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Time Evolution and Thermodynamics for a Nonequilibrium System in Phase-Space

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The integrable system is constrained strictly by the conservation law during the time evolution, and the prethermal state from the nearly integrable system is also constrained by the conserved parameters (the constants of motion) with the corresponding generalized Gibbs ensemble (GGE) which is indubitability a powerful tool in the prediction of the relaxation dynamics. For stochastic evolution dynamics with considerable noise, the two-point correlation of local operators (like the density of kinks or transverse magnetization correlators) which don't exhibit the thermal features, display the behaviors of nonthermalization and an asymptotic GGE. In fact it's an asymptotic quasi-steady state with an infinite temperature, therefore the required distance to the nonthermal steady state is in an infinite time average. In this paper, we unambiguously investigate the relaxation of a nonequilibrium system in a canonical ensemble for integrable and nonintegrable systems. Temporal behavior of the many-body quantum system and the corresponding linear-coupling between the harmonic oscillators are discussed. The matrix-method in entropy ensemble is utilized to discuss the boundary and the diagonalization algebraically. The approximation results for nonintegrable system under the considerable perturbations are also presented.

1 Introduction

The investigation of the time evolution of nonequilibrium system is important for particle physics, condensed matter physics and even the cosmology (like the entropy of Bekenstein-Hawking black hole[1]), and especially in the many-body theory prediction by, e.g., the trapped ultracold atomic gases which have the weak enough interaction with environment and therefore allow the observation of the unitary time evolution[2]. For nonequilibrium system, the most common glass-form of the materials can be blocked by the pinning field[3] and produce a glass transition like the process of ergodic-to-nonergodic by the fluctuation-dissipation theorem. In replica theory, since the homogeneous liquid given by the replica symmetry has an inhibitory effect for the entropy production, the broken replica symmetry will leads to an increase of the overlap of replicas. With the increase of degree of such overlap which can be realized by enlarging the system size, the number of metastable states (or the hidden one[3]) grows exponentially, and the entropy grows logarithmically in the mean time. For a typical nonintegrable system, the chaotic classical system requires the computational resource which increase exponentially with time and the Kolmogorov entropy, is exponentially sensitive to the initial state. The Kolmogorov entropy here dosen't depends on the error tolerance and it's as small as 1.1[4] for the nonintegrable case, while for the integrable system which is effectively solvable by the Bethe ansatz[5], the required computational resource increase polynomially.

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The time evolution of quantum entangled state may cause the decoherence effect which has been widely found in the condensate system and it takes an important role in the quantum information processing, quantum computation and metrology, quantum teleportation, quantum key agreement[6, 7], and even the decoherence in neural network[8]. The entanglement is mostly produced by the dynamical evolution with the nonlinear interaction[9] and the non-destructive measurement, like the Dzyaloshinskii-Moriya interaction[10, 11]. Usually the quantum entanglement in nonequilibrium quantum systems is studied by the two-qubit system[12, 10], but sometimes the qutrit system[13, 11] is needed for the special dissipative configuration. In nonequilibrium and nonstationary open system, the coarse graining which connecting numerous subsystems' degrees of freedom makes more possible to realizing this process[14], and the thermal entropy is a good measurement for the effect of coarse graining. The quantum spaces' dimension increases exponentially with particle number due to the tensor-productor[4], and similarly, the number of metastables which are the subsystems of the whole spin glasses system, increase exponentially with system size at high temperature[15], and the phase transition or critical fluctuation will occur when it from one kind of the subsystems into another. Such destruction and restoration of symmetry within this process will also affects the properties of materials[16], like the dielectric constant, etc. In solid-state quantum system, the spin is the best candidate among various microscopic atom intrinsic degrees of freedom in thermal entanglement which has a higher stability compared to other entanglements due to its relatively long decoherence time[17] and it's also closely connected to the local free energy. The long coherence time in many-body system is useful to detecting the unitary dynamics, e.g., the Hubbard-type model, and it's important to detecting the coherent nonequilibrium dynamics for the multiple phase-transition. Further, when the system is mapped to the one with spinless free fermions through the Wigner-Jordan transformation, it shows a in-phase fermion liquid state[18] and has a stationary behavior in such an equilibrium integrable model which is considered as a powerful tool to obtain the exact solution[19]. The first implementation that using the density matrices in prediction of many-body system (equilibrium or nonequilibrium) is the Ref.[23]. It discussed the situation similar to the quantum irreversible process in an energy- and information-lossy system. The numerical method of time-dependent density matrix renormalization group (t-DMRG) has show that the matrix product operator is simulation-inefficently for nonintegrable model which is similar to the tensor-productor, but it's efficient for the integrable and locally disordered one[20]. Except that, the method of matrix product wave function is also a good tool to dealing with this time-evolving one-dimension quantum system[21]. For time evolution of free fermions or bosons, when the time scale to infinity, the thermal average of spin z-component S^z is zero and the spin states are half-filled[18]. In this case, the interaction between particles is strongest due to the zero-polarization[20], and the entanglement entropy is also increase and becomes more extensive[22].

A fact that the many-body quantum system will tends to equilibrium has been verified by many experiments, like the trapped ultracold atoms in optical lattices or the interactions with optical resonance, while for nonequilibrium system, the relaxation and thermal entanglement and the stochastic force also attract a lot of attention[24, 25]. Furthermore, the system may relax to analogue of thermal state if the initial state is ground state[24]. The method of fluctuation-dissipation relation (FDR) and quantum state diffusion (QSD) is utilized in this work to investigating the evolutions to the steady states in integrable system whose final states are constrained by the conserved law (indeed, it's the scattering process of particles which constrained strictly by the conserved law) and with a finite speed of algebraically relaxation and information transfer under the thermodynamic limit (i.e., the large-N limit, and note that the speed here will not bounded by the speed of light just like in the relativistic quantum theory, but bounded by a well known Lieb-Robinson group velocity[26] (see Sec.7,8)). The integrable system of quantum Newton's cradle with ground-breaking is an explicit example

for the conserved-restrictions[27], and the classical systems also have the same results, like the Fermi-Pasta-Ulam (FPU) theorem[28] and Kolmogorov-Arnold-Moser (KAM) theorem[29]. While for the nonintegrable system, the constants of motion can be expressed by second quantized operator[30] and without the limit like the integrable one. For local observable system, the stationary and linear value may exist (like thermal state), but for integrable system whose time evolution found no thermalization and it will tends to a distribution of GGE with an important fundamental hypothesis for statistical ensemble that it has maximized entropy which is constrained by local conservation law[31], (e.g., the conservation quantity of momentum occupation number), and hence restrict the ergodicity which makes it couldn't reaches the thermal state. For a framework of macroscopic system in finite dimension, it's also important to introducing the quantum field theory for both the equilibrium and nonequilibrium state in the open system[14] for investigating its time-dependent nature and coupling in local and nonlocal case as well as the dynamical fluctuation in short distance. And it's also necessary to considering such a quantum field when the Hilbert space is too large to implement a well numerical simulation[32]. The importance of entangled states for quantum computation is well known, to reduce the confusions from the decoherence, there is a topological way that storing the quantum information non-localized[33] or through the non-Abelian braiding statistics which also support the Majorana fermions[35, 34] by the Majorana-edge modes in finite wire[36], and it can better solve the problem of infinite dilution of the stored information in local area[24]. Such problem also investigated in this work.

Since for nearly integrable system, the behavior of relaxation is under the crossover effect of prethermalization and thermalization which is associate with the thermal correlation and the speed of information transfer, and the prethermalized state can also be well described by the GGE[37], i.e., may be viewed as a integrable system. The suppressed thermalization can be freed by enough perturbation to break the integrability. This crossover effect affects both the nonintegrable systems as well as the open systems. Through the study of this paper, we know that the recurrence will appear for long-enough time evolution. In the configuration considered here, part of the mixed system which is of interest coupling with the environment (i.e., not isolated), and hence the degrees of freedom of the environment (i.e., the counterpart of the target-part) can be traced out in the canonical ensemble[29], i.e., tracing over the variables outside the target-region. This provides support to the matrix-method presented in Sec.10. Such a large number of degrees of freedom is also an important precondition to implementing the global relaxation with the thermodynamic limit[39]. For nonlocal operators in equilibrium state, the dynamical parameters display an effective asymptotic thermal behavior (follow the Gibbs distribution)[40] during equilibrium time evolution with a determined temperature and decay with an asymptotic exponential law, while the model what we focus on is towards the asymptotic quasisteady state with an infinite temperature, which decay with an asymptotic power law[5] and acted by a diffusion term (see Sec.11). The prethermalization will shares the same properties with the nonthermal steady state due to the dynamical parameters which makes the model move close to the integrable points (or superintegrable point) after quenching. But in fact, for integrable quenching, the stationary behavior for both the local and nonlocal observables can be well described by the corresponding GGE, and the particles scattering which constrained by conserved law is purely diagonal[41, 42].

This paper is organized as follows: We introduce the model of two-coupled subsystem in Sec.2, and the bare coupling is further discussed in Appendix.A. The evolution in non-dissipation system is discussed in Sec.3, and the quenching for many-body system is discussed in Sec.4. A system-environment partition is mentioned in these two sections. In Sec.5, we study the dissipation of nonlocal model. In Sec.6, the time evolution and thermal entanglement of Heisenberg XXZ model are investigated. In Sec.7, the correlation and transfer speed of information in quantum system is discussed where we take the one-dimension chain model

as an explicit example. The relations between thermal behavior and the integrability are also discussed in this Section. We present the nonequilibrium dynamics with strong and weak interaction in Hubbard model in Sec.8. In this section, we investigate the phase transition of nonintegrable Hubbard model, and the relaxation of double occupation as well as the kinetic energy. We also use the method of nonequilibrium dynamical mean-field theory (DMFT) to detect the evolution by mapping the lattice model to the self-consistent single-site problem which can be solved numerically. In Sec.9, we discuss the relaxation to a Gaussian state and in Sec.10, we resort to the matrix method, the properties of the boundary and the transfer speed are also mentioned. In Sec.11, we discuss the relaxation of nonequilibrium system with stochastic dynamical variables in a free energy surface, the quantum dissipation in the damp-out process is also discussed. The diagonal contribution to symplectic spectrum of covariance matrix is further explored in Appendix.B. The bulk-edge-coupling (Bulk-edge correspondence) type materials which are related to the spectrum gap are presented in Sec.10 and Appendix.C, and the perturbation theory as well as the diagonalized Hamiltonian are also discussed in Appendix.C.

2 Model Introduction and the Coupling in Quantum Field Theory

We begin with the perturbation theory in space-time dimension, which is related to the strong coupling model[29] or the weak-perturbation limit of nonintegrable system, and even with the broken ergodicity[3]. In dimension of $(d+1)$ space-time, since the particles obtain mass from the broken of non-Abelian gauged symmetry, the coupling constant g is dimension-dependent, except the bare coupling g^b which vanish in $d+1=4$ limit[43]. The broken translational symmetry also make the spin liquid state solidified rapidly and turn it into the crystalline structure[16, 44]. Then we define two d-dimension system ψ_i and ψ_j with potential ϕ_i and ϕ_j, respectively. In weak coupling condition which suitable for the perturbation calculation[16], there exist a spin density wave (SDW) which in a Fourier expression is $\psi_i = L^{-d}\sum_i e^{-iqr_i}\phi(x-r_i)$, and ψ_j is as the same form. Although L here is constrained by the model dimension d, but L itself could be dimensionless with the dimensionless length scale and time scale (see Ref.[45]). The ϕ here describes the fluctuation as a function in arbitrary dimension, and it's also useful for quantum fluctuation or even the vacuum fluctuation. The dimension of ϕ may even up to ten according to D-branes of string theory[46]. In our model, we set coupling in each direction in a range of 0 to n, and the top value is $n = 2^{d/2}N$ in $SO(d)\times SU(N)$ system[47]. So a continuous phase transition with weak coupling pertubative RG under the time evolution can be expressed by $S=\int d^dx\mathcal{L}$ which appear in the imaginary-time path integral $Z=\int D\psi_i^\dagger D\psi_i D\psi_j^\dagger D\psi_j e^{-S}$ [48].

The nonrelativistic Lagrange function $\mathcal{L}$ is[43, 46]

$$\mathcal{L}=\int_\tau^{\tau'} d\tau[(i\psi_i^\dagger\partial_\tau\psi_i+\frac{1}{2\mu}\psi_i^\dagger\nabla\psi_i-\mu\psi_i^\dagger\psi_i)+(i\psi_j^\dagger\partial_\tau\psi_j+\frac{1}{2\mu}\psi_j^\dagger\nabla\psi_j-\mu\psi_j^\dagger\psi_j)], \tag{1}$$

where τ and τ' are the initial and final time, ∇ is the Laplace operator, and μ is the chemical potential. This time evolution Lagrange function ignore the interactions, e.g., the impurity-induced long range order[46, 49]. The fermion system in this model can be expressed as $H=\sum_{a=x,y,z}J_a\sum_{\langle i,j\rangle_a}\psi_i\psi_i^a\psi_j\psi_j^a$, $(a=x,y,z)$, with spin operators $S_i=i\psi_i\psi_i^a$ and $S_j=i\psi_j\psi_j^a$. Such definition also consistent with the properties of gauge field. Then we have $H=-\sum_{a=x,y,z}J_a\sum_{\langle i,j\rangle_a}S_iS_j$. In Eq.(1) we take the imaginary time approach where the quantum Monte Carlo (QMC) method is utilized [50], the differential symbol ∂_τ has below relation according to the definition of Bernoulli number[51]

$$n\partial_\tau(\frac{\tau^{1-z}}{1-z})=\frac{\tau^{1-z}}{1-z}\sum_{n=0}^{\infty}B_n\frac{(-\partial_\tau)^n}{n!}, \tag{2}$$

and the differential symbol for mass μ is as the same form

$$n\partial_\mu(\frac{\mu^{1-z}}{1-z}) = \frac{\mu^{1-z}}{1-z}\sum_{n=0}^{\infty} B_n \frac{(-\partial_\mu)^n}{n!}. \tag{3}$$

The Gardner transition which the critical dimension $d_c = 3$ is an important object in the study of coupling properties of amorphous solids [52]. In (3+1) space-time dimension using the renormalized coupling, since the bare coupling is absent in the dimension of $d+1=4$, the resulting dimensionless bare action with unbroken Quantum electrodynamics (QED) symmetry is

$$S = \int dx \left\{ \frac{1}{2}\sum_{x,y=0}^{n}[(\partial_\mu\phi_{xy})^2 + r\phi_{xy}^2] - \frac{1}{3!}(g_i^b \sum_{x,y=0}^{n}\phi_{xy}^3 + g_j^b \sum_{x,y,z=0}^{n}\phi_{xy}\phi_{xz}\phi_{yz}) \right\}, \tag{4}$$

and the action of Landau-Ginzburg-Wilson (LGW) Hamiltonian with N-component O(N) symmetry and noncollinear order is[53]

$$\begin{aligned} S = \int d^d x \int_\tau^{\tau'} d\tau \Big\{ & \frac{1}{2}\sum_{x,y=0}^{n}[(\partial_\mu\phi_{xy})^2 + r\phi_{xy}^2] \\ & + \frac{1}{4!}[g_i(\sum_{x,y=0}^{n}\phi_{xy}^2)^2 + g_j \sum_{x,y,z=0}^{n}\overline{[(\phi_{xy}\phi_z)^2 - \phi_{xy}^2\phi_z^2]}] \Big\}. \end{aligned} \tag{5}$$

The summation index x y z range from 0 to $(n-1)$ correspond to the parameter space setted above, and here the average term $\sum_{x,y,z=0}^{n}\overline{[(\phi_{xy}\phi_z)^2 - \phi_{xy}^2\phi_z^2]}$ exhibit the correlation between these two fluctuation functions ϕ_{xy} and ϕ_z. Using the method of time dependent density matrix RG which have been proved valid for particles at a fixed time evolution[20]. The fermion system shown as $T_{ij}\delta_{ij} = \mathrm{Tr}\{S_i S_j\}$ where T_{ij} is the interaction tensor, the S_i and S_j are spin operators for ψ_i and ψ_j respectively and $\delta_{ij} = \{c_i c_j^\dagger\}$. This expression is indeed taking the diagonal part of T_{ij}. Ref.[43] put forward a valuable view which connecting the bare coupling to the renormalized coupling by an infinite cutoff, and then the mass-independent bare coupling can be written as [43]

$$\begin{aligned} g^b = \mu^{3-d}\Big\{ & g + \delta_{11}\frac{g^3}{3-d} \\ & + \delta_{21}\frac{g^5}{3-d} + \delta_{22}\frac{g^5}{(3-d)^2} \\ & + \delta_{31}\frac{g^7}{3-d} + \delta_{32}\frac{g^7}{(3-d)^2} + \delta_{33}\frac{g^7}{(3-d)^3} + O(g^9) \Big\}, \end{aligned} \tag{6}$$

which is satisfactory consistent with the series expansion of β function given in Ref.[54]

$$\beta(g) = -\beta_0\frac{g^3}{16\pi^2} - \beta_1\frac{g^5}{(16\pi^2)^2} - \beta_2\frac{g^7}{(16\pi^2)^3} - O(g^9). \tag{7}$$

This β-function is series-expand to the seven-order of coupling, i.e., the three-loop level for the gauge field. The specific quantitative analysis of β-function is presented in the Appendix.A. Fig.1 shows the $\beta(g)$ as a function of g in SU(3) system (i.e. $C_{ij}^{(2)} = 3$ (see Appendix.A)) with different number of fermion multiplets m which setted from 0 to 20 by us. It's obviously to see that the curves shows a drastic non-linear change, and the m-dependent interaction tensor T_{ij} also plays a decisive role in the relation between $\beta(g)$ and g.

3 Evolution Behavior in Non-Dissipation System

Since the long time scales exist in the metastable states which the quantity grows exponentially with system size[16], e.g., the single positive charge state in p-type material[55] or the p-spin model[56]. the imaginary-time path integral can be expressed by the trace of time evolution operator $Z = \mathrm{Tr}(e^{-\beta H})$ with the evolution propagator $U = e^{-\beta H} = \mathrm{Tr}(\sigma_1^i \sigma_2^i \cdots \sigma_n^i \sigma_1^j \sigma_2^j \cdots \sigma_n^j)$, where β is inverse temperature $1/k_B T$. Note that the spin Pauli matrices here contain all the components in finite dimension of Hilbert space and H_{ij} is the nearest neighbor Hamiltonian which can be decomposed by using the Trotter-Suzuki method which mapping the one dimension quantum system into two dimension[57] one and the path integral becomes $Z = \mathrm{Tr}(\Pi_{i,j} e^{-\beta H_{ij}})$. In this way, the long range interaction can be treated locally as a nearest-neighbor pair in this spin isotropic system through a single two-qubit exchange gate $U_{i,i+1} = e^{-H_{i,i+1}\delta\tau}$ due to the iterative nature and acting on two adjacent sites with single time step $\delta\tau$ evolution, it is also meets with the realignment criterion[12], that is, the local field effect. Then we have

$$e^{-\beta H_{i,i+1}} = \prod_i U_{i,i+1}. \tag{8}$$

Except the Andenson localization, the local length may strongly increase and obeys the logarithmic law[58]. The Hamiltonian here was divided by the partition function Z through the temperature interval or external magnetic field h[47]. By investigating the asymptotic behavior of Z, when $\beta \to \infty$, i.e., the temperature decrease with the imaginary time evolution, the $Z \to 0$, and then the system tends to ground state which is $|\psi(0)\rangle = |\psi_1^i\rangle \otimes |\psi_2^i\rangle \cdots \otimes |\psi_n^i\rangle \otimes |\psi_1^j\rangle \otimes |\psi_2^j\rangle \cdots \otimes |\psi_n^j\rangle$. Denoting $\varepsilon_{n'}$ is the energy of n'th level ($n' < n$) in this system above the ground state, then $\varepsilon_{n'} = E_{n'} - E_{n'-1}$. The Pauli operator $\sigma_{n'}^{i/j}$ within the expression of evolution propagator U is $\sigma_{n'}^{i/j} = \sigma^{0 \otimes n'} \otimes \sigma_{n'}^{i/j} \times \sigma^{0 \otimes (n-n')}$[4]. Since within time evolution, the entanglement between particles which is time-dependent rapidly reach the maximum value, the method of time dependent density matrix RG becomes invalid due to the too large growth-speed of entanglement entropy.

The evolution by the evolution propagator U can be expressed as

$$|\psi(\beta)\rangle = U|\psi(0)\rangle, \tag{9}$$

and specifically, in the form with imaginary-time analogue $e^{\tau H(\tau)}$ it has[21, 59]

$$|\psi(\tau)\rangle = \frac{e^{\tau H(\tau)}|\psi(0)\rangle}{||\ e^{\tau H(\tau)}|\psi(0)\rangle\ ||}, \tag{10}$$

where we define the imaginary-time as $\tau = t + i0^+$, while for the evolution Hamiltonian, it is $H(\tau) = e^{\alpha H} H e^{-\alpha H}$ where $\alpha = \beta + i0^+$. Since $\partial_\beta \psi(\beta) = H\psi(\beta)$, we have $\beta \propto (\partial_\tau)^n$, which is also shown in the Eq.(2). For thermal average of an imaginary-time-dependent quantity $\mathcal{F}$, its expectation value which describes the ensemble average can be written as

$$\langle \mathcal{F}_\tau \rangle = \frac{\langle \psi(\tau)|\mathcal{F}|\psi(\tau)\rangle}{\langle \psi(\tau)|\psi(\tau)\rangle}, \tag{11}$$

where $\langle \psi(\tau)|\psi(\tau)\rangle$ is the partition function here, and the accurate value of $\langle \psi(\tau)|\psi(\tau)\rangle$ and $\langle \psi(\tau)|\mathcal{F}|\psi(\tau)\rangle$ can be determined by the method of tensor RG. The cumulative effect is efficiently in this averaging process[24] and often results in a cumulant expansion at the expectation value for the simplified result whose truncation is depends on the detail of dissipation[60]. Through this, a world-line tensor grid RG can be formed by taking the coordinate as horizontal axis, and the time (or temperature) as vertical axis, i.e., forms a tensor network. This tensor network separated by the inverse temperature β and has the spacing $\zeta = \beta/M$ where M is the total number of lattice sites in the network (also called the Trotte number[57]). Such

method which utilize the evolution of time and phase also called Trotterization[61]. Through the theory of t-DMRG, the $\mathcal{F}$ can be treated as a matrix product operator which depends on the time-evolution, $\mathcal{F}_\tau = U(\tau)\mathcal{F}U^\dagger(\tau)$, here $\mathcal{F}_\tau$ and $\mathcal{F}$ base on different basises. With the nonequilibrium time evolution, the integrable system which has the important feature of localization will relax to the stationary state after quantum quenches, i.e., the suddenly change of interaction strengh[24], and the density matrices which constrained by the expectation value will lead to a maximum entropy ensemble[22]. Usually we model the integrable (or superintegrable) model by choosing the special initial state, typically, like the XY spin model which is superintegrable when without the external magnetic field, and it can be affected deeply by the constants of motion in the integrable (superintegrable) points (like the nonthermal steady state). The density matrices here are depend exponentially on the conserved quantities and the Hamiltonians which related to the initial state. For the matrix product operators which describe the quantum states, the minimal rank D is required to be the maximal one of the the reduced density matrices of the bipartition system[4] (bipartition of the target one and its enviroment) and it needed to truncated by the method of singular values decomposing to keep the size of D polynomially increase which is local and time-computable, and we keep only the largest singular value after the truncation, i.e., only keep the basis states[62]. In fact, for dissipation system, the linear or nonlinear dissipation coupling accompanied by the phase noise[63] (like the Wiener noise (see Sec.11)) or the white noise or colored one[64] also have the inhibition effect on the exponential increase.

In Schrödinger picture, the observables of thermal states are achieved by carrying the integrable system into the nonintegrable one (by the perturbations) and in the mean time the energy-level spacing distribution is evolves from the Poisson distribution with diagonal matrices to the Gaussian one (wigner-Dyson type) with level repulsion and random symmetric matrices[65] (there are also symmetrically ordered operators in quantum dynamics by the Wigner representation[45]). It's possible to back to the Possion distribution by applying a series of single gates which prevent the exponential increase of rank D but introduces the norm error[20]

$$\eta = \sum_{i=0}^{n-1}(1 - \sum_{j=0}^{D-1}\lambda_j^2(U_i)), \tag{12}$$

where U_i is a single gate and $\lambda_j(U_i)$ is the decreasing ordered singular values after removing the maximum one, and the maximum entropy is accessible through the local relaxation similar to the entanglement. Although for nonintegrable system the growth of D is founded to be exponential, there exist methods like the diagonalization which keep the size of matrices always proportional to the time (or the system size), like the Bogoliubov rotation (see Appendix.C). The procedure of eliminating the small singular values results in a low-rank matrix, and this is also to keep the local free energy

$$E_{\text{free}} = -\frac{1}{\beta}\ln(\sum_i \lambda_i^2) \tag{13}$$

smallest (λ_i is the singular value), and to enhance the equilibrium characteristics which treated as a thermodynamics anomaly in glass system[66]. This equation also explicitly shows the measurement of erengies in units of inverse temperature. To solve the problem of density matrix in the t-DMRG, one introduce a way to solve the rank minimization problem which makes this method valid even for the low rank matrices (see Ref.[67]), and it's helpful to reducing the error and keeping computational cost low at the same time. On the other hand, that also provides the convenience that making the matrix nondecreasing and so that the maximum rank is always appear in the final step of the algorithm.

Since we have implement the system-environment partition, in a full quantum dynamics, we can yield a well approximation in the weak-coupling regime by the low-order truncation, e.g.,

the Wigner truncation approximation which truncate in the power of one-order[45]. In such a phase space, the coupled two subsystems have the relation $\sum_{k_i,k_j}(-k_i!/A^{k_i})g^{k_i}(-k_j!/A^{k_j})g^{k_j} = \sum_k(-k!/A^k)g^{2k}$[16], where k is the number of powers of truncation in phase space (e.g., $k=1$ when truncate in the first-order) and A is the angles which dominate the series expansion of the dimensionless coupling g (see Sec.2).

From the discussion on this Section, we can see that the imaginary-time propagation has similar behavior like the real-time one, it will provides us another way to detecting the decaying progress including the die-out of the excitations, and it's available for similar real-time setups[50], or application to the nonequilibrium problem with stochastic series expansion in integrable system without the constraint of local conservation law. Therefore it's more feasible to detecting the asymptotics phenomenon in time evolution, especially for the low-order perturbation theory with external potential.

4 Quenching in Many-Body Local System

For integrable open system, we imagine the bipartition of the Hilbert space, which into the two formulated finite-dimension linear space (two associated configuration) V_i and V_j which assumed have same spectrum and their reduced density matrices are

$$\mathcal{J}_i = \sum_{R=1}^{R} \lambda_R |\psi_R^i\rangle\langle\psi_R^i|, \tag{14}$$

$$\mathcal{J}_j = \sum_{R=1}^{R} \lambda_R |\psi_R^j\rangle\langle\psi_R^j|, \tag{15}$$

where λ_R is the Schmidt coefficients (the decreasing singular values). The bipartite state $|\psi\rangle \in \mathbb{C}^{d_i} \otimes \mathbb{C}^{d_j}$ which is realized through the Schmidt decomposition via the singular value decomposition, and the Schmidt rank is $\min[d_i, d_j]$[68]. For inseparable case, the reduced density matrix $\mathcal{J}_i^{'}$ (if it's pure state density matrix with feature of unitarily invariant) can be obtained by tracing over the pure states in its extended subsystem (i.e., $\mathbb{C}^{d_j}$), and the product space which formed by two subsystem is $V_i \otimes V_j$. This bipartition can be used in most of the quantum many-body model, like the Ising transverse field model, XXZ model, and kitaev model, ect.

Integrability is usually relies on the localization, especially for the superintegrable one (like the XY spin model) which are fully relies on the localization[31]. For a concrete example, we consider a XY spin two-chain model without the magnetic field, whose bulk Hamiltonian is[69]

$$H_{i,i+1} = \sum_{i=0}^{N-1} \frac{1}{2}(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y) \cdot \exp[\frac{J}{4}\sum_{j=0}^{N-1}(\sigma_{i+\Theta(j-i)}^{2x} + \sigma_{i+\Theta(j-i)}^{2y})], \tag{16}$$

where J is the coupling, i,j stands the different chains, and $\Theta(j-i)$ is the Heaviside step function. The correlation in such a system is[15]

$$\langle s_i, s_j\rangle = \frac{1}{N-1}\sum_{ij} s_i s_j = \frac{1}{2}q_{ij}(N-1), \tag{17}$$

where q_{ij} is the overlap between these two spin configurations. The local quantum integrability in the bounded bulk model can be deriving by the explicit form of the quantum R-matrix as well as the boundary transfer matrices, e.g, see Ref.[70, 69, 31].

For quench behavior due to the perturbation from local operators, which for the out-of-equilibrium protocol is striking, the amplitude from initial state to instantaneous n state is[71]

$$A_n(t) = -\int_{t_i}^{t_f} dt\langle n|\partial_t|0\rangle \exp[i(\varphi_n(t) - \varphi_0(t))] \quad (0 \leq i \leq n), \tag{18}$$

where $\varphi(t)$ is the dynamical phase. Such an amplitude is also the eigenvalue of density matrices in entropy ensemble with the specific heat $\sum_n [E_n - E_0]|A_n(t)|^2$. The sum of square of amplitudes is the excitation probability $P_{ex} = \sum_n |A_n(t)|^2$ for electrons or holes, i.e., quenched away from initial state (ground state) to a new state. Here we suppose the quench is very fast that the initial state ψ_0 and the quenched state ψ_n are almost exist in the same time t_i. Then using the evolution propagator $U(t)$, the amplitude is obtained as[72]

$$\begin{aligned}\langle n|U(t)|0\rangle &= -i\langle n|\int_{t_i}^{t_f} dt H(t)|0\rangle \\ &= -i\langle n|H_{\rm int}|0\rangle \int_{t_i}^{t_f} dt' \exp[i(E_n - E_0)t'] \\ &= -\langle n|H_{\rm int}|0\rangle \frac{\exp[i(E_n - E_0)t] - 1}{E_n - E_0},\end{aligned} \tag{19}$$

where E_0 is the energy in the initial state ψ_0. Through the fermi golden rule, where $H_{\rm int}$ is the interaction Hamiltonian with scattering amplitude A_i, which is

$$H_{\rm int} = \frac{U(t)(E_n - E_0)}{\sqrt{2 - 2\cos[(E_n - E_0)t]}}. \tag{20}$$

For further detect the perturbation from local operators, we present in Fig.2 (a) the energy difference between the excited state and initial one with different staggered magnetic field h_s in different dimension D of a quantum lattice model, and (b) the excitation probability as a function of the temperature, it's clearly that the probability distribution obeys a Gaussian form. Since the quantum noise comes from the random initial state, we define a Gaussian white noise in the initial state which has a zero mean and therefore the initial probability distribution is Gaussian. Then the probability distribution in the process of relaxation is[73]

$$P = \sum_{m,n} \delta[\Delta E - (E_m(t') - E_n(t_i))]|\langle\psi_m(t')|U(t)|\psi_n(t_i)\rangle|^2 |\langle\psi_n(t_i)|\psi_0(t_i)\rangle|, \tag{21}$$

where δ is the amplitude of the Gaussian (see Sec.11) and ΔE is the energy-difference between the initial and final state of relaxation. In phase space, such a relaxation can be expressed by the density matrix

$$\mathcal{J}(t) = \sum_{\mathbf{k},\mathbf{k+q}} \exp[-\phi(\mathbf{k})t]\mathcal{J}(0) = \sum_{\mathbf{k},\mathbf{k+q}} \exp[-(E_{\mathbf{k+q}} - E_{\mathbf{k}})t]\mathcal{J}(0), \tag{22}$$

where $\mathbf{k}$ and $\mathbf{k}+\mathbf{q}$ are the momentum before and after scattering. For slow quench where the time scales to infinity, the non-diagonal contribution to $\mathcal{J}(t)$ (i.e., the part of $\mathbf{q} \neq 0$) is vanishing due to the fast oscillation of Fourier kernel $\exp[-(E_{\mathbf{k+q}} - E_{\mathbf{k}})t]$.

In fact, the non-diagonal contribution to the mean-field-representation (or the second moments of the distribution of momentum[24]) $\langle c_i c_{i+1}^\dagger\rangle = \int d^n k f(k)\cos(\varphi(k)t)$ is asymptotically to a fixed value with the time evolution[38]. When a external perturbing-field is considered in the free energy landscape, a perturbing-term should be added to the local free energy, and since the perturbation is bad for the conservation of energy, the quantum system under the

influence of noise variables will not completely isolated even for the closed quantum system. The coupling between this perturbing field and the Hamiltonian is beneficial to enhance the system ergodicity by increasing the coupling of metastates. For closed system which have total energy conservation, the ergodicity for observables under the long-time limit can be large enough to expect the time average to the thermal average[30], but there are restrictions on the observables like the bound of the von Neumann entropy, and hence prevent it closing the thermal state. (Note that here the correlation between each distinguishable particle and the environment is still localized.) The entropy of pinning field is increase with the overlap in a metastate, can associate with the hidden glass states, and it's confirmed equal to the mean field potential of glass system[74]. Both the entropy S_{hidden} (not the diagonal one) and its free energy as well as the non-diagonal contribution vanishes in the final status of the relaxation to a steady equilibrium state, e.g., the commensurate superfluid state.

Since for the integrable system, most solvable Hamiltonian can be mapped to the effective noninteracting Hamiltonian[29]

$$H_{\text{eff}} = \sum_{i}^{N-1} \epsilon_i P_i \tag{23}$$

with the eigenenergy ϵ_i and conserved quantity P_i, and the maximum entropy ensemble after quenching with local conserve-law can be written using the density matrix as

$$\mathcal{J}_{\text{quenched}} = \frac{1}{Z}\exp(-\sum_i P_i Y_i), \tag{24}$$

where the conserved observable quantity P_i has the form $P_i = a_i^\dagger a_i$ where a_i is the annihilation operator of bosons or fermions and has commute relation $[H, P_i] = [P_i, P_i'] = 0$, the Y_i is a initial state-dependent quantity. The partition function $Z = \text{Tr}[(\exp(-\sum_i P_i Y_i))]$. This is in fact only a local steady state but not canonical steady states for the full system[22]. For integrable system begin with the maximal entropy in GGE, the Y_i here can be replaced by a Lagrange multiplier set $\{\lambda_i\}$[31, 76, 29, 75], (which is[77] $\lambda_i = \ln[(1 - \langle\psi(0)|P_i|\psi(0)\rangle)/\langle\psi(0)|P_i|\psi(0)\rangle]$ and constrained by $\langle n\rangle_{GGE} = \langle\psi(0)|c^\dagger c|\psi(0)\rangle = \text{Tr}(\rho n)$ where n is the conserved number of particles). For integrable systems which are exactly solvable (i.e., all the eigenvalues and eigenfunctions can be obtained), since the ϵ_i is linear eigenenergy, for a simplest conserved quantity, the number of particles n_i, whose eigenstate can be treated as the energy eigenstate $E = \sum_i \epsilon_i n_i$ which is on the eightbasis of $\{n_i\}$[45].

Within the scheme of adiabatic perturbation $\langle \mathbf{k}\cdot\mathbf{p}\rangle$ theory, the asymptotic behavior can be manipulated by the velocity and acceleration of the tuning parameter in quench dynamic[71]. The tuning-dependent Hamiltonian $\psi(\lambda(t))$ ($\lambda(t)$ is the time-dependent tuning parameter) can also takes effect in the adiabatic excitation of the ground state which then recover quickly due to the asymptotic effect of time evolution[50]. The asymptotic freedom of system will preserved until the number of fermion species is too large[54], so this asymptotic state with the scaling theory depends only on the configuration, e.g., the fluctuation of system [15, 50, 78]. The scaling is affected by the fluctuation and it tends to Gibbs value when the momentum vector $\mathbf{q} \to 0$[78]. The equilibrium Gibbs free energy is[3] (without restrictions)

$$E_{\text{Gibbs}} = -\frac{1}{\beta}\ln \int dt e^{-\beta H(t)}, \tag{25}$$

and since the Hamiltonian here is manipulated by the potential field, the free energy also treated as a potential function with the determined weight (probability distribution).

5 Dissipation in Nonlocal Model

For nonlocal model, there is a large difference compared to the local one. The nonequilibrium long-range force is also usually unobservable in localized interaction models[79]. Consider the Yang-Mills theory, the action of field can be expressed as

$$S = \frac{1}{4}\int d^d x \int_t^{t'} dt F_i^{\mu\nu} F^i_{\mu\nu}, \tag{26}$$

where $F_i^{\mu\nu}$ is the field strength tensor (see Appendix.A), $F_i^{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^\nu_\mu - g C_{iab} A^a_\mu A^b_\nu$[80], where A^a_μ and A^b_ν are the vector potential of the field and here C_{iab} is for introduce the SU(3) structure factor which is $C_{iab} = \gamma^{iab} F^a F^b$, where γ^{iab} is the SU(3) structure constant and F^a is the group generator. The relation between the Lie group structure constant C and quadratic Casimir operator is $\sum_{ab} C_{iab} C_{jab} = C^{(2)}_{ij}\delta_{ij}$[81].

The dissipative effect which is derived from the macroscopic entangled system gives rise the reservior problem and accompanied by a process of coarse-graining by the isometries that integrating the degrees of freedom of subsystems[25] and with a dimension smaller than the maximum one of Hilbert space[82]. The nonlocal correlation between the nearest neighbors can be treated locally by using the matrix product operator with determined rank and the unitary transformation with time-evolution operator (see below). For the localized interaction between nearest neighbor spin accompanied by the local field effect in homogeneous electron system, since the relatively large coupling constant and long time configuration, it's priority to use the nonperturbative method[60], but for nonlinear quantum dissipation it is more acceptable to use the perturbative one (like the perturbative RG). The Gaussian probability distribution exist in the linear case, but doesn't exist in the nonlinear case anymore, and the dimension of density matrices is also grows non-linearly with time[4], but there are still some linear relations, e.g., the entanglement entropy changes linearly with time under a staggered magnetic field in the disordered case[20].

In a open quantum system, the thermal average of observable $\mathcal{F}$ can be written as (here τ is the complex-time for propagators)

$$\langle \mathcal{F} \rangle_\tau = \frac{\mathrm{Tr}(e^{-\beta H} e^{-H\tau} \mathcal{F} e^{H\tau})}{\mathrm{Tr}(e^{-\beta H})}. \tag{27}$$

For integrable system, this equation which describes the thermal average in Gibbs ensemble[19] is equal to the energy of initial state of relaxation process after quenching which evolution with time τ. Such thermal average is also meaningful in thermodynamics description for quasi-equilibrium state[66]. Base on Eq.(8) and using the second order Trotter-Suzuki formation, the evolution propagator can be decomposed as $e^{-\beta H} = e^{-\beta H_x} e^{-\beta H_y} e^{-\beta H_z} + O(\tau^2)$[62], and Eq.(8) can be rewritten as

$$e^{-\beta H_{ij}} = \prod_{i=0,j=0}^{n} U_{i,i+a;j,j+a} \quad (a = x, y, z). \tag{28}$$

To study the dissipation of the remaining degrees of freedom in subsystems after coarse granulation in macroscopic model, the reservoir is very important. To introducing FDR to the steady state, we rewrite the Eq.(27) by the method of path integral as

$$\langle \mathcal{F}_\tau \rangle = \int D\psi(\tau) e^{\tau H} \frac{\langle \psi(0+\varepsilon^+)|\mathcal{F}(0)|\psi(0+\varepsilon^-)\rangle}{\langle \psi(\tau+\varepsilon^+)|\mathcal{F}(\tau)|\psi(\tau+\varepsilon^-)\rangle}, \tag{29}$$

with $\varepsilon \to 0$, and $\psi(\tau) = U\psi(0)$ where U is the time-evolution operator, $U = T_\tau \exp(-\int_\tau^{\tau'} d\tau H(\tau))$. For statistical linear dissipation system, the correlation between reservoirs $\langle R_i R_j \rangle \neq 0$, the

method of unperturbed linear dissipation is also suitable for perturbed macroscopic model if the perturbation Hamiltonianis is linear with the reservoir $H_p = \sum_i f(i)R_i$ where $f(i)$ is a linear term and therefore the collective responses to the perturbation is mostly linear[60]. This form of H_p is suitable for all the integrable or nonintegrable linear dissipation model. While for the non-linear dissipation case, since the reserviors in different subsystems is independent with each other, so we constraint the reservoir states in the Liouville spaces, and have $\langle R(0)|H_{SR}|R(\tau)\rangle = 0$[14], where H_{SR} is the interaction term between system and reservoirs and there exist shared influence function for all constituent[25].

For non-dissipation system, the propagation along time scale can be expressed by the initial Hamiltonian and the observable conserved quantity (i.e., Eq.(24)), whereas for linear dissipation, it needs a stochastic term to compensate the lost energy, and it has a history-independent potential term $\partial_\tau\psi(\tau) = H_0(\tau)\psi(\tau) - \sum_i f(i)q(i)\psi(\tau)$, where $q(i)$ is the stochastic force or the noise. For nonlinear-dissipation system, the states of reservoir variables spans only in the Liouville space[14]. Both the linear-dissipation and nonlinear-dissipation contain a friction force term but the nonlinear-dissipation has a complex memory term which is history-dependent[83] in the evolution while the linear one hasn't.

6 Time Evolution and Thermal Entanglement in Integrable Heisenberg XXZ Model

We already know that for non-dissipation system the antiferromagnetic Ising chain[4], XY spin chain[69, 4] and the bulk model[69] are integrable and can be exactly solved. The Heisenberg XXZ model is also suggested integrable and own the local conserved quantity, e.g., the observable microscopic quantity like the S^z or the observable macroscopic quantity like energy or number of particles. To investigate the imaginary-time evolution in Heisenberg XXZ model, we firstly need to use a c-number representation which depict a shift of $-i\hbar\alpha$ in the axis of Imτ (see, e.g., Ref.[60]). Then we introduce the Heisenberg XXZ model with spin 1/2 antiferromagnetic free fermions interaction, the n-component anisotropy Heisenberg Hamiltonian of this system contains a homogeneous external field h

$$H = \sum_{i=0}^{n-2}(JS_i^xS_{i+1}^x + JS_i^yS_{i+1}^y + J_zS_i^zS_{i+1}^z) + \sum_{i=0}^{n-1}(h_iS_i^z), \tag{30}$$

where J and J_z are the coupling, and $S_i^\alpha = \frac{1}{2}\sum_i \sigma_i^\alpha$ $(\alpha = x, y, z)$ is the total spin in α-component. The important coupling ratio can be defined as

$$\frac{J_z}{J} = \begin{cases} \cos\gamma, & J_z \le J, \\ \cosh\mu, & J_z > J, \end{cases} \tag{31}$$

where the tilted angle γ and μ are enlarged with the increased degrees of anisotropy. We focus on the $J_z/J = \cos\gamma$ case. In the case of $J_z = 0$, i.e., becomes the noninteracting spinless fermion system with strongly correlated electronic characteristics under the Wigner-Jordan (WJ) transformation which turns the regular integrable terms into the chaotic one[4]. In this case, the fermion representation of the gapless bilinear fermionic system is

$$H_{\rm bf} = \sum_i (c_ic_{i+1}^\dagger + c_i^\dagger c_{i+1} + h_in_i), \tag{32}$$

where $\Delta_i = \langle c_ic_{i+1}^\dagger\rangle$ stands for a mean-field and the covalent bonding of WJ fermions[18], and this is also the tight-binding fermionic model with dispersion relation $\kappa = \pm 2\cos k$[22] in π-phase (the phase difference between neighbor site is π). In this case, this Heisenberg Hamiltonian

becomes a strongly correlated electronic system with a finite entropy (will saturation)[69, 20]. The operator of number of the spinless particles is $n_i = c_i^\dagger c_i$, and the electron correlation is $J_z n_i n_{i+1}$. To investigate the nonlinear-dissipation in this spinless fermions chain model, we need to introduce the master equation with system density matrix $\mathcal{J}$ [32],

$$\partial_t \mathcal{J} = -i[H, \mathcal{J}] + \mathcal{K}\sum_i [O_i \mathcal{J} O_i^\dagger - \frac{1}{2}(O_i^\dagger O_i \mathcal{J} + \mathcal{J} O_i^\dagger O_i)] \equiv \mathcal{L}\mathcal{J}, \tag{33}$$

where $\mathcal{J}$ corresponds to the pure state or mixed state and O_i is the Lindblad operator describing the bath coupling. The right-hand side of this equation contains two terms, the first one is the unitary part of the Liouvillean, while the second one is the dissipative term and $\mathcal{K}$ is the coupling strength within the dissipation scenario. We consider the damping here due to the nonlinear-dissipation. The Gaussian area arrived in time evolution has $\partial_t \mathcal{J} = 0$. In this case $\mathcal{K}$ is almost vanish and produce a zero dissipative area, that suggest that the observables exponentially fast approach the steady state[36], and the density matrix is close to the diagonal one with only the main diagonal entries.

To introduce the thermal entanglement in the evolution, we define the generate and annihilate operator for sites i as

$$c_i^\dagger = e^{i\varphi_i} S_i^+ \ , \ c_i = e^{-i\varphi_i} S_i^-. \tag{34}$$

The operators obey the commutation relation $[c_i, c_j^\dagger]_\alpha = \delta_{ij}$ (boson operator and fermion operator for $\alpha = 1$ and -1, respectively), and $c_i^\dagger c_j + c_j c_i^\dagger = \delta_{ij}$ ($\alpha = -1$) under the WJ transformation. The time-involve phase φ_i has

$$\varphi_{i+1} - \varphi_i = c n_i, \tag{35}$$

where c is a c-number-correlated factor which is defined as the imaginary part of $\mathrm{In}(\tau' - \tau)$, and with the phase function $\varphi_i = \sum_i c_i^\dagger c_i c$. Then the Hamiltonian (Eq.(30)) can be represented as

$$H = \begin{bmatrix} J_z/2 + h & 0 & 0 & 0 \\ 0 & -J_z/2 & J & 0 \\ 0 & J & -J_z/2 & 0 \\ 0 & 0 & 0 & J_z/2 - h \end{bmatrix}, \tag{36}$$

when $|J| < h - J_z$, the ground state is disentangled state $|0,0\rangle$ which with the eigenvalue $J_z/2 - h$; when $|J| > h - J_z$, the ground state is entangled state $\frac{1}{\sqrt{2}}(|0,1\rangle - |1,0\rangle)$ for $J > 0$ or $\frac{1}{\sqrt{2}}(|0,1\rangle + |1,0\rangle)$ for $J < 0$ which with the eigenvalue $-J_z/2 - |J|$, and this entangled state will goes to maximal by long-time evolution. Thus, the entanglement increase with the enhancement of coupling J and J_z no matter they are both greater than zero (ferromagnetic) or both less than zero (antiferromagnetic), but it's always symmetry compared to the case of inhomogeneous magnetic field. We can obtain the relaxations in a long-time scale after the sudden quenching of J and J_z, and regulate the entanglement by the quenching of magnetic field h. In equilibrium case, the density matrix of this thermal state can be written as[84]

$$\mathcal{J} = \frac{1}{Z}\exp(-\beta H) = \frac{1}{Z}\begin{bmatrix} e^{-(J_z/2+h)/T} & 0 & 0 & 0 \\ 0 & e^{J_z/2T}\cosh(|J|/T) & -s & 0 \\ 0 & -s & e^{J_z/2T}\cosh(|J|/T) & 0 \\ 0 & 0 & 0 & e^{-(J_z/2-h)/T} \end{bmatrix}, \tag{37}$$

where $Z = e^{-(J_z/2+h)/T}(1 + e^{2h/T}) + 2e^{(J_z+h)/T}\cosh(|J|/T)$ and $s = Je^{J_z/2T}\sinh(|J|/T)/|J|$. Usually, we can creating strong entanglement by raising the ratio of J_z/J, or by raising the degree of inhomogeneity of magnetic field h, or properly lower the temperature through the

previous study[17, 84, 85]. Sometimes the lower temperature which can be implemented by increasing the system size[85] can decrease the eigenvalue of the density matrix (Eq.(37)).

7 Correlation and Transfer Speed in One-dimension Chain Model

In this section, we focus on the two-point spin correlation in $S = 1/2$ Heisenberg chain and $S = 1$ Ising chain, and define that the J_1 and J_2 as the nearest neighbor coupling and next-nearest neighbor coupling in the chain, respectively. The β(inverse temperature)-dependent magnetic susceptibility can be written as $\chi(\beta, t, i) = \beta 2^{-n} \sum_{i=1}^{n-1} \langle S_0^z S_i^z \rangle$ for a n-qubit chain, the latter term in this expression is the spin-spin correlation function for the Heisenberg model[20]. Fig.3 shows the spin correlation C and inverse correlation length ξ^{-1} for (a) $S = 1$ Ising spin chain and (b) $S = 1/2$ Heisenberg chain with different J_2 at different site i. We show that the nonlocal order parameter decay exponentially due to the perturbations from the long-range spin-spin interaction which breaks the integrability and therefore exhibits an effectively asymptotic thermal behavior, though the latter one is exactly solvable (i.e., all eigenvalues can be obtained by the method of Bethe ansatz in thermodynamic Bethe ansatz (TBA)[86]) beforce the perturbation. Such an exponentially decay for the nonlocal operators in the nonintegrable model has been widely observed, e.g., the order parameters in transverse field Ising chain for ferromagnetic/paramagnetic state[87] or the number of quasiparticles in the time evolution for a quantum spin chain[88], etc. We also can see that the ξ^{-1} is tending to saturated with the increase of distance which obey the equilibrium law, and in fact it's equivalent to the coherent state with coherent amplitude in terms of a exponential form, and therefore the phase coherence rate will display a similar behavior with the correlation length. Fig.4 shows the spin correlation for $S = 1/2$ Heisenberg chain as a function of temperature with different J_2. We can see that, with the increase of J_2, the spin correlation also increase. We also make the comparision for the spin correlation C at different temperature for $S = 1$ Ising chain and $S = 1/2$ Heisenberg chain in the Fig.5. It's obviously that the $S = 1/2$ Heisenberg chain is earlier becoming saturated compare to the Ising one. Further, we present the correlation (a) for $S = 1/2$ Heisenberg chain which is obtained by the method of Bethe ansantz and make a comparison on the results of correlation in low-temperature for $S = 1/2$ Heisenberg chain between the methods of Bethe ansatz and renormalization group (b) in Fig.6.

Since the equal time spin correlation C has the relation

$$C(r, t) = \langle S(0, t) S(r, t) \rangle \propto \exp(-r/\xi), \tag{38}$$

with the correlation length $\xi^{-1} = -\lim_{L\to\infty} \ln\langle S_i S_{i+L} \rangle$. Here the distance r can be specified as i which stands the different coordinates in the spin chain and ξ is the correlation length. Note that this expression for equal time two-point correlation is well conform for the ordered phase in the long-time limit, while for disordered phase, the ξ has more complicated form[87]. Now that this spin correlation function displays an effective asymptotic thermal behavior as introduced in Sec.1, and correlation length ξ is related to the quantum quench protocol[37], the thermal behavior for a nondissipation system after quench can also have the relation which mentioned above (Eq.(38)), but note that although this spin correlation is in an exponential form, the correlation length does not follows the thermal distribution but follows a nonthermal distribution[37] and guided by GGE. This is because the correlation length is a local quantity which behave nonthermally. Similar behavior appear in the correlators like the transverse magnetization and so on. We still need to note that though for infinite system which follows the effective thermal distribution is mostly nonintegrable, but the initial state of integrable system which dictated by the noninteracting Hamiltonian may still follows the thermal distribution[89] since without the effect of interactional quench Hamiltonian. Further, if we mapping to the

Fourier space, the equal time correlation (Eq.(38)) for the spin-1/2 square lattice model has a more specific form in the low-momentum case[90]

$$\langle S_i(0)S_j(r)\rangle e^{ikr} \propto \frac{e^{-r/\xi}}{r^4}(1+\frac{r}{\xi})\delta_{ij}, \tag{39}$$

which follow the power law decay when $r \ll \xi$ and exponentially decay when $r \gg \xi$.

Since the pinning field play an important role in the process of ergodic to non-ergodic transition which plug the correlation between subsystems and even the velocity of spin wave v_s[91], which associate with the slope of the dispersion relations in momentum space. For the case of $J_z/J = \cos\gamma$, v_s can be written as[91]

$$v_s = \frac{J\pi}{2}\frac{\sin\gamma}{\gamma} \tag{40}$$

which is consistent with the slope of dispersion relation $\partial_k\kappa = \mp\sin k$. Then a question arisen that if the speed of information transfer which govern the relaxation time of a post-quench state relate to the speed of spin wave in a spin-interaction model? The answer is yes. A direct evidence is the Lieb-Robinson type boundary (the details about the Bose-Hubbard model is presented in next section). In fact, the spin wave is also related to the momentum transfer[92] and even the damping of oscillation of the superfluid regime (see Sec.8 or Ref.[93]). We know that the missing of symmetry is related to the influence of initial states, and the collapse of physical phenomenas like the interference pattern[38] or the collective excitation[94, 95] by inhomogeneous oscillation in condensate with a density wave order which act like a single phase wave or standing wave[96], is revives in the latter time of relaxation. The transfer of correlation with a finite velocity also construct a line-cone which well describe the relaxation behavior.

8 Double Occupation and The Interaction Quench in Nonintegrable Hubbard Model Near The Phase Transition Point

We next construct the Bose-Hubbard lattice model by the Hamiltonian as a explicit example

$$H = -\mathcal{P}\sum_{i=0}^{n-2}(b_i^\dagger b_{i+1} + H.c.) + U\sum_{i=0}^{N-1} n_i n_{i+1} - \mu_i\sum_{i=0}^{n-1} n_i \tag{41}$$

where $\mathcal{P}$ is the hopping constant, U is the chemical potential and μ_i is the local potential of each particle. The interaction between the next-nearest neighbor is assumed to be zero in this model and only the nearest neighbor pairs have nonzero interaction, thus it is integrable. A dimensionless reduced coulping is defined here as

$$g_{\text{red}} = \frac{UN}{\mathcal{P}} \tag{42}$$

where N is the number of interactional particles. We can implement the phase transition from Mott-insulator to the condensed state or superfluid by modulating the value of g_{red}, and it has been implemented experimentally[97, 98, 99]. Even for systems which without hopping at all (i.e., $\mathcal{P} = 0$), the phase transition from metallic state to the Mott insulator is also realizable by the interaction quenching of U, and in this case the oscillations with the collapse-and-revival pattern are periodic with period $2\pi U/\hbar$ [39] The Table.A shows the time scale of relaxation and the period of collapse-and-revival pattern for several models. In fact, most many-body systems can exhibit quantum phases with different entanglement structures in the complex mixed dynamics, and it's usually realizable by tuning the strength of this competing

interaction[9]. The fluctuation of correlation amplitude due to the fast oscillation of phase factor are related to the distribution of the initial state, and the short-range correlation also shows distinguishable differences for different configurations of initial states.

In this model we next define a hopping-determined operator $\mathcal{R} := it\mathcal{P}$. This periodic-time-dependent evolution operator for a single-site can be expanded as[24]

$$e^{\mathcal{R}} := e^{it\mathcal{P}} = \sum_{k\geq d_r} \frac{(it\mathcal{P})^k}{k!} \leq \sum_{k\geq d_r} \frac{(6\mathcal{P}t)^k}{k^k} \tag{43}$$

where k denotes the unit vector in phase space and d_r is the distance between sites i and $i+r$. There exist a upper bound for d_r as $d_r < 6\mathcal{P}t/e$ where e is the natural constant since there is a insurmountable maximum speed for information transfer in this model. The summation of the part which beyond the distance d_r follows the above relation. Thus we also have

$$e^{\mathcal{R}} \leq \frac{(6\mathcal{P}t)^d}{d^d - 6\mathcal{P}t \cdot d^{d-1}}, \tag{44}$$

which requires $d_r > 6\mathcal{P}t$ while the critical distance d_c which corresponds to the upper bound is nearly equals to $6\mathcal{P}t$. If we treat the conserved particles-number $\mathcal{P}$ as a matrix, then it has the operator norm $\|\mathcal{P}\mathcal{P}^*\|_{\text{op}} = 1$ and $\mathcal{P}^\dagger\mathcal{P} = \mathcal{P}\mathcal{P}^\dagger = \mathbf{I}$ where $\mathbf{I}$ is an identity operator. This is related to the case mentioned in Ref.[39] that n_i only have the two eigenvalues 0 and 1, and here the maximal eigenvalue 1 is nondegenerate for our scenario, while other eigenvalues approaches to 1 smoothly in the long-time limit.

Since in long-time limit the relaxation will removes the non-diagonal part of the density matrix, the differece between the density matrices and its diagonal one is $\Delta\mathcal{J} = \mathcal{J}(t) - \mathcal{J}_G$, thus for the hopping matrix which mentioned above, its trace norm is

$$\frac{(6\mathcal{P}t)^{d_c}}{d_c^{d_c} - 6\mathcal{P}t \cdot d_c^{d_c-1}} > \|\Delta\mathcal{J}\|. \tag{45}$$

Note that here the critical value d_c is independent of the size of system.

We have present the upper bound of speed for the information transfer by a form of suppressed exponent. Since the nondiagonal contribution won't vanish until $t \to \infty$ (which corresponds to $\Delta\mathcal{J} = 0$), and it's decay in a time scale as $1/t$[30, 22], i.e., the dephasing process, (note that for large-size system, the inequality of Eq.(45) will become more obvious, and the vanished nondiagonal contribution will reappear if the size is large enough, which called "rephasing"), the phase can be expressed as $\varphi(\mathbf{k}) = \varphi(0) + \mathbf{q}^\ell + O(\mathbf{q}^{\ell+1})$[22] where ℓ is a tunable parameter in phase space. The contribution in such a dephasing with scale $1/t$ in phase space is

$$\mathbf{k}^\ell = \int d\mathbf{k}^\ell e^{i\varphi(k)} \frac{\mathbf{k}^{1-\ell}}{\ell} \int d^{d-1}\mathbf{k} f(k), \tag{46}$$

where $\varphi(k) = \varphi_0 + \mathbf{k}^\ell$.

Next we form the the Bessel formula to show the reducing property of the evolution operator $e^{i\mathcal{P}t}$ which with large size N and can be viewed as the Riemann sum approximation of the following function with phase number α [24],

$$\begin{aligned} J_\alpha(x) =& \frac{1}{2\pi i^\alpha} \int_0^{2\pi} \exp[i(\alpha\varphi + x\ \cos\varphi)] d\varphi \\ =& \frac{1}{2\pi} \int_0^{2\pi} \exp[i(\alpha\varphi - x\ \sin\varphi)] d\varphi, \end{aligned} \tag{47}$$

which is shown in the Fig.7. Through this, the maximum rate for the system to relaxation to the Guassian state is obtained as $(2\mathcal{P}t)^{-N/3}$ for a N-site system.

For this one dimension bosonic system what we are discussing, the Mott gap $U - U_c$ is allowed to exist during the relaxation process[39], (for a experiment, see Ref.[100]). For coupled bose-lattice model, one forms the time-dependent continuous variable $n(t)$ to describe the quasiperiodic decaying[101], the semiclassical motion equation which is in a continuum bath of harmonic potential and additively applying a confining parabolic potential, reads

$$\frac{d^2n}{dt^2} + 4n + 4g_{\rm red}n\left[\cos(\varphi(0)) + \frac{g_{\rm red}n^2}{2}\right] = 0, \tag{48}$$

where $\varphi(0)$ is the initial phase. Thus the double occupation $\langle n^2(t)\rangle$ (also the double momenta occupation number in momentum space) under the quenches from different Mott insulator initial state (with different initial phase) to weak interaction one (with weak $g_{\rm red}$) is[101] (ignore the influence of high-order U here)

$$\langle n^2(t)\rangle \approx n^2(0) - \frac{1}{2\pi}\int_0^\pi \sin^2\varphi(0)\cos[4t\sqrt{1+g_{\rm red}\cos\varphi(0)}]d\varphi(0), \tag{49}$$

where $n^2(0) = 1/4$ here is an effective approximation for two uncoupled systems in semiclassical theory. The $n^2(t)$ with weak $g_{\rm red}(< 1)$ according to above equation is shown in Fig.8. Note that since the critical value of interaction for superfluid-to-Mott insulator phase transition in the Bose-Hubbard lattice model requires $U/\mathcal{P} \approx 16.7$ [102], and the reduced coupling $g_{\rm red} \sim N^2$, so the ground state of this system will keep this superfluid regime in a large range of $g_{\rm red}$ if without excitation like the quench behavior. But this expression doesn't works for the region of $g_{\rm red} > 1$, e.g., see (d) and (e) in Fig.8. For long-time behavior with very weak $g_{\rm red}$, the asymptotic behavior of Eq.(49) is

$$\langle n^2(t)\rangle \approx n^2(0) - \frac{1}{\sqrt{16\pi g_{\rm red}t}}[\cos(4t\sqrt{g_{\rm red}+1} + \frac{\pi}{4}) + \cos(4t\sqrt{1-g_{\rm red}} - \frac{\pi}{4})], \tag{50}$$

which is presented in Fig.9. We can see that the amplitude fluctuation is increase with the reduction of $g_{\rm red}$, and in long-time limit the undulate of oscillation becomes more flat but no completely governed by the time-independent Hamiltonian. This corresponds to the superfluid regime with obvious amplitude fluctuation, the recurrences and interference pattern will also occur (not shown). For the case of initial $g_{\rm red} = N$, when the quenched $g_{\rm red} \gtrsim 7N$, this nonequilibrium system will enter into the nonthermal steady state according to the results shown in Ref.[38] though it's a nonintegrable system.

For one-dimension nonintegrable case of hard-core bosons (where the generalized eigenstate thermalization occurs[103]), a typical model of $1/r$ Hubbard chain also have the feature of collapse-and-revival oscillations[29] like the nonintegrable one, but it's dispersion-linear, i.e., it can be effectively solved by Eq.(23) while the nonintegrable one can not. Now we consider large $g_{\rm red}$ into the strong-coupling perturbation in a two-dimension version of $1/r$ Hubbard model-the lattice fermions Hubbard model, the double occupation $d(t) = \langle n_\uparrow n_\downarrow\rangle/N$ can be written as[104]

$$d(t) = d(0) + \sum_{i=0}^{N-1}\frac{1}{g_{\rm red}}\langle c_i^\dagger c_{i+1}(n_i - n_{i+1})^2\rangle + O(\frac{V^2}{U^2}). \tag{51}$$

whose graphs have been presented in the Fig.2 of Ref.[104]. This is corresponds to the state of Mott insulator with strong interaction and have

$$\mathcal{P}\langle c_i^\dagger c_{i+1}(n_i(0) - n_{i+1}(U))^2\rangle = 2\sum_i[\kappa_i(n_i(0) - n_{i+1}(U))], \tag{52}$$

where κ_i is the dispersion relation related to the kinetic energy $T_{\rm kin}$. The prethermalization regime is also exist in this case for one-dimension or two-dimension Bose-Hubbard model[38],

but this prethermalization regime as well as the general collapse-and-revival oscillations vanish in a little range before the critical value U_c which origin from the discontinuous momentum distribution in Fermi surface due to the quenching.

We show the bandwidth-dependent kinetic energy of $1/r$ Hubbard chain with different bandwidth: $W = 1$, $W = 4$, and $W = 1/2$ (which has been obtained by the method of local density approximation (LDA)[105]) in Fig.10. It's obviously to see that the amplitude of hopping is increases with the bandwidth W (see the inset) and the $T_{\rm kin}$ decay rapidly with the increasing distance along the chain. When it quenches to a large U , the oscillations of Eq.(51) makes a difference[104] $\Delta d = \mathcal{P}\pi(1-2n/3)/U$ which is halved in Falicov-Kimball model for nonequilibrium dynamical mean-field theory (DMFT) due to the vanishing of $\mathcal{P}$ and one of its two spin species, therefore only one spin specie contributes to the kinetic energy. In DMFT, this kinetic function due to the considerable noise (see Sec.10, Appendix.C) yields a single-site Green's function

$$G(t,t') = i\langle c(t)c^\dagger(t')\rangle, \tag{53}$$

where the contour-order correlation $\langle c(t)c^\dagger(t')\rangle$ has

$$\langle c(t)c^\dagger(t')\rangle = \frac{\mathrm{Tr}[e^{\beta H_G}T_{\mathcal{C}}e^S c(t)c^\dagger(t')]}{\mathrm{Tr}[e^{\beta H_G}T_{\mathcal{C}}e^S]}, \tag{54}$$

where $T_{\mathcal{C}}$ is the contour-order temperature, and the single-site action[106]

$$S = \int_{\mathcal{C}} dtdt' c^\dagger(t)\Lambda(t,t')c(t') + \int_{\mathcal{C}} dtV(t), \tag{55}$$

where $\Lambda(t,t')$ is a hybridization between the sites with the rest of the lattice (environment).

By the nonequilibrium DMFT, which well describes the time evolution of an interacting many-body system (fermions lattice Hubbard model), we can map the lattice model to a single-site impurity model as shown above. Unlike Eq.(51), the method of DMFT is nonperturbative, but since we consider the perturbation from noise into the Green's function, the resulting Green's function is

$$G(t,t') = G_0(t,t') + G_0(t,t_i)\Sigma_{ij}G(t_j,t'), \tag{56}$$

where G_0 is the unperturbed Green's function, and it has[107]

$$\frac{e^V - 1}{e^V - iG_0(e^V - 1)} * G_0(t,t') = \Sigma * G(t,t'), \tag{57}$$

where $V = H - H_G$ is the non-Gaussian part of the Hamiltonian, i.e., the interaction term $U(t)n_\uparrow n_\downarrow$ which is noncommuting[108]. Thus to linearizing the rest part of Hamiltonian, we need to tend the partial function which is the denominator of Eq.(54) into the interacting representation with decomposed Boltzmann operator using the method of Hubbard-Stratanovich transformation which require the convergency of the Gaussian integrals[109]. This partial function select all the possible configuration of single-site along the contour $\mathcal{C}$, which make it possible to be decouped by a auxiliary-field quantum Monte Carlo methods[108, 106], (Note that the integrable lattice model for soft-core bosons , the non-Gaussian distribution is origin from the off-site hopping[110] term unlike the case what we are discussing here). The single-energy variables s_i along the contour $\mathcal{C}$ have[108] $e^V_\sigma = \mathrm{diag}(e^{\gamma\sigma s_1}, e^{\gamma\sigma s_2}, \cdots, e^{\gamma\sigma s_i})$ where σ denote the spin order $\sigma = \pm 1$ and γ here is a temperature- and interaction-dependent parameter. This equation means that eigenvalues (which can be specialized as the band energy ϵ_k in Hubbard model) of hopping matrix V can be diagonalized by the diagonal matrices which shown in the bracket of above expression.

Since the total Hamiltonian must be conserved within the evolution, the kinetic energy of $1/r$ Hubbard chain is suppressed by the term $E_{\rm pot} = Ud(t)$. For half-filling Hubbard Hamiltonian

$(n_\uparrow = n_\downarrow = 1/2)$ with a semielliptic density of state $\rho_{hf} = \sqrt{4\mathcal{P}^2 - \epsilon_k^2}/(2\pi\mathcal{P}^2)$, the kinetic energy per lattice site[104] is $T_{\rm kin} = 2\int d\epsilon_k \rho_{hf}(\epsilon_k) n(\epsilon_k, t)\epsilon_k$, (with the band energy ϵ_k) which obeys the Dyson equation in lattice model with the Green's function $G_k(t, t')$

$$G_k(t, t')(i\partial_t + \mu - \epsilon_k - \Sigma) = 1,\ t = t' \tag{58}$$

where the convolution product of local self-energy Σ and G_k yields the equal time double occupation in a homogeneity phase and the self-consistency local Green function has[111] $G_k(t, t') = \int d\epsilon_k \rho(\epsilon_k) G_k(t, t')$, where $G_k(t, t')$ is diagonal. The approximation of Hartree-Fock which works well for the single-particle problem, affects the chemical potential μ which has a zero mean, by the particle number in canonical ensemble

$$\overline{\langle n_\uparrow n_\downarrow \rangle} = \frac{1}{N^2}\sum_{k,k'}\langle n_{k\uparrow} n_{k'\downarrow}\rangle = \frac{n^2}{4}, \tag{59}$$

and it contributes to self-energy by the diagonalized Hartree-Fock Hamiltonian and provides a precise result in half-filling case, but since the Hartree-Fock is sensitive to the spin-correlations[112], it fails when the spin degrees of freedom disappear. In this case, one gives the second-order contribution to the self-energy in the form of[106]

$$\Sigma(t, t') = -U(t)U(t')G_{0\sigma}(t, t')G_{0,\bar{\sigma}}(t', t)G_{0,\bar{\sigma}}(t, t'). \tag{60}$$

Here the unperturbed Green's function $G_{0\sigma}$ can be replaced by the full interacting one: G_σ, and the interaction U can be viewed as an evolution propagator.

Since the fact[112] that the phase transition of metal-to-insulator in half-filling $1/r$ Hubbard chain occurs when $U = W$ where we set $W = 4$ here, we obatin $U_c = 4$. Note that the band energy ϵ_k is closely related to the continuity of momentum distribution, e.g., it's discontinuity when $\epsilon_k = 0^-$ and $\epsilon_k = 0^+$ in each side of critical value U_c. When it approaches to critical value U_c, $d(0) = 1/8$ after quenching, since we set the $n = 1$ and the critical value is $U_c = 4$, the one-dimension half-filling $1/r$ Hubbard model has the double occupation as

$$\begin{aligned} d_{hf}(t) &= \frac{1}{8} - \frac{(4-U)^2}{16U} - \frac{(16-U^2)^2}{16U^2}\ln\left|\frac{4-U}{4+U}\right| - \frac{\cos(Ut)\cos(4t)}{2Ut^2},\ \text{for quench from 0 to U;} \\ d_{hf}(t) &= \frac{1}{8U} + \frac{(4-U)^2}{16U^2} + \frac{(16-U^2)^2}{16U^3}\ln\left|\frac{4-U}{4+U}\right| + \frac{\cos(Ut)\cos(4t)}{2U^2t^2},\ \text{for quench from } \infty \text{ to U,} \end{aligned} \tag{61}$$

while for the quenching that reaches the U_c, the behavior of double occupation is described by

$$\begin{aligned} d_c(t) &= \frac{1}{8} - \frac{1}{512}\left[\frac{48\sin(8t)}{t^3} + (\frac{6-32t^2}{t^4})(\cos(8t)-1)\right] - \frac{3}{32t^2},\ \text{for quench from 0 to U;} \\ d_c(t) &= \frac{1}{32} + \frac{1}{2048}\left[\frac{48\sin(8t)}{t^3} + (\frac{6-32t^2}{t^4})(\cos(8t)-1)\right] + \frac{3}{128t^2}.\ \text{for quench from } \infty \text{ to U.} \end{aligned} \tag{62}$$

Fig.11 shows the graphs of $d_{hf}(t)$ in Mott insulator for quenches from 0 to U and from ∞ to U (according to Eq.(61)), we can see that the later one is roughly the inverse version of the former one, and a significant feature is the fast-saturation. The larger the interaction U is, the faster the curve tends to saturated. Note that the double occupation here is indeed related to the realistic physical quantity of global correlation for bosons system, and the discussion above is for a prediction about the behavior in long-time limit, i.e., the stationary results, which consistent with the thermal values[39]: 1/4 for interaction quenches from 0 to ∞, 1/6 for interaction quenches from ∞ to 0, 1/8 for interaction quenches from 0 or ∞ to U_c, (we set $n = 1$ here). The collapse of oscillations are scale as $1/\sqrt{g_{\rm red}}$, i.e., the amplitude are continually

decaying along the long-time scale which cover the phase transition, and $d(t)$ will shows strictly periodic behavior in the noninteracting regime with $g_{\rm red} = 0$ (not shown in the Fig.11). For quenches from 0 to finite U, the prethermalization regime also shows large agreement with the stationary values of $d(t)$ in long-time limit. The effect of damping on the amplitude of collapse-and-revival oscillations is always exist in the long-time scale, and has important influence on the relaxation. It produces the "overdamp" in the regime with sufficiently large U, which nearly reduce the amplitude to 0 after instantly tends to saturate. The process of damping is related to the velocity of spin wave in Goldsone model that for zero frequency Goldstone mode it followed by a additional standing spin waves[92, 101]. By setting a list of interactions in Fig.11, we found that, for quench from 0 to an infinite interaction U, closer the quenches to critical value U_c, closer the $d_{hf}(t)$ to quasistationary value, which is obtained from the Fig.12 as 0.125 (see the bottom inset of Fig.12(a)): the U which close to U_c in Fig.11(a) is setted as 3.299, and the long-time result for quench to this U is 0.12499, which is very close to the stationary prediction 1/8, and it's reasonably differ from the thermal prediction of 0.098 by the equilibrium result[113]. While for the quench from ∞ to U, we obtain the same conclusion: the result of quench to $U = 3.299$ is d =0.032 which is very close to the stationary value 0.0312 as shown in the bottom inset of Fig.12(b). That is the long-time behavior of nonequilibrium system which shows well agreement with the result of quasistationary value in phase transition point (this conclusion will always exist in the time scale of $1/|\mathcal{P}| \ll t \ll U/\mathcal{P}^2$).

While for the anharmonicity case, the coupling $g_{\rm red}$ still usable by the form of a symmetrical anharmonic term (see Sec.10), the bare action of quantum system with N-component bosonic field ϕ_α in ϕ^4 field theory (where $g_{\rm red}$ close to the critical value with U_c) is[114, 101]

$$S = \int d^d r d\tau \frac{1}{2}[(\nabla_r \phi_\alpha)^2 + \frac{(\partial_\tau \phi_\alpha)^2}{c^2} - (r_c + r)\phi_\alpha^2 + \frac{\lambda x^4}{N}\phi_\alpha^4], \tag{63}$$

where $\alpha = 1 \cdots N$, c is the velocity, λx^4 is the quartic nonlinear coupling term, and the critical value r_c is reached at $r = 0$. For the case of quenches from large U to a small one which is close to zero, i.e., from the Mott insulator initial state to the superfluid or metallic state, we introduce the vectors $k_1 = 2\pi n_1/N$ and $k_2 = 2\pi n_2/N$ which obey the periodic boundary condition (see Appendix.C) and have $n_1 \neq n_2 < N$, then when the coupling is close to zero, the time-dependent nearest-neighbor correlation in the bath with harmonic potential is given as[101]

$$\langle n_r(t) n_{r+1}(t)\rangle = \frac{2g_{\rm red}}{N}\sum_r^{N-1}\frac{\sin^2 \mathcal{G}t}{\mathcal{G}}, \tag{64}$$

where the periodic correlator $\mathcal{G} = 1 + \cos k_1 - \cos k_2 - \cos(k_1 - k_2)$. This utilizes the periodicity of harmonic oscillators in superfluid regime and exclude the high-frequency part due to the periodic boundary condition, i.e., keeps the stable low-frequency only.

For many-body system, the dispersion relation κ of this bosonic model is oscillating as a function of k with the period π (see Fig.13). From Fig.14, it's obvious to see that the periodic dispersion relation results in a degeneracy of energy. In the process of relaxation of two-point correlation, the relevant parameter is assumed changes linearly. By setting the dispersion relations κ before and after quench, the corresponding relaxation of correlations between the bosons is shown in the Fig.14, we see that the oscillations approach to quasisteady state with small (but non-zero) frequency, and with the increasing of dispersion relation the amplitude of correlation is decreased and the required-relaxation time is shorter. In fact this conclusion is always correct for all the many-body system in phase-space.

9 Investigation of Relaxation of Chain Model to Gaussian State By the Transfer Matrices

We then define the transfer matrix

$$t(x) = \mathrm{Tr}(\prod_{l}^{a} T_l(x)), \tag{65}$$

where $T_l(x) = R^l_{n-1}(x)R^l_{n-2}(x)\cdots R^l_0(x)$ is the monodromy matrix with n-site R-matrices and x is the spectral parameter. Employing this transfer matrix representation, the initial state can be written as

$$\mathcal{F}_0(x) = \lim_{n\to\infty}\frac{1}{n-1}\frac{\partial}{\partial x}\langle\psi(0)|t(x)t^\dagger(x)|\psi(0)\rangle \tag{66}$$

where the total number of particles N is a integer multiple of number of transfer matrices $\mathrm{num}(t_1(x))$. Based on this, the localed free energy per spin (or per grid point in the network) is

$$E_{\mathrm{free}} = -\frac{\mathrm{num}(t_1(x))}{N}\frac{1}{\beta}\lim_{M\to\infty}\ln\lambda_{max} \tag{67}$$

where M is the number depends on how many parts the temperature divided into (i.e., the Trotter number), and λ_{max} is the maximum eigenvalue of transfer matrix and in the limit of $N\to\infty$ it has

$$\lambda^N_{max} = \lim_{M\to\infty}\mathrm{Tr}\ t_1^{\mathrm{num}(t_1(x))}(x), \tag{68}$$

i.e., in the case of infinity-system-size the maximum eigenvalue equals to the trace of transfer matrices. Further, we deduce that

$$\lim_{N\to\infty}\frac{\ln(\lambda^N_{max})}{N} = \lim_{M\to\infty}\frac{\ln\lambda_{max}}{N}\cdot\mathrm{num}(t_1(x)), \tag{69}$$

which can be easily confirmed by numerical methods. In the framework of auxiliary space which estabished above, one can define the matrix A_i which acts on the auxiliary space[19], then the wave function of ground state can be redefined as

$$|\psi(0)\rangle = \sum_{s_i}\mathrm{Tr}(\prod_{i=0}^{n-1}A_i)|\prod_{i=0}^{n-1}s_i\rangle \tag{70}$$

where $|\prod_{i=0}^{n-1}s_i\rangle$ denotes a normalized computational basis state[20], while the set of unnormalized part forms a projective space $\mathbb{P}$ is in a dimension of d_id_j-1[68].

Since in normalization case the expectation value of initial state is $\langle\psi(0)|\mathcal{J}_i|\psi(0)\rangle$ with $\langle\psi(0)|\psi(0)\rangle = 1$, the transfer matrices in two subspaces can be obtained by the algebraic Bethe ansatz[76]

$$\begin{aligned} t(i+\mathcal{R}) &= \mathrm{Tr}(A_{n-1}(\mathcal{R})A_{n-2}(\mathcal{R})\cdots A_0(\mathcal{R})),\\ t^\dagger(i+\mathcal{R}) &= \mathrm{Tr}(A^\dagger_{n-1}(\mathcal{R})A^\dagger_{n-2}(\mathcal{R})\cdots A^\dagger_0(\mathcal{R})), \end{aligned} \tag{71}$$

where $\mathcal{R}$ is a constants of motion and the matrices A and $A^\dagger$ are isomorphic with the bipartite space of $\mathbb{C}^{d_i}\otimes\mathbb{C}^{d_j}$. In convex hull construction for nuclear norm, a direction of subgradient is consist of the orthogonal set $\{s_i\}$ and $\{s_i\}_\perp$[115], and it's well known that the Schmidt rank R is invariant by local operations and classical communication (LOCC) but variable when the bipartite state is mixed[68, 116]. For localized quantum communication, Eqs.(43,44) exponentially suppress the transfer of information, which can be reflected by the the exponentially fast quantum propagation in branched tree graph and exponentially slow down in latter-time

motion on the quantum graph[117], for which the information flow toward the random path in local relaxation process.

In the above Bose-Hubbard model, using the Wigner representation which is generally negative definite[45] we also have the characteristic function of density matrix $\mathcal{J}_i$ as[24]

$$\mathrm{Tr}[\mathcal{J}_i e^{\alpha b_i^\dagger - \alpha^* b_i}] = e^{-\frac{|\alpha|^2}{2}} \prod_{d_r} L_m(|\alpha|^2 e^{2it\mathcal{P}}(d_r)), \tag{72}$$

with the translation operator $e^{\alpha b_i^\dagger - \alpha^* b_i} = e^{\alpha b_i^\dagger} e^{-\alpha^* b_i} e^{-|\alpha|^2/2}$ where the state of c-number variable $|\alpha\rangle = e^{-|\alpha|^2/2}(\alpha b_i^\dagger - \alpha^\dagger b_i)$[32], and L_m is the Laguerre polynomial. The translation operator here is non-interactive and can be utilized to express the boundary conditions of parameter space. The density matrix has $\mathcal{J}_i = \mathrm{Tr}(|\psi\rangle\langle\psi|)$ and $b_i^\dagger b_i = -(\frac{\partial}{\partial\alpha} + \frac{\alpha^*}{2})(\frac{\partial}{\partial\alpha^*} + \frac{\alpha}{2})$, $b_i b_i^\dagger = (\frac{\alpha}{2} - \frac{\partial}{\partial\alpha^*})(\frac{\partial}{\partial\alpha} - \frac{\alpha^*}{2})$. After the local relaxation (dephasing) to a steady state ensemble with stationary state $\overline{\rho}_i$, the Eq.(72) tends to the Gaussian form with $e^{-(\overline{\rho}_i + 1/2)\alpha^\dagger \alpha}$[24] where $\overline{\rho}_i$ is the average of initial states for finite system and reach the maximum entanglement which related to the second moments. The Hamiltonian has $\lim_{t\to\infty}\langle\psi(0)|e^{\tau H} H_\tau e^{-\tau H}|\psi(0)\rangle = \mathrm{Tr}(\overline{\rho} H_\tau)$. For integrable homogeneous system (like the one we present in the Sec.6), the translation invariance in the transition states is related to the local conservation law and it's also meaningful in the investigation of relaxation of degrees of freedom,

The small displacement of coordinates due to the local potential produce a negative Hessian eigenvalue[118], and since the one-site shift invariance which corresponds to the noninteracting scenario may be broken by the local conservation law, the incompatible case may appear in the integrable model[31]. The result of Ref.[24] shows that the local relaxation is always preserves the full information of initial state, which shows that the information of initial state is not or at least not only be recorded by the factors of Lagrange multipliers[31], and this is consistent with the above result in Gaussian form which contains the term about initial states. While for inhomogeneous case (like most of the damped or polarized model), since the translation invariance is broken, the thermal behaviors and scattering is very different from the homogeneous one, and the prediction of GGE to the final state is also inadequate[119]. Further, the relaxed result for nonequilibrium system can be constructed as a sum of Gaussians which is associated to the related collective variables[120] or canonical variables which can be utilized to diagonalize the inhomogeneous model[121]. Note that this Gaussian state is quasifree and contains only second moments, i.e., the redistribution by the scattering. We will further represent this process by matrix method in the next section. When the system has already been relaxed to the equilibrium distribution, the dynamic is well described by a stochastic partial differential equation, e.g., the quantum Langevin equation[122]. For this equilibrium state under large time evolution, the diffusion has a non-negligible influence on system and produce the recurrences which occur in a time scale larger that the relaxation time (i.e., the diffusion time is larger than the relaxation time), and the recurrences period also depends on the transfer velocity of information.

10 Matrices Processing

The density matrices of Eqs.(14,15) can be represented by the Schmidt decomposition of bipartite state

$$|\psi\rangle = \sum_R^R \sqrt{\lambda_R} |\mathcal{J}_{iR}\rangle \otimes |\mathcal{J}_{jR}\rangle \tag{73}$$

where λ_R is the maximum eigenvalues of density matrix for each R. If we set the the maximum rank is $\mathbb{R}$, then it have $\sum_R^{\mathbb{R}} \lambda_R = 1$ and $(\sum_R^{\mathbb{R}} \sqrt{\lambda_R})^2 \leq \mathbb{R}$, Definition[116] shows that the Schmidt rank is just $\mathbb{R}$ under the condition $\mathbb{R} - 1 < (\sum_R^{\mathbb{R}} \sqrt{\lambda_R})^2 \leq \mathbb{R}$. In the case of $(\sum_R^{\mathbb{R}} \sqrt{\lambda_R})^2 < \mathbb{R}$,

only the eighnvector which has maximum rank $\mathbb{R}$ is needed, that also explain why the singular values decomposition reserved only the largest singular value (Eq.(12)). The set spaces $\mathbb{S}$ with convex constrution always have $\mathbb{S}_R \subset \mathbb{S}_{\mathbb{R}}$. In the zero-entanglement case, the square root of eigenvalue of $\mathcal{J}\mathcal{J}^*$ has $\sqrt{\lambda_R} = (V\mathcal{J}V^\dagger)_{ij}$ with another index j when $(V\mathcal{J}V^\dagger)_{ij}$ is diagonal, and in another expression is $\langle A_i|\sigma_y A_j^*\rangle = \lambda_R\delta_{ij}$, where $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and A is the matrix-product state. Here we consider the spin flip in term $\sigma_y A_j^*$, and it also has $\langle A_i|\sigma_y A_j^*\rangle = \mathrm{Tr}[(\sigma_y A_j^*)^\dagger A_i] = \mathrm{Tr}[(A_j^T\sigma_y)A_i]$ (not the scalar product). In such a flip in tilted state scheme[19] we let the eigenvalue $\lambda_i = e^{S_i^z}$, and $A = e^{i\theta\sum_i S_i^z}$, i.e., spin flip when the $\theta = \pi$. It's found that $\sum_i e^{2i\theta}\lambda_i = 0$ in the zero-entanglement case[123].

A prerequisite to satisfy this formula is the zero-entanglement, i.e., the two subsystem i and j are separable (or distillable). The density matrix $\mathcal{J}$ here is assumed has the eigenvalue λ_R and it diagonalized by matrix V when $\mathcal{J}$ is symmetry, and in this case, the eigenvalues of $\mathcal{J}\mathcal{J}^*$ is non-negative. Assuming V is a $m \times n$ matrix with n orthonormal columns and $m < n$, thus V acts periodic with period of square of number of column n. Let Σ is the $m \times m$ diagonal matrix which made up of singular values of $\mathcal{J}$, then its nuclear norm can be expressed as $||\mathcal{J}||_* = \mathrm{Tr}(V\Sigma V^T)$. If here V is replaced by the matrix A which appear in Eq.(71), then the trace norm of A which equals the sum of square root of eigenvalues $\sum_i \sqrt{\lambda_i}$ have $||A||_{\mathrm{Tr}} = \mathrm{Tr}\sqrt{AA^\dagger}$ and $||A \otimes A^\dagger||_{\mathrm{Tr}} = ||A||_{\mathrm{Tr}} \cdot ||A^\dagger||_{\mathrm{Tr}} = ||A||_{\mathrm{Tr}}^2$. In normalized case with $\langle\psi|\psi\rangle = 1$, the operator norm of A has the similar property with Hermitian conjugate matrices: $||A^* \otimes A||_{\mathrm{op}} = ||A||_{\mathrm{op}}^2 = 1$. This corresponds to an absolute value of the maximal eigenvalue which is normalized and it's found nondegenerate for $S = 1/2$ Heisenberg model[57]. For separable case, $\sum_R^{\mathbb{R}} \lambda_R \leq 1$ due to the convertibility and the decomposition of entangled state into unentangled pure states in the case that the maximum eigenvalue is smaller than the sum of rest eigenvalues[123], i.e., $\lambda_1 < \lambda_2 + \lambda_3 + \cdots\lambda_n$ (here set the $i = 1, 2, \cdots, n$). For pure state we have

$$\frac{n-1}{n} \geq 1 - \sum_i \lambda_i^2 \geq \frac{4}{n(n-1)}(\sum_{i<j}\sqrt{\lambda_i\lambda_j}). \tag{74}$$

A general bound of dimension of subspace is that the largest dimension of space is almost $d_i \times d_j$ and the smallest one is $(d_i - \mathbb{R} + 1)(d_j - \mathbb{R} + 1)$, and the dimension of these subspaces is within this range, i.e., the rank $R < \mathbb{R}$ can be represented by the affine variety[68]. Since a precondition of increase of the Schmidt rank is incresing the dimension of subspace, and the degree of entanglement is also reaches maximally when it grows into the largest subspace, we can obtain that in most case, the largest subsystem (which almost is full rank) has the almost maximal entanglement, except for the pure state which is unmixed[116]. The largest subspace forms the largest-probability set with the constants of motion which is proportional to the dimension of the corresponding Hilbert space or projector onto its eigenvalues or its integer powers of Hamiltonian[75].

Without losing general, for distillable state, the upper bound of entanglement entropy formed by the logarithmic negativity[124] $S_N = \ln||\mathcal{J}^\Gamma||_{\mathrm{Tr}}$, where $\mathcal{J}^\Gamma$ is the partial transpose of density matrix $\mathcal{J}$ and the corresponds covariance matrix is $\gamma^\Gamma = P\gamma P$ with γ the covariance matric and the diagonal matrix $P = (-\mathbf{I}_i) \oplus \mathbf{I}_j$ ($\mathbf{I}$ is the identities matrices). Let V the nonsingular and skew-sysmetric column vector, and it's real. Then we have $V^T\mathcal{J}V = \mathcal{J}$ (i.e., $\mathcal{J}$ is diagonal), so the nonincreasing ordered symplectic eigenvalues λ^Γ with symplectic matrix $\Omega = \begin{pmatrix} 0 & \mathbf{I}_d \\ -\mathbf{I}_d & 0 \end{pmatrix}$ which describe the reduced Gaussian state [124, 125] has

$$\ln||\mathcal{J}^\Gamma||_{\mathrm{Tr}} = \sum_i \ln(\max[1, (\lambda_i^\Gamma)^{-2}]) \leq \sum_i((\lambda_i^\Gamma)^{-2} - 1), \tag{75}$$

while the normal eigenvalue of $\mathcal{J}$ is λ_i which equal to $(\lambda_i^{\Gamma})^2$ (see Appendix.B for the detail).

11 Relaxation of Nonequilibrium System With Stochastic Dynamical Variables

Since for mixed system, if the initial state is homogeneous, the second moments is conserved and it prevent the system to relax to the thermal state[24], so the effective disentanglement is impossible in this case, and therefore some microstates are inaccessible since the final state is constrained by the conserved constants of motion no matter the system is integrable or not. Like the integrable system which guided by the corresponding GGE with maximal entropy $S_{ij} = -\mathrm{Tr}(\rho_{ij}\ln\rho_{ij})$, will reach nonthermal steady states and share the similar properties with the prethermalization plateaus in the long-time limit[29], which also called prerelaxation in the time evolution of GGE, and this has been founded in the isolated or open quantum system[37, 126], while for the nonintegrable system, it's thermalized directly[75]. In the inhomogeneous case like most of the damping model, the conserved law is no more exist and then the thermal state is achievable directly. The local minima free energy which separated by barriers in free energy surface is connected along the steepest descent path in the scenario of discretized evolution[83] and therefore updates the collective coordinates. This is a powerful way to obtain the symmetric tensor in the flattened space macroscopically, and even the supersymmetry system with global minimal potential energy. In these special points, the gradient of free energy as well as the potential energy vanish, and the energy is rised by the little displacement of coordinates[118].

Defining $\mathcal{Z}$ as a collective variable with coordinate x, then for harmonic oscillators with mass m, in the free energy surface, the distribution of Gaussians can be described by the biasing potential which is guided by the difference of free energy $E(\mathcal{Z}) - E_G(\mathcal{Z}, \tau)$[127]

$$V_{\mathrm{bias}} = w \sum_{\mathrm{Gaussians}} \exp(-[\mathcal{Z}(x) - \mathcal{Z}(x_G)]^2/2(\delta\mathcal{Z})^2), \tag{76}$$

where w and $\delta\mathcal{Z}$ are the height (amplitude) and width of the Gaussians and x_G is the positon of Gaussians, and in the limit of $w \to 0$, it has $\int d\mathcal{Z} e^{-\beta E(\mathcal{Z})} = e^{-\beta E_G(\mathcal{Z},\tau)}$. Such a biasing potential is indeed a history-dependent term which appear in the non-Markovian dynamics equation and as a biased estimator for the free energy, while the unbiased estimate require a Markovian one[120]. An experiment completed recently[128] about the one-dimension Tomonage-Luttinger liquid model detected the the Gaussians propagation, which is adjusted by microwave and along the one-dimension trajectorys ("tubes") and accompanied by a negative perturbation in the time evolution of w and $\delta\mathcal{Z}$. That also shows the stability in chaotic scenario. This biasing potential is indeed a bias estimator of the quantum states with multiple phases, and we can see that it follows the Gaussian decaying. Here the summation symbols is used due to the discretized evolution. Note that this expression is for harmonic oscillators, i.e., the system is linear response. While for anharmonicity oscillators, which produced by, .eg., the detuned Gauusian laser[129, 130] (blue-shift or red-shift) or the (two-photon) Raman detuning[131], this potential need to modified by adding some variational parameter which describing the asymmetry (three-order term) or symmetry (quartic term) anharmonic[94, 132] to the exponent part of Eq.(76). The coupling in this case is nonlinear, like the scenario in FPU theorem. The free energy is $E_{\mathrm{free}} = -V_{\mathrm{bias}}$, and it's govern by the force $F = -\partial_G E_{\mathrm{free}}$[83]. After the flatting process on free energy surface (for a intuitive schematic view, see, e.g., the Ref.[66]), the change of the distribution makes the new Gaussains which govern by a new Hamiltonians, and hence reach a new equilibrium states, but that can only happen in the inhomogeneous situation. After the local minimums of difference of free energy were mostly eliminated, the probability distribution is nearly uniform, and the remaining corrugations are independent of

the $E(\mathcal{Z}) - E_G(\mathcal{Z}, \tau)$. The action describing this dynamic of evolution in complex time scale is (τ is the complex time here)

$$S(\mathcal{Z}) = \frac{m}{2}\int_{\tau}^{\tau'} d\tau[\dot{\mathcal{Z}(\tau)})^2 - \mu^2\mathcal{Z}^2(\tau)], \tag{77}$$

where μ is the natural frequency. Note that for macroscopic model, the actions of harmonic oscillators which are viewed as matter fields coupled with the reservoir or the external electric field is not stationary and therefore it belongs to the nonequilibrium dynamic, and the corresponding kernel functions are also in a nonequilibrium form, (see Ref.[106]).

The correlation matrix Γ_G which obey the Gaussian distribution is

$$\Gamma_G(\tau) = \langle\mathcal{Z}(\tau')\mathcal{Z}(\tau)\rangle_G = \frac{\delta\mathcal{Z}}{2}\langle R_i(\tau')R_j(\tau)\rangle, \tag{78}$$

where R is the coupling operators between the states with dissipation scenario (e.g., the reservoir), and the evolution is $\Gamma_G(\tau) = e^{-\tau H_G}\Gamma(0)$. The coupling is fadeout in damping system through this evolution. Then we have the action function

$$S(\mathcal{Z}) = \int_{\tau}^{\tau'} d\tau g(\mathcal{Z}, \mathcal{Z}(\tau)), \tag{79}$$

which contains the non-Markovian kernel $g(\mathcal{Z}, \mathcal{Z}(\tau))$. In the classical limit approximately, the harmonic motion can be described by

$$M\ddot{\mathcal{Z}} + s\dot{\mathcal{Z}} = -\frac{d}{d\mathcal{Z}}V(\mathcal{Z}) + F_n(\tau), \tag{80}$$

where s is a friction parameter and $s = M\int_{\tau}^{\tau'} d\tau\mathcal{S}(\tau' - \tau)$ where $\mathcal{S}$ is the friction kernel, and F_n is the noise force. Due to the Markovian noise which obeys the Markovian evolution and can be well fitted to the master equation Eq.(33), we then need to replace the history-independent potential term which mentioned above by the form of Eq.(78), i.e., taking the bath coupling R as the noise sourse which is real and Gaussian, and then it has $\langle R_i(t')R_j(t)\rangle = \delta_{ij}\delta_{t'-t}$. That is because that for the harmonic oscillator, using Wick theorem, the density matrix can be diagonalized with a quadratic Gaussian potential (see Ref.[5]), and then the Green's function with infinite imaginary-time becomes[133]

$$G(\mathcal{Z}_i, \mathcal{Z}_j; \tau) = \int_{\mathcal{Z}(\tau)}^{\mathcal{Z}(\tau')} Ds(t)\exp(-S_{\text{eff}}(\mathcal{Z}(t))/\hbar), \tag{81}$$

where the Euclidean effective action is

$$S_{\text{eff}}(\mathcal{Z}(t)) = \int_{\tau}^{\tau'}(\frac{1}{2}M\dot{\mathcal{Z}}(t) + V(\mathcal{Z}))dt - \int_{\tau}^{\tau'} dt\delta(\mathcal{Z} - \mathcal{Z}(t)) + V_0 \tag{82}$$

with V_0 the time-independent potential. In this expression, the state in next time step depends only on the state in this time, i.e., variables satisfy the Marcovian evolution, and more importantly, the contributions of noise in the imaginary axis is vanishing, that also match the real noise sourse, so we only need to consider the noise in real part. Then the time derivative of $\mathcal{Z}$ has the form

$$\frac{d}{dt}\mathcal{Z} = A(t) + B(t)F_n(t), \tag{83}$$

with the $2d \times 2d$ positive definite diffusion matrix $D = BB^T$ which is symmetry in Wigner representation and both A and B are positive and real matrix. By the way, in this case, the

quantum Fisher information matrix satisfies its saturation condition[134]. This Markovian stochastic evolution can be expressed by the second-order Fokker-Planck equation in a stochastic description

$$\frac{\partial}{\partial t}E=\left[-\sum_{i}\frac{\partial}{\partial\mathcal{Z}}A(t)+\frac{1}{2}\sum_{ij}\frac{\partial}{\partial\mathcal{Z}_i}\frac{\partial}{\partial\mathcal{Z}_j}D_{ij}\right]E \tag{84}$$

where E is the free energy of the system influenced by the noise variables. Indeed this expression for the anharmonic case is due to the truncation which discard the asymmetry or symmetry anharmonic terms (see above). While in a probabilistic description, a Laplacian operator equals to the second time-derivative of non-Markovian kernel which is negative definite is contained in the Markovian form Fokker-Planck equation (see Ref.[120]).

Now that in macroscopic system the observables are usually represented by thermal states directly since the error of statistical prediction is negligible[29]. We then investigate the rate of variance of the statistical prediction of observable P_i which belongs to the canonical ensemble, i.e., the relation to the decay rate of Liouvillean relaxation[135]. Writting its statistical prediction as $\mathrm{Tr}(\rho P_i)$ where ρ is the canonical ensemble. As we discussed above, the damp-out process is associate with the decoupling with the dissipation, and therefore we can also define the Hamiltonian here as a damping spectrum of the observable, which been classified discussed here for bosons and fermions, i.e., decompose the P_i into real part and imaginary part. Consider a bath with space $\mathbb{C}^{2d}\otimes\mathbb{C}^{2d}$, then for bosons, the communication relation is $[b_i,b_j^\dagger]=\delta_{ij}$ and for linear bath Hamiltonian which is in a quadratic form (even sector) is $H=u^TH_bu$ where H_b is symmetry, and for fermions $[f_i,f_j^\dagger]_{-1}=\{f_i,f_j^\dagger\}=\delta_{ij}$ with Hamiltonian $H=w^TH_fw$ where H_f is antisymmetry, where u and w are real vectors. Since the real part of prediction can be represented by the covariance matric[125] $(\gamma_b)_{ij}=\frac{1}{2}\mathrm{Tr}\rho P_b$ where $P_b=\{u_i,u_j\}$ and the imaginary part $(\gamma_f)_{ij}=\frac{i}{2}\mathrm{Tr}\rho P_f$ where $P_f=[w_i,w_j]$, and here always have $\gamma_b\geq\sigma_y$. Writting the bath matrix as $M=\sum_i l_i\otimes l_i^\dagger$ with l_i the vector with dimension $2d$ and describing the bath coupling, then we have[125, 36]

$$\partial_t\gamma=X^T\gamma+\gamma X-Y \tag{85}$$

where for fermions it has $X=2\mathrm{Re}M$ and $Y=4\mathrm{Im}M$; while for bosons it has $X=2\mathrm{Im}M$ and $Y=4\mathrm{Re}M$. This Sylvester matrix equation also clarify the FDR.

Through this, the bulk-edge-coupling (bulk-edge-correspondence) type materials like the topological insulators or topological superconductors with the quantum spin Hall effect, have the full pairing gap inside the bulk and the gapless edge states which protected by the time-reversal invariance in the edge[34] can decoupling with the bulk part, i.e., without dissipation at the sample boundary[136] and the subspaces of edge and bulk will separated through the long-enough time evolution (The closing of gap is due to the effect of off-diagonal term here and often leads to the phase transition, e.g., which follows the power law decay with the system size N in the a spinor condensate system[137]). For example, the chiral superconductor with $d+id'$ pairing phase[138] may with the broken time-reversal symmetry (it's realizable by, e.g., applying a strong magnetic field[139]), or the non-Abelian statistical in the Majorana zero model[140]. For most the bulk-edge-coupling type model which is the spinless fermions model, the time evolution is presented in the Appendix.C.

We already know that the integrable system in the homogeneous phase can only relaxes to the nonthermal steady state, but there are some models for which we can't find the thermalization (e.g., can only to the generalized canonical), like the soft-core bosons model (e.g., the Mott insulator[38]), spinless fermions model, integrable Luttinger model[89], etc. This kind of model can't be effectively predicted by the form of Eq.(24). While for the models which nearly integrable (like the Hubbard model) or nonintegrable, the expectation will relax to thermal equilibrium finally, the resulting quasistationary state of this kind of model is nonthermal[30].

The final state which not be thermalized is quasisteady due to the off-diagonal contribution. But there are still some integrable system which have the features of thermalization for some specific variables whose final state is described by the Gibbs ensemble, like the hard-core bosons system[30, 141], so the integrablility is not the only criterion of the thermodynamic behavior, the varied or conservative observables which have nonnegligible effect and their off-diagonal contribution as well as the integrability broken (broken of integrals of motion[142])(see Appendix.C) are also important to consider. The required distance to the nonthermal steady state is in an infinite time average, and the required distance away from integrable point for thermalization to occur is infinitesimal[29], while for a nonintegrable system, the thermalization will gradually ("smoothly") broken when approaches to an integrable point[143] with an infinite time scale.

12 Conclusion

This work mainly investigate the time evolution of quantum many-body system as well as the thermodynamics of macroscopical system with the non-Markovian processes in the free-energy surface for which the steepest descent is used to find the minimal coupling (similar to the method of covariant derivatives). The condition of the presence of thermalization in a relaxation process of quantum many-body system is discussed in this work as well as the entropy and entanglement in the harmonic and anharmonic system. The main model of our investigation is the non-isolated system and so that the degrees of freedom can be traced out from the discussed canonical ensembles (or the microcanonical one), and therefore the ergodic is suppressed, the detail investigation is presented above. Although the integrable system which governed by the corresponding GGE keeps the expectation value of observables in initial state while the chaotic one keeps the initial memory less, and it helps to understand the quenches towards the stationary state in the ordered phase or disordered phase in thermodynamic limit or scaling limit respectively, the required numerical computation is more demanding and the eigenstate thermalization hypothesis is failure[143]. We also obtain that, the integrability is not only affected by the constants of motion, but some other important considerable factors which constitute the integrability breaking term (see Appendix.C). In applications, e.g., the frequency-dependent noise which is induced by the current-current correlation in nonequilibrium Josephson setup leads to the 4π period of the Josephson current, due to the existence of Majorana bound states[144, 145]. However, such current-current correlation-induced noise will be exponentially suppressed at low-temperature, e.g., lower than the superconducting critical temperature. That also implies that the dissipation (which related to the effective mass of the band gap[146, 147, 148, 149]) plays a important role during the quasiparticle transportation in spintronics and valleytronics [150, 151, 152, 153, 154].

To investigate the approaching to Gaussian state with maximum local entropy within the relaxation process, an estimator in terms of trace norm is presented in the Sec.8 which related to the matrix method. The open quantum system is discussed deeply in the above sections, while for a closed quantum system which begin with a pure state with $\mathrm{Tr}\rho^2 = 1$ (ρ is the square root of eigenvalue of the density matrix), it never relax to the thermal state with $\mathrm{Tr}\rho^2 < 1$ which corresponds to $\sum_R^{\mathbb{R}} \lambda_R \leq 1$ as discussed in Sec.10. For the diagonal Hamiltonian which makes the observables tend to diagonal form with the infinite time average, it can be implemented by the methods like Bogoliubov transformation and a fast relaxation to diagonal ensemble (reach a quasisteady state) requires the system spectrum is nondegenerate[143] where we have exclude the accidental degeneracies of diagonal ensemble. In this case, the eigenenergies is linear like the one mentioned in Sec.4, and the globally observable follows the relation Eq.(126) in the long-time limit. Note that such a nondegenerate will not be long-live due to the irregular dispersion in boundaries or the degeneracies generated by the nonlinear waveguide.

13 Appendix A : Deduction of β-function and the coupling in perturbed system

The β-function can be defined as $\beta = \mu\frac{\partial}{\partial\mu}g = \frac{d}{d(\ln\lambda)}g$, where $\ln\lambda = \frac{1}{2}\mu^2$. When $\lambda \to +\infty$, the $g \to 0$[155]. Since the bare coupling g^b is independent of the mass, so $\frac{d}{d\mu}g^b = 0$, according to the relation given in the Ref.[43]

$$\mu\frac{d}{d\mu}g^b = (\mu\frac{\partial}{\partial\mu} + \mu\frac{d}{d\mu}g\frac{\partial}{\partial g})g^b. \tag{86}$$

We can deduce that $-\partial_\mu g^b = \frac{d}{d\mu}g\frac{\partial}{\partial g}g^b \neq 0$, according to the asymptotic series expansion which given in the Ref.[43]

$$\mu\frac{d}{d\mu}g^b = \varepsilon g^b - \varepsilon g\frac{\partial}{\partial g}g^b + (b_3g^2 + b_5g^4 + b_7g^6 + O(g^8))g\frac{\partial}{\partial g}g^b, \tag{87}$$

when the $\varepsilon \to 0$, i.e., dimension $n \to n_c$,

$$\mu\frac{d}{d\mu}g^b \to \mu\frac{d}{d\mu}g\frac{\partial}{\partial g}g^b. \tag{88}$$

The coefficient of Eq.(7) is [54, 155]

$$\begin{aligned}
\beta_0 &= \frac{11}{3}C_{ij}^{(2)} - \frac{4}{3}T_{ij} \\
\beta_1 &= \frac{34}{3}(C_{ij}^{(2)})^2 - \frac{20}{3}C_{ij}^{(2)}T_{ij} - 4C_F^{(2)}T_{ij} \\
\beta_2 &= \frac{2857}{54}(C_{ij}^{(2)})^3 - \frac{5033}{162}C_{ij}^{(2)}T_{ij} + \frac{2925}{864}C_F^{(2)}T_{ij}^2,
\end{aligned} \tag{89}$$

where $C_{ij}^{(2)}$ is the quadratic Casimir operator acting on the adjacent nodes, which equals to N for SU(N) system[155], $C_F^{(2)}$ is the quadratic Casimir operator acting on fermions, and has the relation with mass as $\frac{1}{4}C_{ij}^{(2)}\dim(T_{ij})$=m[54, 155], where m is the number of fermion multiplets[80]. With the increase of m, there will be a lot of novel nature in fermion stand model which we don't discuss here, for a reference can see the Ref.[156].

According to the supersymmetry SU(3) Yang-Mills theory in Ref.[157, 158], the quadratic Casimir operator which have $C_{ij}^{(2)} = F^{\mu\nu}F_{\mu\nu}$ where $F^{\mu\nu}$ is the field strength tensor or the SU(N) generate meta (here is the group generator of SU(3)) which have the below relation with the coupling g

$$\frac{\beta(g)}{g}F\tilde{F} = -\frac{11}{4}\partial_\mu(\psi^\dagger(x)\gamma^\mu\gamma_5\psi(x)), \tag{90}$$

where $F\tilde{F} = \varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$, $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol. Note that this relation is correct for l-loop order where $l \geq 2$ since it's gauge-independent for $\beta(g)$ in one-loop order. It's easy to obtain that

$$\begin{aligned}
F\tilde{F} &= -\frac{11}{4}\frac{g}{\beta(g)}\partial_\mu(\psi^\dagger(x)\gamma^\mu\gamma_5\psi(x)), \\
C_{ij}^{(2)} &= \frac{16\pi^2}{g}[-\frac{8}{33}(\frac{\beta(g)}{g})^2 - \frac{1}{3}\frac{\beta(g)}{g}],
\end{aligned} \tag{91}$$

where $\frac{\beta(g)}{g}$ in SU(3) system obeys [157].

$$\frac{\beta(g)}{g} = \frac{-3C_{ij}^{(2)}}{16\pi^2 - 2C_{ij}^{(2)}} = \frac{-9}{16\pi^2 - 6}, \tag{92}$$

here utilize the virtue of invariance of γ_5 as $\Lambda_{\frac{1}{2}}\gamma_5\Lambda_{\frac{1}{2}}^{-1} = \gamma_5$.

14 Appendix B : The Supplement of Covariance Matrix

Firstly we consider the Minkowski space function[159]

$$Z = \mathrm{Tr}e^{-iHt} = \int DAe^{-iS(t)}, \tag{93}$$

where A is the vector potential in quantum field, and the partition function $Z(\beta) = Z(-i\beta) = \mathrm{Tr}e^{-\beta H}$. We consider the canonical ensemble here as

$$\rho(\beta) = \frac{e^{-\beta H}}{Z(\beta)}. \tag{94}$$

We take the Hamiltonian of components of decomposed covariance matrix $\gamma = (H_1 \oplus H_2)/2$, where $H_1 = V^{-1/2}$ and $H_2 = V^{1/2}$ and V is the potential matrix, into the blocks of $1/\beta$. Then the free energy in entropy ensemble is[160]

$$\begin{aligned} E(\beta) &= \mathrm{Tr}H_2(\beta) \\ &= \sum \ln\gamma(\beta) \\ &= \sum \ln\frac{H_1(\beta) \oplus H_2(\beta)}{2} \end{aligned} \tag{95}$$

where $H_1(\beta) = V^{-1/2}[\mathbf{I}_d + 2(\exp(\beta H_2) - \mathbf{I}_d)^{-1}]$ and $H_2(\beta) = V^{1/2}[\mathbf{I}_d + 2(\exp(\beta H_2) - \mathbf{I}_d)^{-1}]$. Then the Eq.(75) can be represented as

$$\ln||\mathcal{J}^{\Gamma}||_{\mathrm{Tr}} = \sum_i \ln(\max[1, \lambda_i^{-1}]) \leq ||\lambda_i^{-1} - 1||_{\mathrm{Tr}} \leq 2(e^{\beta H_2} - \mathbf{I_d})^{-1} \tag{96}$$

here $e^{\beta H_2} = -\Omega^{-1/2}\gamma^{\Gamma}\sigma_y\gamma^{\Gamma}(-\Omega)^{1/2}$ is the blocks of H_2 and indeed it play a key role in the coupling between the target region with the rest. The maximal l_1-norm[161] of $(e^{\beta H_2} - \mathbf{I_d})$ is linear bounded[124] by the size of target region, (linear with the number of degrees of freedom of boundary of ρ), but it's independent of size of the total size (contain the nontarget-region).

For a explicit example, we take the equation of stochastic-description dynamics (Eq.(83)) into consider and let the parameters A, B, F be the matrices. For quantitative analysis, we form a new potential matrix $Q = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$. Through the mathematical method, we have[160]

$$S^{-1}\begin{pmatrix} A & B \\ B & A \end{pmatrix} S = (A + BF) \oplus (A - BF) \tag{97}$$

where $S = (P + F)/\sqrt{2}$ and $S^{-1} = S$, and we have $A + BF = (A - BF)^{-1}$. The Hamiltonian which describe the conserved observable becomes

$$H = \mathrm{Tr}(F\mathrm{In}Q) = \mathrm{Tr}(\mathrm{In}\frac{A + BF}{A - BF}), \tag{98}$$

then the determinant $\det[A + BF] = \exp(-p\mathrm{Tr}(F\mathrm{In}Q))$, where p is the probability within the canonical ensemble $\rho = \sum p|\psi\rangle\langle\psi|$. Through Jacobi's formula, we have

$$\begin{aligned} \partial_t \det[A + BF] &= \mathrm{Tr}(\mathrm{adj[A + BF]} \cdot \partial_{\mathrm{t}}[\mathrm{A + BF}]) \\ &= \exp(-p\mathrm{Tr}(F\mathrm{In}Q)) \cdot \partial_t Q, \end{aligned} \tag{99}$$

where adj[$\cdot$] denotes the adjoint matrix.

15 Appendix C : The Perturbation Theory Applied to Diagonalized Ising Chain Hamiltonian and The Discuss of Off-Diagonal Contribution Term

We next taking the Ising chain model $H=-J\sum_{i=0}^{N-2}\sigma_i^x\sigma_{i+1}^y-gJ\sum_{i=0}^{N-2}\sigma_i^z$ with quenching of magnetic field $g_0\to g$, as an example to detect the effect of perturbation theory in diagonalization. For fermion quasiparticles with quasimomentum[162] which have even parity and even fermion number $N_{\rm even}$, they obey the antiperiodic boundary conditions $\psi(r+N)=-\psi(r)$ with essential vectors $k=\pi(2n-1)/N$ (n is a integer); while for the odd parity one which have a odd fermion number $N_{\rm odd}$, they obey the periodic boundary conditions $\psi(r+N)=\psi(r)$ with essential vectors $k'=2\pi n/N$. Note that these two sectors can well describe the stationary phase-space probability distribution by the WKB spectrum[101, 163]. Then the WJ fermions $c_r^\dagger$ satisfy

$$\sigma_r^+=\frac{\sigma_{r'}^x+i\sigma_r^y}{2}=c_r^\dagger e^{i\pi N}, \tag{100}$$

$$\sigma_r^-=\frac{\sigma_{r'}^x-i\sigma_r^y}{2}=c_r^\dagger e^{i\pi N}, \tag{101}$$

with r and r' satisfy the anticommute relation $\{c_r,c_{r'}\}=\delta_{r,r'}$ [164]. We introduce the Guassian white noise to this in this model, then the conserved observables follow the Guassian distribution after the quenching, which with the Gaussian amplitude $\omega=1/(\delta\mathcal{Z}\sqrt{2\pi})$ (see Eq.(76)).

For the currents which is proportional to the diagonalization[121], the antiperiodic boundary conditions which also called the Neveu-Schwarz sector[87] corresponds to the left current J_L^c, and the periodic boundary conditions corresponds to the right current J_R^c, which are

$$J_R^c(k)=\sum_k\psi_R^\dagger(k+k')\psi_R(k')+\psi_R^\dagger(k+k')\psi_R^\dagger(k')+\text{H.c.} \tag{102}$$

$$J_L^c(k')=\sum_{k'}\psi_L^\dagger(k+k')\psi_L(k)+\psi_L^\dagger(k+k')\psi_L^\dagger(k)+\text{H.c.} \tag{103}$$

The lagerest current is appear in the ground state, i.e., the $J^c(0)$, and the net current $J_{\rm net}=N_R-N_L$ which is conserved. The observable A in long-time limit has

$$\lim_{t\to\infty}\langle\psi(t)|A|\psi(t)\rangle=\lim_{t\to\infty}\frac{\langle\psi_R(t)|A|\psi_R(t)\rangle+\langle\psi_L(t)|A|\psi_L(t)\rangle}{2}, \tag{104}$$

and

$$\frac{\langle\psi_R(t)|\psi_R(t)\rangle}{\langle\psi_L(t)|\psi_L(t)\rangle}=1+O(e^{-nt}), \tag{105}$$

where n is a constant associate with the $J_{\rm net}$, i.e., the wave function in the pictures of left current and right current are nearly equivalence if $J_{\rm net}$ is small enough.

Mapping the fermi field into the Fourier space for simplicity through the transformation $\sigma_r^z=1-2c_r^\dagger c_r$ and $\sigma_r^x=-\prod_{r'=0}^{r-1}(1-2c_{r'}^\dagger c_{r'})(c_r+c_r^\dagger)$, we have $\psi_r(k)=\frac{1}{\sqrt{N}}\sum_k\psi_k e^{ikr}$ for even parity, and $\psi_r(k')=\frac{1}{\sqrt{N}}\sum_{k'}\psi_{k'}e^{ik'r}$ for odd parity. We then obtain the quadratic Hamiltonian (but no diagonalized)

$$H=2\sum_{k>0}\mathbf{c}_k^\dagger\mathbf{H}_k\mathbf{c}_k, \tag{106}$$

where Nambu vector $\mathbf{c}_k^\dagger=\begin{pmatrix}c_k^\dagger\\ c_{-k}\end{pmatrix}$, and $\mathbf{H}_k=\mathbf{H_0}+R(t,k)\sigma_z$ where $R(t,k)\sigma_z$ is the term associate to the noise and $\mathbf{H_0}$ is the Hamiltonian without the noise which is

$$\mathbf{H_0}=\begin{pmatrix}2J(g-\cos k) & -2Ji\sin k\\ 2Ji\sin k & -2J(g-\cos k)\end{pmatrix} \tag{107}$$

To make the Hamiltonian diagonal in a nonperturbative treatment, we use the Bogoliubov transformation (rotation) to obtain the expression of Bogoliubov quasiparticles with Bogoliubov angle $\theta(k)$ (assuming the lattice spacing $\bar{a}=1$)

$$c(k)=\cos\theta(k)c_0(k)+i\sin\theta(k)c_0^\dagger(-k), \tag{108}$$

$$c^\dagger(k)=i\sin\theta(k)c_0(-k)+\cos\theta(k)c_0^\dagger(k), \tag{109}$$

with the gap $\Delta=\epsilon_0=2J|1-g|$ which vanish in the phase transition point (quantum critical point k_c=1) where the interactions of quasiparticle become more effective. The excitation probability of quasiparticles becomes $\langle\psi(0)|c^\dagger(k)c(k)|\psi(0)\rangle=\tan^2[(\theta(k)-\theta(0))/2]$ and obeys the nonthermal distribution. When $g\gg 1$, the ground state is strictly a paramagnetic, while when $g\ll 1$, the ground states are two degenerate ferromagnetic. If we ignore the noise term, the diagonalized Hamiltonian after the transformation is

$$H_0=2\sum_k \epsilon_k(c_0^\dagger(k)c_0(k)-c_0(-k)c_0^\dagger(-k)-1), \tag{110}$$

where the linear dispersion ϵ_k dependents on the $\mathbf{H}_0$, and $\epsilon_k=\sqrt{|\mathbf{H}_0|}=2J\sqrt{g^2-2g\cos k+1}$. This a noninteracting Hamiltonian and has the accidental degeneracies due to the periodic dispersion which has being mentioned above. This procedure is also available for the phonon field operators, whose Hamiltonian can be exactly diagonalized in harmonic-oscillator[89, 165]. If we consider the noise term, the density matrix of diagonalized Hamiltonian which satisfy the master equation (Eq.(33)) can be written as

$$\mathcal{J}(k)=\begin{pmatrix} c_0^\dagger(k)c_0(k) & c_0^\dagger(k)c_0^\dagger(-k) \\ c_0(-k)c_0(k) & c_0(-k)c_0^\dagger(-k)\end{pmatrix}, \tag{111}$$

where the two elements in the main diagonal stands for the number of levels in momentum space which is invariant under the time evolution, and the two elements in the vice diagonal describe the coherence which will decay exponentially under time evolution and finally lead the system to the mixed state with decoherence superposition. For example, we denote the element $c_0(-k)c_0(k)$ as c_{10}, then $c_{10}(t)=e^{-\mathcal{K}t}c_{10}(0)$, i.e., it vanishes when $t\gg 1/\mathcal{K}$, this result is obey the thermal Glauber dynamics[37]. So it has $\partial_t\mathcal{J}(k)\neq 0$. Base on the Bogoliubov transformation introduced above, the initial state before the quench can be written as[73]

$$|\psi(g_0)\rangle=N\prod_{k,k'>0}[1+i\tan\Delta\theta\ c^\dagger(k)c^\dagger(-k)]|\psi(g)\rangle, \tag{112}$$

where the difference of Bogoliubov angle $\Delta\theta(k)=\theta(k;g)-\theta(k;g_0)$ for the left current regime or $\Delta\theta(k)=\theta(k';g)-\theta(k';g_0)$ for right current regime, and $N=\exp[-\frac{1}{2}\sum_{k,k'>0}\ln(1+\Delta\theta(k))]$. More parameterized, the difference of Bogoliubov angle $\Delta\theta(k)$ has[87]

$$\cos\Delta\theta(k)=\frac{\epsilon_k^2(g_0g)}{\epsilon_k(g_0)\epsilon_k(g)}, \tag{113}$$

where $\epsilon_k(g_0g)=2J\sqrt{g_0g-(g_0+g)\cos k+1}$.

Since the WJ fermions is exist here, it's spinless and therefore the thermalization can't be found in this model, which it's similar to the one mentioned in the Ref.[30]. For the setups of model mentioned in Sec.10 which have a damping model with damping spectrum, the result is different with what disscussed above. In integrable case for this Majorana fermions setup, the Hamiltonian can be simplified as $H=-i\mathcal{P}_f(i\gamma_L+\gamma_R)$ where γ are the Majorana models

and $\mathcal{P}_f$ is the hopping between nearest-neighbor fermions. The Majorana model in the edge of sample is nonlocal and decoherence, the total edge localized model is

$$c_M(k) = \frac{1}{2}(i\gamma_L(k) + \gamma_R(k)), \tag{114}$$

i.e., the conserved currents couples to the Majorana models. This combination process cost energy $2\mathcal{P}_f$ and form a dissipative gap with the bulk (this gap requires that the on-site interaction $U < 2\mathcal{P}_f$ [35]). Since the damping feature, the bulk part of density matrix (not the Eq.(111)) decay with the time evolution, and its time derivative has the same form with Eq.(85), while the edge part is not, i.e., both the main diagonal and vice diagonal are decay with time exponentially, so the final state becomes a pure state ($\mathcal{J} = |\psi\rangle|\langle\psi|$) with coherence superposition (in a similar way to Eq.(112)).

In perturbation theory, with the variables driven by time-dependent white noise, the correlation matrix becomes $\Gamma(t) = \frac{\mathcal{K}}{2}\langle R_i(t')R_j(t)\rangle = \frac{\mathcal{K}}{2}\delta_{ij}\delta_{t'-t}$, i.e., the coupling strength $\mathcal{K}$ is associate with the dephasing effect of noise which accelerate the relaxation in a time scale of order $1/\mathcal{K}$[37] while the diverging length scale is $1/\Delta$. We take the approximation $H = H_0 + gH_1$, where $H_0 = \sum_k \epsilon_k c^\dagger(k)c(k)$ and $H_1 = \sum_k \frac{\delta_k}{2}c^\dagger(k)c^\dagger(k)c(k)c(k)$ where H_1 is second quantized and δ_k is a nonlinear two-body interaction potential unlike the linear eigenenergy ϵ_k. Then we introduce the anti-Hermitian operator s as $s = gs_1 + \frac{1}{2}g^2 s_2 + O(g^3)$ where g is time-dependent parameter and diagonalize the Hamiltonian through canonical transformation have been presented in the Ref.[29]

$$\begin{aligned} H_d &= H_0 + gH_d^{(1)} + g^2 H_d^{(2)} + O(g^3) \\ &= H_0 + g(H_1 + [s_1, H_0]) + g^2(\frac{1}{2}[s_2, H_0] + [s_1, H_1] + \frac{1}{2}[s_1, [s_1, H_0]]) + O(g^3) \end{aligned} \tag{115}$$

then the conserved observable P_i have $[H_d, P_i] = O(g^3)$. In this way, the diagonalized quasiparticles are $c^\dagger(k,t) = e^{iH_d t}c^\dagger(k)e^{-iH_d t}$ and $c(k,t) = e^{iH_d t}c(k)e^{-iH_d t}$. In the range of $1/|g| \ll$ time scale $\ll 1/g^2$ [29], the pure state have the same expectation value with the mixed state, i.e., the main diagonal and vice diagonal of diagonalized Hamiltonians' density matrix have the same degree of decaying.

In the case of $g^2 \ll 1$, the s can be viewed as gs_1, then since $H_d(t) = e^{gs_1}He^{-gs_1}$, we obtain

$$\frac{d}{dg}H_d(t) = e^{gs_1}[s_1, H]e^{-gs_1}, \tag{116}$$

$$\frac{d^2}{dg^2}H_d(t) = e^{gs_1}[s_1, [s_1, H]]e^{-gs_1}, \tag{117}$$

$$\cdots$$

then we further obtain

$$\frac{d}{dg}H_d(t) = e^s[\frac{s}{g}, H]e^{-s}, \tag{118}$$

$$\frac{d^2}{dg^2}H_d(t) = e^s[\frac{s}{g}, [\frac{s}{g}, H]]e^{-s}, \tag{119}$$

$$\cdots$$

For a globally conserved observable $A = \prod_i P_{\alpha_i}$, apply H_d to it with the GGE average, we have[39]

$$\langle A\rangle_{GGE} = \sum_{\alpha_1\cdots\alpha_n} \tilde{A}_{\alpha_1\cdots\alpha_n} \prod_{i=1}^{n} \langle P_{\alpha_i}\rangle_{GGE}, \tag{120}$$

where $\tilde{A}_{\alpha_1\cdots\alpha_n}$ is the perturbation-averaged matrix elements which is utilized to diagonalize the P_{α_i} here and it have the property of

$$\langle A\rangle_{GGE} = \langle\prod_{i=1}^{n} P_{\alpha_i}\rangle_{\mathrm{GGE}} = \prod_{i=1}^{n}\langle P_{\alpha_i}\rangle_{\mathrm{GGE}} = \langle\prod_{i=1}^{n} P_{\alpha_i}\rangle_0 = \prod_{i=1}^{n}\langle P_{\alpha_i}\rangle_0 + O(g^3) \tag{121}$$

we have[29]

$$\begin{aligned}\langle A(t)\rangle &= \langle\psi(0)|e^{iHt}\mathcal{F}e^{-iHt}|\psi(0)\rangle \\ &= \langle\psi(0)|e^{-s}e^{iH_dt}e^{s}\mathcal{F}e^{-s}e^{-iH_dt}e^{s}|\psi(0)\rangle,\end{aligned} \tag{122}$$

which is diagonalized, and with $s(t) = e^{iH_dt}se^{-iH_dt}$. This transformation uses the formula $e^{iHt} = e^{-s}e^{iH_dt}e^{s}$, we define the $e^{-s}e^{iH_dt}e^{s} = e^{e^{R_{-s}}}e^{iH_dt}$ where the real linear map $R_{-s} := ad_{-s}$[166], and have[167]

$$-s\cdot(iH_dt) = -s + \frac{R_{-s}(iH_dt)}{1-e^{R_{-s}}}, \tag{123}$$

then it's easy to obtain

$$\mathrm{In}(e^{-s}e^{iH_dt}) \approx -s + \frac{s^{-1}e^{R_{-s}}s}{e^{R_{-s}}-1}. \tag{124}$$

A estimator for the integrability breaking is given by the Ref.[142] that adding the integrability broken term into the expression of observable

$$A(t) \approx \mu A_{\mathrm{initial}} + (1-\mu)A_{\mathrm{thermal}}. \tag{125}$$

The system is completely integrable when $\mu = 1$, and the system expectation value is the same as the initial one in this case, and it's different from the thermal expectation value of microcanonical ensemble in completely chaotic case (nonintegrable), which can be well described by the standard statistical mechanics[143]. The latter case appear when $\mu \ll 1$ and it average over the initial states, and leads to the thermal state, $\langle\psi(0)|A_{\mathrm{thermal}}|\psi(0)\rangle = \langle\psi(t)|A_{\mathrm{thermal}}|\psi(t)\rangle$, all these eigenstates are within the relevant energy windows with different weight[77]. That allow the precise prediction for the thermal state in long-time limit with the energy close to the initial one. So the thermalization require a large number of coarse-grained observables[39]. As predicted in the classical system by KAM theorem, it's a crossover of regular and chaotic regime[142], and the achievement of thermalization require enough integrability breaking (otherwise the ergodicity is ineffective and the thermalization is suppressed) and a long-time process ($\sim 1/g^3$ in our limit), or e.g., an infinite time which average to the diagonal ensemble and then fluctuate around it in the latter time[143], which can be shown as (not consider the possible degeneracies here)

$$\begin{aligned}\langle A(t)\rangle &= \lim_{t\to\infty}\frac{1}{t}\int_0^t dt\mathrm{Tr}(A\rho(t)) = \langle\psi(t)|A|\psi(t)\rangle_{\mathrm{diag}} \\ &= \sum_{\alpha}|\langle\alpha|\psi(0)\rangle|^2\langle\alpha|A|\alpha\rangle,\end{aligned} \tag{126}$$

where $|\alpha\rangle = \sum_b[(|b\rangle\langle b|gH_1|\alpha\rangle)/(E_\alpha - E_b)]$. This equation gives the long-time average, and keeps the diagonal term only. This long-time average will equal to the GGE expectation value or the one which dominated by the conserved P_i. For Eq.(122), when the state ρ which can be described by the Hamiltonian $H = H_0 + gH_1$ is nondiagonal while the observable A is diagonal (i.e., $[A, H_d] = 0$), it becomes[29]

$$\begin{aligned}\langle A(t)\rangle &= -\langle\psi(0)|(s(t)-s)A(s(t)-s)|\psi(0)\rangle + O(g^3) \\ &= -2(\langle\psi(0)|sAs|\psi(0)\rangle - \mathrm{Re}\langle\psi(0)|sAs(t)|\psi(0)\rangle),\end{aligned} \tag{127}$$

where the term $-\mathrm{Re}\langle\psi(0)|sAs(t)|\psi(0)\rangle$ is due to the off-diagonal contribution:

$$-\mathrm{Re}\langle a|sAs(t)|a\rangle = \mathrm{Re}\sum_b \frac{|\langle a|gH_1|b\rangle|}{(E_a-E_b)^2}\langle b|A|b\rangle e^{-i(E_a-E_b)t} + O(g^3), \tag{128}$$

where we simplify the initial state $\psi(0)$ as a and the quenched state $\psi(t)$ $(t>0)$ as b. But in the case of both ρ and A are off-diagonal, this off-diagonal contribution term becomes

$$-2\mathrm{Re}\sum_b \frac{(|\langle a|gH_1|a\rangle| - |\langle a|gH_1|b\rangle| - |\langle b|gH_1|b\rangle|)^2}{(E_a-E_b)^2}\langle a|A|b\rangle e^{-i(E_a-E_b)t} + O(g^3). \tag{129}$$

While the diagonalized state is

$$\rho_{\mathrm{diag}}(|b\rangle) = \sum_a P_a\rho_0 P_a, \tag{130}$$

where the prejector $P_a = |a\rangle\langle a|$ projects ρ_0 onto the subspace of the initial state $|a\rangle$.

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16 Tables

Table.A:

Model	Time scale of relaxation	Period of collapse and revival	Ref(s).
Falicov-Kimball	$\hbar$/bandwidth	$h/U^{\S}$	[111]
Bose-Hubbard	$1/\mathcal{P}$	h/U	[38]
Spin glasses	macroscopical and with a very broad range	-	[168],[169]
Tomonaga-Luttinger	2∼3 orders of time	h/J (J is the coupling of nearest-neighbor)	* [5],[30]
Hubbard	$\rho_F^{-1}U^{-2} \sim \rho_F^{-3}U^{-4\dagger}$	-	[170]
One-dimension hard core bosons	$1/\mathcal{P}_{\mathrm{f}}{}^{\ddagger}$	h/U	[143],[30]

* Here taking the decaying of time derivative of initial Hamiltonian as the criterion of relaxation.
§ h is the Planck constant and U is the strength of nearest interaction (The belows are also follow this).
† ρ_F is the density of states at the Fermi level.
‡ $\mathcal{P}_{\mathrm{f}}$ is the hopping of finial state after quench.

17 Figures

Fig.1

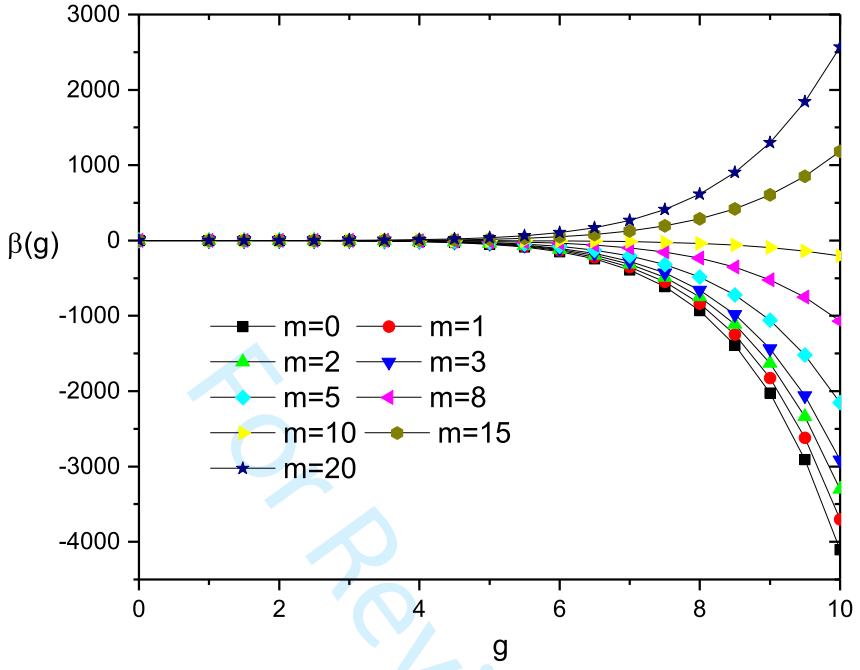

Figure 1: (Color online)$\beta(g)$ as a function of g in SU(3) system (i.e. $C_{ij}^{(2)} = 3$ (see Appendix.A) with the number of fermion multiplets $m = 0, 1, 2, 3, 5, 8, 10, 15, 20$, i.e., the 0-plet, 1-plet,$\cdots$, 20-plet fermion configration.

Fig.2

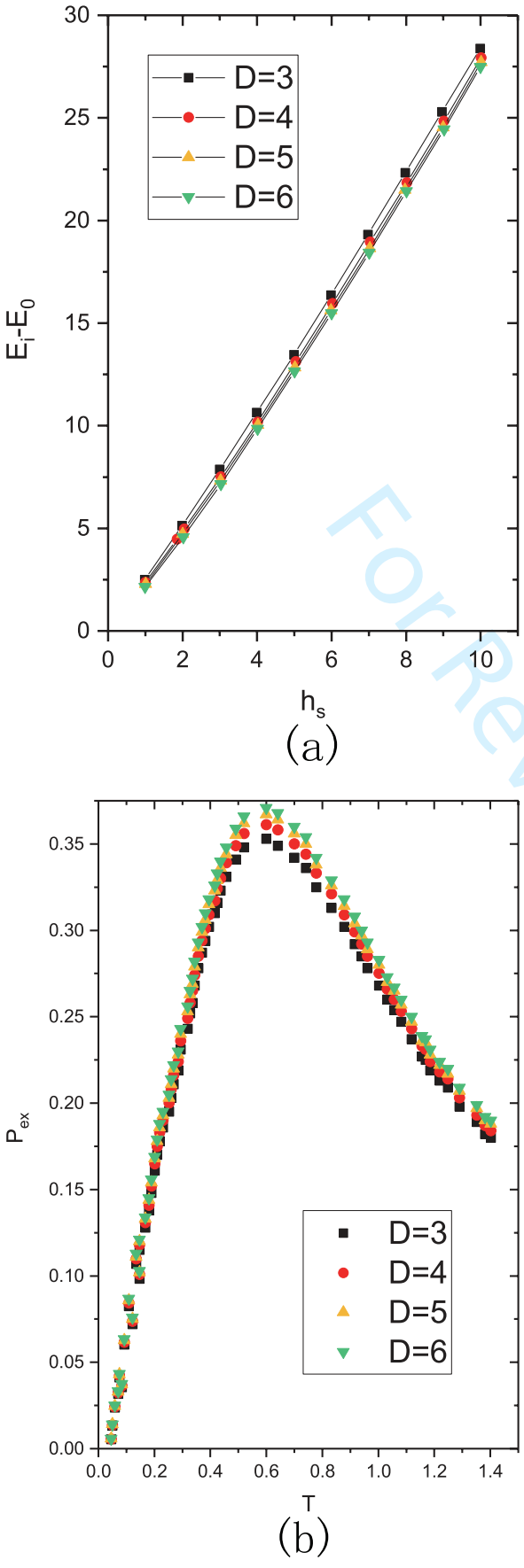

Figure 2: (Color online) (a) Energy difference between the excited state and initial state as a function of staggered magnetic field h_s for different dimensions of matrix. (b) Probability of excitation P_{ex} as a function of temperature for different dimensions.

Fig.3

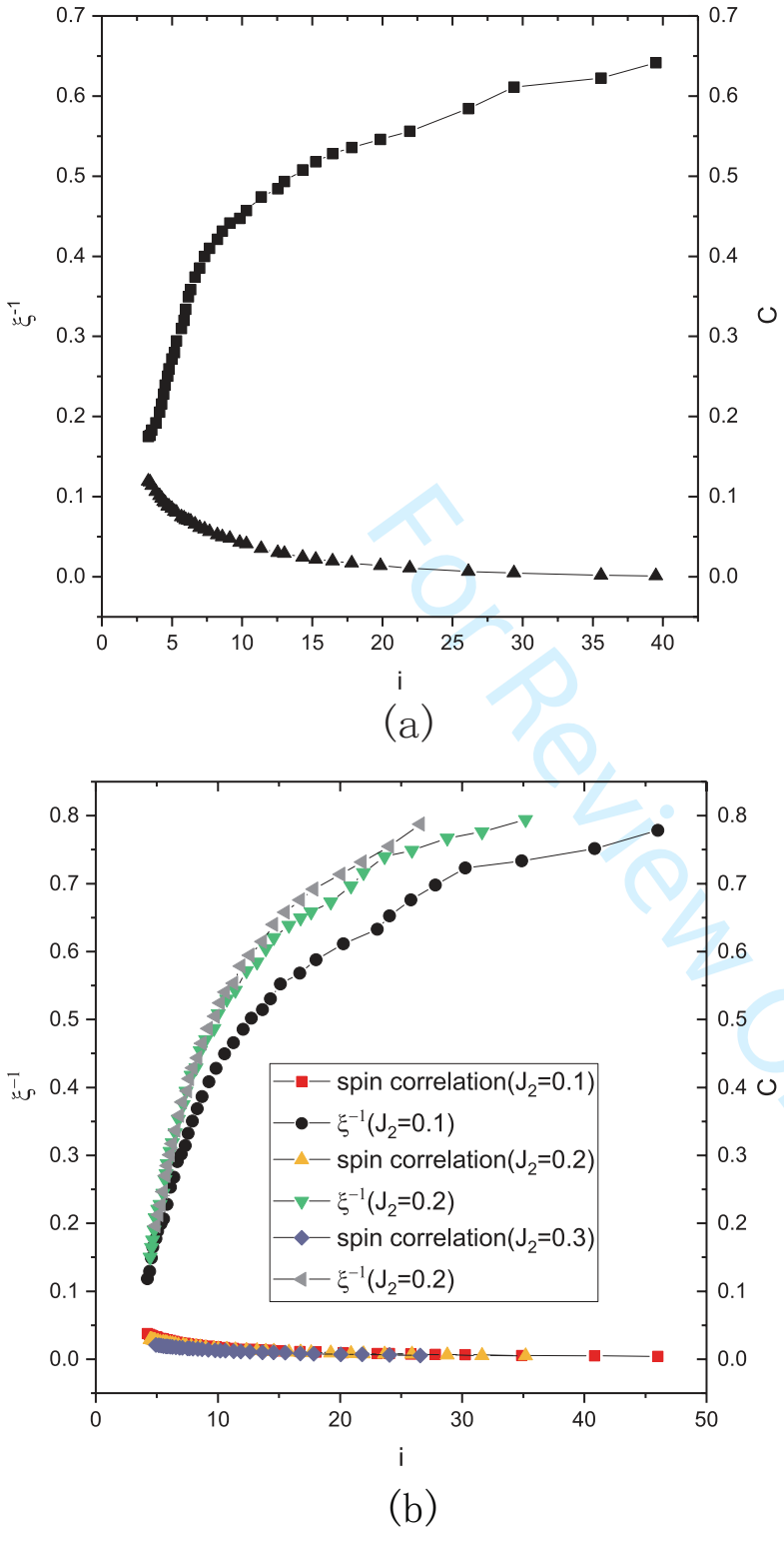

Figure 3: (Color online)(a)Inverse spin correlation length (square) and spin correlation (triangle) for $S = 1$ Ising spin chain at different site i. (b)Inverse spin correlation length and spin correlation for $S = 1/2$ Heisenberg spin chain at different site i for different J_2. The J_1 here is setted as 0.7.

Fig.4

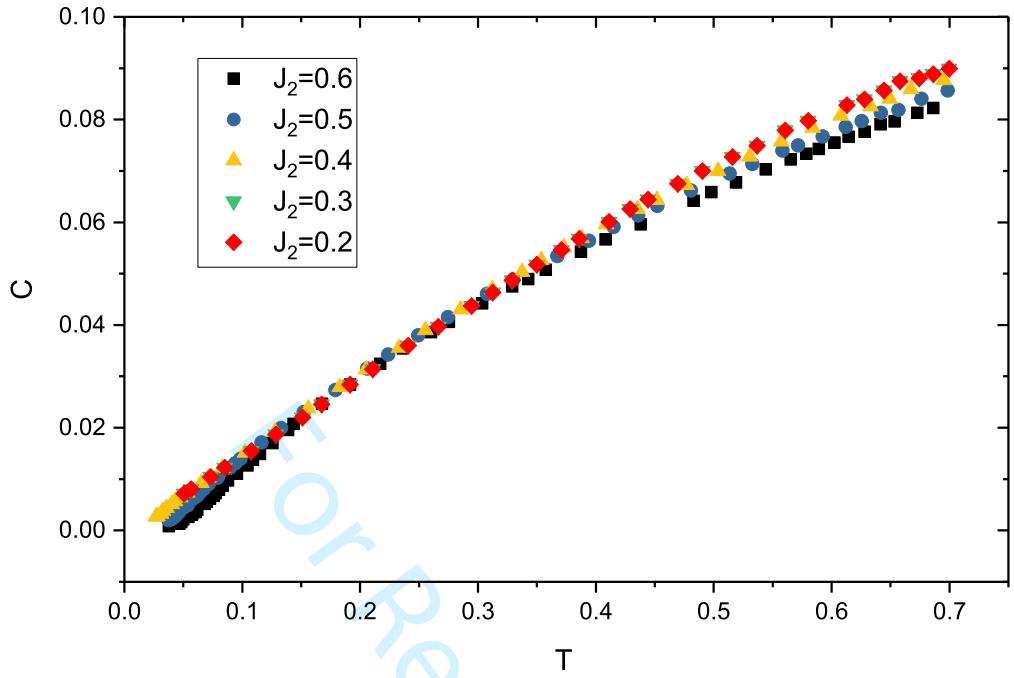

Figure 4: (Color online)Spin correlation for $S = 1/2$ spin chain as a function of temperature for different next-nearest neighbor coupling J_2.

Fig.5

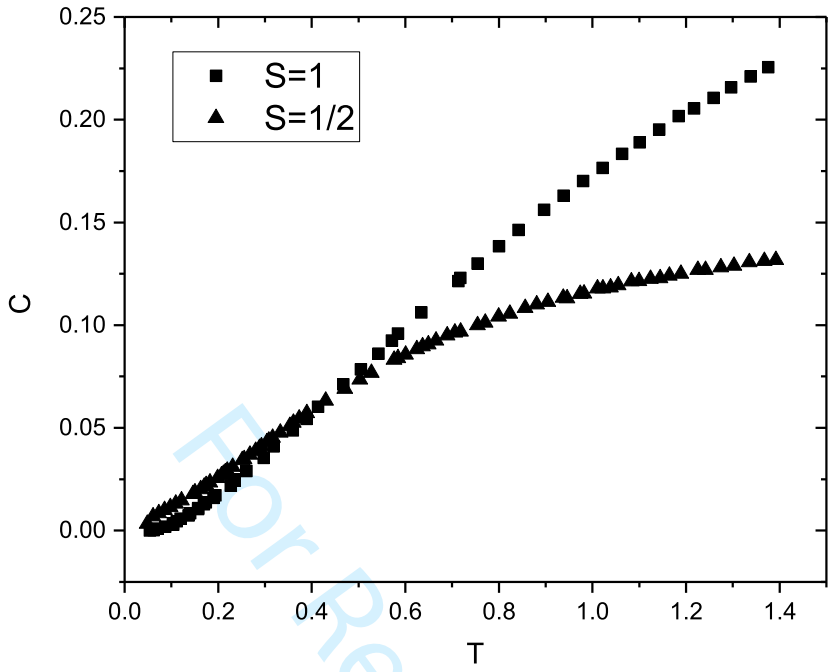

Figure 5: Spin correlation for $S = 1$ Ising spin chain and $S = 1/2$ Heisenberg chain as a function of temperature.

Fig.6

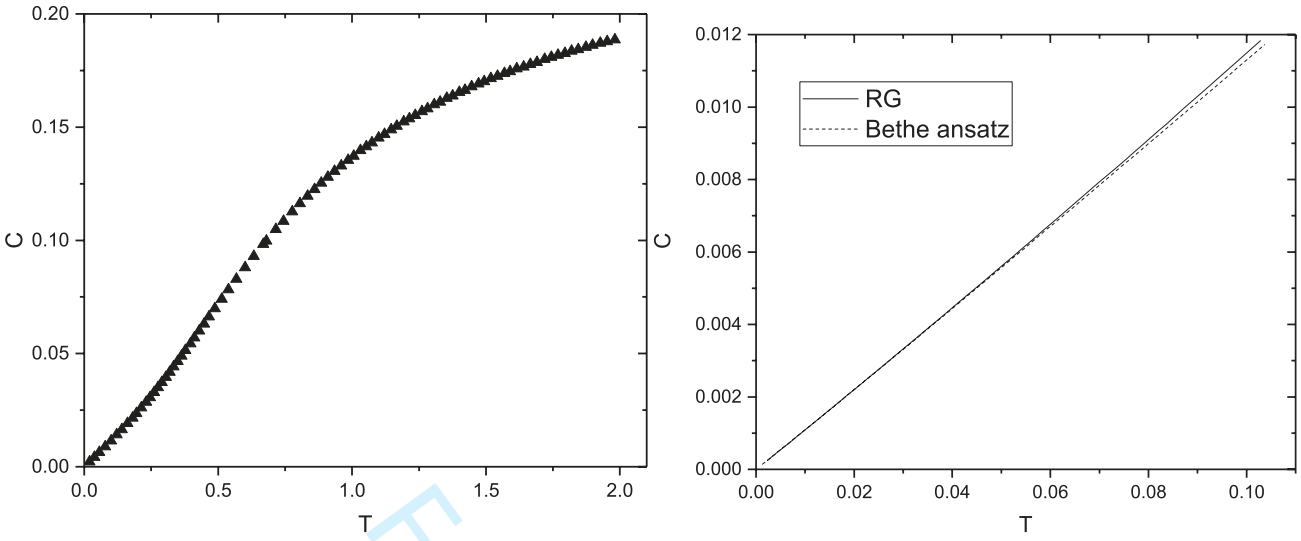

Figure 6: (left) Spin correlation as a function of temperature by the method of Bethe ansantz; (right) Comparison of the results of spin correlation under low temperature between Bethe ansatz and renormalization group (RG).

Fig.7

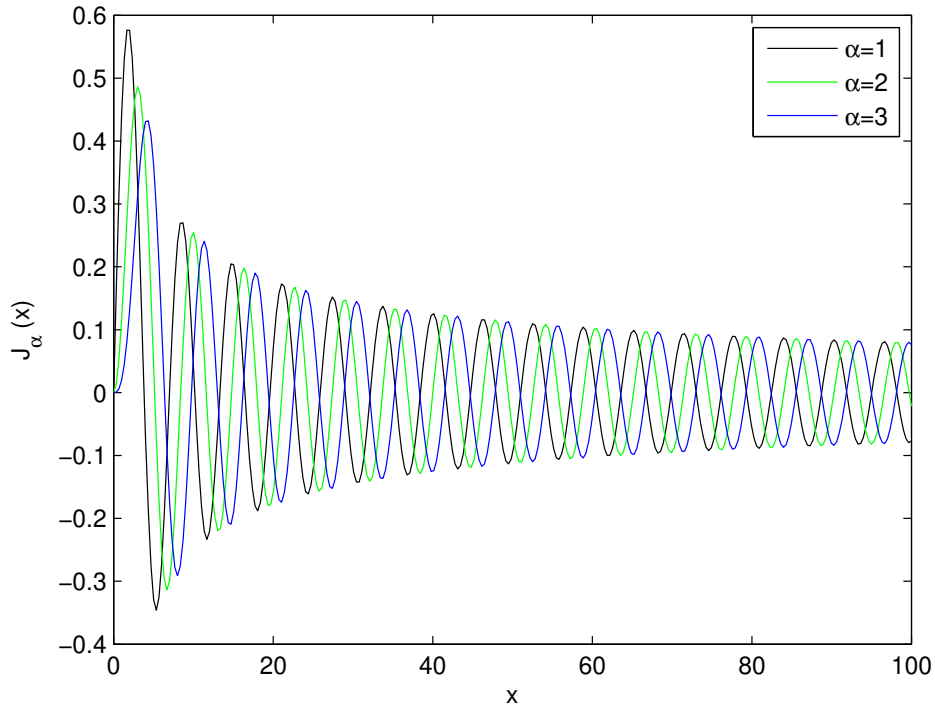

Figure 7: (Color online)Graph of Eq.(47) with phase $\alpha = 1, 2, 3$. It's obviously to see that the contours is bounded by a power function.

Fig.8

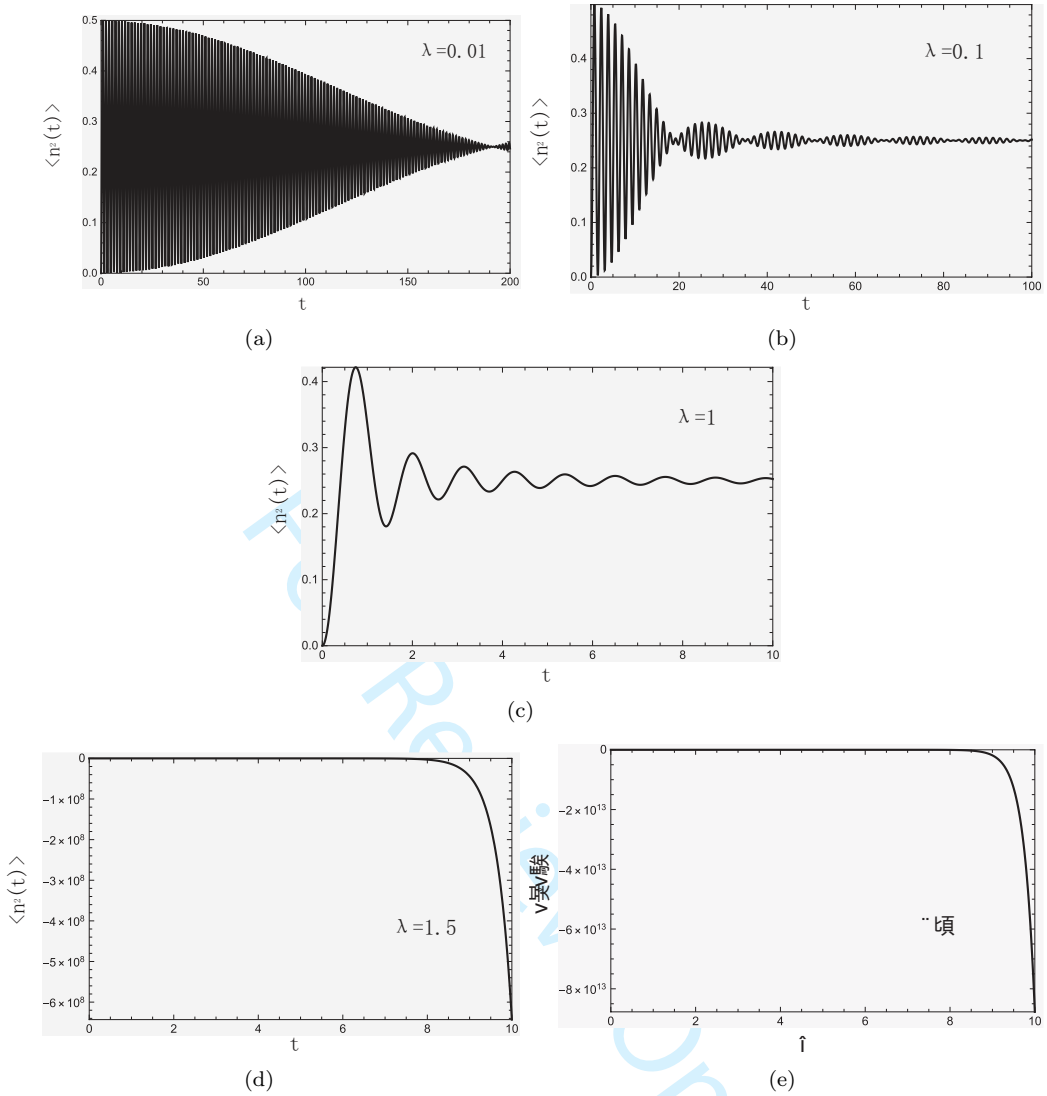

Figure 8: The graphs of $\langle n^2(t)\rangle$ as a function of t (Eq.(49)). The reduced coupling $g_{\rm red} = 0.01, 0.1, 1, 1.5, 2$ from (a) to (e), respectively.

Fig.9

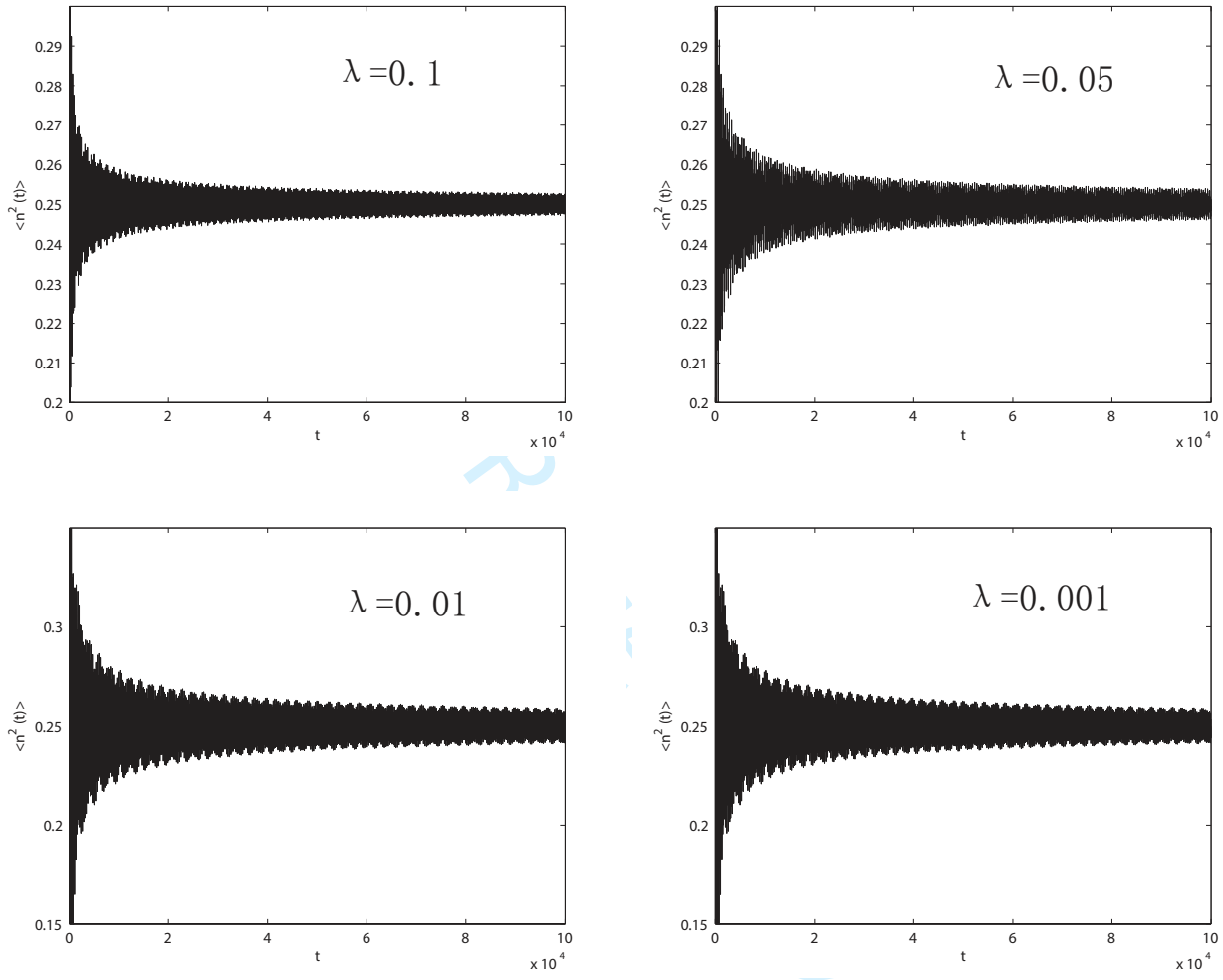

Figure 9: The large time behavior of $\langle n^2(t)\rangle$ with coupling $g_{\rm red} = 0.1, 0.05, 0.01, 0.001$ from left to right (Eq.50).

Fig.10

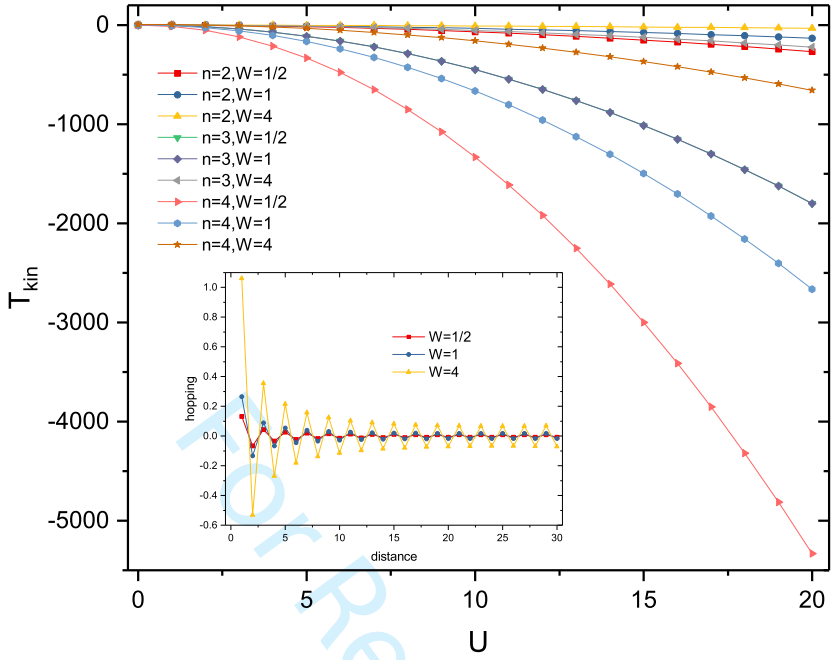

Figure 10: (Color online) Kinetic energy of $1/r$ Hubbard chain as a function of U with different n and bandwidth $W = 1/2, 1, 4$. The bandwidth-dependent hopping constants of $1/r$ Hubbard chain as a function of distance is shown in the inset.

Fig.11

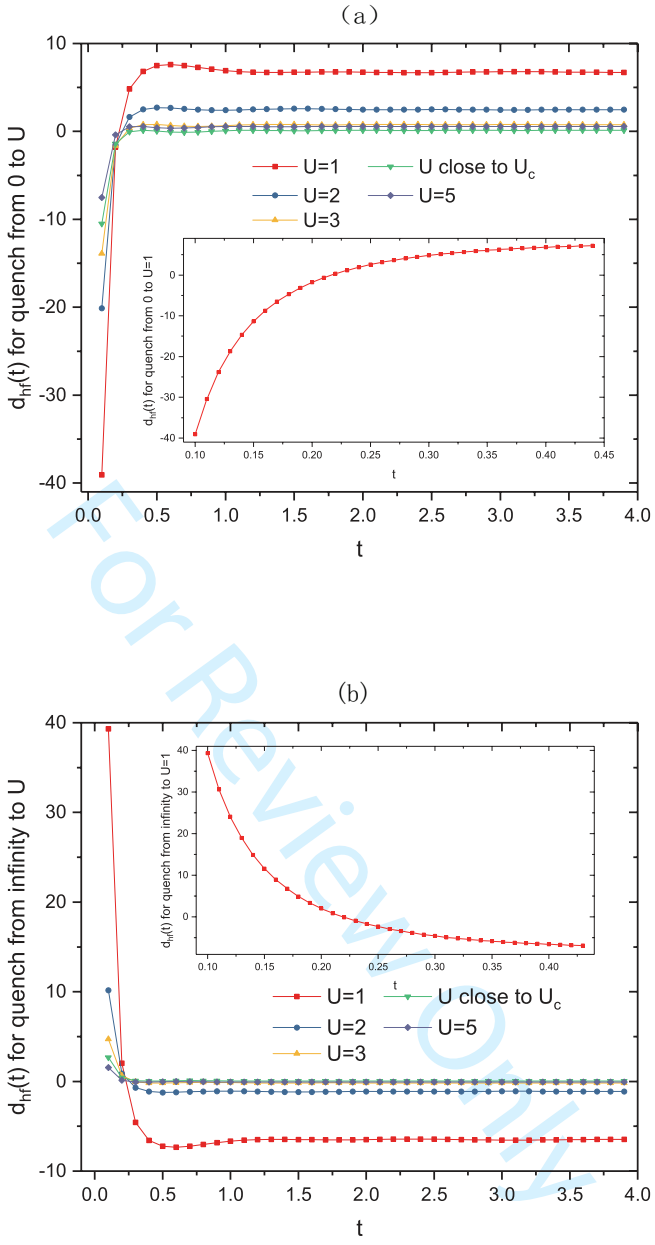

Figure 11: (Color online) Double occupation for half-filling Mott insulator $d_{hf}(t)$ quenches from $U = 0$ to U (a) and from ∞ to U (b). The insets show the enlarged views of the $d_{hf}(t)$ for quenches to $U = 1$.

Fig.12

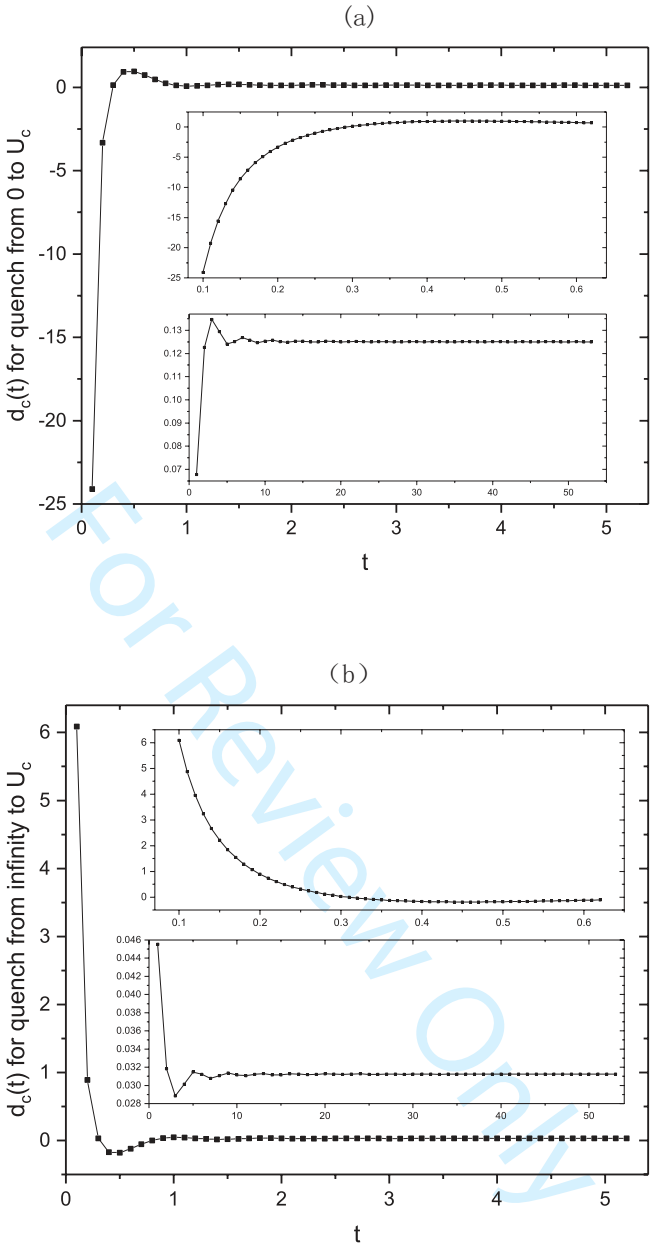

Figure 12: Double occupation for quenches from 0 to critical value U_c (a) and from ∞ to U_c. The top insets show the enlarge views on short-time scale, while the bottom insets show the enlarge views on large-time behavior.

Fig.13

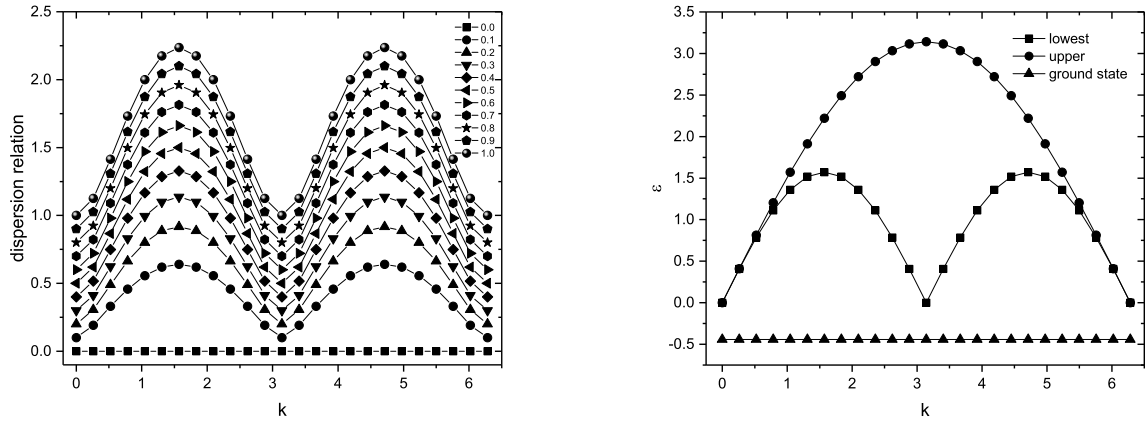

Figure 13: (left)The dispersion relation in k space with different regulatory paramater (0 to 1 from bottom to the upper); (right)The upper, lowest, and ground state energy in a same space according to Ref.[57].

Fig.14

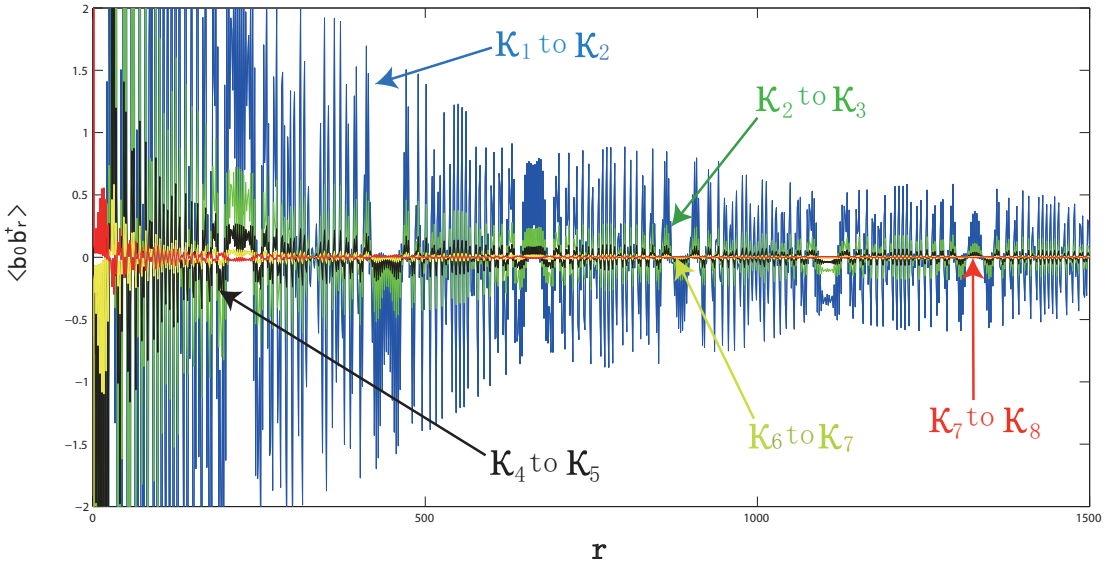

Figure 14: (Color online) Correlations $\langle b_0 b_r^\dagger \rangle$ as a function of distance r for different quench of dispersion relations. The curves with different colors from outside to inside corresponds to κ_1 to κ_2,κ_2 to κ_3,κ_4 to κ_5, κ_6 to κ_7, and κ_7 to κ_8, respectively. The dispersion relations are setted as $\kappa_1 = 0.191820018, \kappa_2 = 0.331662479, \kappa_3 = 0.45825757, \kappa_4 = 0.5, \kappa_5 = 0.619656837, \kappa_6 = 0.866025404, \kappa_7 = 1.118033989, \kappa_8 = 1.322875656$.