Time-Domain Finite-Element Modeling of Dispersive Media

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***Abstract*—A general formulation is described for time-domain finite-element modeling of electromagnetic fields in a general dispersive medium. The formulation is based on the second-order vector wave equation and incorporates the dispersion effect of a medium via a recursively evaluated convolution integral. This evaluation is kept to second order in accuracy using linear interpolation within each time step. Numerical examples are given to validate the proposed formulation.**

***Index Terms*—Dispersive medium, finite-element method.**

I. Introduction

FOR any time-domain based numerical method to accurately perform wide-band electromagnetic simulations, one has to incorporate the effect of medium dispersion in its formulation. Over the past decade, several approaches have been proposed for the finite-difference time-domain (FDTD) method [1]–[6]. Little work has been reported on the dispersion modeling in the time-domain finite-element method (TDFEM) since TDFEM is not as well developed as FDTD. This situation, however, is changing rapidly; much interest has recently been attracted to TDFEM because of its modeling accuracy and flexibility [7]–[9]. In this work, a general formulation is developed to model the dispersion effect in TDFEM. This TDFEM is based on the second-order vector wave equation, in contrast to most FDTD schemes that solve the first-order Maxwell's equations. The required convolution integral is evaluated recursively without a need to store the fields of all past time steps. This evaluation is ensured of second order in accuracy by adopting a linear interpolation for the fields within each time step.The proposed formulation is shown to be valid for plasma, Debye, and Lorentz media with a single or multiple poles. Three-dimensional (3-D) numerical examples are given to demonstrate its efficacy.

II. Formulation

The electric field in a general dispersive medium satisfies the second-order wave equation

$$\nabla \times \mu_r^{-1} \nabla \times \mathbf{E}(\boldsymbol{r},t) + \mu_0 \epsilon \partial_t^2 \mathbf{E}(\boldsymbol{r},t) + \mathcal{L}(\mathbf{E}(\boldsymbol{r},t)) = -\mu_0 \partial_t \mathbf{J}_s(\boldsymbol{r},t) \tag{1}$$

where $\mathbf{J}_s(\boldsymbol{r},t)$ denotes the source current density and $\mathcal{L}$ represents an operator on the field $\mathbf{E}$. For plasma

$$\begin{aligned}\mathcal{L}(\mathbf{E}(\boldsymbol{r},t)) &= \mu_0\epsilon_0\omega_p^2\{1-\nu_c\varphi(t)*\}\mathbf{E}(\boldsymbol{r},t)\\ \varphi(t) &= e^{-\nu_c t}\bar{u}(t)\end{aligned} \tag{2}$$

where

- ω_p plasma frequency;
- ν_c damping frequency;
- $\bar{u}(\cdot)$ unit step function;
- $*$ convolution.

For a Debye medium

$$\begin{aligned}\mathcal{L}(\mathbf{E}(\boldsymbol{r},t)) &= \mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\tau^{-3}\{\tau^2\partial_t - \tau + \varphi(t)*\}\mathbf{E}(\boldsymbol{r},t)\\ \varphi(t) &= e^{-t/\tau}\bar{u}(t)\end{aligned} \tag{3}$$

where τ is the relaxation time, ϵ_s and ϵ_∞ denote the relative dielectric constants at zero (dc) and infinite frequencies, respectively. For a Lorentz medium

$$\begin{aligned}\mathcal{L}(\mathbf{E}(\boldsymbol{r},t)) &= \mu_0\epsilon_0(\epsilon_s-\epsilon_\infty)\omega_0^2\\ &\quad\times\{G-\alpha^{-1}\left[2\delta\partial_t+\omega_0^2\right]\varphi(t)*\}\mathbf{E}(\boldsymbol{r},t)\\ \varphi(t) &= e^{-\delta t}\sin(\alpha t)\bar{u}(t)\end{aligned} \tag{4}$$

where

- $\delta = \nu_c/2$ damping constant;
- ω_0 resonant frequency;
- $\alpha = \sqrt{\omega_0^2-\delta^2}$, and G coefficient weighting the contribution from the induced polarization currents.

To illustrate the finite element solution of (1), we assume a mixed boundary condition on the surface of the volume of interest as

$$\mu_r^{-1}\hat{n}\times[\nabla\times\mathbf{E}(\boldsymbol{r},t)]+c^{-1}\partial_t\hat{n}\times[\hat{n}\times\mathbf{E}(\boldsymbol{r},t)] = \mathbf{U}(\boldsymbol{r},t). \tag{5}$$

The corresponding weak-form solution is then given by

$$\begin{aligned}&\iiint_V \big\{\mu_r^{-1}[\nabla\times\mathbf{N}_i(\boldsymbol{r})]\cdot[\nabla\times\mathbf{E}(\boldsymbol{r},t)]\\ &\qquad+\mu_0\epsilon\mathbf{N}_i(\boldsymbol{r})\cdot\partial_t^2\mathbf{E}(\boldsymbol{r},t)+\mathbf{N}_i(\boldsymbol{r})\cdot\mathcal{L}(\mathbf{E}(\boldsymbol{r},t))\\ &\qquad+\mu_0\mathbf{N}_i(\boldsymbol{r})\cdot\partial_t\mathbf{J}_s(\boldsymbol{r},t)\big\}dV\\ &+\iint_S\big\{c^{-1}[\hat{n}\times\mathbf{N}_i(\boldsymbol{r})]\cdot\partial_t[\hat{n}\times\mathbf{E}(\boldsymbol{r},t)]\\ &\qquad+\mathbf{N}_i(\boldsymbol{r})\cdot\mathbf{U}(\boldsymbol{r},t)\big\}dS = 0\end{aligned} \tag{6}$$

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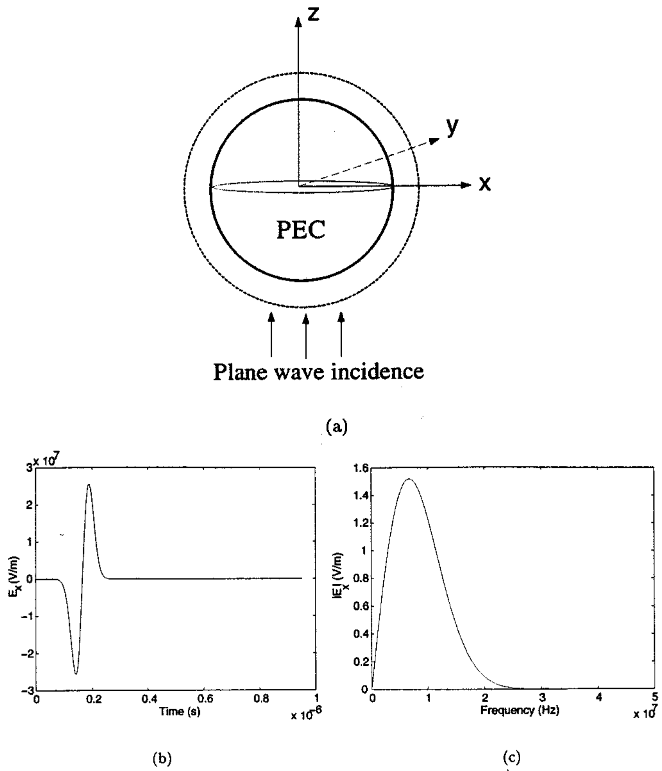

Fig. 1. The coated sphere and the incident electric field. (a) Geometry. (b) E_x versus time. (c) $|E_x|$ versus frequency.

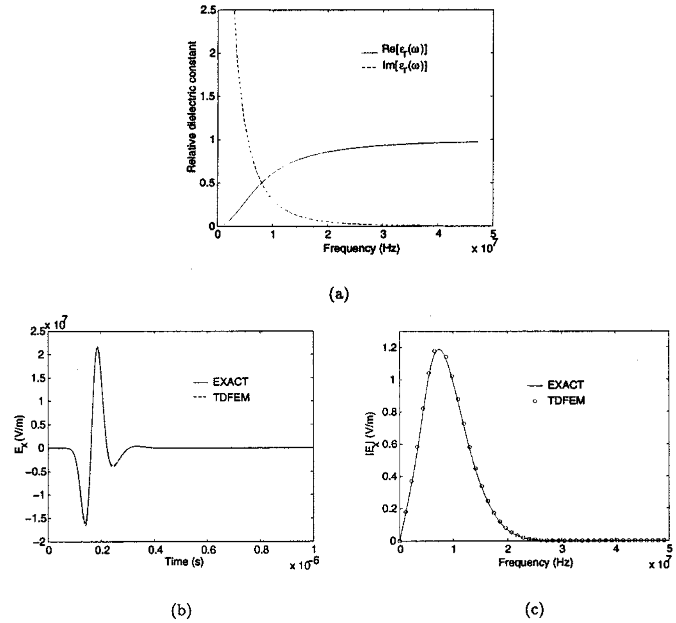

Fig. 2. Results for a metallic sphere coated with plasma. (a) Relative dielectric constant($\omega_p = \nu_c = 50$ Mrad/s). (b) E_x versus time. (c) $|E_x|$ versus frequency.

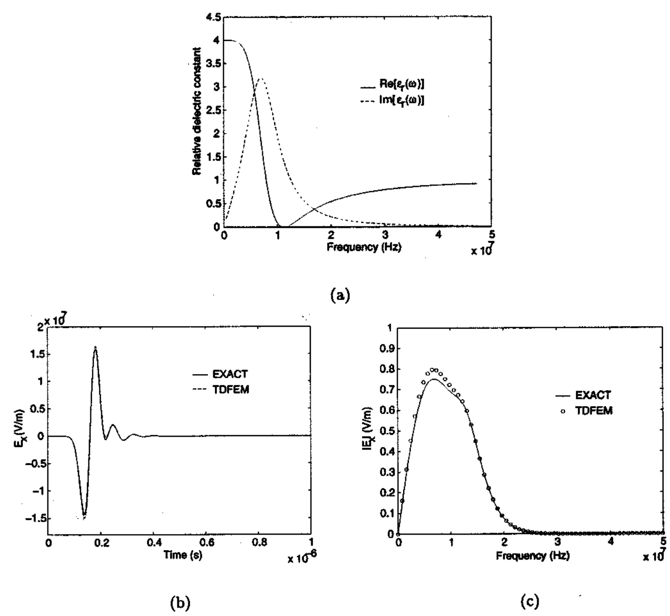

Fig. 3. Results for a metallic sphere coated with a Lorentz medium. (a) Relative dielectric constant ($\omega_1 = 2\delta = 50$ Mrad/s, $\epsilon_s = 4.0$, $\epsilon_\infty = 1.0$, $G = 1$). (b) E_x versus time. (c) $|E_x|$ versus frequency.

where $\mathbf{N}_i(\boldsymbol{r})$ denotes the vector basis functions. Expanding the electric field as

$$\mathbf{E}(\boldsymbol{r},t)=\sum_{j=1}^{N}u_j(t)\mathbf{N}_j(\boldsymbol{r}) \tag{7}$$

with N denoting the total number of unknowns, and substituting into (6), we obtain the ordinary differential equation

$$\mathbf{T}\frac{d^2u}{dt^2}+\mathbf{R}\frac{du}{dt}+\mathbf{S}u+\mathbf{Y}\frac{d\psi}{dt}+\mathbf{Z}\psi+w=0 \tag{8}$$

where $\mathbf{T}$, $\mathbf{R}$, $\mathbf{S}$, $\mathbf{Y}$, and $\mathbf{Z}$ denote matrices whose elements can be identified from (6). Also, u is a vector given by $u=[u_1,u_2,\ldots,u_N]^{\mathrm{T}}$, ψ is a vector whose elements are given by

$$\psi_i(t)=\varphi(t)*u_i(t) \tag{9}$$

and finally, w is a vector contributed by $\mathbf{J}_s(\boldsymbol{r},t)$ and $\mathbf{U}(\boldsymbol{r},t)$.

Since the susceptibility function of a general dispersive medium can be expressed as a rational function in frequency domain, its time-domain counterpart inherits the feature of exponential functions. Without loss of generality, we can write $\varphi(t)$ as

$$\varphi(t)=\mathrm{Re}[ae^{-bt}\bar{u}(t)] \tag{10}$$

where

$a=1$ and $b=\nu_c$	plasma;
$a=1$ and $b=\tau^{-1}$	Debye medium;
$a=-j$ and $b=\delta-j\alpha$	Lorentz medium.

As a result, the convolution in (9) can be evaluated recursively as

$$\begin{aligned}\psi_i^{n+1}=&\mathrm{Re}[\hat{\psi}_i^{n+1}]\\ \hat{\psi}_i^{n+1}=&e^{-b\Delta t}\hat{\psi}^n+ae^{-b(n+1)\Delta t}\int_{n\Delta t}^{(n+1)\Delta t}e^{b\tau}u_i(\tau)d\tau.\end{aligned} \tag{11}$$

Instead of assuming $u_i(t)$ to be constant within the time interval $(n\Delta t,(n+1)\Delta t)$, we employ linear interpolation to guarantee

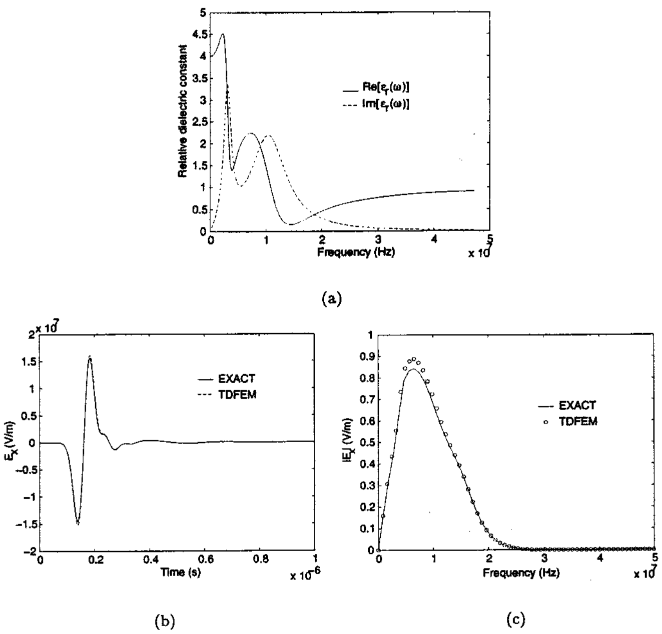

Fig. 4. Results for a metallic sphere coated with a second-order Lorentz medium. (a) Relative dielectric constant ($\omega_1 = 70$ Mrad/s, $2\delta_1 = 50$ Mrad/s, $\omega_2 = 20$ Mrad/s, $2\delta_2 = 10$ Mrad/s, $G_1 = G_2 = 0.5$, $\epsilon_s = 4.0$, $\epsilon_\infty = 1.0$). (b) E_x versus time. (c) $|E_x|$ versus frequency.

the second order in accuracy. As a consequence, we obtain the recursive relation

$$\hat{\psi}_i^{n+1} = e^{-b\Delta t}\hat{\psi}_i^n + 0.5a\Delta t(u_i^{n+1} + e^{-b\Delta t}u_i^n). \qquad (12)$$

The above strategy can easily be extended to a general dispersive medium with an arbitrary order.

III. Numerical Examples

To validate the proposed formulation, we consider a metallic sphere coated with either plasma, a Lorentz, or a second-order Lorentz medium. The metallic sphere has a radius of 0.8 m and the coating has a thickness of 0.2 m. This coated sphere is illuminated by an x-polarized incident plane wave, whose electric field is plotted in Fig. 1, propagating along the z-direction. The boundary integral equation is used to truncate the finite-element mesh accurately [9].

Figs. 2–4 display the calculated electric field E_x at the observation point $\boldsymbol{r} = 0.1\hat{\mathbf{y}} - 1.18\hat{\mathbf{z}}$ m as a function of time and frequency. It is seen that the calculated results agree very well with the exact solution obtained from the Mie series.

IV. Conclusion

A general approach was proposed to incorporate the dispersion effect in the TDFEM modeling of electromagnetic fields in a general dispersive medium. This approach employs a recursive evaluation of convolution integrals to avoid the storage of all past fields and adopts linear interpolation within each time step to achieve a second-order accuracy. Three-dimensional numerical examples were given to demonstrate its validity.

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