

# Time-dependent natural orbitals and occupation numbers

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received 3 June 2010; accepted in final form 1 October 2010

published online 15 November 2010

PACS 31.15.E- – Density-functional theory

PACS 31.70.Hq – Time-dependent phenomena: excitation and relaxation processes, and reaction rates

PACS 32.80.Qk – Coherent control of atomic interactions with photons

**Abstract** – Using the equations of motion for the occupation numbers of natural spin orbitals we show that adiabatic extensions of common functionals employed in ground-state reduced-density-matrix-functional theory have the shortcoming of leading to occupation numbers *independent of time*. We illustrate the exact time-dependence of the natural spin orbitals and occupation numbers for two strongly nonlinear cases, namely electron-ion scattering and atoms in strong laser fields. In the latter case, we observe strong variations of the occupation numbers in time.



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**Introduction.** – The description of quantum many-body systems out of equilibrium has become an important research topic in nuclear, plasma and condensed-matter physics. The common interest of different fields in non-equilibrium quantum evolution is mainly driven by experimental and technological progress which raises questions such as how many-body systems evolve on transient or non-adiabatic time-scales, how they thermalize or which kind of transport phenomena are to be expected in extreme environments.

The Kadanoff-Baym equations [1–3] provide a rigorous basis to investigate the dynamics of non-equilibrium many-body systems. Although the equations are known for some decades, so far only *ab initio* solutions for very small atomic or molecular systems under special assumptions have been achieved [4,5]. For a more comprehensive *ab initio* treatment of non-equilibrium systems with many degrees of freedom they are still out of scope of present-day computing facilities.

An alternative for the study of non-equilibrium processes in many-body systems is provided by time-dependent density-functional theory (TDDFT) [6]. TDDFT is currently the method of choice for many-electron systems out of equilibrium because it generally achieves decent accuracy at affordable computational

cost. All physical observables are in principle functionals of the time-dependent density [7,8]. However, in practice it is sometimes rather cumbersome to find approximate expressions for the observables of interest. Prominent examples are double and above-threshold ionization of atoms and molecules in strong laser fields, where no accurate functionals for the observables are known [9].

Recently, there has been renewed interest in reduced-density-matrix-functional theory (RDMFT) [10,11]. RDMFT is a promising candidate to treat long-standing problems present in traditional density-functional theory (DFT). RDMFT describes very accurately the cohesion and dissociation of diatomic molecules which presents a difficult problem for DFT methods due to the importance of static correlation [12,13]. The theory has also been successfully applied to the calculation of the fundamental gap [14,15].

RDMFT has recently been generalized to the time-domain [16] and successfully explored in the linear response regime where the focus is the determination of molecular excitation spectra from the poles of the frequency dependent density-matrix response [17,18].

In the present paper, we focus on the strongly nonlinear regime found in electron-atom scattering and in atom-laser interaction. Such systems have been very successfully studied with 1D soft Coulomb particle-interactions [19–21]. The Hamiltonian we consider takes

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—for two-body systems— the form

$$\hat{H}_\lambda = \frac{\hat{p}_1^2}{2} + \frac{\hat{p}_2^2}{2} - \frac{2}{\sqrt{\hat{x}_1^2 + 1}} - \frac{2}{\sqrt{\hat{x}_2^2 + 1}} + \frac{\lambda}{\sqrt{(\hat{x}_1 - \hat{x}_2)^2 + 1}}. \quad (1)$$

To vary the degree of correlation in the many-body wave function, we introduce a coupling constant  $\lambda$  in the Hamiltonian which controls the strength of the electron-electron interaction. Systems evolving in time under model Hamiltonians of the form of eq. (1) can be propagated essentially exactly with numerical techniques on a grid [22]. Once the exact correlated wave function has been calculated one can determine from it the time-dependent one-body reduced-density matrix  $\gamma_1(11'; t)$ . By diagonalizing  $\gamma_1(11'; t)$  at each fixed point in time one obtains eigenvectors  $\varphi_j(1; t)$  and eigenvalues  $n_j(t)$  which we term, in analogy to static RDMFT, time-dependent natural orbitals and time-dependent occupation numbers, respectively [16]. The spectral representation of the one-body matrix then takes the form

$$\gamma_1(11'; t) = \sum_k n_k(t) \varphi_k(1; t) \varphi_k^*(1'; t). \quad (2)$$

To monitor how correlations evolve in time we employ the correlation entropy [23]

$$s(t) := -\frac{1}{N} \sum_k n_k(t) \ln n_k(t) \quad (3)$$

as a measure for the strength of correlation. The correlation entropy  $s$  defined in this way is identical to zero for noninteracting particles and grows with increasing correlation in the system.

The idea is to deduce from the exact time-dependent two-body wave function the exact occupation numbers and exact natural orbitals in this strongly nonlinear case. To date the time-dependence of occupation numbers and natural orbitals for atomic and molecular systems is totally unknown. Taking the example of a laser-induced  $(1s)^2 \rightarrow (1s, 2p)$  transition in the helium atom (with a laser pulse optimized such that the occupation of the final  $(1s, 2p)$  two-body excited state is close to 100%) two extreme examples are conceivable: a) The orbitals stay constant in time while the occupations change:  $n_{2p}$  from 0 to 1,  $n_{1s}$  from 1 to 0. b) The occupation numbers stay constant while only the orbitals change in time. The latter case corresponds to the TDKS approach. So the question is: Will the exact natural orbitals and occupation numbers be close to the TDKS case or should we expect strongly time-dependent occupation numbers? This question will be answered below (including the proper spin degrees of freedom).

**Equation of motion.** – The time evolution of the reduced density matrices is given by the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy of equations

of motion [24]. The first equation of the hierarchy has the form

$$i\partial_t \gamma_1(11'; t) = (\hat{h}_0(1; t) - \hat{h}_0(1'; t)) \gamma_1(11'; t) + \int (v_{ee}(12) - v_{ee}(1'2)) \gamma_2(121'2'; t) |_{2'=2} d2, \quad (4)$$

where the bare single-particle Hamiltonian is given by  $\hat{h}_0(1; t) = -\nabla_1^2/2 + v_{\text{ext}}(1)$  and  $v_{ee}(12)$  denotes the particle-particle interaction. The coordinates are written as combined space-spin variables  $1 = (\mathbf{r}_1, \sigma_1)$  and we use  $\int d1 = \sum_{\sigma_1} \int d\mathbf{r}_1$ . In general, similar equations in the BBGKY hierarchy connect the time evolution of the  $N$ -th-order reduced density matrix  $\gamma_N$  to the density matrix of order  $N+1$ . Since  $2N$  coordinates appear in the equation of order  $N$ , the propagation of the complete hierarchy is even more involved than solving the underlying time-dependent Schrödinger equation (TDSE). Hence, truncations of the hierarchy are performed in practice. To close the system of evolution equations at a given order  $N$  it is therefore necessary to express the matrix of order  $N+1$  in terms of matrices with order less than or equal to  $N$ . This is the central point where approximations enter in a time-dependent reduced density matrix theory. Note, that in the case of two-body interactions  $v_{ee}(12)$  it is sufficient for the calculation of the ground-state energy to know the diagonal of the two-body reduced density matrix  $\gamma_2(1212; t=0)$ . As can be seen directly from eq. (4), this is in contrast to the time-dependent case, where also off-diagonal matrix elements  $\gamma_2(121'2'; t \geq 0)$  enter the equation of motion.

Since the eigenfunctions  $\varphi_j(1; t)$  form a complete set at each point in time, we can also express the two-body reduced density matrix in the basis of natural orbitals of the one-body matrix

$$\gamma_2(12, 1'2'; t) = \sum_{ijkl} \gamma_{2,ijkl}(t) \varphi_i(1; t) \varphi_j(2; t) \varphi_k^*(1'; t) \varphi_l^*(2'; t). \quad (5)$$

With eq. (2) the time-dependent natural orbitals and occupation numbers have been introduced by a diagonalization procedure. It is now interesting to observe that the occupation numbers obey evolution equations which have the following form:

$$i\dot{n}_k(t) = \sum_{ijl} \gamma_{2,ijkl}(t) \langle ij | v_{ee} | kl \rangle(t) - \text{c.c.}, \quad (6)$$

where we have used  $\langle ij | v_{ee} | kl \rangle(t)$  as shorthand for the Coulomb integrals

$$\int \varphi_i(1; t) \varphi_j(2; t) v_{ee}(12) \varphi_k^*(1; t) \varphi_l^*(2; t) d1 d2. \quad (7)$$

Next, we separate the coefficients  $\gamma_{2,ijkl}(t)$  into a mean-field and a cumulant part  $\lambda_{2,ijkl}(t)$  [25]:

$$\gamma_{2,ijkl}(t) = n_i(t) n_j(t) (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + \lambda_{2,ijkl}(t). \quad (8)$$

Inserting (8) into the equation of motion for the occupation numbers (6) one finds, that Hartree and exchange-like contributions cancel. Consequently, we obtain that only the remaining *cumulant part* of the two-body reduced density matrix determines the time evolution of the occupation numbers

$$i \dot{n}_k(t) = \sum_{ijl} \lambda_{2,ijkl}(t) \langle ij|v_{ee}|kl \rangle(t) - \text{c.c.} \quad (9)$$

For the closure of the BBGKY hierarchy up to level  $N=1$  an approximation of the two-body reduced density matrix in terms of the one-body reduced density matrix is required. One might be tempted to extend common functionals used in static RDMFT to the time-dependent case in a spirit similar to the adiabatic local density approximation of TDDFT [26], *i.e.* for a slowly varying time-dependence in the Hamiltonian the ground-state functional is evaluated at the time-dependent density. In the case of time-dependent reduced density matrices this corresponds to replacing the ground-state occupation numbers by their time-dependent counterparts. Although this replacement is rather *ad hoc*, the hope would be that such functionals perform also reasonably well in non-adiabatic situations.

The functional form of most commonly used ground-state functionals in RDMFT, written in the basis of the natural orbitals, can be summarized with the following expression:

$$\gamma_{2,ijkl} = f_{ijkl} \delta_{ik} \delta_{jl} - g_{ijkl} \delta_{il} \delta_{jk}, \quad (10)$$

which contains Hartree ( $\delta_{ik} \delta_{jl}$ ) and exchange-like ( $\delta_{il} \delta_{jk}$ ) terms. As example, for the Müller functional [27] we have  $f_{ijkl} = 1/2 n_i n_j$ ,  $g_{ijkl} = 1/2 \sqrt{n_i n_j}$ , the self-interaction corrected functional of Goedecker and Umrigar [10] reads  $f_{ijkl} = 1/2 (n_i n_j - n_i^2 \delta_{ij} \delta_{ik} \delta_{il})$ ,  $g_{ijkl} = 1/2 (\sqrt{n_i n_j} - n_i \delta_{ij} \delta_{ik} \delta_{il})$  and the BBC1 functional of Baerends *et al.* [13] has the form  $f_{ijkl} = 1/2 n_i n_j$ ,  $g_{ijkl} = \sqrt{n_i n_j} (1/2 - \delta_{il} \delta_{jk} (1 - \delta_{ij}) \Theta(i - N - \epsilon) \Theta(j - N - \epsilon))$ , where  $\Theta$  denotes the usual Heaviside step function and  $0 < \epsilon < 1$ . In a similar fashion the BBC2/BBC3 functionals can be written in the form of eq. (10). Note, that all functionals have the symmetry  $g_{ijkl} = g_{jilk}$  and all matrices  $f_{ijkl}$ ,  $g_{ijkl}$  are real valued.

Replacing the static occupation numbers which appear in eq. (10) by their time-dependent counterparts and inserting this approximation for the two-body matrix into the equation of motion for the occupation numbers (6) we obtain

$$i \dot{n}_k(t) = \sum_j (f_{kjjk}(t) - f_{kjjk}^*(t)) \langle kj|v_{ee}|kj \rangle(t) + \sum_j (g_{jkkj}^*(t) - g_{jkkj}(t)) \langle jk|v_{ee}|kj \rangle(t), \quad (11)$$

which shows that all functionals of the form (10) with real-valued matrices  $f_{ijkl}$ ,  $g_{ijkl}$  cause a zero right-hand

side in (11). Hence, if this class of approximations is used for the time evolution of the one-body reduced density matrix  $\gamma_1$ , the occupation numbers stay *constant in time*. A similar result was known before in the linear-response regime [17], *i.e.*  $\delta \dot{n}_k(t) = 0$  was known to hold for the *first-order* change,  $\delta n_k$ , of the occupation numbers. Our derivation shows that this result is also true in the strongly nonlinear regime, *i.e.* in all orders of perturbation theory. The lacking time-dependence in the occupation numbers is a severe shortcoming of an adiabatic extension of present functionals of static RDMFT which needs to be addressed in the development of future functionals. Possible functional forms that lead to a non-vanishing right-hand side in (11) would be  $\gamma_{2,ijkl}(t) = g(f_{ik}(t)f_{jl}(t) - f_{il}(t)f_{jk}(t))$  or  $\gamma_{2,ijkl}(t) = g((f_{ij}(t) - f_{ji}(t)) \times (f_{kl}(t) - f_{lk}(t)))$ , where  $f_{ij}$  is a non-diagonal real-valued matrix and  $g$  some Taylor-expandable function. Also, functionals with complex valued matrices could be employed.

An alternative route towards RDMFT functionals which give rise to non-constant occupation numbers in time is based on antisymmetrized-geminal-power (AGP) wave functions. Wave functions of AGP form have an explicit cumulant contribution to the two-body reduced density matrix which can be expressed in terms of the underlying AGP geminal [28]. A connection to RDMFT can be established readily: the AGP geminal is a two-electron wave function and hence can be expressed exactly in terms of natural orbitals and occupation numbers as shown by Löwdin and Shull [29]. Combining the Löwdin-Shull form of the geminal with the AGP ansatz for the wave function results in a RDMFT functional which includes terms beyond the usual Hartree and exchange-like contributions to the total energy. As shown by Bratož and Durand the total energy can be expressed in this case by terms which involve a recursion-relation based on the natural occupation numbers [30]. Work along investigating and improving such AGP-based approaches in RDMFT is in progress, both, for the ground state and for the time evolution of natural orbitals and occupation numbers [31].

**Model.** – How do occupation numbers actually change in time? In the discussion above we have seen that present functionals of static RDMFT would lead to time-independent occupation numbers. Is this a sensible approximation? To investigate the exact time-dependence of natural orbitals and occupation numbers we consider two prototypical cases: i) e-He<sup>+</sup> scattering and ii) the above-mentioned transition of the helium atom from the ground state to the first excited state induced by an optimized short laser pulse.

Case i) —*electron-ion scattering*. For the initial state of the propagation we consider an antisymmetric spin singlet product wave function formed from a Gaussian wave packet  $\psi(x) = \exp(-(x-x_0)^2/\sigma^2) \exp(-k_0 x)$  and the ground state  $\phi_0(x)$  of the He<sup>+</sup>-ion. For all calculations we place the packet initially at a distance of  $x_0 = 15$  a.u.

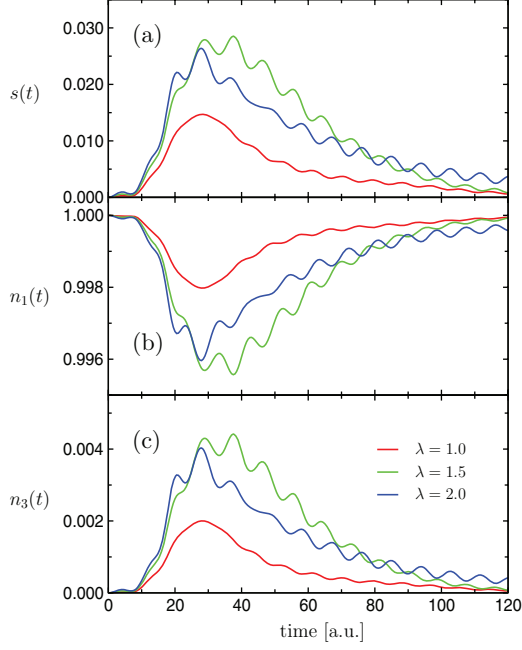


Fig. 1: (Colour on-line) (a) Correlation entropy of e-He<sup>+</sup> scattering for different interaction strengths  $\lambda$ . (b), (c) Occupation numbers for the largest natural orbitals, respectively.

away from the ionic core and give the scattering electron a momentum of  $k_0 = 0.3$  a.u. which is pointing towards the ion. In fig. 1 we plot the occupation numbers and the correlation entropy as function of time for different values of the electron-electron interaction strength  $\lambda$ . After about  $t = 10$  a.u. the wave packet has approached the ionic core and is passing the atomic nucleus. During this time the degree of correlation in the wave function is enhanced. After the collision the transmitted and reflected waves are leaving the ionic center and the correlation entropy starts to decrease. As indicated by the correlation entropy for long times after the scattering event, the many-body wave function is again well represented by a single Slater determinant. The occupation numbers deviate most strongly from their determinantal values (0 or 1) for  $\lambda = 1.5$ . For  $\lambda = 2.0$  the electron-electron repulsion is already so strong that the incident wave packet is mainly reflected back.

Case ii) — *optimal control*. To study laser-induced transitions in the helium atom we add an external dipole laser field of the form  $\hat{V}(x_1, x_2) = (\hat{x}_1 + \hat{x}_2) E(t)$  to the Hamiltonian  $\hat{H}_{\lambda=1}$ . We use standard optimal control theory (OCT) [32–34] to find the optimal laser pulse with amplitude  $E(t)$  which drives the atom in a finite time-interval  $[0, t_{\text{end}}]$  from the ground state to the first excited state. The full solution of the time-dependent many-body Schrödinger equation shows, that the actual transition is a mixture of both anticipated scenarios: the natural orbitals *as well as* the occupation numbers change as a function of time during the transition. The occupation numbers undergo large changes which reflects the multi-reference

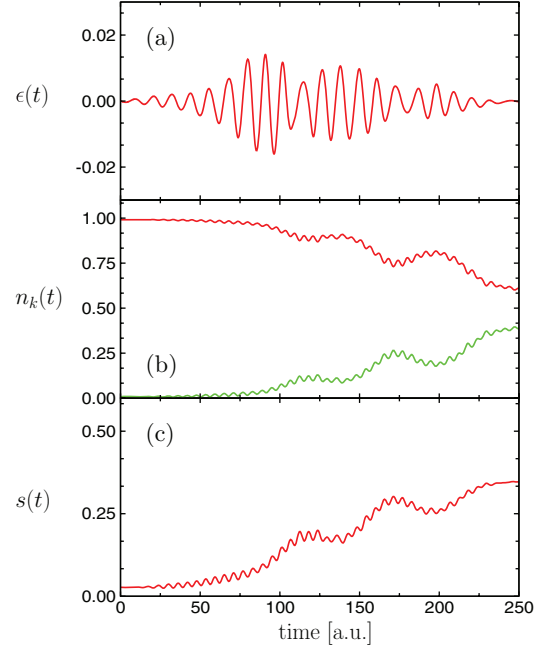


Fig. 2: (Colour on-line) (a) Optimal laser pulse for the transition of the Helium atom from the ground state to the first excited singlet state. The overlap of the propagated wave function and the target state was 98.59% after 12 OCT iterations. (b) The two largest occupation numbers and (c) the correlation entropy as a function of time.

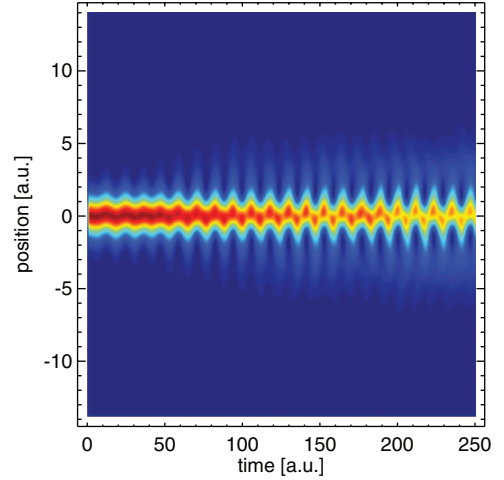


Fig. 3: (Colour on-line) Time evolution of the orbital density of the first natural orbital of helium during the excitation from the ground state to the first excited state. The quiver motion of the electrons is nicely reflected in the density profile of the natural orbital.

nature of the first excited state while the orbitals nicely reflect the quiver motion of the electrons in the laser field. This is displayed in fig. 2 and fig. 3, where we plot optimal laser pulse, correlation entropy, occupation numbers and the orbital density of the natural orbital with the largest occupation number.



**Conclusions.** – In summary, we have presented the exact time evolution of natural orbitals and occupation numbers for electron-ion scattering and for the helium  $1s$ - $2p$  transition. The exact analysis shows that sizable changes in the occupation numbers can occur during the time evolution of the system. With help of the equations of motion for the occupation numbers of the natural spin orbitals we have shown for all orders in perturbation theory that the adiabatic extension of present ground-state functionals of RDMFT to the time-dependent domain yields always occupation numbers which stay constant in time. Approximations beyond the ones used for static RDMFT will therefore be necessary to reasonably capture the time evolution of the one-body reduced density matrix.

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We would like to thank N. HELBIG, I. V. TOKATLY and N. N. LATHIOTAKIS for useful discussions. Partial financial support by the EC Nanoquanta NoE and the EXC!TING network is gratefully acknowledged.

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