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TIME FRACTIONAL ADVECTION-DISPERSION EQUATION

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ABSTRACT. A time fractional advection-dispersion equation is obtained from the standard advection-dispersion equation by replacing the first-order derivative in time by a fractional derivative in time of order α ($0 < \alpha \leq 1$). Using variable transformation, Mellin and Laplace transforms, and properties of H-functions, we derive the complete solution of this time fractional advection-dispersion equation.

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1. Introduction

A space-time fractional partial differential equation, obtained from the standard partial differential equation by replacing the second order space-derivative by a fractional derivative of order $\beta > 0$ and the first order time-derivative by a fractional derivative of order $\alpha > 0$ has been recently treated by a number of authors. Wyss (1986) considered the time fractional diffusion equation and the solution was given in closed form in terms of H-functions. Schneider and Wyss (1989) considered the time fractional diffusion and wave equations. The corresponding Green functions were obtained in closed form for arbitrary space dimensions in terms of H-functions and their properties were exhibited. Gorenflo, Iskenderov, and Luchko (2000) used the similarity method and the Laplace transform method to obtain the scale-invariant solution of the time-fractional diffusion-wave equation in terms of the Wright function. Benson, Whearcraft and Meerschaert (2000a,b) considered space fractional advection-dispersion equation. They gave an analytical solution featuring the α -stable error function. Liu, Anh and Turner (2002) presented a numerical solution of the space fractional advection-dispersion equation. Mainardi, Luchko and Pagnini (2001) considered the space-time fractional diffusion equation and provided a

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general representation of the Green functions in the terms of Mellin-Barnes integrals in the complex plane. Anh and Leonenko (2000) proposed scaling laws for fractional diffusion-wave equations with singular data. Angulo, Ruiz-Medina, Anh and Grecksch (2000) introduced a fractional heat equation, where the diffusion operator is the composition of the Bessel and Riesz potentials. Wyss (2000) considered a fractional Black-Scholes equation and gave a complete solution of this equation. Some partial differential equations of space-time fractional order were successfully used for modelling relevant physical processes (Giona and Roman, 1992; Mainardi, 1994; Hilfer, 1995; Caputo, 1996; Benson, 2000a,b; El-Sayed and Aly, 2002; Basu and Acharya, 2002).

In this paper, we consider the time fractional advection-dispersion equation. This equation is obtained by replacing the time-derivative in the advection dispersion equation by a generalized derivative of order α with $0 < \alpha \leq 1$. We consider

$$\frac{\partial^\alpha C(x, t)}{\partial t^\alpha} = -\nu \frac{\partial C(x, t)}{\partial x} + D \frac{\partial^2 C(x, t)}{\partial x^2},$$

$$x > 0, t > 0, 0 < \alpha \leq 1 \quad (1)$$

with the initial condition

$$C(x, 0) = C_0(x), \quad x \geq 0 \quad (2)$$

where $\nu \geq 0$, $D > 0$ and $\frac{\partial^\alpha C(x, t)}{\partial t^\alpha}$ is a fractional derivative.

Properties and more details about the fractional derivatives can be found in Samko, Kilbas and Marichev (1993). Using variable transformation, the time fractional advection-dispersion equation is reduced to a more familiar form. Using Schneider and Wyss's (1989) and Wyss's (2000) techniques we derive in this paper the complete solution of this time fractional advection-dispersion equation (TFADE).

2. The reduced time fractional advection dispersion equation

Eq. (1) can be expressed by the following integral equation (Wyss, 2000; Schneider and Wyss, 1989):

$$C(x, t) = C(x, 0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \left[-\nu \frac{\partial C(x, \tau)}{\partial x} + D \frac{\partial^2 C(x, \tau)}{\partial x^2} \right] d\tau \quad (3)$$

with $n - 1 < \alpha \leq n$, $n = 1$. To reduce (3.1) to a more familiar form, let

$$C(x, t) = u(\xi, t) \exp\left(\frac{\nu \xi}{2\sqrt{D}}\right), \quad x = \frac{\xi}{\sqrt{D}} \quad (4)$$

with the initial condition:

$$C(x, 0) = C_0(x) = u(\xi, 0) \exp\left(-\frac{\nu\xi}{2\sqrt{D}}\right), \quad \xi > 0. \quad (5)$$

Let $\mu^2 = \frac{\nu^2}{4D}$, this leads to the equation

$$u(\xi, t) = u(\xi, 0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} \left[\frac{\partial^2 u(\xi, \tau)}{\partial \xi^2} - \mu^2 u(\xi, \tau) \right] d\tau \quad (6)$$

and

$$u(\xi, 0) = C_0\left(\frac{\xi}{\sqrt{D}}\right) \exp\left(-\frac{\nu\xi}{2\sqrt{D}}\right). \quad (7)$$

3. The Green function for the reduced TFADE

According to the properties of the Laplace transform and (49), the Laplace transform of (6) with respect to t gives for $p > 0$

$$\tilde{u}(\xi, p) = p^{-1} u(\xi, 0) + p^{-\alpha} \left[\frac{\partial^2 \tilde{u}(\xi, p)}{\partial \xi^2} - \mu^2 \tilde{u}(\xi, p) \right] \quad (8)$$

or

$$\frac{\partial^2 \tilde{u}(\xi, p)}{\partial \xi^2} - [\mu^2 + p^\alpha] \tilde{u}(\xi, p) = -p^{-1+\alpha} u(\xi, 0). \quad (9)$$

Letting $w^2 = \mu^2 + p^\alpha$, we have the equation

$$\frac{\partial^2 \tilde{u}(\xi, p)}{\partial \xi^2} - w^2 \tilde{u}(\xi, p) = -p^{-1+\alpha} u(\xi, 0). \quad (10)$$

According to Schneider and Wyss (1989), Eq. (10) has the solution

$$\tilde{u}(\xi, p) = \int_0^\infty \tilde{G}_\mu^\alpha(|\xi - y|, p) u(y, 0) dy, \quad (11)$$

where

$$\tilde{G}_\mu^\alpha(r, p) = p^{\alpha-1} \kappa(r, w) = p^{\alpha-1} \kappa(r, \sqrt{\mu^2 + p^\alpha}) \quad (12)$$

and

$$\kappa(r, w) = \left(\frac{r}{2\pi w}\right)^{\frac{1}{2}} K_{\frac{1}{2}}(wr). \quad (13)$$

A direct transition to the time domain (i.e., inverting the Laplace transform) does not seem to be feasible. This difficulty is circumvented by passing through the intermediate step of the Mellin transform (33), connected with the Laplace transform by (50). Thus, to invert the Laplace transform of Eq. (12) to find

Green's function $G_\mu^\alpha(r, t)$, we first compute its Mellin transform according to Eq. (50):

$$G_\mu^{*\alpha}(r, s) = \frac{1}{\Gamma(1-s)} \int_0^\infty p^{-s} \tilde{G}_\mu^\alpha(r, p) dp \quad (14)$$

or

$$G_\mu^{*\alpha}(r, s) = \left(\frac{r}{2\pi}\right)^{\frac{1}{2}} \frac{1}{\Gamma(1-s)} \int_0^\infty p^{-s+\alpha-1} [\mu^2 + p^\alpha]^{-\frac{1}{4}} K_{\frac{1}{2}}(r\sqrt{\mu^2 + p^\alpha}) dp. \quad (15)$$

Letting $q = p^{\frac{\alpha}{2}}$, we obtain

$$G_\mu^{*\alpha}(r, s) = \frac{2}{\alpha} \left(\frac{r}{2\pi}\right)^{\frac{1}{2}} \frac{1}{\Gamma(1-s)} \int_0^\infty q^{[2-\frac{2s}{\alpha}]-1} [\mu^2 + q^2]^{-\frac{1}{4}} K_{\frac{1}{2}}(r\sqrt{\mu^2 + q^2}) dq. \quad (16)$$

Using formula (42) with $b = \mu, \nu = \frac{1}{2}, a = r$, we have

$$G_\mu^{*\alpha}(r, s) = \frac{1}{\alpha\sqrt{\pi}} \left(\frac{2\mu}{r}\right)^{\frac{1}{2}} \left(\frac{2\mu}{r}\right)^{-\frac{s}{\alpha}} \frac{\Gamma(1-\frac{s}{\alpha})}{\Gamma(1-s)} K_{\frac{s}{\alpha}-\frac{1}{2}}(\mu r). \quad (17)$$

From the H-function representation (52), we have

$$H_{1,1}^{*1,0}(-s) = H_{1,1}^{*1,0} \left(z \left| \begin{matrix} (1, 1) \\ (1, \frac{1}{\alpha}) \end{matrix} \right. \right) (-s) = \frac{\Gamma(1-\frac{s}{\alpha})}{\Gamma(1-s)}. \quad (18)$$

Let $\varphi(p) = \frac{1}{2} \exp[-\frac{1}{2}\mu r(p^\alpha + p^{-\alpha})]$ and $\phi(p) = p^{-\frac{\alpha}{2}} \varphi(p)$. From the properties of the Mellin transform and Eqs. (38) and (41), we have

$$\varphi^*(s) = \frac{1}{\alpha} K_{\frac{s}{\alpha}}(\mu r), \quad \phi^*(s) = \frac{1}{\alpha} K_{\frac{s}{\alpha}-\frac{1}{2}}(\mu r). \quad (19)$$

Thus the Mellin transform $G_\mu^{*\alpha}(r, s)$ can be written as

$$G_\mu^{*\alpha}(r, s) = \frac{1}{\sqrt{\pi}} \left(\frac{2\mu}{r}\right)^{\frac{1}{2}} \left(\frac{2\mu}{r}\right)^{-\frac{s}{\alpha}} \phi^*(s) H_{1,1}^{*1,0}(-s). \quad (20)$$

From (38), (40) and (43), we have

$$\int_0^\infty \phi\left(\left(\frac{2\mu}{r}\right)^{\frac{1}{\alpha}} \frac{t}{z}\right) H_{1,1}^{1,0}(z^{-1}) \frac{dz}{z} \xleftrightarrow{M} \left[\left(\frac{2\mu}{r}\right)^{\frac{1}{\alpha}}\right]^{-s} \phi^*(s) H_{1,1}^{*1,0}(-s). \quad (21)$$

Letting $\zeta = \frac{1}{z}$, we have

$$\int_0^\infty \phi\left(\left(\frac{2\mu}{r}\right)^{\frac{1}{\alpha}} \frac{t}{z}\right) H_{1,1}^{1,0}(z^{-1}) \frac{dz}{z} = \int_0^\infty \phi\left(\left(\frac{2\mu}{r}\right)^{\frac{1}{\alpha}} \zeta t\right) H_{1,1}^{1,0}(\zeta) \frac{d\zeta}{\zeta}. \quad (22)$$

Therefore, the inverse Mellin transform leads to

$$G_\mu^\alpha(r, s) = \frac{1}{\sqrt{\pi}} \left(\frac{2\mu}{r}\right)^{\frac{1}{2}} \int_0^\infty \phi\left(\left(\frac{2\mu}{r}\right)^{\frac{1}{\alpha}} \zeta t\right) H_{1,1}^{1,0}(\zeta) \frac{d\zeta}{\zeta} \quad (23)$$

or

$$\begin{aligned} G_\mu^\alpha(r, s) &= \frac{1}{\sqrt{\pi}} \left(\frac{2\mu}{r}\right)^{\frac{1}{2}} \int_0^\infty \left(\left(\frac{2\mu}{r}\right)^{\frac{1}{\alpha}} \zeta t\right)^{-\frac{\alpha}{2}} \varphi\left(\left(\frac{2\mu}{r}\right)^{\frac{1}{\alpha}} \zeta t\right) H_{1,1}^{1,0}(\zeta) \frac{d\zeta}{\zeta} \\ &= \frac{1}{2\sqrt{\pi}\sqrt{t^\alpha}} \int_0^\infty \frac{1}{\sqrt{\zeta^\alpha}} \exp[-\mu^2 t^\alpha \zeta^\alpha - \frac{r^2}{4} t^{-\alpha} \zeta^{-\alpha}] H_{1,1}^{1,0}(\zeta) \frac{d\zeta}{\zeta}. \end{aligned} \quad (24)$$

Letting $\sigma = \zeta^\alpha$ and using the properties (61) and (62) of the H-function, we obtain

$$\begin{aligned} G_\mu^\alpha(r, t) &= \frac{1}{2\alpha\sqrt{\pi}\sqrt{t^\alpha}} \int_0^\infty \sigma^{-\frac{3}{2}} \exp[-\mu^2 t^\alpha \sigma - \frac{r^2}{4} t^{-\alpha} \sigma^{-1}] H_{1,1}^{1,0} \left(\sigma^{\frac{1}{\alpha}} \left| \begin{matrix} (1, 1) \\ (1, \frac{1}{\alpha}) \end{matrix} \right. \right) d\sigma \\ &= \frac{1}{2\sqrt{\pi}\sqrt{t^\alpha}} \int_0^\infty \sigma^{-\frac{3}{2}} \exp[-\mu^2 t^\alpha \sigma - \frac{r^2}{4} t^{-\alpha} \sigma^{-1}] H_{1,1}^{1,0} \left(\sigma \left| \begin{matrix} (1, \alpha) \\ (1, 1) \end{matrix} \right. \right) d\sigma \quad (25) \\ &= \frac{1}{2\sqrt{\pi}\sqrt{t^\alpha}} \int_0^\infty \exp[-\mu^2 t^\alpha \sigma - \frac{r^2}{4} t^{-\alpha} \sigma^{-1}] H_{1,1}^{1,0} \left(\sigma \left| \begin{matrix} (1 - \frac{3\alpha}{2}, \alpha) \\ (-\frac{1}{2}, 1) \end{matrix} \right. \right) d\sigma. \end{aligned}$$

Hence, the inverse Laplace transform of Eq. (11) leads to

$$u(\xi, t) = \int_0^\infty G_\mu^\alpha(|\xi - y|, t) u(y, 0) dy \quad (26)$$

4. The complete solution of TFADE

In this section, we find the exact solution of the initial problem (1), (2) with $C_0(x) = C_0$. Using the Green function (25) and the initial condition (7), we have

$$\begin{aligned} u(\xi, t) &= C_0 \int_0^\infty G_\mu^\alpha(|\xi - y|, t) \exp(-\mu y) dy \\ &= \frac{C_0}{2\sqrt{\pi t^\alpha}} \int_0^\infty \exp(-\mu y) dy \\ &\quad \times \int_0^\infty \exp(-\mu^2 t^\alpha \sigma - \frac{1}{4}(\xi - y)^2 t^{-\alpha} \sigma^{-1}) H_{1,1}^{1,0} \left(\sigma \left| \begin{matrix} (1 - \frac{3\alpha}{2}, \alpha) \\ (-\frac{1}{2}, 1) \end{matrix} \right. \right) d\sigma \\ &= \frac{C_0}{2\sqrt{\pi t^\alpha}} \exp(-\mu \xi) \int_0^\infty H_{1,1}^{1,0} \left(\sigma \left| \begin{matrix} (1 - \frac{3\alpha}{2}, \alpha) \\ (-\frac{1}{2}, 1) \end{matrix} \right. \right) d\sigma \\ &\quad \times \int_0^\infty \exp\left\{-\left[\frac{\xi - y - 2\mu t^\alpha \sigma}{2\sqrt{t^\alpha \sigma}}\right]^2\right\} dy. \end{aligned} \quad (27)$$

Letting $\eta = \frac{2\mu t^\alpha \sigma - \xi + y}{2\sqrt{t^\alpha \sigma}}$, we get

$$u(\xi, t) = \frac{C_0}{2} \exp(-\mu\xi) \int_0^\infty \sqrt{\sigma} H_{1,0}^{1,0} \left(\sigma \left| \begin{matrix} (1 - \frac{3\alpha}{2}, \alpha) \\ (-\frac{1}{2}, 1) \end{matrix} \right. \right) d\sigma \left[\frac{2}{\sqrt{\pi}} \int_{\eta_1}^\infty \exp\{-\eta^2\} d\eta \right] \quad (28)$$

where $\eta_1 = \frac{2\mu t^\alpha \sigma - \xi}{2\sqrt{t^\alpha \sigma}}$.

Using the property (61) of the H-function, we obtain

$$u(\xi, t) = \frac{C_0}{2} \exp(-\mu\xi) \int_0^\infty H_{1,1}^{1,0} \left(\sigma \left| \begin{matrix} (1 - \alpha, \alpha) \\ (0, 1) \end{matrix} \right. \right) [\operatorname{erfc}(\eta_1)] d\sigma \quad (29)$$

where the complementary error function, $\operatorname{erfc}(x)$, is defined as

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-x^2) dx. \quad (30)$$

Using Eq. (4), we obtain the complete solution for time fractional advection dispersion equation with $0 < \alpha \leq 1$:

$$C(x, t) = \frac{C_0}{2} \int_0^\infty H_{1,1}^{0,1} \left(\sigma \left| \begin{matrix} (1 - \alpha, \alpha) \\ (0, 1) \end{matrix} \right. \right) [\operatorname{erfc}(\eta_1)] d\sigma. \quad (31)$$

The function

$$H_{1,1}^{1,0} \left(z \left| \begin{matrix} (1 - \alpha, \alpha) \\ (0, 1) \end{matrix} \right. \right)$$

is a probability density. From Eqs (59), (60) and (56), we have

$$H_{1,1}^{1,0} \left(z \left| \begin{matrix} (1 - \alpha, \alpha) \\ (0, 1) \end{matrix} \right. \right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1 - \alpha - \alpha k)} \frac{z^k}{k!}, \quad 0 < \alpha \leq 1. \quad (32)$$

5. Conclusions

The time fractional advection-dispersion equation is obtained from the classical advection-dispersion equation by replacing the first-order time derivative by a fractional derivative of order α ($0 < \alpha \leq 1$). Using variable transformation, the intermediate steps of Mellin and Laplace transforms, we derive the complete solution of this time fractional advection-dispersion equation. Its Green function includes a probability density function and a complementary error function.

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Appendix: Preliminaries

For the reader's convenience we present here certain ideas and the essential notions and notations concerning the Mellin and Laplace transforms, which are used in the paper.

Appendix A: The Mellin transform

The Mellin transform of a sufficiently well-behaved function φ on R^+ is defined as follows (Samko, Kilbas and Marichev, 1993):

$$\varphi^*(s) = M\{\varphi(t); s\} = \int_0^\infty t^{s-1}\varphi(t)dt \quad (33)$$

and its inverse is given by the formula

$$\varphi^*(t) = M^{-1}\{\varphi(t)^*(s); t\} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \varphi^*(s)t^{-s}ds, \quad \gamma = \text{Re}(s). \quad (34)$$

The Mellin convolution relation is defined as

$$(h \circ \varphi)(t) = \int_0^\infty h\left(\frac{t}{z}\right)\varphi(z)\frac{dz}{z}. \quad (35)$$

The Mellin convolution theorem with respect to (35) takes the form

$$(h \circ \varphi)^*(s) = h^*(s)\varphi^*(s). \quad (36)$$

Substituting the expression (36) instead of $\varphi^*(s)$ in (34) and taking (35) into account we obtain the Parseval relation

$$\int_0^\infty h\left(\frac{t}{z}\right)\varphi(z)\frac{dz}{z} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} h^*(s)\varphi^*(s)t^{-s}ds. \quad (37)$$

If we denote by \xleftrightarrow{M} the correspondence between a function and its Mellin transform, then the following relations of the general types hold:

$$\varphi(at) \xleftrightarrow{M} a^{-s}\varphi^*(s), \quad a > 0, \quad (38)$$

$$t^a\varphi(t) \xleftrightarrow{M} \varphi^*(s+a), \quad (39)$$

$$\varphi(t^m) \xleftrightarrow{M} \frac{1}{|m|}\varphi^*\left(\frac{s}{m}\right), \quad m \neq 0, \quad (40)$$

$$\begin{aligned} \exp(-at^h - bt^{-h}) &\xleftrightarrow{M} \frac{2}{h}\left(\frac{b}{a}\right)^{\frac{s}{2h}} K_{\frac{s}{h}}(2\sqrt{ab}), \\ \text{Re}(a) > 0, \text{Re}(b) > 0, h > 0, \end{aligned} \quad (41)$$

$$(t^2 + b^2)^{-\frac{1}{2}\nu} K_\nu[a(t^2 + b^2)^{\frac{1}{2}}] \xleftrightarrow{M} a^{-\frac{s}{2}} 2^{\frac{s}{2}-1} b^{\frac{s}{2}-\nu} \Gamma\left(\frac{s}{2}\right) K_{\nu-\frac{s}{2}}(ab),$$

$$\operatorname{Re}(a) > 0, \operatorname{Re}(b) > 0, \operatorname{Re}(s) > 0, \quad (42)$$

$$\int_0^\infty h\left(\frac{t}{z}\right)\varphi(z)\frac{dz}{z} \xleftrightarrow{M} h^*(s)\varphi^*(s), \quad (43)$$

where $K_\nu(z)$ denotes the modified Bessel function of the second kind (*Erde'lyi et al.*, 1954). The Mellin transform formulae of some functions and the properties of the Mellin transform can be found in *Erde'lyi et al.* (1954), Samko, Kilbas and Marichev (1993).

Appendix B: The Laplace transform

The Laplace transform of a function $\varphi(t)$, $0 < t < \infty$, is defined as follows (Samko, Kilbas and Marichev, 1993):

$$\tilde{\varphi}(p) = L\{\varphi(t)\} = L\{\varphi(t); p\} = \int_0^\infty e^{-pt}\varphi(t)dt, \quad (44)$$

and its inverse is given by the formula

$$\varphi(t) = L^{-1}\{\tilde{\varphi}(p); t\} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\tilde{\varphi}(p)dp, \quad \gamma = \operatorname{Re}(p) > p_0. \quad (45)$$

It follows that

$$L\{t^\nu; p\} = \frac{\Gamma(\nu+1)}{p^{\nu+1}}, \quad \nu > -1. \quad (46)$$

One of the most useful properties of the Laplace transform is embodied in the convolution theorem (Churchill, 1944), which states that the Laplace transform of the convolution of two functions is the product of their Laplace transforms. Thus if $\tilde{h}(s)$ and $\tilde{\varphi}(s)$ are the Laplace transforms of $h(t)$ and $\varphi(t)$, respectively, then

$$L\left\{\int_0^t h(t-\tau)\varphi(\tau)d\tau\right\} = \tilde{h}(p)\tilde{\varphi}(p). \quad (47)$$

Now if φ is continuous, the fractional integral of order α of φ is

$$D^{-\alpha}\varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1}\varphi(\tau)d\tau, \quad \alpha > 0 \quad (48)$$

which is a convolution integral. From (46), (47) and (48), we have

$$L\{D^{-\alpha}\varphi(t)\} = \frac{1}{\Gamma(\alpha)}L\{t^{\alpha-1}\}L\{\varphi(t)\} = p^{-\alpha}\tilde{\varphi}(p), \quad \alpha > 0. \quad (49)$$

The Mellin and Laplace transforms are related to each other by (Schneider and Wyss, 1989)

$$\varphi^*(s) = \frac{1}{\Gamma(1-s)} \int_0^\infty p^{-s}\tilde{\varphi}(p)dp. \quad (50)$$

Appendix C: The H-function

An H-function is defined in terms of a Mellin-Barnes type integral as follows (Mathai and Saxena, 1978, Anh and Lenenko, 2001):

$$\begin{aligned} H_{p,q}^{m,n}(z) &= H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_1, \alpha_1) \dots (a_p, \alpha_p) \\ (b_1, \beta_1) \dots (b_q, \beta_q) \end{matrix} \right. \right] \\ &= \frac{1}{2\pi i} \int_L H_{p,q}^{*m,n}(s) z^{-s} ds, \end{aligned} \quad (51)$$

where m, n, p and q are nonnegative integers such that $0 \leq n \leq p, 1 \leq m \leq q$ and empty products are interpreted as unity. The parameters $\alpha_1, \dots, \alpha_p$ and β_1, \dots, β_q are positive real numbers, whereas a_1, \dots, a_p and b_1, \dots, b_q are complex numbers. The Fox or H-function is characterized by its Mellin transform:

$$H_{p,q}^{*m,n}(s) = H_{p,q}^{*m,n} \left[z \left| \begin{matrix} (a_j, \alpha_j) j = 1, \dots, p \\ (b_i, \beta_i) i = 1, \dots, q \end{matrix} \right. \right] (s) = \frac{A(s)B(s)}{C(s)D(s)}, \quad (52)$$

where

$$\begin{aligned} A(s) &= \prod_{j=1}^m \Gamma(b_j + \beta_j s), \quad B(s) = \prod_{j=1}^m \Gamma(1 - a_j - \alpha_j s), \\ C(s) &= \prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j s), \quad D(s) = \prod_{j=n+1}^p \Gamma(a_j + \alpha_j s). \end{aligned} \quad (53)$$

In (51) $z^{-s} = \exp\{-s \cdot \log|z| - i \arg(z)\}$ and $\arg(z)$ is not necessarily the principal value. The parameters are restricted by the condition $P(A) \cap P(B) = \emptyset$, where

$$\begin{aligned} P(A) &= \{\text{poles of } \Gamma(1 - a_i + \alpha_i s)\} = \left\{ \frac{1 - a_i + k}{\alpha_i} \in C; i = 1, \dots, n, k \in N_0 \right\}, \\ P(B) &= \{\text{poles of } \Gamma(b_i + \beta_i s)\} = \left\{ \frac{-b_i - k}{\beta_i} \in C; i = 1, \dots, m, k \in N_0 \right\}, \\ N_0 &= \{0, 1, \dots\}. \end{aligned}$$

The integral (51) converges if one of the following conditions holds (Hilfer, 2000; Anh and Lenenko, 2001):

$$L = L(c - i\infty, c + i\infty; P(A), P(B)), |\arg z| < \frac{w\pi}{2}, w > 0;$$

$$L = L(c - i\infty, c + i\infty; P(A), P(B)), |\arg z| < \frac{w\pi}{2}, w \geq 0, cR < -\text{Re}(\gamma);$$

where

$$\begin{aligned} w &= \sum_{j=1}^n \alpha_j - \sum_{j=n+1}^p \alpha_j + \sum_{j=1}^m \beta_j - \sum_{j=1}^q \beta_j, \\ R &= \sum_{j=1}^q \beta_j - \sum_{j=1}^p \alpha_j, \end{aligned}$$

$$\gamma = \sum_{j=1}^q b_j - \sum_{i=1}^p a_i + \frac{p-q}{2} + 1.$$

The H-functions are analytic for $z \neq 0$ and multivalued (single-valued on the Riemann surface of $\log z$). The H-functions may be represented as (Hilfer, 2000; Anh and Lenenko, 2001):

$$H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_1, \alpha_1) \cdots (a_p, \alpha_p) \\ (b_1, \beta_1) \cdots (b_q, \beta_q) \end{matrix} \right. \right] = \sum_{i=1}^m \sum_{k=0}^{\infty} c_{ik} \frac{(-1)^k}{k! \beta_i} z^{\frac{b_i+k}{\beta_i}} \quad (54)$$

where

$$c_{ik} = \frac{\prod_{j=1, j \neq i}^m \Gamma(b_j - (b_i + k) \frac{\beta_j}{\beta_i}) \prod_{j=1}^n \Gamma(1 - a_j + (b_i + k) \frac{\alpha_j}{\beta_i})}{\prod_{j=m+1}^q \Gamma(1 - b_j + (b_i + k) \frac{\beta_j}{\beta_i}) \prod_{j=n+1}^p \Gamma(a_j - (b_i + k) \frac{\alpha_j}{\beta_i})} \quad (55)$$

whenever $R \geq 0$ and the poles in $P(A)$ are simple. Similarly,

$$H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_1, \alpha_1) \cdots (a_p, \alpha_p) \\ (b_1, \beta_1) \cdots (b_q, \beta_q) \end{matrix} \right. \right] = \sum_{i=1}^m \sum_{k=0}^{\infty} c_{ik} \frac{(-1)^k}{k! \alpha_i} z^{-\frac{1+a_i+k}{\alpha_i}}, \quad (56)$$

where

$$c_{ik} = \frac{\prod_{j=1}^m \Gamma(b_j + (1 - a_i + k) \frac{\beta_j}{\alpha_i}) \prod_{j=1, j \neq i}^n \Gamma(1 - a_j + (1 - a_i + k) \frac{\alpha_j}{\alpha_i})}{\prod_{j=m+1}^q \Gamma(1 - b_j + (1 - a_i + k) \frac{\beta_j}{\alpha_i}) \prod_{j=n+1}^p \Gamma(a_j + (1 - a_i + k) \frac{\alpha_j}{\alpha_i})} \quad (57)$$

whenever $R \leq 0$ and the poles in $P(A)$ are simple.

In particular, if $R > 0$, we obtain from (54) that

$$\begin{aligned} H_{p,q}^{1,n}(z) &= H_{p,q}^{1,n} \left[z \left| \begin{matrix} (a_1, \alpha_1) \cdots (a_p, \alpha_p) \\ (b_1, \beta_1) \cdots (b_q, \beta_q) \end{matrix} \right. \right] \\ &= \frac{1}{\beta_1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{B(s_k)}{C(s_k)D(s_k)} z^{-s_k}, \end{aligned} \quad (58)$$

where $s_k = -b_1 + k/\beta_1$. In the case $R < 0$, we obtain from (54) that

$$\begin{aligned} H_{p,q}^{m,1}(z) &= H_{p,q}^{m,1} \left[z \left| \begin{matrix} (a_1, \alpha_1) \cdots (a_p, \alpha_p) \\ (b_1, \beta_1) \cdots (b_q, \beta_q) \end{matrix} \right. \right] \\ &= \frac{1}{\alpha_1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{A(s_k)}{C(s_k)D(s_k)} z^{-s_k}, \end{aligned} \quad (59)$$

where $s_k = (k+1 - \alpha_1)/\alpha_1$.

The following identities for the H-function are well-known:

$$\begin{aligned} & z^k H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (a_1, \alpha_1) \dots (a_p, \alpha_p) \\ (b_1, \beta_1) \dots (b_q, \beta_q) \end{array} \right. \right] \\ &= H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (a_1 + k\alpha_1, \alpha_1) \dots (a_p + k\alpha_p, \alpha_p) \\ (b_1 + k\beta_1, \beta_1) \dots (b_q + k\beta_q, \beta_q) \end{array} \right. \right], \end{aligned} \quad (60)$$

$$\begin{aligned} & H_{p,q}^{m,n} \left[z \left| \begin{array}{c} (a_1, \alpha_1) \dots (a_p, \alpha_p) \\ (b_1, \beta_1) \dots (b_q, \beta_q) \end{array} \right. \right] \\ &= k H_{p,q}^{m,n} \left[z^k \left| \begin{array}{c} (a_1, k\alpha_1) \dots (a_p, k\alpha_p) \\ (b_1, k\beta_1) \dots (b_q, k\beta_q) \end{array} \right. \right]. \end{aligned} \quad (61)$$

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