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TIME-DEPENDENT INTERNAL SOLUTIONS FOR SPHERICALLY SYMMETRICAL BODIES IN GENERAL RELATIVITY

II. Adiabatic Radial Motions of Uniformly Dense Spheres

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Summary

A differential equation is derived for the radius of a spherically symmetrical body of uniform density as a function of time t, for any arbitrary equation of state of the material at the centre, the equation of state elsewhere being determined by that at the centre through the condition $\rho = \rho(t)$. This differential equation can be applied to cases where the body may oscillate or continually contract or expand. In Paper I of this series we overlooked the possibility of asymptotic contraction to a radius $R > 9GM/4c^2$, where M is the mass. If, however, a stage is reached for which $R < 9GM/4c^2$, collapse to zero volume must occur. We have applied our analysis to bodies in which the material at the centre obeys a polytropic equation of state and obtained general formulae for determining the radial motion uniquely for any polytropic index and initial conditions.

1. *Introduction.* In the latter part of the previous paper (1) we considered the adiabatic contraction of a spherically symmetrical body the density of which is uniform throughout at each instant t. We confined attention to the case in which the body continually contracts, but we overlooked* the possibility of asymptotic contraction to a radius greater than $9GM/4c^2$. In the present paper we adopt a completely general approach to the problem, except for the condition that $\rho = \rho(t)$. Physically, this means that we are free to choose the equation of state of the material at any one given point.

2. *Fundamental equations.* From the previous paper, we find that the internal metric can be expressed in the form

$$ds^2 = -e^{\lambda}dr^2 - e^{\mu}d\Omega^2 + e^{\nu}dt^2, \tag{1}$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2,$$

and μ is given by

$$e^{-\mu/2} = Bf + C/f, \tag{2}$$

where B, C are arbitrary functions of t whose physical significance will be discussed below and f is an arbitrary† function of r, depending only on the definition of the radial coordinate,

$$\lambda = \mu - a, \tag{3}$$

$a = a(r)$ being related to f by

$$f = \exp\left(-\int e^{-a/2}dr\right), \tag{4}$$

* After we had come to the conclusion that our analysis following equation (34) was faulty, we found that our mistake had been discovered independently by H. Bondi.

† We could take $f(r) = 1(r)$, and hence $e^{\lambda} = e^{\mu}/r^2$, without loss of generality.

and

$$e^{\nu} = \dot{\mu}^2 F(t), \tag{5}$$

where F is an arbitrary function of t depending only on the definition of the time coordinate. For convenience, we identify t as the proper time of an observer permanently situated at the centre, so that F is given by

$$F = 1/\dot{\mu}^2(0, t) = B^2/4\dot{B}^2, \tag{6}$$

in accordance with equation (31) of Paper I.

According to equation (29) of Paper I, the pressure is given by

$$p = \rho\{\exp \tfrac{1}{2}(\nu(r_s) - \nu) - 1\}, \tag{7}$$

where the suffix s refers to the surface value. Since the pressure at the surface is taken to be zero, it follows that there is a relation between B and C, given by equation (34) of Paper I, namely

$$\left(\frac{\dot{B}}{B}\right)^2 + 4BC = K^2(B\kappa^2 + C)^3, \tag{8}$$

where

$$\kappa = f(r_s), \tag{9}$$

and K is a constant.

We shall now give a more thorough and accurate analysis replacing that following equation (34) in Paper I. In particular, we shall show that the equation of state of the material at the centre provides a further equation relating B and C, hence enabling us to obtain a unique solution given the arbitrary initial conditions* of the problem.

3. *Conditions for a realistic model.* We begin by recalling that for the space–time to be locally Minkowskian near the centre, B must be positive.

We shall now investigate what other restrictions must be placed on B, C in order to obtain a realistic model. We shall impose the following conditions,† except possibly at the end-point of collapse:

(a) $0 < \rho < \infty$,

(b) $0 \leqslant p < \infty$,

(c) $-\infty < p' \leqslant 0$,

(d) $e^{\mu/2} = R(r, t) > 0$, for $r > 0$,

where the prime symbol denotes, as previously, differentiation with respect to r.

It is convenient to introduce a new variable $y(t)$ defined by the equation

$$y = \frac{C}{B} + \kappa^2, \tag{10}$$

and to work from now on in terms of B and y, instead of B and C. From equation (2), it follows that

$$e^{-\mu/2} = \frac{1}{R} = Bf + C/f = \frac{B(y + f^2 - \kappa^2)}{f}, \tag{11}$$

* The reference in the Summary of Paper I to 'an initial non-zero velocity distribution' is unnecessary.

† Professor Bondi has drawn our attention to the fact that condition (c) could be violated without infringing the fundamental laws of physics, but we shall not consider this possibility here.

and so, by equation (9),

$$R(r_s, t) = R_s = \kappa/By. \tag{12}$$

Since

$$f' = -e^{-a/2} \exp\left(-\int e^{-a/2}dr\right) < 0,$$

and B must be positive, it follows that a necessary and sufficient condition for R to be positive is that y is positive. On rewriting equation (33) of Paper I in the form

$$8\pi\rho = 3K^2B^3y^3, \tag{13}$$

we see that since $B > 0$, necessary and sufficient conditions that $0 < \rho < \infty$ are that $0 < B < \infty$ and $0 < y < \infty$.

Since we shall impose the condition that $p(r_s, t) = 0$, condition (b) will automatically follow from condition (c).

On substituting for ν from equation (5) in equation (7) we get

$$p = \rho\left\{\frac{\dot{\mu}(r_s, t)}{\dot{\mu}(r, t)} - 1\right\}. \tag{14}$$

It immediately follows from this equation that

$$p' = -\frac{\dot{\mu}'}{\dot{\mu}}(p+\rho).$$

From equation (11) we deduce that

$$-\tfrac{1}{2}\dot{\mu}(r, t) = \frac{\dot{B}(y+f^2-\kappa^2+B\dot{y}/\dot{B})}{B(y+f^2-\kappa^2)}, \tag{15}$$

and hence

$$\dot{\mu}'(r, t) = \frac{4ff'\dot{y}}{(y+f^2-\kappa^2)^2}. \tag{16}$$

Consequently,

$$p' = \frac{2ff'B(p+\rho)}{(y+f^2-\kappa^2)(y+f^2-\kappa^2+B\dot{y}/\dot{B})}\frac{\dot{y}}{\dot{B}}.$$

Since $f' < 0$ and $f \to \infty$ as $r \to 0$, it follows that, since $p + \rho > 0$ at $r = r_s$, a necessary and sufficient condition that $-\infty < p' \leqslant 0$ for all r is that $0 \leqslant \dot{y}/\dot{B} < \infty$.

We therefore conclude that the following conditions on B and y are both necessary and sufficient to ensure that the model is ' realistic ' in the sense defined by (a), (b), (c) and (d) above:

(i) $0 < B < \infty$,

(ii) $0 < y < \infty$,

(iii) $0 \leqslant \dot{y}/\dot{B} < \infty$.

4. *Inequalities for models that are instantaneously static.* With the aid of equation (10), equation (8) can be rewritten as

$$\frac{\dot{B}^2}{B^4} - 4\kappa^2 + 4y = K^2By^3, \tag{17}$$

and this cubic equation for y is to be solved subject to conditions (i), (ii) and (iii) above. If we write equation (17) in the standard form

$$y^3 + p^* y + q^* = 0, \tag{18}$$

then

$$p^* = -\frac{4}{K^2 B}, \tag{19}$$

and

$$q^* = \frac{4\kappa^2}{K^2 B} - \frac{\dot{B}^2}{K^2 B^5}. \tag{20}$$

The nature of the solutions of (18) depends on the sign of the discriminant D given by

$$D = -4p^{*3} - 27q^{*2}. \tag{21}$$

If $D > 0$, there are three distinct real roots given by

$$y = 2\sqrt{\left(-\frac{p^*}{3}\right)} \cos\left(\frac{\theta}{3} + \frac{2n\pi}{3}\right), \tag{22}$$

where

$$0 < \theta = \tan^{-1}\left\{-\frac{2}{q^*}\sqrt{\left(\frac{D}{108}\right)}\right\} < \pi,$$

and $n = 0, 1, 2$. If $D = 0$, y is again given by equation (22), but with $\theta = 0$ or π, depending on the sign of q^*, and there are only two distinct roots. If $D < 0$, there is only one real root and this is given by

$$y = \left\{-\frac{1}{2}q^* + \sqrt{\left(\frac{-D}{108}\right)}\right\}^{1/3} + \left\{-\frac{1}{2}q^* - \sqrt{\left(\frac{-D}{108}\right)}\right\}^{1/3}. \tag{23}$$

We now consider models that are initially static, i.e. models for which $\dot{B}$, $\dot{y} = 0$ for some value t_0 of t. We shall use the suffix zero to denote values at $t = t_0$. From equation (20) we see that $q_0^* > 0$, and hence we deduce that $D_0 \geqslant 0$, for if $D_0 < 0$ it would follow from equation (23) that y_0 would be negative, contrary to condition (ii) for a realistic model. Also, since $q_0^* > 0$, it follows that $\pi/2 < \theta_0 \leqslant \pi$, and hence, from equation (22),

$$y_0^2 = -\alpha^2 p_0^*,$$

where

$$0 \leqslant \alpha^2 < 1.$$

From this inequality, we can obtain another inequality that is analogous to the well-known inequality $R_s^2 \leqslant 1/(3\pi\rho)$, or equivalently $M/R_s \leqslant 4/9$, that applies to the permanently static (equilibrium) uniform density Schwarzschild model. For, on substituting for $R_s(t)$ and $\rho(t)$ from equations (12) and (13), respectively, we find that the total mass M given by*

$$M = \tfrac{4}{3}\pi\rho(t)\, R_s^3(t)$$

becomes

$$M = \tfrac{1}{2}\kappa^3 K^2. \tag{24}$$

Hence, using equation (19) with $y_0^2 = -\alpha^2 p_0^*$, it follows that

$$\frac{M^2}{R_s^2(t_0)} = \tfrac{1}{4}\kappa^4 K^4 B_0^2 y_0^2 = \alpha^2 \kappa^4 K^2 B_0. \tag{25}$$

* Justification of this formula is given in the Appendix.

However, from equations (19), (20) and (21), it is clear that $D_0 \geqslant 0$ implies that

$$\kappa^4 K^2 B_0 \leqslant \frac{16}{27}.$$

Hence, we deduce from equation (25) that

$$\frac{M^2}{R_s^2(t_0)} \leqslant \frac{16\alpha^2}{27}, \tag{26}$$

and since $0 \leqslant \alpha^2 < 1$, it follows, for any model that is static at $t = t_0$, that

$$\frac{M}{R_s(t_0)} < \frac{4}{3\sqrt{3}}. \tag{27}$$

Since

$$M = \frac{4\pi}{3}\,\rho(t)\,R_s^3(t),$$

we can write equation (27) in the form

$$R_s^2(t_0) < \frac{1}{\sqrt{3}\pi\rho_0}. \tag{28}$$

Furthermore it is easily shown that, if initially the model contracts slowly through a sequence of quasi-equilibrium configurations, so that $\dot{B} \sim 0$, $\ddot{B} \sim 0$, the condition (27) can be sharpened by replacing the right hand side by 4/9. For, on differentiating equation (18) and using equations (19), (20) and (17), we obtain

$$\begin{aligned} \dot{y}(3y^2 + p^*) &= -\dot{p}^* y - \dot{q}^* \\ &= \frac{\dot{B}}{K^2 B}\left(\frac{2\ddot{B}}{B^4} - \frac{4\dot{B}^2}{B^5} - K^2 y^3\right). \end{aligned} \tag{29}$$

Since, for $t \sim t_0$, we have postulated that $\dot{B} \sim 0$, $\ddot{B} \sim 0$, and since conditions (i), (ii) and (iii) hold generally, it follows from equation (29) that

$$3y_0^2 + p_0^* < 0.$$

Since $y_0^2 = -\alpha^2 p_0^*$, and $p_0^* < 0$, it follows that $\alpha^2 < \frac{1}{3}$. Consequently, from equation (26) we deduce that condition (27) can be replaced by the sharper condition

$$\frac{M}{R_s(t_0)} < \frac{4}{9}, \tag{30}$$

in agreement with the result obtained by H. Bondi who, in considering the slow contraction of models of uniform density, found that γ, defined by

$$\frac{\delta p}{p} = \gamma\,\frac{\delta\rho}{\rho},$$

diverged to infinity as $M/R \to 4/9$.

5. *A sufficient condition for collapse.* We will now show that the inequalities

$$\frac{4}{3\sqrt{3}} > \frac{M}{R_s(t_0)} > \frac{4}{9}$$

provide a sufficient condition that collapse to a point-singularity of infinite density must occur in a finite time. For, in this case $\frac{1}{3} < \alpha^2 < 1$ and, since $y_0{}^2 = -\alpha^2 p_0{}^*$, it follows that $3y_0{}^2 + p_0{}^* > 0$. By condition (iii), $\dot{B}$ and $\dot{y}$ must be of the same sign. Consequently, from equation (29) we must have $\ddot{B}_0 > 0$. Therefore, B and y are increasing functions, since initially $\dot{B}_0 = 0 = \dot{y}_0$. Thus, it follows from equation (19) that $3y^2 + p^* > 0$ for all $t > t_0$, and hence, from equation (29), we see that

$$\frac{2\ddot{B}}{B^4} - \frac{4\dot{B}^2}{B^5} - K^2 y^3 > 0,$$

for all $t > t_0$. From this inequality we deduce that

$$\left(\frac{\dot{B}}{B^2}\right)^{\cdot} > \frac{1}{2} K^2 y^3 B^2 > 0,$$

and hence, since $\dot{B} = 0$ for $t = t_0$, that

$$\frac{\dot{B}}{B^2} > \beta_1 > 0,$$

for some positive constant β_1, for all $t > t_1$, where t_1 is some (arbitrarily) chosen value of t later than $t = t_0$.

If we now assume that $B \to B_\infty$ as $t \to \infty$, where B_∞ may be positively finite or infinite, we obtain a contradiction. For the inequality for $\dot{B}/B^2$ implies that

$$\int_{B_0}^{B_\infty} \frac{dB}{B^2} > \int_{t_0}^{\infty} \beta_1 dt,$$

which is impossible, since the right hand side is infinite (because $\beta_1 > 0$) and the left hand side is finite. Hence, since B is an increasing function, the only possibility is that B diverges to infinity for some finite value τ of t. Consequently, from equation (12), it follows that $R_s(\tau) = 0$ and so, from equation (13), $\rho(\tau)$ is infinite.

We therefore see that, if the body contracts to within a surface radius of less than $9GM/4c^2$, it must collapse to a point singularity within a finite proper time.

6. *Differential equation for the radius of the body in the general case.* Since we are considering the general problem of radial motion of a spherically symmetrical body of uniform density, it is clear that we are free to choose the equation of state of the material at any one point, e.g. the centre. This will in fact provide the other equation relating B and y (or, alternatively, B and C), besides equation (17). These two equations will then yield a unique solution, given the initial conditions of the problem.

We take the equation of state at the centre to be of the form

$$u = u(\rho)$$

where $u = \rho(t)/p(0, t)$, or equivalently for a given M,

$$u = u(R_s),$$

since $\rho = 3M/4\pi R_s{}^3$. Consequently, we can eliminate B, y and obtain a differential equation involving only the variables of immediate physical significance: R_s, u and M. From now on we shall drop the suffix s and write R for R_s.

From equation (15), since $f^2(0)$ is infinite, it follows that

$$\frac{\dot{\mu}(r_s, t)}{\dot{\mu}(0, t)} = 1 + \frac{\dot{y}B}{y\dot{B}}, \tag{31}$$

and hence, from equation (14), it follows that

$$\frac{p(0, t)}{\rho(t)} = \frac{\dot{y}B}{y\dot{B}}. \tag{32}$$

From equation (32), u is given by

$$u = u(\rho) = u(R) = \frac{\rho(t)}{p(0, t)} = \frac{y\dot{B}}{\dot{y}B}, \tag{33}$$

and since, by equation (12), $R = \kappa/By$, it follows that

$$\dot{R} = -\frac{\kappa\dot{B}(1+u)}{yB^2u}, \tag{34}$$

and hence that

$$\frac{\dot{B}}{B} = -\frac{yBu\dot{R}}{\kappa(1+u)} = -\frac{u\dot{R}}{R(1+u)}. \tag{35}$$

On multiplying equation (17) by B^2R^2 and substituting for $\dot{B}^2/B^2$ from equation (35), we obtain

$$\frac{u^2\dot{R}^2}{(1+u)^2} - 4\kappa^2B^2R^2 + 4yB^2R^2 = K^2B^3y^3R^2 = \frac{8\pi}{3}\rho R^2 = \frac{2M}{R},$$

by equation (13). On eliminating y, by equation (12), it follows that

$$\frac{u^2\dot{R}^2}{(1+u)^2} - 4\kappa^2B^2R^2 + 4\kappa BR = 2M/R.$$

This equation, regarded as a quadratic in κBR_s, can be solved to give

$$\kappa BR = \frac{1}{2}\left\{1 \pm \sqrt{\left(1 - \frac{2M}{R} + \frac{u^2\dot{R}^2}{(1+u)^2}\right)}\right\}. \tag{36}$$

On taking the logarithm of each side and differentiating with respect to the time, we obtain

$$\frac{\dot{B}}{B} + \frac{\dot{R}}{R} = \frac{d}{dt}\log\left\{1 \pm \sqrt{\left(1 - \frac{2M}{R} + \frac{u^2\dot{R}^2}{(1+u)^2}\right)}\right\}.$$

Finally, on eliminating $\dot{B}/B$ by means of equation (35), we find that

$$\frac{\dot{R}}{R(1+u)} = \frac{d}{dt}\log\left\{1 \pm \sqrt{\left(1 - \frac{2M}{R} + \frac{u^2\dot{R}^2}{(1+u)^2}\right\}}\right. \tag{37}$$

Clearly, given $u = u(R)$ and the initial values of R, $\dot{R}$ and the constant M, equation (37) determines $R(t)$.

7. *Some immediate consequences of equation* (37). On carrying out the differentiation indicated in equation (37), we obtain

$$\frac{1}{1+u} = \frac{\pm\left\{\frac{M}{R} + \frac{uR\dot{u}\dot{R}}{(1+u)^3} + \frac{\ddot{R}Ru^2}{(1+u)^2}\right\}}{\left\{1 \pm \sqrt{\left(1 - \frac{2M}{R} + \frac{u^2\dot{R}^2}{(1+u)^2}\right\}}\right.\sqrt{\left\{1 - \frac{2M}{R} + \frac{u^2\dot{R}^2}{(1+u)^2}\right\}}}. \tag{38}$$

From this equation, we can draw some immediate consequences of interest that provide useful checks on the validity of equation (37):

(i) Clearly, since R, $M > 0$, it follows that, if $\dot{R} \sim 0 \sim \ddot{R}$, we must take the positive signs on the right hand side of equation (38). The value $\bar{u}$ of u required for the body to be in equilibrium, so that $\dot{R} = 0 = \ddot{R}$, is given by

$$\frac{1}{1+\bar{u}} = \frac{M/R}{\{1+\sqrt{(1-2M/R)}\}\sqrt{(1-2M/R)}},$$

and this yields

$$\bar{u} = \frac{3\sqrt{(1-2M/R)}-1}{1-\sqrt{(1-2M/R)}}, \tag{39}$$

in agreement with the value given by the standard solution of the Schwarzschild problem. (We note that $M/R = 4/9$ gives $\bar{u} = 0$ and so $p(0, t)$ infinite.)

(ii) For a body that is instantaneously static, but not in equilibrium, at $t = t_0$, the radial acceleration at this instant is readily found from equation (38) to be given by

$$\frac{1+\bar{u}}{1+u_0} = \frac{M/R+\ddot{R}Ru_0{}^2/(1+u_0)^2}{M/R},$$

where u_0 is the (initial) value of u, $\bar{u}$ is given by equation (39) and R, $\ddot{R}$ denote values at $t = t_0$. Hence,

$$\ddot{R} = \frac{M(1+u_0)(\bar{u}-u_0)}{u_0{}^2R^2}. \tag{40}$$

From this equation, it follows that, according as $u_0 \gtrless \bar{u}$, i.e. as $p(0)$ is less than or greater than the value required to maintain equilibrium, so $\ddot{R} \lessgtr 0$, and (as we should expect) the body begins either to contract or expand.

(iii) We observe that, in the pressure-free case, since u_0 in equation (40) is infinite, it follows that

$$\ddot{R} = -M/R^2,$$

in accordance with standard theory. Moreover, on putting u infinite in equation (37), we get

$$1-\frac{2M}{R}+\dot{R}^2 = \text{constant},$$

which is the well-known formula for radial motion in the pressure-free model.

8. *Relativistic central-polytropes.* We propose now to apply our analysis to the case where the material *at the centre* obeys the polytropic equation of state

$$pV^\gamma = \text{constant}, \tag{41}$$

where V is a differential element of volume and γ is constant. This equation reduces to the form $p\rho^{-\gamma} = $ constant when $p \ll \rho c^2$, since ρV is constant in Newtonian theory. However, in relativity, the work done by the pressure contributes to the density. Since

$$(\rho V)^{\cdot} = -p\dot{V},$$

it follows that

$$\delta p = \frac{\gamma p}{\rho+p}\,\delta\rho, \tag{42}$$

where δp, $\delta\rho$ are changes in the pressure and density due to compression or expansion. Incidentally, it would appear that we should take equation (41) as the relativistic polytropic equation of state, since $p\rho^{-\gamma}$ = constant leads to a velocity of sound that increases indefinitely with increasing ρ and so ultimately exceeds the local velocity of light.

Since $u = \rho(t)/p(0, t)$, it follows from equation (42) that

$$\gamma = (1+u)\frac{d}{d\rho}\left(\frac{\rho}{u}\right) = \frac{1+u}{u}\left(1-\frac{\rho\dot{u}}{\dot{\rho}u}\right) = \frac{1+u}{u}\left(1+\frac{R\dot{u}}{3\dot{R}u}\right), \tag{43}$$

since $\rho = 3M/4\pi R^3$. Because γ is a constant, equation (43) can be integrated to give

$$R^3 = \frac{\beta^3\{u(\gamma-1)-1\}^{\gamma/(\gamma-1)}}{u} = \frac{\beta^3(u/n-1)^{n+1}}{u}, \tag{44}$$

where β is an arbitrary constant and n is the polytropic index given by $\gamma = 1+1/n$. Equations (37) and (44) enable the solution to be found.

On substituting for $\dot{R}/R$ from equation (43) in the left hand side of equation (37), we obtain

$$\frac{\dot{u}}{3u(u/n-1)} = \frac{d}{dt}\log\left\{1\pm\sqrt{\left(1-\frac{2M}{R}+\frac{u^2\dot{R}^2}{(1+u)^2}\right)}\right\}.$$

On integration this equation gives

$$\frac{u/n-1}{u} = k^3\left\{1\pm\sqrt{\left(1-\frac{2M}{R}+\frac{u^2\dot{R}^2}{(1+u)^2}\right)}\right\}^3, \tag{45}$$

where k is a constant. On substituting for R and $\dot{R}$ in terms of u and $\dot{u}$ from equations (43) and (44), we obtain a first order separable differential equation for u as a function of t. It is convenient to replace u by v, where

$$v = u/n-1, \tag{46}$$

and equation (45) reduces to the form

$$\dot{v} = \pm\frac{3}{n\beta}\sqrt{f}, \tag{47}$$

where

$$f(v) = \frac{1}{k^2}v^{2(1-n/3)} - \frac{2n^{1/3}}{k}(v+1)^{1/3}v^{(5-2n)/3} + \frac{2Mn}{\beta}(v+1)\,v^{(1-n)}. \tag{48}$$

These equations give $v(t)$, and $R(t)$ is given by equation (44) which can be rewritten in the form

$$R^3 = \frac{\beta^3 v^{n+1}}{n(v+1)}. \tag{49}$$

We see from equation (47) that $\dot{v}$ and f vanish together, and from equation (49) that the vanishing of $\dot{v}$ implies the vanishing of $\dot{R}$ and conversely. Consequently the vanishing of f provides a necessary and sufficient condition that the radial motion is instantaneously arrested. In general, this will mean that at the epoch in question the radial motion is reversed.

In any particular model where the material at the centre is subject to a polytropic equation of state (equation (41)), we need only consider equations (47), (48) and (49) with the appropriate values of the various parameters.

9. *Examples of relativistic central polytropes.* To illustrate the method, we shall consider two particular values of γ, namely $\gamma = 3/2$ and $\gamma = 4/3$. In the former case, the motion will depend on the initial conditions and the mass; whereas, in the latter case, a general result is obtained.

(I) $\gamma = 3/2$ $(n = 2)$ *and the motion starts from rest*

In this case, equation (48) gives

$$f(v) = \frac{1}{k^2}\, v^{2/3}\left\{1 - 2^{4/3}k\left(1+\frac{1}{v}\right)^{1/3}\right\} + \frac{4M}{\beta}\left(1+\frac{1}{v}\right). \tag{50}$$

The constants k, β depend on the initial conditions and we shall consider two particular examples:

(a) the motion starts from rest at a radius R_0, given by

$$l = \sqrt{(1-2GM/c^2R_0)} = 0\cdot 99,$$

so that $R_0 = 20\,000/199\ GM/c^2$. We take central pressure to be such that $v_0 = 125$ and hence $u_0 = 252$;

(b) the motion starts from rest at a radius R_0, given by $l = 0\cdot 9$, so that $R_0 = (200/19)\ GM/c^2$, and in this case we shall take $v_0 = 8$, and hence $u_0 = 18$.

In case (a), formula (39) gives $\bar{u} = 197$ and therefore the central pressure is insufficient to maintain equilibrium. The initial inward acceleration is given by equation (40). Since the motion starts from rest, $f(v_0) = 0$, and the constant k is given from equation (45) by

$$k = \frac{1}{\{n(1+1/v_0\}^{1/3}}\left\{\frac{1}{(1+l)}\right\}, \tag{51}$$

and on putting $v_0 = 125$, $n = 2$, and $l = 0\cdot 99$, we can determine k. We get β from equation (49), so that equation (50) gives

$$f(v) = 6\cdot 324\, v^{2/3}\left\{1 - 1\cdot 0023\left(1+\frac{1}{v}\right)^{1/3}\right\} + 0\cdot 7875\left(1+\frac{1}{v}\right). \tag{52}$$

Values of f have been computed for values of v from 10 to 150 at intervals of 10. From this table we see that starting at $v_0 = 125$, for which $f(v) = 0$, the body

Table I

v	$f(v)$
10	−0·151933
20	−0·048124
30	−0·002209
40	+0·020593
50	0·031646
60	0·035853
70	0·035710
80	0·032640
90	0·027506
100	0·020877
110	0·013135
120	0·004547
130	−0·004689
140	−0·014430
150	−0·024565

contracts with surface velocity given by equations (47) and (49), until $v \sim 30$, when the contraction is halted and the body begins to expand again. The body thereafter oscillates between these extreme values of v. The amplitude of the oscillations is given by

$$\frac{R_{\rm max}}{R_{\rm min}} \sim \left(\frac{125}{30}\right)^{2/3} \sim 2 \cdot 6.$$

To obtain an idea of the order of magnitude of the period of oscillation we note that, on taking $v+1 \sim v$ in equation (49), we get

$$\dot{R} = \frac{2^{2/3}}{3} \beta \frac{\dot{v}}{v^{1/3}}.$$

Equation (47) gives

$$\dot{v} = \frac{3}{2\beta} \sqrt{f}, \tag{53}$$

and so, on introducing c explicitly,

$$\dot{R} = \frac{c\sqrt{f}}{(2v)^{1/3}}. \tag{54}$$

From equation (53) it follows that the period of oscillation in seconds is given by

$$\frac{4\beta}{3c} \int_{30}^{125} \frac{dv}{\sqrt{f}},$$

where the lower limit is approximate. (The maximum surface velocity occurs when $v \sim 60$ and is about $0 \cdot 04c$.) We find that in the case of a mass equal to 10^8 times that of the Sun, the period of oscillation is about $3 \cdot 25 \times 10^6$ seconds, i.e. about a month. The mean radius of this model is approximately $1 \cdot 4 \times 10^4$ times the radius of the Sun, and the mean density is approximately $0 \cdot 5 \times 10^{-4}$ g cm^{-3}. (This low density makes the assumption of uniform density somewhat unrealistic in the case of this particular example.)

In case (b), we have $\bar{u} = 17$, and since $v_0 = 8$ it follows that $u_0 = 18$, leading to a small inward acceleration. In general, the body will 'bounce' or collapse depending on whether the inward motion is halted before the radius becomes too small. By the same method as before, we find that in this case,

$$f(v) = 6 \cdot 12\, v^{2/3} \left\{1 - 1 \cdot 012 \left(1+\frac{1}{v}\right)^{1/3}\right\} + 1 \cdot 16 \left(1+\frac{1}{v}\right). \tag{55}$$

Values of f have been computed for various values of v. It is found that for $v > 8$, f is negative (at least in the neighbourhood of $v = 8$) and for $v < 8$, f is always positive and diverges positively as v tends to zero. Hence the body collapses to a point singularity of infinite density. The total time of collapse for a body of mass 10^8 times (and radius 2×10^3 times) that of the Sun can be calculated by the method used for case (a) and we find that it is about 5×10^4 seconds, i.e. about 14 h.

For comparison, we note that for the same mass the time of pressure-free collapse (from the same initial radius) is about $4 \cdot 5$ h.

(II) $\gamma = 4/3$ ($n = 3$) *and the motion starts from rest*

Equation (48) becomes

$$f(v) = \frac{1}{k^2} - \frac{2(3)^{1/3}}{k} \left(1+\frac{1}{v}\right)^{1/3} + \frac{6M}{\beta} \left(1+\frac{1}{v}\right) \frac{1}{v}, \tag{56}$$

where k is given by equation (51) with $n = 3$. We have, from equation (49),

$$\frac{6M}{\beta} = 3^{2/3}(1-l^2)\frac{v_0}{(1+1/v_0)^{1/3}}. \tag{57}$$

On substituting for k and β from equations (51) and (57) in (56), we get

$$\frac{f(v)(1+1/v_0)^{1/3}}{3^{2/3}(1+l)} = \left(1+\frac{1}{v}\right)\left\{(1-l)\frac{v_0}{v} - 2\left(\frac{1+1/v_0}{1+1/v}\right)^{2/3} + \left(\frac{1+1/v_0}{1+1/v}\right)(1+l)\right\}. \tag{58}$$

From equation (39), the value $\bar{u}$ of u_0 to maintain equilibrium is given by

$$\bar{u} = (3l-1)/(1-l),$$

and hence

$$\bar{v} = \tfrac{1}{3}\bar{u} - 1 = (6l-4)/3(1-l).$$

Consequently,

$$l = \frac{3\bar{v}+4}{3\bar{v}+6}. \tag{59}$$

Therefore, on writing

$$w = \frac{1+1/v_0}{1+1/v}, \tag{60}$$

equation (58) becomes

$$\frac{f(v)(1+1/v_0)^{1/3}}{3^{2/3}(1+l)(1+1/v)} = 2w - 2w^{2/3} - \frac{2(w-1)(w+1+v_0)}{3w(\bar{v}+2)} = g(w), \tag{61}$$

say.

The sign of $f(v)$ is that of $g(w)$, the derivative of which is given by

$$\frac{dg}{dw} = 2 - \frac{4}{3w^{1/3}} - \frac{2}{3(\bar{v}+2)}\left\{1 + \frac{1+v_0}{w^2}\right\}. \tag{62}$$

The behaviour of this derivative depends on whether $v_0 \gtrless \bar{v}$, i.e. on whether $p_0 \lessgtr \bar{p}$, where $\bar{p}$ denotes the central pressure required to maintain equilibrium. Putting $w = 1$ in equation (62) we get

$$\frac{dg}{dw} = \frac{2}{3}\left\{1 - \frac{v_0+2}{\bar{v}+2}\right\}.$$

If $v_0 > \bar{v}$, then this derivative at $w = 1$ $(v = v_0)$ is negative and hence the body initially contracts, as we would expect. From the form of dg/dw in equation (62), we immediately see that the negative slope of g becomes steeper as w decreases and the body collapses.

On the other hand, if we have $v_0 < \bar{v}$, so that the system initially expands, then dg/dw is an increasing function for increasing values of w and the expansion is never halted.

These results are independent of the initial value of M/R and depend only on v_0, and hence on the initial central pressure.

10. *Stability of relativistic central-polytropes.* We can readily show that *an equilibrium model (of uniform density with a polytropic equation of state at the centre) is stable if, at the value of v concerned,*

$$\frac{d^2f}{dv^2} < 0, \tag{63}$$

and otherwise is unstable.

First, we observe that $\dot{R} = 0 = \ddot{R}$, the conditions that characterize an equilibrium model, imply that $\dot{v} = 0 = \ddot{v}$. Also, we have seen that these conditions imply that

$$f = 0 = \frac{df}{dv}.$$

From equation (47), we must always have $f \geqslant 0$. If, at the value v_0 of v for which and df/dv vanish, we have

$$\frac{d^2f}{dv^2} < 0,$$

then any curve in the (v, f)-plane arbitrarily close to that of f, corresponding to slightly different initial conditions from those of equilibrium, can be positive only for a small range of v surrounding $v = v_0$. From this we deduce that, if slightly disturbed from equilibrium, the body can only oscillate with a small amplitude and hence the equilibrium model is stable against small radial disturbances. Similarly, if condition (63) does not hold for the value of v in question, then the body is unstable.

We can use this result to recover a result given by Bondi (2) concerning the critical value of γ (at the centre) below which the equilibrium model would be unstable. We require the value of n for which d^2f/dv^2 vanishes when f and df/dv vanish.

We rewrite equation (48) in the form

$$\frac{f(v)(1+1/v_0)^{1/3}}{n^{2/3}(1+l)} = v^{2(1-n/3)}\left(1+\frac{1}{v}\right) g(w),$$

where w is given by equation (60), and

$$g(w) = 2w - 2w^{2/3} + \frac{2}{n(\bar{v}+1)+3}\left\{\left(\frac{1+(1-w)\,v_0}{w}\right)^{n/3} - w\right\}.$$

We can readily verify that $g'(w) = 0$, for $w = 1$, if and only if $v_0 = \bar{v}$ (equilibrium condition). Since, corresponding to

$$\bar{u} = (3l-1)/(1-l),$$

we find that, in general,

$$\bar{v} = \frac{(3+n)\,l-(1+n)}{n(1-l)},$$

if follows that if $g''(w) = 0$ for $w = 1$ then n must be given by

$$\frac{1}{n} = \frac{-1+14l-9l^2}{3(1-3l)^2}.$$

Consequently, the critical value of γ at the centre is given by

$$\gamma = 1+\frac{1}{n} = \frac{2}{3}+\frac{8}{3}\frac{l}{(3l-1)^2}. \tag{64}$$

For values of γ exceeding this, the body is stable and for smaller values it is unstable.

We observe that, since $l = \sqrt{(1-2GM/c^2R)}$ if we put $l = 1$ in equation (64) corresponding to c infinite (or R infinite), then we recapture the Newtonian critical value $\gamma = 4/3$. It is also possible to obtain Bondi's value of γ for any point of the body.

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APPENDIX

To show that we are justified in taking the total mass to be given by the formula

$$M = \tfrac{4}{3}\pi \rho(t)\, R_s^3(t), \tag{65}$$

we must consider the external metric. This will be a case of the general form given by equation (1) of Paper I, namely

$$ds^2 = -e^\lambda dr^2 - e^\mu d\Omega^2 + e^\nu dt^2.$$

For $r > r_s$, both p and ρ vanish and so, if m is defined by equation (11) of Paper I, namely

$$8m = \dot{\mu}^2 e^{3\mu/2-\nu} + 4e^{\mu/2} - \mu'^2 e^{3\mu/2-\lambda}, \tag{66}$$

equations (12) and (13) of Paper I imply that outside the sphere

$$m' = 0 = \dot{m}.$$

Hence, in this region

$$m(r, t) = \text{constant} = m_s,$$

if m is assumed to be continuous at $r = r_s$.

By Birkhoff's theorem the external metric can be expressed in the static Schwarzschild form. Hence, for $r > r_s$, we can choose the coordinates so that, in particular, $\dot{\mu} = 0$. Equation (7) of Paper I then implies that outside the sphere

$$0 = \tfrac{1}{2}e^{-\lambda}(\tfrac{1}{2}\mu'^2 + \mu'\nu') - e^{-\mu}, \tag{67}$$

and equation (66) becomes

$$8m_s = 4e^{\mu/2} - \mu'^2 e^{3\mu/2-\lambda}. \tag{68}$$

On writing R for $e^{\mu/2}$, so that $\mu' = 2R'/R$, we find that equation (68) gives

$$1 - 2\, m_s/R = R'^2 e^{-\lambda},$$

and hence

$$e^\lambda dr^2 = \frac{dR^2}{1 - 2\, m_s/R}.$$

Also, from equation (67), we obtain

$$\nu' = R'\left(\frac{2m_s/R^2}{1 - 2\, m_s/R}\right),$$

and it follows that

$$e^\nu dt^2 = (1 - 2\, m_s/R)\, dt^2.$$

Comparison with the Schwarzschild metric shows that the total mass of the sphere is given by $M = m_s$.

Inside the sphere equation (12) of Paper I, namely

$$m' = 2\pi\rho\mu' e^{3\mu/2},$$

signifies that

$$m' = 4\pi\rho R^2 R', \tag{69}$$

where $R = e^{\mu/2}$. From the definition of m, it follows that $m = 0$ when R vanishes. Hence, at any given time t, equation (69) implies that

$$M = m_s = \int_{R=0}^{R=R_s} dm = \tfrac{4}{3}\pi\rho R_s^3,$$

which is the required formula.