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TIME-DEPENDENT INTERNAL SOLUTIONS FOR SPHERICALLY SYMMETRICAL BODIES IN GENERAL RELATIVITY

I. Adiabatic Collapse

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Summary

The gravitational collapse of a spherically symmetrical distribution of matter, following an initial non-zero velocity distribution, is investigated on the basis of Einstein's field equations for the adiabatic (radiationless) case with a non-vanishing internal pressure gradient. In particular, it is shown that, if the density is uniform throughout the body at each instant (and so is a function of time only) and if the body continually contracts, it must, as in the pressure-free case, collapse to a point-singularity of infinite density in a finite time, as reckoned by a co-moving observer. To an external observer, the body would appear to contract asymptotically to its gravitational radius. The consequences of relaxing the adiabatic condition will be considered in a later paper.

1. *Introduction.* In recent years, following the discovery of quasars and attempts to account for their enormous output of energy if they are at cosmological distances, there has been a great revival of interest in the internal solutions of the problem of a spherically symmetrical body in general relativity. It is the object of the present paper to make a further contribution to this subject, which is of interest in itself apart from its possible astrophysical applications. To put our work in its proper setting, we begin with a brief survey of previous investigations.

The simplest problem concerns a static sphere of uniform density (without radiation-transfer). This was solved in a classic paper as early as 1916 by K. Schwarzschild who obtained both the external and the internal solutions. In 1964, J. A. Wheeler and his co-workers (**1**) examined exhaustively the case of a static sphere of non-uniform density (without radiation-transfer) subject to an equation of state thought to represent material at the termination of thermonuclear evolution (Fe^{56}). Their results indicated that for such material there exists no equilibrium configuration for a body of mass exceeding a certain critical value of approximately 1·2 times the mass of the Sun. Investigation of the stability of the equilibrium models suggests that there are only two possible ranges of central density for the terminal states of stars that are almost burnt out, corresponding to white-dwarf stars and ‘ neutron stars ’. Stars with masses exceeding the critical value are thought to collapse catastrophically under their own gravitational fields, when they can no longer support themselves by thermonuclear reactions, and produce supernovae.

The theory of the collapse of a spherically symmetric mass was studied in 1939 by Oppenheimer & Snyder (**2**). They confined attention to pressure-free collapse, and made use of co-moving co-ordinates in their analysis, but did not obtain the most general solution. This line of approach was continued in 1964 by McVittie (**3**) who introduced a constant non-zero internal pressure. Because there must be zero

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pressure at the surface, it was not possible to join the internal and external solutions continuously. In 1966, McVittie (4) introduced a pressure-gradient running from zero at the surface to infinity at a certain distance from the centre (Schwarzschild-type singularity). The co-ordinate radius of this singular region increases during the process of collapse.

In the present paper, we also assume that co-moving coordinates can be used. The following analytical solutions can then be obtained for adiabatic collapse, i.e. for cold (radiationless) bodies: (i) the zero-pressure case, (ii) a set of cases which includes that of uniform density. We shall show that in the uniform density case with non-uniform pressure, as in the pressure-free case, the body collapses in a finite time, as reckoned by a co-moving observer, to a point-singularity of infinite density. Physically, this would appear to mean that the matter concerned ultimately crushes itself out of existence, leaving only a gravitational field with a singularity. However, before this stage is attained quantum effects, which are here ignored, would presumably have to be taken into account; but to deal with them a new theory combining general relativity and quantum concepts is needed.

2. *Method of analysis*. Assuming the use of co-moving coordinates (r, θ, ϕ), where r is a radial coordinate (origin at the centre O of the body) and θ, ϕ are angular coordinates, we work with a spherically symmetric metric of the form

$$ds^2 = -e^{\lambda}dr^2 - e^{\mu}d\Omega^2 + e^{\nu}dt^2, \tag{1}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2,$$

in standard notation, t denotes time, λ, μ, ν are functions of (r, t), and units are chosen so that G, c are each unity. For an isotropic fluid of energy-momentum tensor with terms $(-p, -p, -p, \rho)$ down the main diagonal and all other terms zero, there are four independent field equations connecting the five functions λ, μ, ν, p, ρ, and hence a further equation is needed to obtain a solution. This would normally be an equation of state relating p and ρ, but this approach leads to analytically intractable equations that can only be solved numerically (5).

Instead, in the present paper, we specify some relation between the components of the metric tensor, choosing one that not only leads to an analytical solution but also seems physically plausible. A relation satisfying these conditions is

$$\dot{\lambda} = \dot{\mu}, \tag{2}$$

where the dot denotes differentiation with respect to time. Physically, this relation implies that the ratio of the distances AB and AC, as measured by a local observer, where A, B are *neighbouring* particles on a sphere r = constant, and A, C are *neighbouring* particles on the same radius vector through O, remains the same throughout the motion.

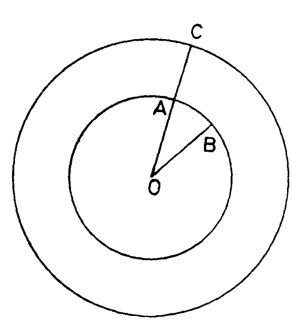

Fig. 1.

In particular, if the motion is such that, at any given instant, the total energy-density ρ is uniform throughout the sphere, condition (2) must be satisfied, i.e. $\rho=\rho(t)$ is a sufficient (but not a necessary) condition for the validity of equation (2), as will be shown below.

3. *The basic equations.* Taking units such that $c=1$, $G=1$, the field equations are

$$8\pi T_q^p = G_q^p,$$

where $G_q{}^p$ is the Einstein tensor. On evaluating $G_q{}^p$ for the metric (1), and using a prime to denote differentiation with respect to r, we obtain the equations

$$8\pi T_1{}^1 = \tfrac{1}{2}e^{-\nu}(2\ddot{\mu}-\dot{\mu}\dot{\nu}+\tfrac{3}{2}\dot{\mu}^2)-\tfrac{1}{2}e^{-\lambda}(\tfrac{1}{2}\mu'^2+\mu'\nu')+e^{-\mu}, \tag{3}$$

$$8\pi T_2{}^2 = 8\pi T_3{}^3 = \tfrac{1}{2}e^{-\nu}(\ddot{\mu}+\tfrac{1}{2}\dot{\mu}^2-\tfrac{1}{2}\dot{\mu}\dot{\nu}+\tfrac{1}{2}\dot{\mu}\dot{\lambda}-\tfrac{1}{2}\dot{\lambda}\dot{\nu}+\ddot{\lambda}+\tfrac{1}{2}\dot{\lambda}^2)$$
$$-\tfrac{1}{2}e^{-\lambda}(\mu''+\nu''+\tfrac{1}{2}\mu'^2+\tfrac{1}{2}\nu'^2+\tfrac{1}{2}\mu'\nu'-\tfrac{1}{2}\lambda'\mu'-\tfrac{1}{2}\lambda'\nu'), \tag{4}$$

$$8\pi T_4{}^4 = -\tfrac{1}{2}e^{-\lambda}(2\mu''-\lambda'\mu'+\tfrac{3}{2}\mu'^2)+\tfrac{1}{2}e^{-\nu}(\tfrac{1}{2}\dot{\mu}^2+\dot{\lambda}\dot{\mu})+e^{-\mu}, \tag{5}$$

$$8\pi T_4{}^1 = \tfrac{1}{2}e^{-\lambda}(\dot{\mu}\mu'-\dot{\lambda}\mu'+2\dot{\mu}'-\dot{\mu}\nu'), \quad 8\pi T_1{}^4 = -\tfrac{1}{2}e^{-\nu}(\dot{\mu}\mu'-\dot{\lambda}\mu'+2\dot{\mu}'-\dot{\mu}\nu'), \tag{6}$$

and other terms vanishing identically.

Since we are using co-moving coordinates and assuming the fluid to be cold and its pressure p to be spatially isotropic, we have

$$T_1{}^1 = T_2{}^2 = T_3{}^3 = -p,$$

and

$$T_4{}^4 = \rho, \qquad T_4{}^1 = 0 = T_1{}^4.$$

Hence, we arrive at the four independent field equations:

$$8\pi p = -\tfrac{1}{2}e^{-\nu}(2\ddot{\mu}-\dot{\mu}\dot{\nu}+\tfrac{3}{2}\dot{\mu}^2)+\tfrac{1}{2}e^{-\lambda}(\tfrac{1}{2}\mu'^2+\mu'\nu')-e^{-\mu}, \tag{7}$$

$$8\pi p = \tfrac{1}{2}e^{-\nu}(-\ddot{\mu}-\tfrac{1}{2}\dot{\mu}^2+\tfrac{1}{2}\dot{\mu}\dot{\nu}-\tfrac{1}{2}\dot{\mu}\dot{\lambda}+\tfrac{1}{2}\dot{\lambda}\dot{\nu}-\ddot{\lambda}-\tfrac{1}{2}\dot{\lambda}^2)$$
$$+\tfrac{1}{2}e^{-\lambda}(\mu''+\nu''+\tfrac{1}{2}\mu'^2+\tfrac{1}{2}\nu'^2+\tfrac{1}{2}\mu'\nu'-\tfrac{1}{2}\lambda'\nu'-\tfrac{1}{2}\lambda'\mu'), \tag{8}$$

$$8\pi\rho = -\tfrac{1}{2}e^{-\lambda}(2\mu''-\lambda'\mu'+\tfrac{3}{2}\mu'^2)+\tfrac{1}{2}e^{-\nu}(\tfrac{1}{2}\dot{\mu}^2+\dot{\lambda}\dot{\mu})+e^{-\mu}, \tag{9}$$

$$0 = \dot{\mu}\mu'-\dot{\lambda}\mu'+2\dot{\mu}'-\dot{\mu}\nu'. \tag{10}$$

We shall now obtain a further set of equations that are equivalent to equations (7), (8), (9) and (10) but are physically more illuminating. We first introduce a new variable $m(r, t)$ defined by the following equation

$$8m = \dot{\mu}^2 e^{3\mu/2-\nu}+4e^{\mu/2}-\mu'^2 e^{3\mu/2-\lambda}. \tag{11}$$

Equations (7) and (9) with the aid of (10) then give

$$2\pi\rho\mu' e^{3\mu/2} = m', \tag{12}$$

$$2\pi p\dot{\mu} e^{3\mu/2} = -\dot{m}, \tag{13}$$

and it follows that $m(r, t)$ denotes the mass from O out to r at time t. For, if we write R for $e^{\mu/2}$ in equation (12), we find that $m'=4\pi\rho R^2 R'$. Equation (13) is the energy equation for the rate of work of the pressure. From equations (7), (8), (9 and (10) we may derive the equations

$$p' = -\tfrac{1}{2}\nu'(p+\rho), \tag{14}$$

$$\dot{\rho} = -(\dot{\mu}+\tfrac{1}{2}\dot{\lambda})(p+\rho). \tag{15}$$

Alternatively, these two equations can be derived from the conservation equations,

$$T^{\mu}_{\nu\,;\,\mu} = 0.$$

They can be interpreted physically as follows. An element of volume bounded by the same particles is $\delta v = e^{\lambda/2+\mu}\delta r\delta\Omega^2$, and hence $(\delta v)^{\cdot} = (\dot{\mu}+\frac{1}{2}\dot{\lambda})\delta v$. Consequently, equation (15) signifies that $(\rho\delta v)^{\cdot} + p(\delta v)^{\cdot} = 0$, i.e. conservation of mass-energy. Equation (14) corresponds to conservation of linear momentum.

Equations (11), (12), (13), (14) and (15) are equivalent to equations (7), (8), (9), and (10), for from the former set we can recover the latter, provided that $\dot{\mu} \neq 0$. For, from equations (12), (13), (14) and (15) we can recover equation (10). Using equations (10) and (11), we find that equations (12) and (13) reduce to (9) and (7), respectively. Equation (8) is then recovered from these equations with the aid of either (14) or (15). In the case $\dot{\mu} = 0$, the model is static, as is easily seen from equations (13) and (12), and this case we are not considering.

Two equations that are particularly useful in the case when $\dot{\lambda} = \dot{\mu}$ can be derived as follows. First, by equating the respective right hand sides of equations (7) and (8) and substituting for ν' from equation (10), we find that

$$4\{(\mu' e^{-\lambda/2})' e^{\mu}\}^{\cdot} + (1-\dot{\lambda}/\dot{\mu})(\mu'^2 e^{-\lambda})^{\cdot} e^{\mu+\lambda/2} + 4\dot{\mu}e^{\lambda/2} + 2\dot{\mu}e^{-\nu/2}\{(\dot{\mu}-\dot{\lambda})e^{\mu+\lambda/2-\nu/2}\}^{\cdot} = 0, \tag{16}$$

which reduces to an easily integrable equation when $\dot{\lambda} = \dot{\mu}$. Second, we shall find that

$$[3e^{\mu+\lambda/2}(\bar{\rho}-\rho)]^{\cdot} = \dot{\bar{\rho}}(1-\dot{\lambda}/\dot{\mu})e^{\mu+\lambda/2}, \tag{17}$$

where $\bar{\rho}$ is defined by

$$m = \tfrac{4}{3}\pi\bar{\rho}e^{3\mu/2}. \tag{18}$$

To obtain equation (17), we combine (18) and (13) to get

$$2\pi p\dot{\mu}e^{3\mu/2} = -\dot{m} = -\frac{4\pi}{3}(\dot{\bar{\rho}} + \tfrac{3}{2}\dot{\mu}\bar{\rho})e^{3\mu/2},$$

whence

$$-p = \bar{\rho} + 2\dot{\bar{\rho}}/3\dot{\mu}.$$

Hence,

$$\frac{2\dot{\bar{\rho}}}{3\dot{\mu}}(\dot{\mu}+\tfrac{1}{2}\dot{\lambda}) + \bar{\rho}(\dot{\mu}+\tfrac{1}{2}\dot{\lambda}) = -p(\dot{\mu}+\tfrac{1}{2}\dot{\lambda}).$$

On rewriting equation (15) in the form

$$-p(\dot{\mu}+\tfrac{1}{2}\dot{\lambda}) = \dot{\rho} + \rho(\dot{\mu}+\tfrac{1}{2}\dot{\lambda}),$$

it is easily seen that

$$(\bar{\rho}-\rho)(\dot{\mu}+\tfrac{1}{2}\dot{\lambda}) + (\dot{\bar{\rho}}-\dot{\rho}) = \tfrac{1}{3}\dot{\bar{\rho}}(1-\dot{\lambda}/\dot{\mu}),$$

and from this equation we can immediately obtain equation (17).

It can now be shown that the condition that $\dot{\lambda} = \dot{\mu}$ implies that

$$\rho' e^{3\mu/2} = \text{function of } r, \tag{19}$$

and conversely. For, $\dot\lambda = \dot\mu$ implies that

$$(\bar\rho - \rho)e^{\mu+\lambda/2} = \text{function of } r,$$

and hence, since $\lambda = \mu +$ some function of r, we find that

$$(\bar\rho - \rho)e^{3\mu/2} = \text{function of } r.$$

Hence,

$$m - \frac{4\pi}{3}\rho e^{3\mu/2} = g(r),$$

say. Consequently,

$$m' - \frac{4\pi}{3}(\rho' + \tfrac{3}{2}\mu'\rho)e^{3\mu/2} = g'(r)$$

and so, using equation (12) for m', we get

$$\frac{4\pi}{3}\rho' e^{3\mu/2} = -g'(r),$$

signifying that $\rho' e^{3\mu/2}$ is a function of r.

Conversely, if we suppose that

$$\rho' e^{3\mu/2} = \text{function of } r,$$

then, using equation (12),

$$m' - \frac{4\pi}{3}(\rho' + \tfrac{3}{2}\mu'\rho)e^{3\mu/2} = \text{function of } r.$$

Hence, since $m(0) = 0$ because $e^{\mu(0)} = 0$, we find that

$$m - \frac{4\pi}{3}\rho e^{3\mu/2} = h(r), \tag{20}$$

say. From this it follows by (18) that

$$(\bar\rho - \rho)e^{3\mu/2} = 3h/4\pi,$$

and hence that

$$3(\bar\rho - \rho)e^{\mu+\lambda/2} = (9h/4\pi)e^{(\lambda-\mu)/2}.$$

On differentiating with respect to t, we obtain

$$[3(\bar\rho - \rho)e^{\mu+\lambda/2}]^{\cdot} = (9h/8\pi)(\dot\lambda - \dot\mu)e^{(\lambda-\mu)/2}. \tag{21}$$

We now show that, unless $\dot\lambda = \dot\mu$, we obtain a contradiction. For, if $\dot\lambda \neq \dot\mu$, it follows on comparing (21) with (17) that

$$-(9h/8\pi)e^{(\lambda-\mu)/2} = (\dot{\bar\rho}/\dot\mu)e^{\mu+\lambda/2},$$

and hence that

$$-(9h/8\pi)\dot\mu e^{-3\mu/2} = \dot{\bar\rho}.$$

Consequently,

$$(3h/4\pi)e^{-3\mu/2} = \bar\rho + k(r),$$

and so, using (18),

$$h = m + \frac{4\pi}{3}k(r)e^{3\mu/2}.$$

But equation (20) states that

$$h = m - \frac{4\pi}{3}\rho e^{3\mu/2}.$$

From this it would follow that $\rho = -k(r)$, and hence that the sphere is static, implying that $\dot{\lambda} = 0 = \dot{\mu}$, which contradicts the assumption that $\dot{\lambda} \neq \dot{\mu}$.

Hence, $\dot{\lambda} = \dot{\mu}$ implies that $\rho' e^{3\mu/2}$ is a function of r, and conversely. A particular case is given by $\rho = \rho(t)$, for then $\rho' = 0$ and so the condition is satisfied. We thus find that $\rho = \rho(t)$ is a sufficient, but not a necessary, condition that $\dot{\lambda} = \dot{\mu}$.

On inserting the condition $\dot{\lambda} = \dot{\mu}$ in equation (10), we find that

$$e^{\nu} = \dot{\mu}^2 F(t), \tag{22}$$

where $F(t)$ is an arbitrary function of integration. Inserting the condition $\dot{\lambda} = \dot{\mu}$ in equation (16) leads to the equation*

$$2\mu'' - \mu'^2 + 4e^{-a} + \mu' a' = be^{-\mu/2}, \tag{23}$$

where $a = a(r) = \mu - \lambda$, $b = b(r)$ are also arbitrary functions of integration. Equation (23) is an ordinary differential equation for μ which requires only the specification of a, b to yield a solution.

To see the physical significance of a and b, we evaluate ρ' from equation (9) and obtain the equation

$$8\pi\rho' = -\tfrac{1}{2}(be^{a})' e^{-3\mu/2}. \tag{24}$$

If, for example, the radial coordinate r were chosen so that $e^{\mu} = r^2$ at an initial epoch $t = 0$ at which the energy–density distribution is known (but can be arbitrary), then equations (23) and (24) would suffice to determine the functions a, b.

4. *Uniform-density models.* We now consider a particular solution of equation (23) for which the *initial* density-distribution is uniform, i.e. $\rho'(r, 0) = 0$. Then, from equation (24),

$$be^{a} = \alpha, \tag{25}$$

where α is a constant. Clearly, the motion will be such that the density remains uniform throughout the body, although it is a function of time. The actual value of α does not depend on the initial conditions (density and velocity distributions) but only on the choice of coordinates. If coordinates are chosen so that e^{λ}, $e^{\mu}r^{-2}$ and e^{ν} are all analytic near the centre, it can be shown that

$$\left.\begin{aligned} e^{a} &= r^2(1 + a_1 r + a_2 r^2 + \ldots), \\ b &= b_0 + b_1 r + b_2 r^2 + \ldots . \end{aligned}\right\} \tag{26}$$

To obtain these results, we substitute the condition $\mu = \lambda + a$ in the form of th metric to get

$$ds^2 = -e^{\lambda}(dr^2 + e^{a} d\Omega^2) + e^{\nu} dt^2.$$

By considering the 3-space cross-section near $r = 0$, it is clear that e^{a} must be of th form

$$e^{a} = r^2(1 + a_1 r + a_2 r^2 + \ldots).$$

* The solution obtained by McVittie (4) corresponds to a particular solution of equati (23).

Also, since e^{λ} must be finite at $r=0$,

$$\lambda = L_0(t) + L_1(t)r + L_2(t)r^2 + \ldots .$$

Hence, using $\mu = \lambda + a$ in equation (23), it follows that b must be of the form

$$b = b_0 + b_1 r + b_2 r^2 + \ldots ,$$

and therefore be^{a} is zero at $r=0$. Thus, it follows that for (25) to be satisfied we must have

$$\alpha = 0,$$

and consequently,

$$b = 0.$$

We thus find that $\rho = \rho(t)$ implies $b=0$. Conversely, from equation (24) we immediately see that $b=0$ implies $\rho=\rho(t)$.

Equation (23), with $b=0$, when integrated twice gives

$$e^{-\mu/2} = Bf + C/f, \tag{27}$$

where

$$f = f(r) = \exp\left(-\int e^{-a/2} dr\right), \tag{28}$$

and $B=B(t)$, $C=C(t)$ are arbitrary functions of integration. Equation (27) yields the general solution in terms of which the parameters λ, ν and p, ρ can be readily evaluated for motion which is such that at each instant the density throughout the body is uniform. The function $f(r)$ is determined from the definition of the radial coordinate (i.e. the value of $\mu(r, 0)$): for example, if

$$e^{\mu(r,0)} = r^2,$$

f is given by

$$\frac{1}{r} = B_0 f + \frac{C_0}{f},$$

where B_0, C_0 are constants.

The functions $B(t)$, $C(t)$ are related by an equation derived from the boundary condition $p(r_s)=0$, where $r=r_s$ at the surface. Although, corresponding to a certain degree of freedom in the choice of equation of state, the functions B, C are not determined uniquely by conditions at the boundary $r=r_s$, there are certain conditions to be imposed on them if a realistic model is sought. In particular, we should expect that p must increase for decreasing values of r, that is $p'<0$. We shall find that this implies that, so long as the model contracts, $(C/B)^{\cdot} > 0$. This condition, together with $p(r_s)=0$, will be shown to imply that collapse to zero volume must occur in a finite time.

From equation (14), rewritten in the form

$$(pe^{\nu/2})' = -\rho(e^{\nu/2})',$$

it follows that, since $\rho=\rho(t)$,

$$pe^{\nu/2} = -\rho e^{\nu/2} + L,$$

where $L=L(t)$ is an arbitrary function of integration. Since $p(r_s)=0$, we find that

$$p = \rho\{\exp \tfrac{1}{2}(\nu(r_s) - \nu) - 1\}. \tag{29}$$

Similarly, the density ρ is found from (9) to be given by

$$8\pi\rho = \tfrac{3}{4}(\dot{\mu}^2 e^{-\nu} + 16BC). \tag{30}$$

If we define the time-coordinate so that

$$e^{\nu(0,\,t)} = 1$$

then, from equation (22), it follows that

$$\dot{\mu}^2 e^{-\nu} = \{\dot{\mu}(0, t)\}^2.$$

From equation (27) we find that

$$\dot{\mu} = -2\left(\frac{\dot{B}f^2 + \dot{C}}{Bf^2 + C}\right).$$

If we now substitute in equation (28) for e^{α} from equations (26), we find that

$$f = S/r,$$

where S is a power series in r, involving only positive and zero powers of r. Hence

$$\dot{\mu}(0, t) = -2\dot{B}/B, \tag{31}$$

and so equation (30) becomes

$$8\pi\rho = 3\{(\dot{B}/B)^2 + 4BC\}. \tag{32}$$

The boundary condition $p(r_s) = 0$ applied to equation (15) yields, with the aid of (27), a relation that may be written in the form

$$8\pi\rho = 3K^2\{Bf^2(r_s) + C\}^3, \tag{33}$$

where K is a constant. On comparing (32) and (33), we get

$$(\dot{B}/B)^2 + 4BC = K^2\{Bf^2(r_s) + C\}^3. \tag{34}$$

This is the equation relating B and C.

Furthermore, it is easily seen that the function B must always be positive. For, since the series for $e^{-\mu/2}$ must begin with a positive term in $1/r$ so that space-time can be locally Minkowskian near the origin, it follows from equations (27) that, as $f > 0$ from equation (28), we must have $B > 0$.

It will now be shown that, because in a realistic model pressure must increase towards the centre, and hence $p' < 0$, we always have

$$\frac{d}{dt}\left(\frac{C}{B}\right) > 0$$

in the case of contraction. This result can be established in the following way.

From equation (14), since p, $\rho > 0$, $p' < 0$ implies that $\nu' > 0$. Because $\dot{\lambda} = \dot{\mu}$, equation (10) gives

$$\nu' = 2\dot{\mu}'/\dot{\mu}.$$

Since we are considering contracting material, $\dot{\mu} < 0$ and it follows that $\dot{\mu}' < 0$, because $\nu' > 0$. Evaluating $\dot{\mu}'$ from equation (27), we find that

$$\dot{\mu}' = -\frac{2}{(Bf^2 + C)^2}\{2ff'(\dot{B}C - \dot{C}B\},$$

where

$$2ff' = (f^2)' = -2e^{-a/2} \exp\{-2\int e^{-a/2}dr\} < 0.$$

Hence

$$\dot{B}C - \dot{C}B < 0,$$

and since $B \neq 0$, (for we have shown that $B > 0$), it follows that

$$\frac{d}{dt}\left(\frac{C}{B}\right) > 0. \tag{35}$$

It will now be shown that equation (34) together with condition (35) imply that, provided that the model continues to contract,* it will collapse to zero volume in a *finite* time, as reckoned by a co-moving observer. To prove this we will show that in the case of continual contraction the assumption that both B and C are finite for all finite values of t would lead to a contradiction.

From condition (35) it follows for indefinitely increasing t that C/B either tends to a constant value k or diverges positively. We consider these two cases in turn. First, if C/B tends to k, we find that we must consider separately the possibilities that $k > 0$, $k = 0$, and $k < 0$:

(i) If $k > 0$, as t increases B will be asymptotic to a solution of the differential equation

$$(\dot{A}/A)^2 + 4kA^2 = K^2A^3\{f^2(r_s) + k\}^3. \tag{36}$$

The general solution of equation (36) can be expressed in the form

$$A = \frac{4k}{K^2\{f^2(r_s)+k\}^3} \sec^2\theta, \tag{37}$$

where

$$\frac{K^2\{f^2(r_s)+k\}^3}{8k^{3/2}}(\theta + \tfrac{1}{2}\sin 2\theta) = t - t_1, \tag{38}$$

t_1 being a constant of integration. We observe that A is infinite for $\theta = (2n+1)\pi/2$ where n is an integer or zero. For these values of θ, equation (38) gives

$$t - t_1 = \frac{K^2\{f^2(r_s)+k\}^3}{8k^{3/2}}(2n+1)\frac{\pi}{2}. \tag{39}$$

It immediately follows that as t increases B will be asymptotic to a function that diverges for each of the infinite sequence of finite values of t given by equation (39). This contradicts our assumption that B is finite for all finite values of t.

(ii) If $k = 0$, B will be asymptotic to a solution of

$$\dot{A}^2A^{-5} = K^2f^6(r_s),$$

and hence to a function A for which we have

$$A^3 = \frac{1}{\{t_2 - \frac{3}{2}Kf^3(r_s)t\}^2}, \tag{40}$$

where t_2 is a constant of integration. For values of t given by

$$t < \frac{2t_2}{3Kf^3(r_s)}, \tag{41}$$

* It seems possible that in some cases the radius of the model may oscillate, but this question will not be considered further in the present paper.

A is an increasing diverging function, and for values of t exceeding the right hand side of (41) it is a decreasing function. If the model continually contracts, we see from (31), by considering points near the centre, that $\dot{B}$ must be positive and hence B cannot be asymptotic to a decreasing function of time. Hence, there is no solution for B consistent with our assumptions.

(ii) If $k<0$, B will be asymptotic to a positive solution of the differential equation (36) with k negative. The general solution of this equation is then given by

$$A=\frac{-16k}{K^2\{f^2(r_s)+k\}^3}\left\{\frac{u^2}{(1-u^2)^2}\right\}, \tag{42}$$

where

$$\tfrac{1}{2}u^2-2\log u-\frac{1}{2u^2}=\frac{16(-k)^{3/2}}{K^2\{f^2(r_s)+k\}^3}(t-t_3), \tag{43}$$

and t_3 is a constant of integration. For values of t in the range $-\infty<t<t_3$, we have $0<u<1$ and A is an increasing function that diverges as t tends to t_3. For $t>t_3$, $u>1$ and A is a decreasing function. Consequently, as in case (ii), there is no solution for B consistent with our assumptions.

Finally, we show that the alternative condition that C/B diverges positively as t increases indefinitely leads to a contradiction. For, in this case, the relation between B and C given by equation (34) is asymptotic to the relation

$$\dot{B}/B=KC^{3/2}, \tag{44}$$

the term on the left being slightly larger than that on the right for large values of t. From equation (31) we have already deduced that $\dot{B}$ must be positive for contraction and hence we must take K positive in equation (44). Moreover, since

$$\dot{B}C-\dot{C}B<0,$$

the behaviour of C for large values of t will be subject to the inequality

$$\dot{C}>KC^{5/2}. \tag{45}$$

Consequently, these must be an epoch T after which (45) is always valid and hence

$$\int_{C(T)}^{\infty}\frac{dC}{C^{5/2}}>\int_{T}^{\infty}K\,dt, \tag{46}$$

since C diverges positively with t. But (46) cannot be valid, since the left hand side is finite and the right hand side is positively infinite. We can therefore rule out the possibility that C/B diverges positively, since it leads to a contradiction.

This completes the proof that, if the model continually contracts, the functions $B(t)$ and $C(t)$ satisfying equation (34) and condition (35) cannot both remain finite for all finite values of t. From (27) and (32) we deduce that the model must collapse to zero volume at some finite value of t.

On the other hand, according to an external observer, in a reference system that is at rest with respect to O, the model will appear to contract asymptotically to its gravitational radius $2M$, where M is its mass. For, if the model continually collapses without radial oscillation, there will be a value t_0 of t such that, corresponding to any given value r_0 of r, $R(r_0, t_0)=2m(r_0, t_0)$, where $R=e^{\mu/2}$ and m is given by equation (11). Consequently, using equation (11), we find that

$$\dot{R}e^{-\nu/2}=R'e^{-\lambda/2},$$

at $r=r_0$, $t=t_0$. If we consider a hypothetical observer P at a fixed distance from O such that the particle of the model coincident with P at $t=t_0$ has its r-coordinate given by r_0, then at time t_0+dt the particle of the model coincident with P will have its r-coordinate given by r_0+dr, where

$$\frac{dR}{dt}e^{-\nu/2}=\frac{dR}{dr}e^{-\lambda/2},$$

and dR/dt, dR/dr are evaluated at (r_0, t_0). Consequently,

$$\frac{dr}{dt}=e^{(\nu-\lambda)/2},$$

and so dr/dt will be equal to the coordinate velocity of light. In particular, by taking $r_0=r_s$, the surface value of r, we deduce that according to the observer at P (and hence according to any external observer) the model will appear to collapse asymptotically to its gravitational, or Schwarzschild, radius.

5. *Conclusion.* If a cold spherically symmetrical distribution of matter acted on only by its own gravitational field and a non-vanishing internal pressure gradient continually contracts in such a way that its density at each instant is uniform, it will collapse to a point-singularity of infinite density in a finite time, according to a co-moving observer, and asymptotically to its gravitational radius according to an external observer in a reference frame not attached to the model.

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