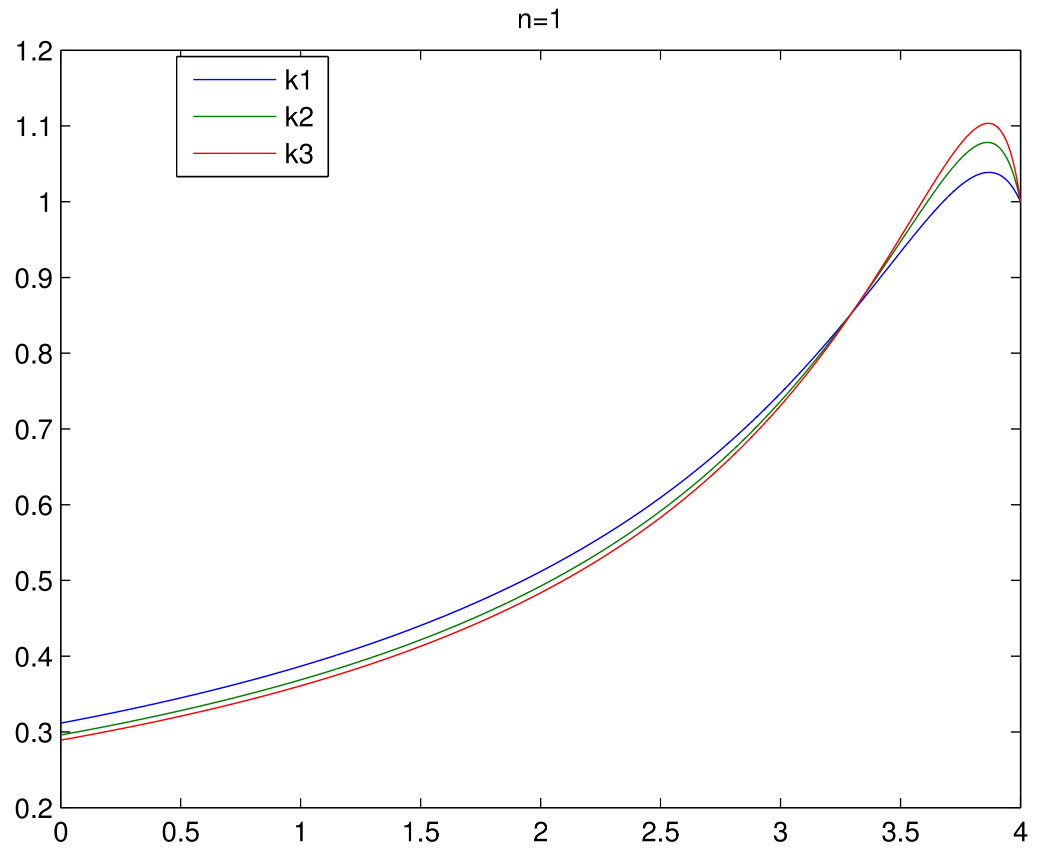
- We shut off some paramters; $m(t) = i(t) = 0$, and lifetime is deterministic T.

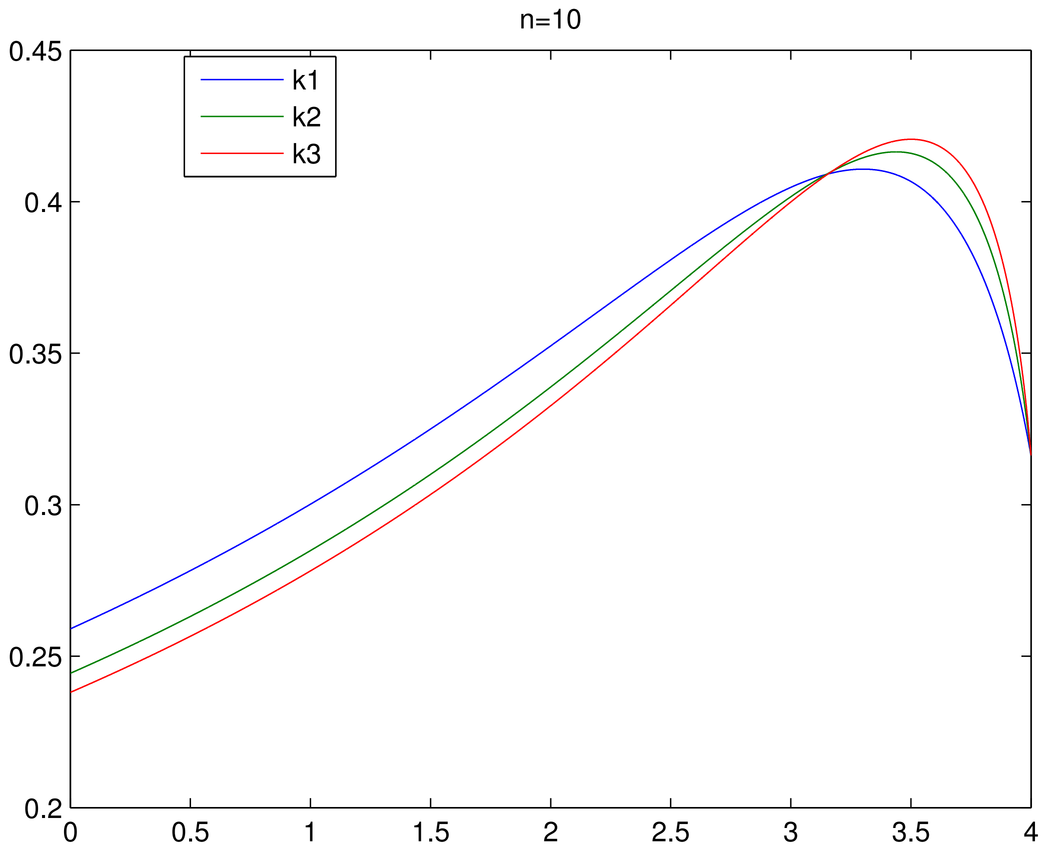
- We consider one stock following a geometric Brownian motion with drift $\alpha = 0.12$, volatility $\sigma = 0.2$, interest rate $r = 0.05$, and the horizon $T = 4$.

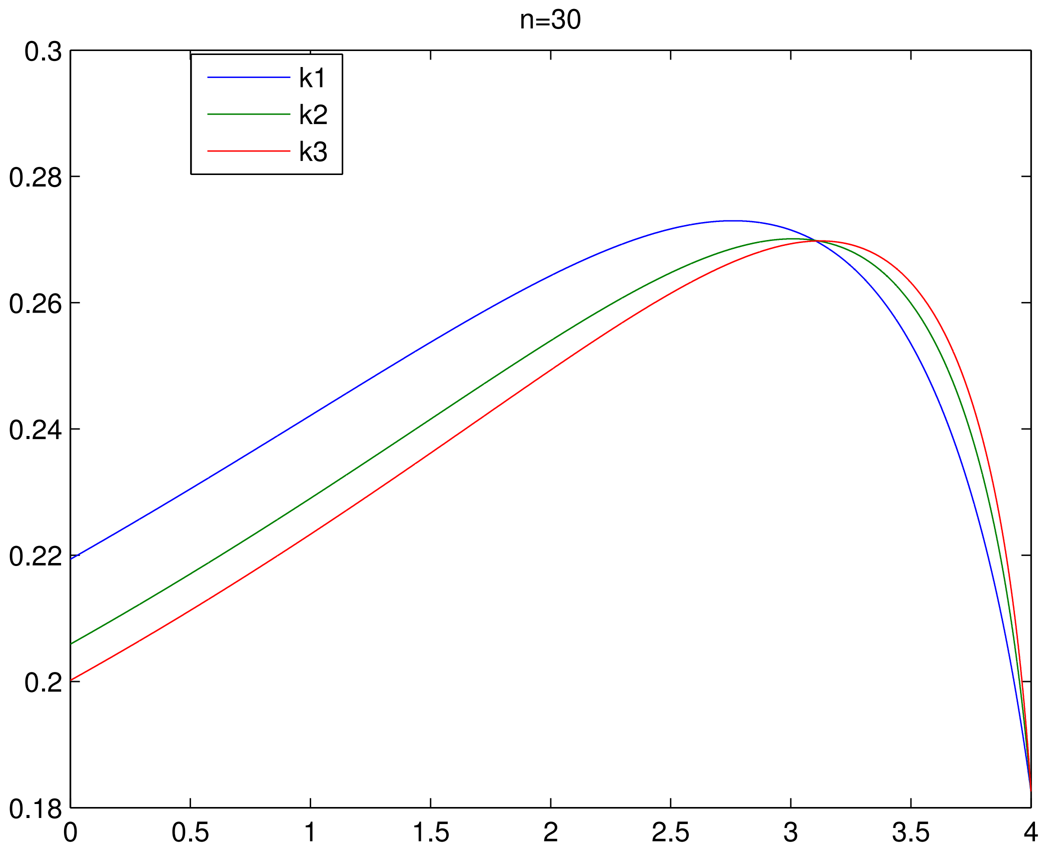
- Let the discount function $h(x) = (1 + k_j x)^{-\frac{b}{k_j}}$ be one of the three choices of hyperbolic discount: case 1. $k_1 = 5$; case 2. $k_2 = 10$; case 3. $k_3 = 15$; and b is chosen such that $h(1) = 0.3$.



- Let us graph equilibrium consumption rates


n=1
k1
k2
k3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0
0.5
1
1.5
2
2.5
3
3.5
4


n=10
k1
k2
k3
0.45
0.4
0.35
0.3
0.25
0.2
0
0.5
1
1.5
2
2.5
3
3.5
4


n=30
k1
k2
k3
0.3
0.28
0.26
0.24
0.22
0.2
0.18
0
0.5
1
1.5
2
2.5
3
3.5
4

The Household consumption Problem

- two members of a family with consumption (denoted $c_1(t)$, and $c_2(t)$) up to some time horizon T (deterministic).
- The expected utility functional is

$$J(t,x,\zeta,c_1,c_2) = \mathbb{E}\bigg[\int_t^T h_1(s-t)U_\gamma(c_1(s))\,ds$$

$$+\int_t^T m(s-t)\ h_2(s-t)U_\gamma(c_2(s))\,ds + h_1(T-t)U_\gamma(X^{\zeta,c_1,c_2}(T))\bigg],$$

- We assume that the function $m(\cdot)$ is decreasing with $m(0) \simeq \infty$, and $m(T) \simeq 0$.
- This capture the situation when one member plans for short time and the other plans for long time.

A Regime Switching Model

- The price process of the bank account and risky asset

$$dB(t) = r(t, J(t))B(t)dt,$$
$$dS(t) = S(t)\left[\alpha(t, J(t))\, dt + \sigma(t, J(t))\, dW(t)\right], \quad 0 \le t \le \infty,$$

- $r(t,i), \alpha(t,i), \sigma(t,i) : i \in \mathcal{S}$, are deterministic.

$$\mu(t,i) \triangleq \alpha(t,i) - r(t,i)$$

  stands for the stock excess return.

- the discount rate is $\rho_{J(t)}$ (it switches between two values)

$$\Theta(t,x,i,\pi,c) \triangleq \mathbb{E}_t^{x,i}\left[\int_t^T e^{-\rho_i(s-t)}U(c(s))\, ds + e^{-\rho_i(T-t)}U(X^u(T))\right].$$

Let $F = (F_1, F_2) : [0,T] \times^{+} \times \mathcal{S} \to {}^{+} \times \mathcal{S}$ be a map such that for any $t, x > 0$ and $i \in \mathcal{S}$

$$\liminf_{\epsilon \downarrow 0} \frac{\Theta(t,x,i,F_1,F_2) - \Theta(t,x,i,\pi_\epsilon,c_\epsilon)}{\epsilon} \geq 0,$$

where

$$\Theta(t,x,i,F_1,F_2) \triangleq \Theta(t,x,i,\bar{\pi},\bar{c}),$$

$$\bar{\pi}(s) \triangleq F_1(s,\bar{X}(s),J(s)), \quad \bar{c}(s) \triangleq F_2(s,\bar{X}(s),J(s)).$$

- The Value Function

$$v(t,x,i) \triangleq \mathbb{E}_t^{x,i}\left[\int_t^T e^{-\rho_i(s-t)} U(F_2(s,\bar{X}(s),J(s)))\,ds + e^{-\rho_i(T-t)} U(\bar{X}(T))\right].$$

$$\begin{aligned} d\bar{X}(s) &= [r(s,J(s))\bar{X}(s) + \mu(s,J(s))F_1(s,\bar{X}(s),J(s)) - F_2(s,\bar{X}(s),J(s))]ds \\ &+ \sigma(s,J(s))F_1(s,\bar{X}(s))dW(s). \end{aligned}$$

$$F_1(t,x,i) \triangleq -\frac{\mu(t,i)\frac{\partial v}{\partial x}(t,x,i)}{\sigma^2(t,i)\frac{\partial^2 v}{\partial x^2}(t,x,i)},\ F_2(t,x,i) \triangleq I\left(\frac{\partial v}{\partial x}(t,x,i)\right),\ \ t \in [0,T].$$

- CRRA Preferences $U(x) = U_\gamma(x) = \frac{x^\gamma}{\gamma}$.

$$v(t,x,i) = g(t,i)\frac{x^\gamma}{\gamma}, \quad x \geq 0$$

$$F_1(t,x,i) = \frac{\mu(t,i)x}{\sigma^2(t,i)(1-\gamma)}$$

$$F_2(t,x,i) = g^{\frac{1}{\gamma-1}}(t,i)x$$

- Here $g(t,i), \bar{g}(t,i),\ i \in \mathcal{S}$: solve

$$\frac{\partial g}{\partial t}(t,i)+[\gamma r(t,i)+\frac{\mu^2(t,i)\gamma}{2\sigma^2(t,i)(1-\gamma)}-\rho_i]g(t,i)+\lambda_{ii}g(t,i)+(1-\gamma)g^{\frac{\gamma}{\gamma-1}}(t,i) = -\lambda_{ij}\bar{g}(t,j)$$

$$\frac{\partial \bar{g}}{\partial t}(t,i)+[\gamma r(t,i)+\frac{\mu^2(t,i)\gamma}{2\sigma^2(t,i)(1-\gamma)}-\rho_j]\bar{g}(t,i)+\lambda_{ii}\bar{g}(t,i)+(1-\gamma)g^{\frac{1}{\gamma-1}}(t,i)\bar{g}(t,i) = -\lambda_{ij}g(t,j)$$

with $g(T,i) = \bar{g}(T,i) = 1$, $i \in \mathcal{S}$.

- There are three important trading strategies:
- the subgame perfect strategy $\{\bar{\pi}(s), \bar{c}(s)\}_{s\in[0,T]}$

$$\bar{\pi}(s) = \frac{\mu(s,J(s))\bar{X}(s)}{\sigma^2(s,J(s))(1-\gamma)}, \quad \bar{c}(s) = g^{\frac{1}{\gamma-1}}(s,J(s))\bar{X}(s),$$

- two pre-commitment strategies corresponding to the two discount rates $\rho_{\mathbf{0}}, \rho_{\mathbf{1}}$ which are optimal at time 0, but fail to remain optimal afterwards

$$\hat{\pi}_k(s) = \frac{\mu(s,J(s))\hat{X}_k(s)}{\sigma^2(s,J(s))(1-\gamma)}, \;\; \hat{c}_k(s) = \hat{g}_k^{\frac{1}{\gamma-1}}(s,J(s))\hat{X}_k(s), \;\; k = 1,2;$$

here $\hat{g}_k(t,i)$, $i,j \in \mathcal{S}$, $k = 1,2$ solve:

$$\frac{\partial \hat{g}_1}{\partial t}(t,i) + [\gamma r(t,i) + \frac{\mu^2(t,i)\gamma}{2\sigma^2(t,i)(1-\gamma)} - \rho_{\mathbf{0}}]\hat{g}_1(t,i) + \lambda_{ii}\hat{g}_1(t,i) + (1-\gamma)\hat{g}_1^{\frac{\gamma}{\gamma-1}}(t,i) = -\lambda_{ij}\hat{g}_1(t,j)$$

$$\frac{\partial \hat{g}_2}{\partial t}(t,i)+[\gamma r(t,i)+\frac{\mu^2(t,i)\gamma}{2\sigma^2(t,i)(1-\gamma)}-\rho_{\mathbf{1}}]\hat{g}_2(t,i)+\lambda_{ii}\hat{g}_2(t,i)+(1-\gamma)\hat{g}_2^{\frac{\gamma}{\gamma-1}}(t,i)=-\lambda_{ij}\hat{g}_2(t,j)$$

with $\hat{g}_1(T,i)=\hat{g}_1(T,i),\ i\in\mathcal{S}$.

- a naive strategy that switches in between the two pre-commitment strategies.
- Numerics. We plot the subgame perfect and pre-commitment consumption rates denoted by

$$\bar{C}(t,J(t))\triangleq\frac{F_2(t,\bar{X}(t),J(t))}{\bar{X}(t)}=g^{\frac{1}{\gamma-1}}(t,J(t)),$$

$$\hat{C}_k(t,J(t))\triangleq\frac{\hat{c}_k(t)}{\hat{X}_k(t)}=\hat{g}_k^{\frac{1}{\gamma-1}}(t,J(t)),\ \ k=1,2.$$

- Subgame perfect and pre-commitment consumption rates for $\mu_{\mathbf{0}}=0.1,\mu_{\mathbf{1}}=0.1,\sigma_{\mathbf{0}}=0.2,\sigma_{\mathbf{1}}=0.2,r_{\mathbf{0}}=0.05,r_{\mathbf{1}}=0.05$; the discount rates $\rho_{\mathbf{0}}=0.3,\rho_{\mathbf{1}}=0.06$.

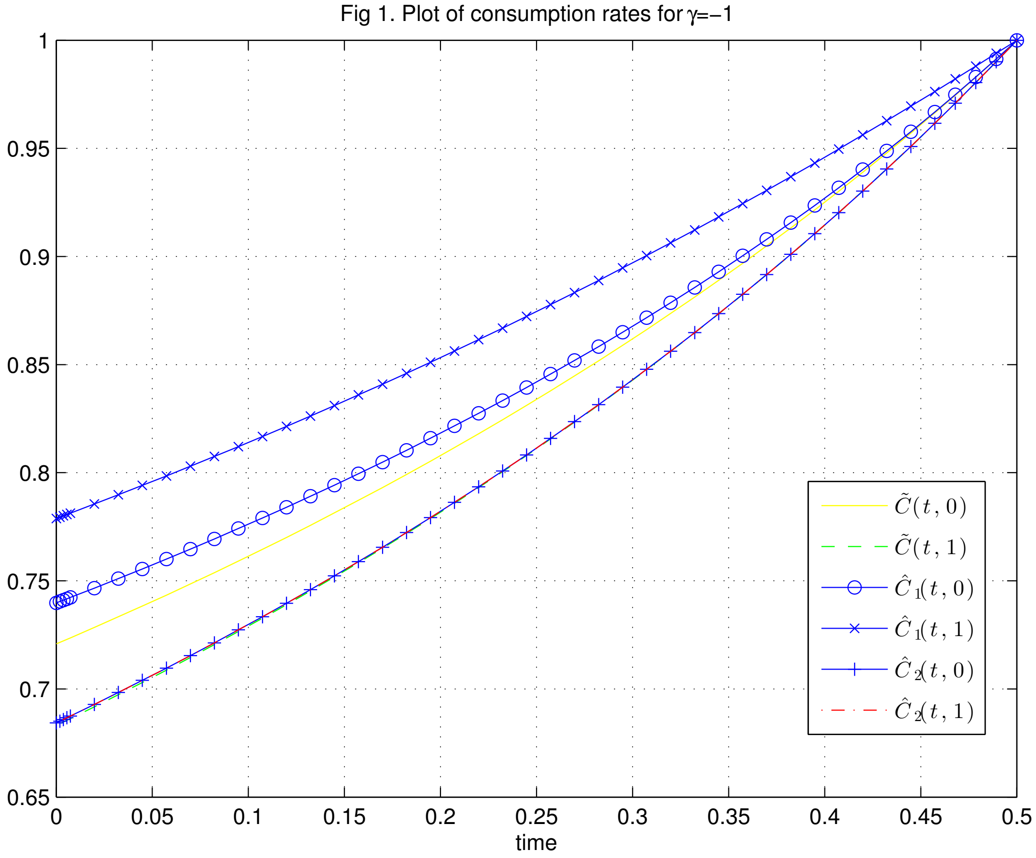


Fig 1. Plot of consumption rates for γ=−1
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
time
$\tilde{C}(t,0)$
$\tilde{C}(t,1)$
$\hat{C}_1(t,0)$
$\hat{C}_1(t,1)$
$\hat{C}_2(t,0)$
$\hat{C}_2(t,1)$

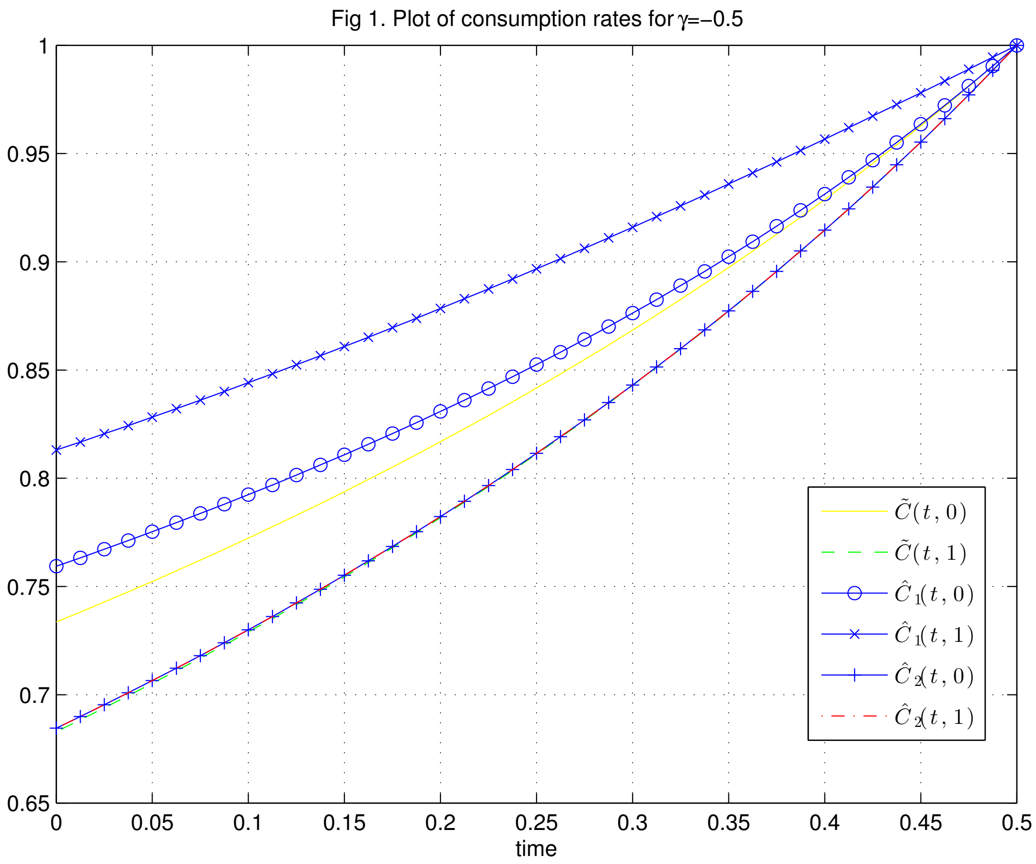


Fig 1. Plot of consumption rates for γ=−0.5
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
time
$\tilde{C}(t,0)$
$\tilde{C}(t,1)$
$\hat{C}_1(t,0)$
$\hat{C}_1(t,1)$
$\hat{C}_2(t,0)$
$\hat{C}_2(t,1)$

Fig 1. Plot of consumption rates for γ=0

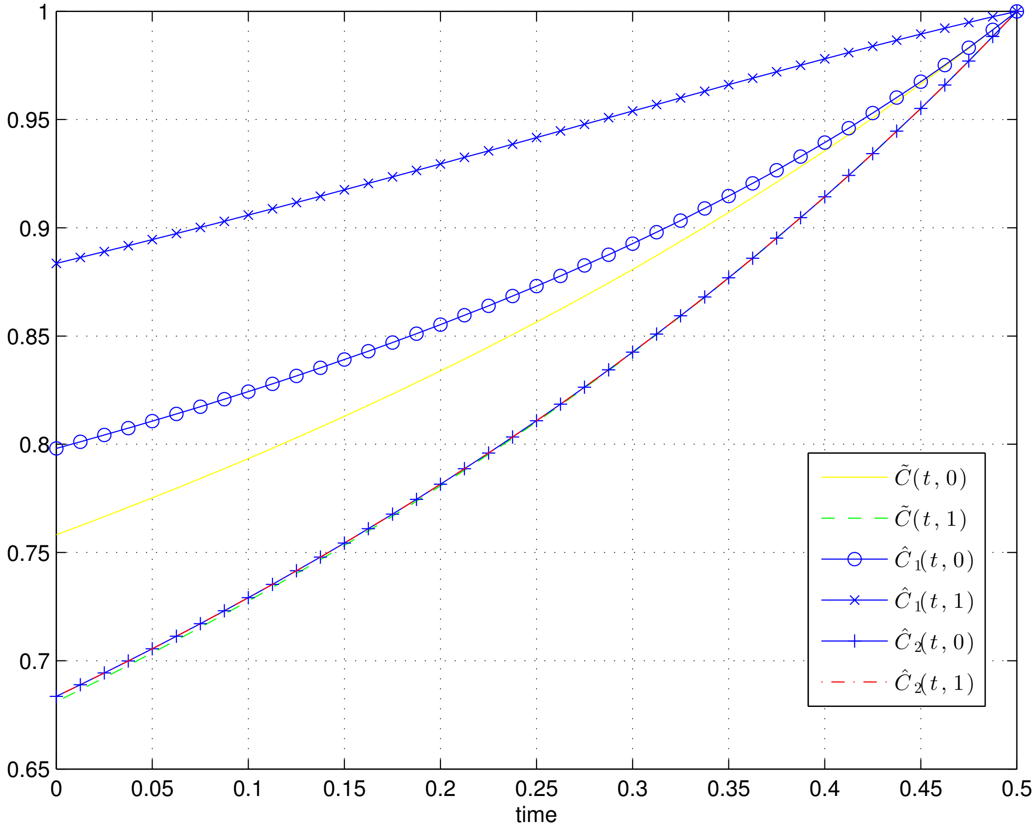
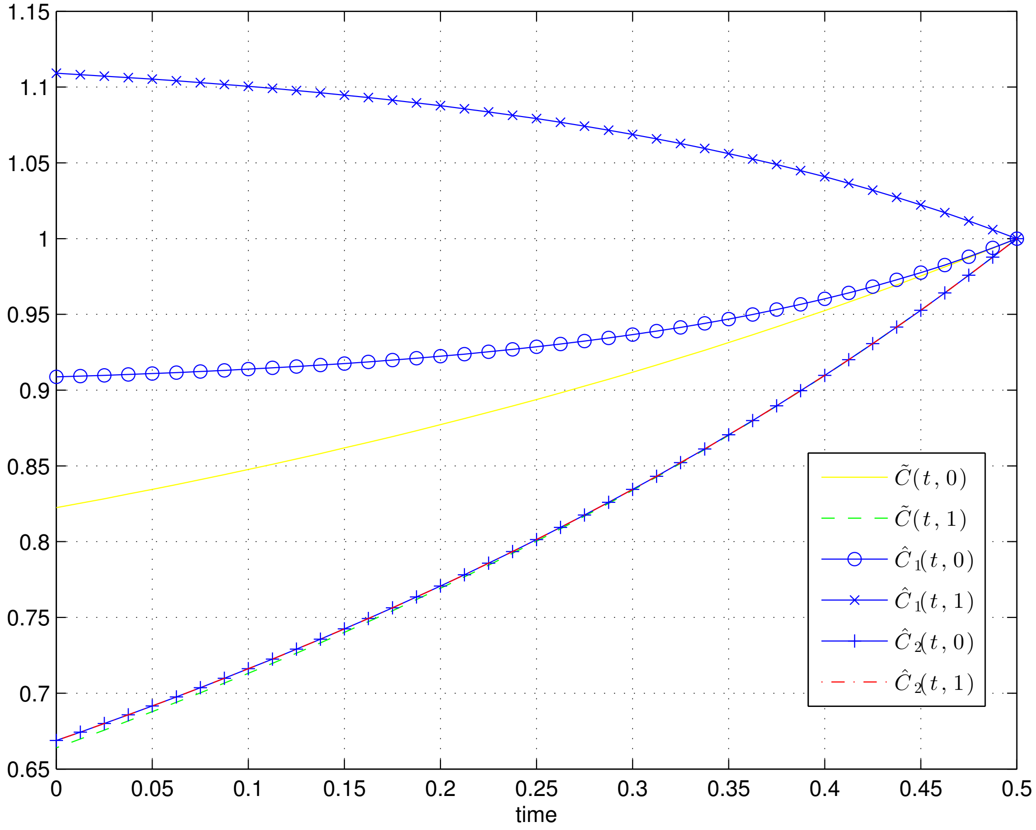
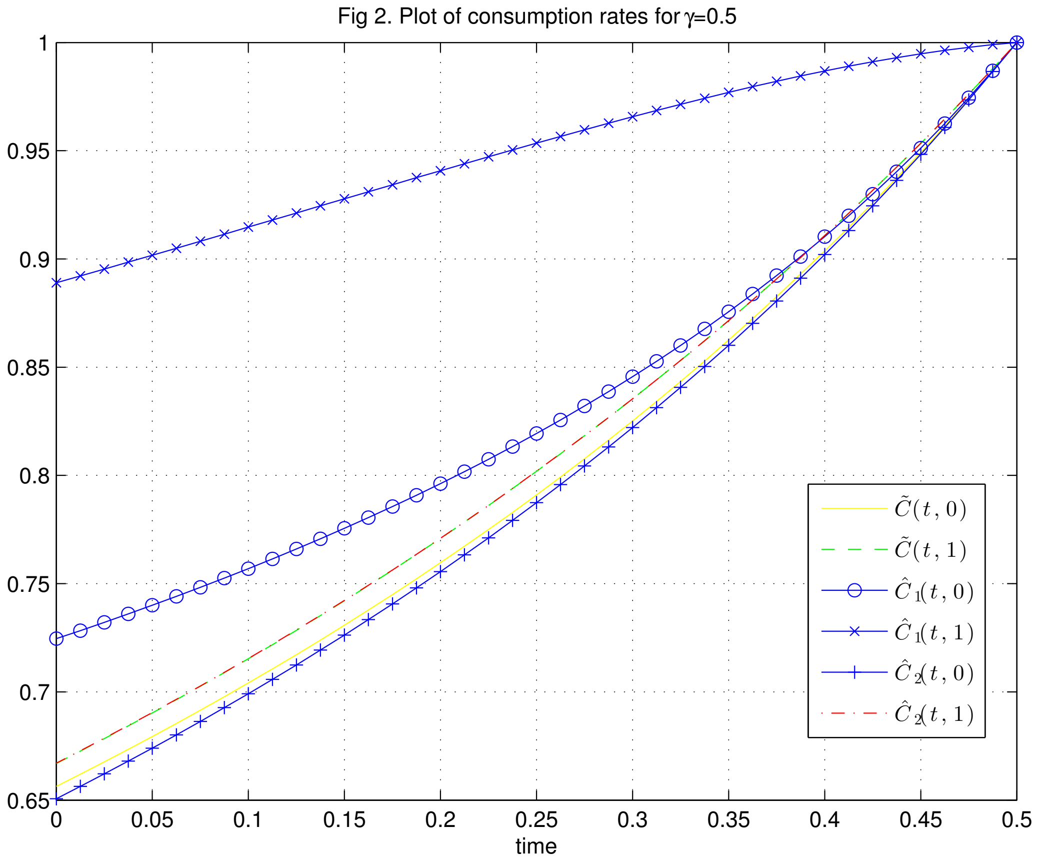


Fig 1. Plot of consumption rates for γ=0.5



- Subgame perfect and pre-commitment consumption rates for $\mu_0 = 0.1, \mu_1 = 0.1, \sigma_0 = 0.2, \sigma_1 = 0.2, r_0 = 0.01, r_1 = 0.09$; the discount rates $\rho_{\mathbf{0}} = 0.07, \rho_{\mathbf{1}} = 0.06.$



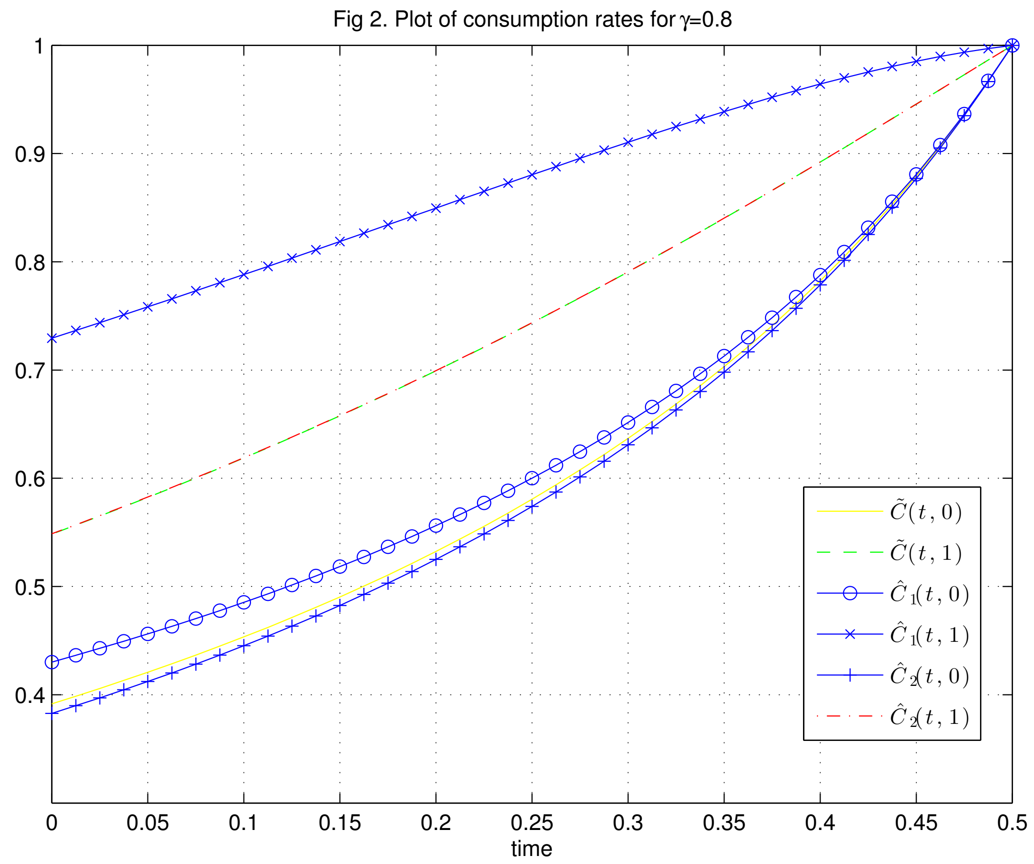


Fig 2. Plot of consumption rates for γ=0.8
1
0.9
0.8
0.7
0.6
0.5
0.4
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
time
$\tilde{C}(t,0)$
$\tilde{C}(t,1)$
$\hat{C}_1(t,0)$
$\hat{C}_1(t,1)$
$\hat{C}_2(t,0)$
$\hat{C}_2(t,1)$

The End!

Thank You!