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TIME-CORRELATION FUNCTIONS AND CRITICAL RELAXATION IN A CLASS OF ONE-DIMENSIONAL STOCHASTIC SPIN SYSTEMS

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Synopsis

The kinetics of the spin $\frac{1}{2}$ Ising chain is studied for a class of master equations describing transitions in the spin system owing to interactions with a heat bath. The case in which only transitions of n successive spins are allowed is called the n-flip model. The time-correlation functions for energy density and magnetization are calculated exactly in the limit of an infinite chain for all values of n. The effect of critical slowing-down near temperature $T = 0$ is studied and it is shown that the critical exponents characterizing the divergence of the relaxation times of energy and magnetization are independent of n.

1. *Introduction.* In equilibrium statistical mechanics the theory of Ising spin systems has led to a wealth of interesting results. The study of the non-equilibrium statistics of Ising systems may likewise be highly instructive. As a starting point it seems advisable to take a modest point of view and to relax the requirement of a fully microscopic, quantum-mechanical treatment of an Ising system with spin-flipping interactions. Instead one may use as a result of non-equilibrium statistical mechanics that in many cases the behaviour in time of large systems may in good approximation be described by a (markoffian) master equation. Thus for the Ising system we postulate that, owing to the interaction with a heat bath, spin configurations may change in the course of time with certain transition probabilities per unit time.

Glauber[1]) has introduced a model in which the only allowed transitions are those in which individual spins flip, with probability depending on the temperature of the heat bath and on the state of the neighbouring spins. For the one-dimensional model he was able to solve exactly for the expectation value of individual spins and of the products of pairs of spins[2]).

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Recently Bedeaux, Shuler and Oppenheim[3]) have solved the equations of motion for a set of correlation functions involving an arbitrary number of spins. One of us (B.U.F.) has given an alternative complete solution of the master equation for this model[4]).

The Glauber model is not exactly soluble in dimensions higher than 1, but various approximate methods have been applied, in particular with a view of studying the behaviour near the critical point[5]). Yahata and Suzuki[6]) have made numerical studies for both 2 and 3 dimensions of exact high-temperature series expansions of the time-correlation functions of energy and magnetization. These studies have shown that the slowing down of the relaxation near the critical point is non-classical and have confirmed conjectures concerning the critical exponents characterizing the divergence of the relaxation times. According to these conjectures the critical exponents are independent of the details of the kinetics and the equilibrium statistics, and are sensitive only to drastic changes, such as a change of dimensionality, change of symmetry or conservation laws, or change from short-range to long-range interactions. Thus for Ising systems with nearest-neighbour interactions in 2 and 3 dimensions the critical exponents have been shown to be independent of lattice structure.

In the present paper we study the Ising chain with nearest-neighbour interactions in the absence of a magnetic field for a class of relaxation models. In each model the only allowed transitions are those in which n neighbouring spins flip simultaneously. The time-correlation functions of the Fourier components of energy density and magnetization are calculated exactly in the thermodynamic limit for all values of n. It is shown that the critical exponents characterizing the divergence of the relaxation times near temperature $T = 0$, which thermodynamically acts as a critical point[7]), are independent of n, thus confirming the above conjecture.

In section 2 the models are described in more detail, and a mathematically convenient formulation is given in section 3. In section 4 a class of transformations of Pauli spin operators to new sets of Pauli operators is presented. In section 5 it is shown that the so-called $\sigma\tau$ transformation of order n may be used to reduce the n-flip model to a collection of n statistically independent 1-flip models. This decomposition is exact in the thermodynamic limit of an infinite number of spins. In section 6 the time-correlation function for Fourier components of the energy density is calculated explicitly, and in section 7 a similar calculation is made for the magnetization. In section 8 the relaxation times for energy and magnetization are studied near $T = 0$ and are shown to exhibit slowing-down with critical exponents independent of n. In section 9 we present some conclusions. In appendix A it is shown by a $\sigma\tau$ transformation of order 2 that the master operator of the 1-flip model is essentially equivalent to the hamiltonian of the XY model. Appendix B contains some computational details.

2. *Description of the model.* We consider a one-dimensional lattice of N spins $\frac{1}{2}$. A microstate of the lattice is specified by the set of N quantum numbers $\boldsymbol{s}_N \equiv (s_1, s_2, \ldots, s_N)$ where s_j can take the values $+1$ and -1. For a cyclic Ising chain with nearest-neighbour interactions an energy

$$E(\boldsymbol{s}_N) = -J \sum_{j=1}^{N} s_j s_{j+1}, \tag{2.1}$$

where $s_{N+1} \equiv s_1$, is associated with each microstate. Thermal equilibrium is characterized by a probability distribution $P_{\mathrm{eq}}(\boldsymbol{s}_N)$ on the set of microstates,

$$P_{\mathrm{eq}}(\boldsymbol{s}_N) = \exp[-\beta E(\boldsymbol{s}_N)]/Z_N(\beta), \tag{2.2}$$

where $\beta = 1/k_{\mathrm{B}}T$ and

$$Z_N(\beta) = \sum_{\boldsymbol{s}_N} \exp[-\beta E(\boldsymbol{s}_N)], \tag{2.3}$$

is the canonical partition function. Non-equilibrium states of the system are specified by a probability distribution $P(\boldsymbol{s}_N, t)$. We postulate that the approach to thermal equilibrium owing to the interactions with a heat bath is governed by a master equation

$$\frac{\partial P(\boldsymbol{s}_N, t)}{\partial t} = \sum_{\boldsymbol{s}_N'} [W(\boldsymbol{s}_N, \boldsymbol{s}_N')\, P(\boldsymbol{s}_N', t) - W(\boldsymbol{s}_N', \boldsymbol{s}_N)\, P(\boldsymbol{s}_N, t)]. \tag{2.4}$$

Glauber[1]) has proposed a model in which the transition probabilities $W(\boldsymbol{s}_N', \boldsymbol{s}_N)$ describe transitions of individual spins with probability per unit time depending on the state of the neighbours of the flipping spin and on temperature. We shall refer to this model as the 1-flip model. In the present paper we discuss more generally the n-flip model for which in each transition n neighbouring spins simultaneously change their state. We may write

$$W_n(\boldsymbol{s}_N', \boldsymbol{s}_N) = \sum_{j=1}^{N} w_j(n), \tag{2.5}$$

where $w_j(n)$ refers to the transition in which the spins $j, j+1, \ldots, j+n-1$ flip. We assume uniformity throughout the chain so that $w_j(n)$ in magnitude is independent of j. In order to be consistent with the thermal equilibrium distribution (2.2) we assume in addition that the transition probabilities satisfy the detailed balance relation

$$W(\boldsymbol{s}_N, \boldsymbol{s}_N')\, P_{\mathrm{eq}}(\boldsymbol{s}_N') = W(\boldsymbol{s}_N', \boldsymbol{s}_N)\, P_{\mathrm{eq}}(\boldsymbol{s}_N). \tag{2.6}$$

Arguing in the same manner as in the 1-flip case[1]) one finds that (2.6) may be satisfied by

$$w_j(n) = \tfrac{1}{2}\alpha_n[1 - \tfrac{1}{2}\gamma(s_{j-1}s_j + s_{j+n-1}s_{j+n})], \tag{2.7}$$

with

$$\gamma = \tanh 2K, \qquad K = \beta J. \tag{2.8}$$

The factor α_n determines the overall transition rate in the n-flip model.

In the sequel we wish to compare various relaxation times for different temperatures of the heat bath. For this reason it is necessary to specify the temperature dependence of α_n on the basis of a physical model. There are three types of transitions, namely transitions in which the spin system absorbs an energy $4J$, transitions in which it loses an energy $4J$, and transitions in which the energy is conserved. We assume that the energy-non-conserving transitions occur due to interactions with a heat bath of phonons (or photons) and that only single phonons are absorbed or emitted. The transition probability for spontaneous emission of a phonon is independent of temperature, say of magnitude

$$|w_j^{\mathrm{sp}}(n)| = A_n, \tag{2.9}$$

whereas the probabilities for stimulated emission and absorption are given by

$$|w_j^{\mathrm{st}}(n)| = |w_j^{\mathrm{abs}}(n)| = A_n(\mathrm{e}^{4\beta J} - 1)^{-1}, \tag{2.10}$$

where the last factor is the average occupation of phonon states with energy $4J$, as given by the Planck distribution. The expressions (2.9) and (2.10) are consistent with (2.7) if we put

$$\alpha_n = A_n/\gamma. \tag{2.11}$$

From (2.7) it follows that the energy-conserving transitions have a probability $\frac{1}{2}\alpha_n$ per unit time. This is an arbitrary limitation of generality with the sole purpose of making the model soluble. In the limit $T \to 0$ the energy-conserving transition probability tends to $\frac{1}{2}A_n$, whereas the probabilities for stimulated emission and absorption tend to zero, with the limiting behaviour conveniently expressed as

$$|w_j^{\mathrm{st}}(n)| = |w_j^{\mathrm{abs}}(n)| = A_n\varepsilon^2 + \mathcal{O}(\varepsilon^4), \tag{2.12}$$

where $\varepsilon = \exp[-2\beta J]$.

3. *Mathematical formulation.* It is mathematically convenient to represent a microstate $\boldsymbol{s}_N$ as a direct product of N spinors

$$\begin{aligned} \alpha_i &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i \qquad \text{for} \qquad s_i = +1, \\ \beta_i &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}_i \qquad \text{for} \qquad s_i = -1. \end{aligned} \tag{3.1}$$

The direct product will be denoted by $|\boldsymbol{s}_N\rangle$ and will be regarded as a basis

vector in a 2^N-dimensional Hilbert space $\mathfrak{H}_N$ endowed with the usual scalar product. The vectors $|s_N\rangle$ satisfy the orthonormality relation

$$\langle s_N | s'_N \rangle = \delta(s_N - s'_N), \tag{3.2}$$

where $\delta(s_N - s'_N)$ is shorthand for a product of N Kronecker deltas. A probability distribution $P(s_N, t)$ may be represented uniquely by a time-dependent vector in $\mathfrak{H}_N$,

$$|P(t)\rangle = \sum_{s_N} P(s_N, t) \, |s_N\rangle. \tag{3.3}$$

It is convenient to introduce the unit state $|1\rangle$ by

$$\begin{aligned} |1\rangle &= \sum_{s_N} |s_N\rangle \\ &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}_1 \times \begin{pmatrix} 1 \\ 1 \end{pmatrix}_2 \times \cdots \times \begin{pmatrix} 1 \\ 1 \end{pmatrix}_N . \end{aligned} \tag{3.4}$$

The norm of this state is

$$\langle 1 | 1 \rangle = 2^N. \tag{3.5}$$

The normalization of the probability distribution $P(s_N, t)$ is expressed by

$$\langle 1 | P(t) \rangle = 1. \tag{3.6}$$

The master equation (2.4) corresponds to an equation of motion for the vector $|P(t)\rangle$ which may be written

$$\frac{\partial}{\partial t} |P(t)\rangle = \mathscr{W} |P(t)\rangle, \tag{3.7}$$

where $\mathscr{W}$ is a linear operator in $\mathfrak{H}_N$. For the n-flip model this operator is given explicitly by

$$\begin{aligned} \mathscr{W}_n = \tfrac{1}{2}\alpha_n \sum_{j=1}^{N} \{ &[1 + \tfrac{1}{2}\gamma(\sigma_{j-1}^z \sigma_j^z + \sigma_{j+n-1}^z \sigma_{j+n}^z)] \sigma_j^x \sigma_{j+1}^x \cdots \sigma_{j+n-1}^x \\ &- [1 - \tfrac{1}{2}\gamma(\sigma_{j-1}^z \sigma_j^z + \sigma_{j+n-1}^z \sigma_{j+n}^z)] \}, \end{aligned} \tag{3.8}$$

where σ_j^x and σ_j^z act as Pauli spin operators on the jth spinor and as unit operators on all other spinors. (We use a cyclic definition for $j = 0$ and $j > N$.)

In the sequel we shall be interested in the calculation of equilibrium time-correlation functions defined by

$$\langle A(t)_n \, B \rangle_{\mathrm{eq}} = \sum_{s_N} \sum_{s_{N'}} A(s'_N) \, B(s_N) \, P_n(s'_N | s_N, t) \, P_{\mathrm{eq}}(s_N), \tag{3.9}$$

where $P_n(s'_N | s_N, t)$ is the conditional probability for the n-flip model, *i.e.*, the solution of the master equation with initial condition

$$P_n(s'_N | s_N, 0) = \delta(s'_N - s_N). \tag{3.10}$$

In the present formulation (3.9) may be written

$$\langle A(t)_n B\rangle_{\mathrm{eq}} = \langle 1\,|A^z\,\mathrm{e}^{\mathscr{W}_n t} B^z|\,P_{\mathrm{eq}}\rangle, \tag{3.11}$$

where A^z and B^z are operators diagonal in the σ^z representation with eigenvalues $A(\boldsymbol{s}_N)$ and $B(\boldsymbol{s}_N)$, respectively. A more symmetric formulation may be found by introducing the density operator

$$\rho_{\mathrm{eq}} = \exp[-\beta\mathscr{H}_N]/Z_N(\beta), \tag{3.12}$$

where $\mathscr{H}_N$ is the Ising hamiltonian

$$\mathscr{H}_N = -J\sum_{j=1}^{N}\sigma_j^z\sigma_{j+1}^z. \tag{3.13}$$

We then have

$$|P_{\mathrm{eq}}\rangle = \rho_{\mathrm{eq}}\,|1\rangle. \tag{3.14}$$

Defining the "vacuum" state $|0\rangle$ by

$$|0\rangle = \rho_{\mathrm{eq}}^{\frac{1}{2}}\,|1\rangle, \tag{3.15}$$

we may write

$$\begin{aligned}\langle A(t)_n B\rangle_{\mathrm{eq}} &= \langle 1\,|A^z\,\mathrm{e}^{\mathscr{W}_n t} B^z\rho_{\mathrm{eq}}|\,1\rangle \\ &= \langle 0\,|A^z\,\mathrm{e}^{\mathscr{W}_n(\beta)t} B^z|\,0\rangle,\end{aligned} \tag{3.16}$$

where we have used that ρ_{eq} commutes with the operators A^z and B^z and where $\mathscr{W}_n(\beta)$ is defined by

$$\mathscr{W}_n(\beta) = \rho_{\mathrm{eq}}^{-\frac{1}{2}}\mathscr{W}_n\rho_{\mathrm{eq}}^{\frac{1}{2}}. \tag{3.17}$$

From the detailed-balance relation (2.6) it follows that in the σ^z representation $\mathscr{W}_n(\beta)$ is a symmetric real operator. From the fact that the thermal equilibrium distribution is a static solution of (2.4) it follows that $|P_{\mathrm{eq}}\rangle$ is an eigenvector of $\mathscr{W}_n$ with eigenvalue 0, or equivalently

$$\mathscr{W}_n(\beta)\,|0\rangle = 0. \tag{3.18}$$

Hence we may also write

$$\langle A(t)_n B\rangle_{\mathrm{eq}} = \langle 0\,|A^z(t)_n B^z|\,0\rangle, \tag{3.19}$$

where

$$A^z(t)_n = \mathrm{e}^{-\mathscr{W}_n(\beta)t} A^z\,\mathrm{e}^{\mathscr{W}_n(\beta)t}. \tag{3.20}$$

Thus we may evaluate time-correlation functions as vacuum expectation values of products of operators whose time evolution is determined by a Bloch-like equation. We shall be interested mainly in the correlation functions of energy and magnetization, in particular as regards their behaviour near $T = 0$, which in many respects acts as a critical point.

4. *The $\sigma\tau$ transformation.* In I the complete solution of the master equation for the 1-flip Glauber model has been given by diagonalizing the operator $\mathscr{W}_1(\beta)$. A similar diagonalization of the operator $\mathscr{W}_n(\beta)$ for the general n-flip model does not seem feasible. We have, however, found a transformation which shows that the n-flip model is macroscopically equivalent to n independent 1-flip models. By "macroscopically equivalent" we mean that except for boundary effects the systems have identical properties. Thus in the thermodynamic limit $N \to \infty$ correlation functions of bulk properties, such as the total energy, the total magnetization or their Fourier components, may be calculated exactly.

We introduce a class of transformations, each of which transforms a set of $3N$ Pauli-spin operators $(\sigma_j^x, \sigma_j^y, \sigma_j^z;\ j = 1, \ldots, N)$ into another set of $3N$ Pauli-spin operators $(\tau_j^x, \tau_j^y, \tau_j^z;\ j = 1, \ldots, N)$. A member of the class will be called the *$\sigma\tau$ transformation of order n*, where n runs through the positive integers. For order n it is convenient to subdivide the lattice into cells each containing n spins (except for the last cell which may have less than n spins). Correspondingly we enumerate the spins as follows

$$j = (m - 1)\, n + l,$$

where (4.1)

$$l = 1, 2, \ldots, n, \qquad m = 1, 2, \ldots, M.$$

Thus m indicates the lattice cell to which the spin belongs and l indicates the location in the cell. Instead of the subscript j we use the pair (m, l). The $\sigma\tau$ transformation of order n reads

$$\begin{aligned}
\tau_{m,l}^x &= (\sigma_{m,l}^x \sigma_{m,l+1}^x \cdots \sigma_{m,n}^x \sigma_{m+1,1}^x \cdots \sigma_{m+1,l-1}^x),\\
\tau_{m,l}^y &= (\sigma_{1,l-1}^z \sigma_{1,l}^z)(\sigma_{2,l-1}^z \sigma_{2,l}^z) \cdots (\sigma_{m,l-1}^z \sigma_{m,l}^y \sigma_{m,l+1}^x \sigma_{m,l+2}^x \cdots \sigma_{m+1,l-1}^x),\\
\tau_{m,l}^z &= (\sigma_{1,l-1}^z \sigma_{1,l}^z)(\sigma_{2,l-1}^z \sigma_{2,l}^z) \cdots (\sigma_{m,l-1}^z \sigma_{m,l}^z),
\end{aligned} \tag{4.2}$$

where we have bracketed products of operators corresponding to uninterrupted sequences of spins. Furthermore, we adopt the conventions

$$\begin{aligned}
&(m, n + 1) \equiv (m + 1, 1), \qquad (m + 1, 0) \equiv (m, n),\\
&\boldsymbol{\sigma}_{m,l} = 1 \qquad \text{for} \qquad j < 1 \text{ and } j > N.
\end{aligned} \tag{4.3}$$

It is easily verified that the operators $\boldsymbol{\tau}_{m,l}$ satisfy all the rules for Pauli-spin operators.

The inverse transformation of order n reads

$$\begin{aligned}
\sigma_{m,l}^x &= (\tau_{m,l}^x \tau_{m,l+1}^x)(\tau_{m+1,l}^x \tau_{m+1,l+1}^x) \cdots (\tau_{M,l}^x \tau_{M,l+1}^x),\\
\sigma_{m,l}^y &= (\tau_{m-1,l+1}^z \cdots \tau_{m,l-1}^z \tau_{m,l}^y \tau_{m,l+1}^x)\,(\tau_{m+1,l}^x \tau_{m+1,l+1}^x) \cdots (\tau_{M,l}^x \tau_{M,l+1}^x),\\
\sigma_{m,l}^z &= (\tau_{m-1,l+1}^z \cdots \tau_{m,l}^z),
\end{aligned} \tag{4.4}$$

with the convention

$$\tau_{m,l} = 1 \qquad \text{for} \qquad j < 1 \text{ and } j > N. \tag{4.5}$$

One may naturally associate with the $\tau_{m,l}$ operators a one-dimensional lattice of N spins $\frac{1}{2}$. We shall call this the τ lattice, and the original lattice the σ lattice. The significance of the transformation is most easily perceived by considering the expression for $\sigma^z_{m,l}$. The eigenvalues ± 1 of this operator correspond to the product of n successive spins on the τ lattice, except for the first lattice cell where $\sigma^z_{1,l}$ is a product of l operators τ^z. Obviously, specification of the signs of the τ spins uniquely determines the signs of the σ spins and *vice versa*.

5. *Application of the transformation.* For convenience of notation we shall henceforth assume that for the n-flip model $N = nM$, *i.e.*, the last lattice cell also contains n spins. Since we are interested only in bulk properties this simplication is clearly allowed. The master operator (3.8) of the n-flip model may then be rewritten

$$\begin{aligned}
\mathscr{W}_n = {} & \tfrac{1}{2}\alpha_n \sum_{m=1}^{M-1} \sum_{l=1}^{n} \{[1 + \tfrac{1}{2}\gamma(\sigma^z_{m,l-1}\sigma^z_{m,l} + \sigma^z_{m+1,l-1}\sigma^z_{m+1,l})] \\
& \times \sigma^x_{m,l}\sigma^x_{m,l+1} \cdots \sigma^x_{m+1,l-1} \\
& - [1 - \tfrac{1}{2}\gamma(\sigma^z_{m,l-1}\sigma^z_{m,l} + \sigma^z_{m+1,l-1}\sigma^z_{m+1,l})]\} \\
& + \tfrac{1}{4}\alpha_n\gamma(\sigma^z_{M,n}\sigma^z_{1,1} - \sigma^z_{1,1})(\sigma^x_{1,1}\sigma^x_{1,2} \cdots \sigma^x_{1,n} + 1) \\
& + \tfrac{1}{2}\alpha_n \sum_{l=1}^{n} \{[1 + \tfrac{1}{2}\gamma(\sigma^z_{M,l-1}\sigma^z_{M,l} + \sigma^z_{1,l-1}\sigma^z_{1,l})] \\
& \times \sigma^x_{M,l} \cdots \sigma^x_{M,n}\sigma^x_{1,1} \cdots \sigma^x_{1,l-1} \\
& - [1 - \tfrac{1}{2}\gamma(\sigma^z_{M,l-1}\sigma^z_{M,l} + \sigma^z_{1,l-1}\sigma^z_{1,l})]\} \\
& + \tfrac{1}{4}\alpha_n\gamma(\sigma^z_{M,n}\sigma^z_{1,1} - \sigma^z_{1,1})(\sigma^x_{M,1}\sigma^x_{M,2} \cdots \sigma^x_{M,n} + 1),
\end{aligned} \tag{5.1}$$

where we have used the convention (4.3). Applying now the $\sigma\tau$ transformation of order n one finds

$$\begin{aligned}
\mathscr{W}_n = {} & \tfrac{1}{2}\alpha_n \sum_{m=1}^{M-1} \sum_{l=1}^{n} \{[1 + \tfrac{1}{2}\gamma(\tau^z_{m-1,l}\tau^z_{m,l} + \tau^z_{m,l}\tau^z_{m+1,l})]\, \tau^x_{m,l} \\
& - [1 - \tfrac{1}{2}\gamma(\tau^z_{m-1,l}\tau^z_{m,l} + \tau^z_{m,l}\tau^z_{m+1,l})]\} \\
& + \text{boundary terms.}
\end{aligned} \tag{5.2}$$

It is mathematically convenient to consider a slightly different operator

$$\begin{aligned}
\mathscr{W}'_n = {} & \tfrac{1}{2}\alpha_n \sum_{m=1}^{M} \sum_{l=1}^{n} \{[1 + \tfrac{1}{2}\gamma(\tau^z_{m-1,l}\tau^z_{m,l} + \tau^z_{m,l}\tau^z_{m+1,l})]\, \tau^x_{m,l} \\
& - [1 - \tfrac{1}{2}\gamma(\tau^z_{m-1,l}\tau^z_{m,l} + \tau^z_{m,l}\tau^z_{m+1,l})]\},
\end{aligned} \tag{5.3}$$

where now, instead of (4.5), we adopt the *cyclic* definition

$$\tau^z_{0,l} \equiv \tau^z_{M,l}, \qquad \tau^z_{M+1,l} \equiv \tau^z_{1,l}. \tag{5.4}$$

It is easily shown that when $\mathscr{W}'_n$ is transformed back to σ operators it differs from $\mathscr{W}_n$ only in boundary terms relating to transitions of spins in the first two and the last cell of the σ lattice. As long as we are interested in bulk properties these terms have negligible effect in the limit $N \to \infty$. As is evident from (5.3), on the τ lattice the master operator $\mathscr{W}'_n$ decomposes into a sum of 1-flip operators, one for each of the n sublattices.

We also apply a $\sigma\tau$ transformation of order n to the hamiltonian (3.13). Again one finds that apart from terms of order unity the eigenvalues of $\mathscr{H}_N$ are identical to the eigenvalues of

$$\mathscr{H}'_N = -J \sum_{m=1}^{M} \sum_{l=1}^{n} \tau^z_{m,l}\tau^z_{m+1,l}, \tag{5.5}$$

where we use the cyclic definition (5.4). Hence in this approximation the hamiltonian also decomposes into a sum of uncoupled hamiltonians for the n τ-sublattices. As a consequence, the thermal equilibrium distribution P'_{eq} corresponding to this hamiltonian is a product of n uncorrelated equilibrium distributions for the different sublattices.

Since we know the complete solution of the 1-flip Glauber model[4]) we now also have the solution of the n-flip model at our disposal, at least as regards the bulk properties. The calculation of non-equilibrium averages is generally rather complicated because the non-equilibrium distribution $P(\boldsymbol{s}_N, t)$ will involve correlations between the τ sublattices. In the calculation of equilibrium time-correlation functions, however, the statistical independence of the τ sublattices may be used with benefit.

6. *Energy–energy correlations.* We consider the time-correlation function of a Fourier component of the energy density

$$\langle E_Q(t)_n E_{-Q}\rangle_{\mathrm{eq}} = \langle 0\,|\mathscr{H}_Q(t)_n \mathscr{H}_{-Q}|\,0\rangle, \tag{6.1}$$

where $\mathscr{H}_Q$ is defined by

$$\mathscr{H}_Q = -JN^{-\frac{1}{2}} \sum_{j=1}^{N} \sigma^z_j\sigma^z_{j+1}\,\mathrm{e}^{-\mathrm{i}Qj}. \tag{6.2}$$

Except for boundary terms this operator is identical to

$$\mathscr{H}'_Q = -JN^{-\frac{1}{2}} \sum_{m=1}^{M} \sum_{l=1}^{n} \tau^z_{m,l}\tau^z_{m+1,l}\,\mathrm{e}^{-\mathrm{i}Q[(m-1)n+l]}. \tag{6.3}$$

We define a Fourier transform on the lth τ sublattice

$$f^{(l)}_q = M^{-\frac{1}{2}} \sum_{m=1}^{M} f_{m,l}\,\mathrm{e}^{-\mathrm{i}qm}, \tag{6.4}$$

where for cyclic boundary conditions q takes the values (mod 2π)

$$q = 0, \pm 2\pi/M, \pm 4\pi/M, \ldots, \pi, \tag{6.5a}$$

while for anticyclic boundary conditions q takes the values (mod 2π)

$$q = \pm\pi/M, \pm 3\pi/M, \ldots, \pm(M-1)\,\pi/M. \tag{6.5b}$$

We assume for convenience that M is even. The relation to the Fourier transform on the complete τ lattice is

$$f_Q = N^{-\frac{1}{2}} \sum_{j=1}^{N} f_j \, \mathrm{e}^{-\mathrm{i}Qj} = n^{-\frac{1}{2}} \, \mathrm{e}^{\mathrm{i}nQ} \sum_{l=1}^{n} f_{nQ}^{(l)} \, \mathrm{e}^{-\mathrm{i}lQ}, \tag{6.6}$$

where Q takes the values (mod 2π)

$$Q = 0, \pm 2\pi/nM, \pm 4\pi/nM, \ldots, \pi, \tag{6.7a}$$

or

$$Q = \pm\pi/nM, \pm 3\pi/nM, \ldots, \pm(nM-1)\,\pi/nM, \tag{6.7b}$$

for cyclic or anticyclic boundary conditions, respectively. Hence if we define

$$\mathscr{H}_q^{(l)} = -JM^{-\frac{1}{2}} \sum_{m=1}^{M} \tau_{m,l}^z \tau_{m+1,l}^z \, \mathrm{e}^{-\mathrm{i}qm}, \tag{6.8}$$

we have

$$\mathscr{H}'_Q = n^{-\frac{1}{2}} \, \mathrm{e}^{\mathrm{i}nQ} \sum_{l=1}^{n} \mathscr{H}_{nQ}^{(l)} \, \mathrm{e}^{-\mathrm{i}lQ}. \tag{6.9}$$

As noted before, in thermal equilibrium the different τ sublattices are statistically independent. Moreover, we shall show in a moment that

$$\langle 0 | \mathscr{H}_q^{(l)}(t)_1 | 0 \rangle_l = 0 \qquad \text{for} \qquad q \neq 0 \pmod{2\pi}, \tag{6.10}$$

where the subscript l indicates that the average is performed over the thermal equilibrium ensemble of the lth τ sublattice. Hence one finds

$$\langle 0 | \mathscr{H}'_Q(t)_n \, \mathscr{H}'_{-Q} | 0 \rangle' = \langle 0 | \mathscr{H}_{nQ}^{(1)}(t)_1 \, \mathscr{H}_{-nQ}^{(1)} | 0 \rangle_1, \tag{6.11}$$

for all values of $Q \neq 0$ (mod $2\pi/n$). Thus we have derived a simple relation between the time-correlation functions of the energy density in the n-flip and the 1-flip model,

$$\langle E_Q(t)_n \, E_{-Q} \rangle_{\mathrm{eq}} \approx \langle E_{nQ}^{(1)}(t)_1 \, E_{-nQ}^{(1)} \rangle_{\mathrm{eq}}. \tag{6.12}$$

The relation is exact in the thermodynamic limit $M \to \infty$. From (6.12) it follows that $\langle E_Q(t)_n \, E_{-Q} \rangle_{\mathrm{eq}}$ is periodic in the first Brillouin zone ($-\pi \leq Q \leq \pi$) with periodicity $2\pi/n$. The exceptional point $Q = 0$ (mod $2\pi/n$) need be of no special concern, as it merely points to the necessity of subtracting equilibrium averages in the definition of the time-correlation func-

tion. As is evident from (6.9) the average $\langle \mathscr{H}'_Q \rangle'_{\mathrm{eq}}$ differs from zero only for $Q = 0$. In the thermodynamic limit the expressions for the properly defined correlation functions in the exceptional points $Q = 0 \pmod{2\pi/n}$ may be found simply by taking a continuous limit.

The correlation function in the 1-flip model may be calculated using the formalism of I. We refer to I for notation and more detailed explanation. In order to evaluate $\langle 0|\mathscr{H}_q^{(1)}(t)_1 \mathscr{H}_{-q}^{(1)}|0\rangle_1$ and $\langle 0|\mathscr{H}_q^{(1)}(t)_1|0\rangle_1$ we note that

$$\mathscr{H}_q^{(1)} = -JM^{-\frac{1}{2}} \sum_k (\xi_k^\dagger - \xi_{-k})(\xi_{-k-q}^\dagger + \xi_{k+q}) \times \exp \mathrm{i}(\tfrac{1}{2}\chi_k + \tfrac{1}{2}\chi_{k+q} + k + q), \tag{6.13}$$

where k and q range over the set of values (6.5b) and where $k + q$ is defined mod 2π. The operators $\xi_k^\dagger$ and ξ_k satisfy the Fermi anticommutation rules and have $|0\rangle_1$ as vacuum,

$$\xi_k |0\rangle_1 = 0. \tag{6.14}$$

The angle χ_k is defined by

$$\exp \mathrm{i}\chi_k = (\cosh K - \mathrm{e}^{-\mathrm{i}k} \sinh K)^2/(\cosh 2K - \sinh 2K \cos k). \tag{6.15}$$

The relation (6.13) is valid only if the operator acts on a state containing an even number of fermions. The transformed 1-flip operator $\mathscr{W}_1(\beta)$ (for M spins) is linear in the fermion number operators $\xi_k^\dagger \xi_k$. In I we have defined two operators $\mathscr{W}_1^+(\beta)$ and $\mathscr{W}_1^-(\beta)$ by

$$\mathscr{W}_1^{\pm}(\beta) = -\sum_k \lambda_k \xi_k^\dagger \xi_k, \tag{6.16}$$

where

$$\lambda_k = \alpha_n(1 - \gamma \cos k) \tag{6.17}$$

and where for $\mathscr{W}_1^+(\beta)$ the sum runs over the values (6.5b) and for $\mathscr{W}_1^-(\beta)$ it runs over the values (6.5a). The fermion operators occurring in $\mathscr{W}_1^-(\beta)$ have a slightly different vacuum state given by

$$|0^-\rangle_1 = \Lambda_M(\cosh K - \tau_{M,1}^z \tau_{1,1}^z \sinh K)\,|0\rangle_1, \tag{6.18}$$
$$\Lambda_M = [1 + (\tanh K)^M]^{\frac{1}{2}}[1 - (\tanh K)^M]^{-\frac{1}{2}}.$$

The eigenvectors of $\mathscr{W}_1(\beta)$ consist of those eigenvectors of $\mathscr{W}_1^+(\beta)$ having even numbers of fermions and those eigenvectors of $\mathscr{W}_1^-(\beta)$ having odd numbers of fermions. The "odd" eigenvectors of $\mathscr{W}_1^+(\beta)$ and the "even" eigenvectors of $\mathscr{W}_1^-(\beta)$ are irrelevant and must be discarded. In appendix A we demonstrate that the $\sigma\tau$ transformation may be used to find an interesting relation between the operator $\mathscr{W}_1(\beta)$ and the hamiltonian of the XY model.

From (3.20), (6.13) and (6.16) it follows that the time-behaviour of $\mathscr{H}_q^{(1)}(t)_1$ is found by replacing the fermion operators in (6.13) by the time-dependent

operators

$$\xi_k(t)_1 = \xi_k \, e^{-\lambda_k t}, \qquad \xi_k^\dagger(t)_1 = \xi_k^\dagger \, e^{\lambda_k t}. \tag{6.19}$$

One immediately confirms the validity of (6.10) with the aid of (6.14). Evaluating the time-correlation function in the same manner one finds for $q \neq 0$ using $\chi_{-k} = -\chi_k$

$$\langle 0 | \mathscr{H}_q^{(1)}(t)_1 \mathscr{H}_{-q}^{(1)} | 0 \rangle_1 = J^2 M^{-1} \sum_k e^{-(\lambda_k + \lambda_{k+q})t} \times [1 - \exp i(\chi_k + \chi_{k+q} + 2k + q)], \tag{6.20}$$

where k ranges over the values (6.5b). In the thermodynamic limit $M \to \infty$ the sum over k may be replaced by an integral. From (6.12) and (6.20) we finally have for the energy–energy correlation function in the limit $M \to \infty$ and for $Q \neq 0$

$$\langle E_Q(t)_n E_{-Q} \rangle_{eq} = \frac{J^2}{2\pi} \int_{-\pi}^{\pi} e^{-(\lambda_k + \lambda_{k+nQ})t} \times [1 - \exp i(\chi_k + \chi_{k+nQ} + 2k + nQ)] \, dk. \tag{6.21}$$

For the time-correlation function of the properly normalized energy fluctuation

$$\delta E_N \equiv N^{-\frac{1}{2}}[E_N - \langle E_N \rangle_{eq}], \tag{6.22}$$

we find by taking the limit $Q \to 0$ in (6.21)

$$\langle \delta E(t)_n \delta E \rangle_{eq} = \frac{J^2}{\pi} \int_{-\pi}^{\pi} e^{-2\lambda_k t} \sin^2(\chi_k + k) \, dk, \tag{6.23}$$

again in the thermodynamic limit $M \to \infty$. As was to be expected this correlation function is independent of n, except for a trivial time-scale dependence through α_n. In section 8 we shall examine some features of the correlation function (6.23).

7. *Magnetization–magnetization correlations.* The time-correlation function of a Fourier component of the magnetization is given by

$$\langle S_Q(t)_n S_{-Q} \rangle_{eq} = \langle 0 | \sigma_Q^z(t)_n \sigma_{-Q}^z | 0 \rangle, \tag{7.1}$$

where σ_Q^z is defined by

$$\sigma_Q^z = N^{-\frac{1}{2}} \sum_{j=1}^{N} \sigma_j^z \, e^{-iQj}. \tag{7.2}$$

Except for boundary terms this operator is identical to

$$\sigma_Q^{z'} = N^{-\frac{1}{2}} \sum_{m=1}^{M} \sum_{l=1}^{n} (\tau_{m-1,l+1}^z \cdots \tau_{m,l}^z) \, e^{-iQ[(m-1)n+l]}, \tag{7.3}$$

with the cyclic definition (5.4). We define the Fourier transform

$$\tau_q^{z(l)} = M^{-\frac{1}{2}} \sum_{m=1}^{M} \tau_{m,l}^{z} \, \mathrm{e}^{-\mathrm{i}qm}, \tag{7.4}$$

where q takes the values (6.5a). Substituting in (7.3) one finds

$$\sigma_Q^{z'} = n^{-\frac{1}{2}} M^{-\frac{1}{2}n} \sum_{\boldsymbol{q}} [\prod_{l=1}^{n} \tau_{q_l}^{z(l)}] \, \delta(\sum_{l=1}^{n} q_l - nQ) \, F(\boldsymbol{q}, Q), \tag{7.5}$$

where $\boldsymbol{q} \equiv (q_1, \ldots, q_n)$ and where

$$F(\boldsymbol{q}, Q) = \sum_{l=1}^{n} \exp\{-\mathrm{i}[\sum_{r=l+1}^{n} q_r - (n-l)\, Q]\}. \tag{7.6}$$

All wavenumbers in (7.5) and (7.6) must be reduced to the first Brillouin zone. Using the statistical independence of different τ sublattices one hence finds for the correlation function

$$\begin{aligned} \langle 0 | \sigma_Q^{z'}(t)_n \, \sigma_{-Q}^{z} | 0 \rangle' &= n^{-1} M^{-n} \sum_{\boldsymbol{q}} \delta(\sum_{l=1}^{n} q_l - nQ) |F(\boldsymbol{q}, Q)|^2 \\ &\times \prod_{l=1}^{n} \langle 0 | \tau_{q_l}^{z(l)}(t)_1 \, \tau_{-q_l}^{z(l)} | 0 \rangle_l. \end{aligned} \tag{7.7}$$

In the thermodynamic limit $M \to \infty$ the magnetization correlation function of a τ sublattice is given by

$$\lim_{M \to \infty} \langle 0 | \tau_q^{z(l)}(t)_1 \, \tau_{-q}^{z(l)} | 0 \rangle_l = G(q) \, \mathrm{e}^{-\lambda_q t}, \tag{7.8}$$

where

$$G(q) = [\cosh 2K(1 - \gamma \cos q)]^{-1}. \tag{7.9}$$

Thus one finds

$$\begin{aligned} \lim_{M \to \infty} \langle S_Q(t)_n \, S_{-Q} \rangle_{\mathrm{eq}} &= n^{-1} (2\pi)^{1-n} \int_{-\pi}^{\pi} \ldots \int_{-\pi}^{\pi} \mathrm{d}_n \boldsymbol{q} \\ &\times \delta(\sum_{l=1}^{n} q_l - nQ) |F(\boldsymbol{q}, Q)|^2 \prod_{l=1}^{n} G(q_l) \exp[-\lambda(q_l)\, t]. \end{aligned} \tag{7.10}$$

In the next section we shall examine some features of this correlation function. The frequency- and wavenumber-dependent susceptibility is related to the correlation function by a Fourier transform [2])

$$\chi_n(Q, \omega) = \chi(Q, 0) - \mathrm{i}\omega\beta \int_0^{\infty} \langle S_Q(t)_n \, S_{-Q} \rangle_{\mathrm{eq}} \, \mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{d}t. \tag{7.11}$$

For the 1-flip model [2]) the function $\chi_1(Q, \omega)$ has a simple pole in the complex ω plane at $\omega(Q) = \mathrm{i}\lambda(Q)$. For the 2-flip model $\chi_2(Q, \omega)$ may also be evaluated explicitly; it exhibits a branch-cut singularity in the ω plane. The

static susceptibility $\chi(Q, 0)$ is given by

$$\chi(Q, 0) = \beta G(Q) = \beta[\cosh 2K(1 - \gamma \cos Q)]^{-1}. \tag{7.12}$$

To conclude this section we note that the energy density–magnetization correlation function $\langle E_Q(t)_n S_{-Q}\rangle_{\mathrm{eq}}$ vanishes identically, since the equilibrium ensemble and the first operator are invariant under reversal of spins, whereas the second operator changes its sign.

8. *Relaxation times.* In order to study qualitatively the behaviour of the correlation functions near $T = 0$ we define characteristic times for the relaxation of energy and magnetization. We shall here consider only the long-wavelength limit $Q \to 0$. A characteristic relaxation time for the fluctuations of the total energy in the n-flip model is defined by[8])

$$\tau_E(n) = \int_0^\infty \langle \delta E(t)_n \, \delta E\rangle_{\mathrm{eq}} \, \mathrm{d}t / \langle (\delta E)^2\rangle_{\mathrm{eq}}. \tag{8.1}$$

From (6.15) and (6.23) it follows that the numerator is given by

$$\begin{aligned}\int_0^\infty \langle \delta E(t)_n \, \delta E\rangle_{\mathrm{eq}} \, \mathrm{d}t &= \frac{J^2}{2\pi\alpha_n \cosh^2 2K} \int_{-\pi}^{\pi} \frac{\sin^2 k}{(1 - \gamma \cos k)^3} \, \mathrm{d}k \\ &= \tfrac{1}{2}\alpha_n^{-1} J^2 (\cosh 2K)^{-2} (1 - \gamma^2)^{-\frac{3}{2}} \\ &= \tfrac{1}{2}\alpha_n^{-1} J^2 (1 + a^2)/(1 - a^2),\end{aligned} \tag{8.2}$$

where $a = \tanh K$. The denominator in (8.1) is given by

$$\langle (\delta E)^2\rangle_{\mathrm{eq}} = J^2(1 - a^2), \tag{8.3}$$

so that

$$\tau_E(n) = \tfrac{1}{2}\alpha_n^{-1}(1 + a^2)/(1 - a^2)^2, \tag{8.4}$$

with limiting behaviour as $T \to 0$

$$\tau_E(n) = \tfrac{1}{16} A_n^{-1} \varepsilon^{-2} + \mathcal{O}(\varepsilon^{-1}), \tag{8.5}$$

where

$$\varepsilon = \exp[-2\beta J] \tag{8.6}$$

and where we have used (2.11). Thus we find that the energy relaxation time diverges as $T \to 0$ with a characteristic singularity which is independent of n, that is, independent of the particular relaxation model. We deal with a case of non-classical slowing-down since the singularity is different from that predicted by the classical theory in which

$$\tau_E^{\mathrm{cl}}(n) = \langle (\delta E)^2\rangle_{\mathrm{eq}} / L_E(n), \tag{8.7}$$

with a nonsingular Onsager coefficient $L_E(n)$. In the present case the classical theory would actually imply that $\tau_E(n)$ tend to zero as $T \to 0$, because of the vanishing of the specific heat.

A characteristic relaxation time for the fluctuations of the total magnetization is defined by[8])

$$\tau_M(n) = \int_0^\infty \langle S_0(t)_n \, S_0 \rangle_{\mathrm{eq}} \, \mathrm{d}t / \langle S_0^2 \rangle_{\mathrm{eq}}. \tag{8.8}$$

In the thermodynamic limit the denominator is given by

$$\langle S_0^2 \rangle_{\mathrm{eq}} = G(0) = [\cosh 2K(1-\gamma)]^{-1}. \tag{8.9}$$

For the numerator in (8.8) we have from (7.10), (6.17), and (7.9)

$$\int_0^\infty \langle S_0(t)_n \, S_0 \rangle_{\mathrm{eq}} \, \mathrm{d}t = \alpha_n^{-1} [\cosh 2K]^{-n} \mathscr{I}_n, \tag{8.10}$$

where

$$\mathscr{I}_n = n^{-1}(2\pi)^{1-n} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} |F(\boldsymbol{q}, 0)|^2 \prod_{l=1}^{n} (1 - \gamma \cos q_l)^{-1} \times [\sum_{l=1}^{n} (1 - \gamma \cos q_l)]^{-1} \, \delta(\sum_{l=1}^{n} q_l) \, \mathrm{d}_n \boldsymbol{q}. \tag{8.11}$$

For $n = 1$ and $n = 2$ the integral may be evaluated explicitly with the results

$$\mathscr{I}_1 = (1-\gamma)^{-2}$$
$$\mathscr{I}_2 = \tfrac{1}{4}(1+\gamma)(2+\gamma)(1-\gamma^2)^{-\frac{1}{2}}. \tag{8.12}$$

For general n the integral is singular as $\gamma \to 1$ owing to the behaviour of the integrand near $\boldsymbol{q} = 0$. Integrating once to get rid of the delta function and expanding the cosines one finds that the dominant term is given by

$$\mathscr{I}_n = 2^{\frac{1}{2}(1-n)} n C_n (1-\gamma)^{-\frac{1}{2}(n+3)} + \mathcal{O}[(1-\gamma)^{-\frac{1}{2}(n+1)}], \tag{8.13}$$

where

$$C_n = \pi^{1-n} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \prod_{l=1}^{n} (1 + x_l^2)^{-1} [\sum_{l=1}^{n} (1 + x_l^2)]^{-1} \times \delta(\sum_{l=1}^{n} x_l) \, \mathrm{d}x_1 \dots \mathrm{d}x_n. \tag{8.14}$$

One easily finds

$$C_1 = 1, \qquad C_2 = \tfrac{3}{16}. \tag{8.15}$$

It is shown in appendix B that C_n is monotonically decreasing in n with

asymptotic behaviour

$$C_n = Bn^{-3} + \mathcal{O}(n^{-4}), \tag{8.16}$$

where B is a constant.

From (8.8), (8.9), (8.10) and (8.13) it finally follows that as $T \to 0$ the asymptotic behaviour of the magnetization relaxation time is given by

$$\tau_M(n) = \tfrac{1}{2}A_n^{-1}nC_n\varepsilon^{-2} + \mathcal{O}(\varepsilon^{-1}). \tag{8.17}$$

Hence for the magnetization the critical exponent is again independent of the particular relaxation model. Only the prefactor is model dependent.

The classical theory would predict slowing-down owing to the divergence of the susceptibility $\chi = \beta G(0)$,

$$\tau_M^{\mathrm{cl}}(n) = \chi / L_M(n), \tag{8.18}$$

with a nonsingular Onsager coefficient $L_M(n)$. Again this would lead to a weaker singularity, namely

$$\tau_M^{\mathrm{cl}}(n) \approx \tfrac{1}{2}J^{-1}L_M^{-1}(n)\,\varepsilon^{-1}\,|\ln \varepsilon|. \tag{8.19}$$

From (8.16) and (8.17) it follows that if the spontaneous emission rate A_n is taken to be independent of n, then at a fixed low temperature the relaxation time $\tau_M(n)$ decreases proportionally to n^{-2} for large n. This is understandable since, as is evident from (3.8), on the average every spin partakes in n times as many transitions per unit time as in the 1-flip model, and every transition is n times as effective in changing the magnetization.

9. *Conclusions.* We have studied time-correlation functions for a class of stochastic relaxation models for the Ising chain. In particular we have derived explicit expressions (6.21) and (7.10) for the time-correlation functions of Fourier components of the energy density and the magnetization. In the last section we have studied the gross features of the correlation functions of total energy and total magnetization and we have shown that the characteristic relaxation times exhibit singularities near $T = 0$. Since for the Ising chain temperature $T = 0$ thermodynamically acts as a critical point the effect may be termed critical slowing-down. We have shown that for our class of models the behaviour is non-classical and we find agreement with conjectures made for the 2- and 3-dimensional Ising model implying that the critical exponents are independent of the detailed dynamics, but are sensitive only to drastic changes, such as change of dimensionality and change of symmetry or conservation laws. In our case the relaxation time for the energy (8.5) is completely independent of the relaxation model, whereas the relaxation time for the magnetization (8.17) exhibits a singularity at $T = 0$ with a model-independent critical exponent, but with a model-dependent prefactor.

Qualitatively the origin of the slowing down may be understood as follows. The behaviour of the energy–energy correlation function for all models is determined by the combined decay of pairs of elementary excitations with wavenumbers q and $-q$ and decay rate $\lambda_q = \alpha_n(1 - \gamma \cos q)$. Near $T = 0$ the dominant contribution comes from a range of wavenumbers around $q = 0$ with a width

$$\Delta q = [2(1 - \gamma)/\gamma]^{\frac{1}{2}} \approx 2\varepsilon, \tag{9.1}$$

which shrinks to zero as $T \to 0$. We note that Δq is approximately equal to the inverse correlation length $|\ln a| \approx 2\varepsilon$. The relaxation time $\tau_E(n)$ is proportional to the characteristic decay time for long wavelength excitations $\lambda_0^{-1} = [\alpha_n(1 - \gamma)]^{-1}$. The same is true for the relaxation of the magnetization. In the n-flip model the magnetization on the average decays through the combined nonlinear effect of n elementary excitations. The main contribution to the relaxation time $\tau_M(n)$ comes from a domain of range $(\Delta q)^{n-1}$ in the $(n - 1)$-dimensional $\boldsymbol{q}$ space. Again the singularity of the relaxation time is essentially proportional to λ_0^{-1}. It may be of interest to compare the above exact calculation with the basic ideas of mode–mode coupling theories.

APPENDIX A

Equivalence of the master operator $\mathscr{W}_1(\beta)$ to the XY hamiltonian. We show that the transformed master operator $\mathscr{W}_1(\beta)$ of the 1-flip model may be related by a $\sigma\tau$ transformation of order 2 to the hamiltonian of the XY model[9]). On a τ lattice of N spins the operator $\mathscr{W}_1(\beta)$ is given by

$$\begin{aligned}\mathscr{W}_1(\beta) = {} & \tfrac{1}{2}\alpha_1 \sum_{j=1}^{N} \{\tfrac{1}{2}\gamma[a^{-1} - \tau_{j-1}^z \tau_{j+1}^z a]\, \tau_j^x \\ & - [1 - \tfrac{1}{2}\gamma\tau_j^z(\tau_{j-1}^z + \tau_{j+1}^z)]\},\end{aligned} \tag{A.1}$$

where $a = \tanh K$ and where we use cyclic boundary conditions. Using the $\sigma\tau$ transformation of order 2 given by (4.2) this may be transformed to

$$\begin{aligned}\mathscr{W}_1(\beta) = {} & \tfrac{1}{2}\alpha_1 \sum_{j=2}^{N-1} [\tfrac{1}{2}\gamma a^{-1}\sigma_j^x\sigma_{j+1}^x + \tfrac{1}{2}\gamma a\sigma_j^y\sigma_{j+1}^y - 1 + \gamma\sigma_j^z] \\ & + \tfrac{1}{2}\alpha_1\{\tfrac{1}{2}\gamma[a^{-1} - (\sigma_3^z \ldots \sigma_N^z)\, a]\, \sigma_1^x\sigma_2^x \\ & + \tfrac{1}{2}\gamma[a^{-1} - (\sigma_2^z \ldots \sigma_{N-1}^z)\, a]\, \sigma_N^x - 2 + \gamma(\sigma_2^z \ldots \sigma_N^z + \sigma_N^z)\}.\end{aligned} \tag{A.2}$$

Hence apart from boundary terms the operator $\mathscr{W}_1(\beta)$ coincides with the hamiltonian of the XY model on the σ lattice for a suitable choice of parameters.

APPENDIX B

Asymptotic behaviour of C_n for large n. In this appendix we study the asymptotic behaviour for large n of the coefficient C_n, defined by (8.14),

$$C_n = \pi^{1-n} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \prod_{l=1}^{n} (1 + x_l^2)^{-1} \times [\sum_{l=1}^{n} (1 + x_l^2)]^{-1} \delta(\sum_{l=1}^{n} x_l) \, dx_1 \dots dx_n. \tag{B.1}$$

The integral may be rewritten

$$\begin{aligned} C_n &= \pi^{1-n} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \int_{0}^{\infty} d\eta \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \prod_{l=1}^{n} (1 + x_l^2)^{-1} \\ &\quad \times \exp[i\xi \sum_{l=1}^{n} x_l - \eta \sum_{l=1}^{n} (1 + x_l^2)] \, dx_1 \dots dx_n \\ &= \int_{0}^{\infty} \int_{0}^{\infty} [f(\xi, \eta)]^n \, d\xi \, d\eta, \end{aligned} \tag{B.2}$$

where

$$\begin{aligned} f(\xi, \eta) &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi x - \eta(1+x^2)}}{1 + x^2} \, dx \\ &= \int_{\eta}^{\infty} (\pi u)^{-\frac{1}{2}} \exp[-u - \xi^2/4u] \, du. \end{aligned} \tag{B.3}$$

The function $f(\xi, \eta)$ is monotonically decreasing in both ξ and η in the first quadrant of the (ξ, η) plane, tending to 0 at ∞, and takes its maximum value $f(0, 0) = 1$ at the origin. Hence it follows that C_n is monotonically decreasing with n and in the limit $n \to \infty$ the dominant contribution to the integral (B.2) comes from the neighbourhood of the origin. One may also write

$$f(\xi, \eta) = e^{-\xi} - \int_{0}^{\eta} (\pi u)^{-\frac{1}{2}} \exp[-u - \xi^2/4u] \, du, \tag{B.4}$$

so that along the ξ axis one has

$$f(\xi, 0) = e^{-\xi}, \tag{B.5}$$

whereas along the η axis

$$f(0, \eta) = \int_{\eta}^{\infty} (\pi u)^{-\frac{1}{2}} e^{-u} \, du. \tag{B.6}$$

For fixed ξ all derivatives of $f(\xi, \eta)$ with respect to η vanish at $\eta = 0$. The second derivative of $f(\xi, \eta)$ with respect to ξ is finite for fixed η at $\xi = 0$. Changing variables in the integral in (B.4) to $v = 4u/\xi^2$ one finds that the behaviour of $f(\xi, \eta)$ for small ξ and η is given by

$$f(\xi, \eta) = \mathrm{e}^{-\xi} - (2\sqrt{\pi})^{-1}\, \xi[g_0(\xi^2/4\eta) - \tfrac{1}{4}\xi^2 g_1(\xi^2/4\eta) + \ldots], \tag{B.7}$$

with

$$g_0(z) = \Gamma(-\tfrac{1}{2}, z), \qquad g_1(z) = \Gamma(-\tfrac{3}{2}, z), \tag{B.8}$$

where $\Gamma(\nu, z)$ is the incomplete gamma function. Hence to lowest order $f(\xi, \eta)$ is approximated near the origin by

$$f(\xi, \eta) \approx 1 - \xi\Phi(\xi^2/4\eta), \tag{B.9}$$

with

$$\Phi(z) = 1 + (2\sqrt{\pi})^{-1}\, \Gamma(-\tfrac{1}{2}, z). \tag{B.10}$$

The leading contribution to the integral (B.2) is therefore given by

$$\int_0^\infty \int_0^\infty \exp[-n\xi\Phi(\xi^2/4\eta)]\, \mathrm{d}\xi\, \mathrm{d}\eta = Bn^{-3}, \tag{B.11}$$

where

$$B = \int_0^\infty (2x^2)^{-1}\, [\Phi(x)]^{-3}\, \mathrm{d}x. \tag{B.12}$$

The higher-order correction terms in (B.7) lead to the estimate

$$C_n = Bn^{-3} + \mathcal{O}(n^{-4}) \qquad \text{as} \qquad n \to \infty. \tag{B.13}$$

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