Time–Frequency Localization Operators: A Geometric Phase Space Approach

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Abstract **—We define a set of operators which localize in both time and frequency. These operators are similar to but different from the low-pass time-limiting operators, the singular functions of which are the prolate spheroidal wave functions. Our construction differs from the usual approach in that we treat the time–frequency plane as one geometric whole (phase space) rather than as two separate spaces. For disk-shaped or ellipse-shaped domains in the time–frequency plane, the associated localization operators are remarkably simple. Their eigenfunctions are Hermite functions, and the corresponding eigenvalues are given by simple explicit formulas involving the incomplete gamma functions.**

I. Introduction

Operators which localize in both time and frequency are of interest for many applications in optics and signal analysis. More generally, the operators may localize in a phase space associated with two sets of complementary variables that may have more than one dimension, such as, e.g., position and spatial frequency. In many cases one can observe signals (time-dependent signals or optical data) only within a certain frequency window W and during a limited time interval T (or, in the case of optical data, in a limited space interval). This implies that one effectively observes only

$$L_{T,W}f = Q_T P_W f \tag{1}$$

where f is the original signal and where Q_T, P_W are projection operators on the relevant intervals in, respectively, time and frequency, i.e.,

$$(Q_T f)(t) = \begin{cases} f(t), & \text{if } |t| \le T \\ 0, & \text{if } |t| \ge T \end{cases}$$

$$(P_W f)(t) = \int_{-\infty}^{\infty} dt' \frac{\sin[W(t-t')]}{\pi(t-t')} f(t').$$

The operator $L_{T,W}$ is effectively a phase space localization operator, i.e., it selects out of the signal f that part which is associated with the rectangle $[-T,T]\times[-W,W]$ in the time–frequency plane and filters out the rest. The singular functions of the operator $L_{T,W}$, or equivalently, the eigenfunctions of $L^*_{T,W}L_{T,W} = P_W Q_T P_W$, are the prolate spheroidal wave functions. They have been extensively studied in a series of excellent papers by Slepian and Pollak [1], Landau and Pollak [2], [3], and Slepian [4], [5]. For a review, see [6], [7]. These functions are extremely useful in the discussion of optimal recovery of information from data restriction in the sense of (1) (see, e.g., [6], [8]).

As eigenfunctions of a phase space localization operator, the prolate spheroidal wave functions can also be used to filter out noise from given (noisy) signals. Applications of this type can be found in [9]–[11].

Time–frequency localization operators other than $L_{T,W}$ are sometimes considered, in which Q_T and P_W are replaced by smoother versions. More explicitly, one may take

$$(\tilde{Q}_T f)(t) = g(t/T) f(t) \tag{2}$$

where g is a smooth positive function centered around $t = 0$ and tending to zero for $t \to \pm\infty$. A similar change may be made in $\tilde{P}_W$. In optics, for instance, this change corresponds to a nonuniform illumination of the object [12], [13]. The resulting operators $\tilde{L}_{T,W} = \tilde{Q}_T \tilde{P}_W$, and the associated singular values and singular vectors can again be used in the recovery and in the filtering of signals (see, e.g., [12], [13]).

The same kind of analysis can of course be done in more than one dimension. See [4] for a study of the operator $L_{T,W}$ in the case where the two conjugated variables are both two-dimensional, and where the intervals $[-T,T],[-W,W]$ are replaced by disks.

As stated, the operator $L_{T,W}$ and its generalizations are operators which "project" onto a subset of phase space or, more generally, weight different parts of phase space differently. Clearly, this phase space (or time–frequency plane) picture is important in signal analysis. This is also illustrated by the use of the Wigner distributions in signal analysis. Originally, the Wigner distribution was introduced as a means to keep track of quantum mechanical phenomena in phase space. Precisely because the Wigner distribution is a phase space concept, it turns out to be useful also for signal analysis; by means of the Wigner distribution one can convert any time-dependent signal into a detailed representation in the time–frequency plane. See [14] for an example of this kind of analysis applied to the study of characteristics of loudspeakers.

We propose here a family of operators, different from $L_{T,W}$ and its natural generalizations, which also localize in

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the time–frequency plane or, more generally, in phase space. Their generalization to more than one dimension is straightforward. The construction of these operators involves the so-called "coherent states." This terminology was coined in a quantum optics context where the coherent states do indeed express coherence. They have since spilled over to many other areas in physics, where their meaning is often different. The quantum optics terminology has stuck, however. As we shall see, the coherent states are again linked closely to the phase space, and, as such, are a natural tool in the construction of phase space localization operators. Coherent states have been used extensively in theoretical physics and in many different applications; for a review see [15]. The remainder of this paper is organized as follows.

In Section II we give a precise definition of coherent states, and we review some of their properties. In Section III we give our construction of phase space localization operators, using these coherent states. In Section IV we single out an important family of special cases. For these special operators the eigenvectors are the Hermite functions, and the eigenvalues are given by simple, explicit formulas. The nonuniform Gaussian filter (in which Q_T, P_W are replaced by $\tilde{Q}_T$, $\tilde{P}_W$, as defined in (2), where g is a Gaussian), discussed in, e.g., [13], [18], turns out to be a special case of our construction. Other examples correspond to a "projection" onto a bounded subset of phase space and are thus closer in spirit to the nonsmoothed $L_{T,W}$ itself, which singles out the subset $[-T,T]\times[-W,W]$ in the time–frequency plane. The subsets of the time–frequency plane singled out by our localization operators are disks; the corresponding eigenvalues are given by incomplete gamma functions (see Section IV). We also give a short discussion of the asymptotic behavior of the eigenvalues and eigenvectors of the localization operators constructed here. In Section V we show how our disk-type operators can be generalized to deal with elliptical and other subsets of phase space.

II. Coherent States

Coherent states are L^2 functions (i.e., square integrable functions) labeled by phase space points. Since we want to treat functions depending on n-dimensional Cartesian variables, the associated phase space is $\mathbb{R}^n\times\mathbb{R}^n$.

To construct a family of coherent states, one starts by choosing one vector (sometimes called the "fiducial vector"; see [15]) ϕ in $L^2(\mathbb{R}^n)$. The associated coherent states are then generated from ϕ by phase space translations. More precisely, for any phase space point $(p,q)\in\mathbb{R}^n\times\mathbb{R}^n$, the associated coherent state $\phi_{p,q}$ is defined by

$$\phi_{p,q}(x)=e^{ipx}\phi(x-q). \tag{3}$$

In the one-dimensional case, $n=1$, this does indeed correspond to a translation by q in time (if x is taken to be time), and to a shift by p in frequency.

The function $\phi=\phi_{0,0}$ may be chosen arbitrarily in $L^2(\mathbb{R}^n)$. A "canonical" choice for ϕ is

$$\phi^0(x)=\pi^{-n/4}\exp(-x^2/2). \tag{4}$$

The resulting coherent states $\phi^0_{p,q}$ are often called canonical coherent states in the physics literature [15] or Gabor wave functions in the engineering literature (after Gabor [16]). The $\phi^0_{p,q}$ are localized, in phase space, around (p,q), i.e.,

$$\int\cdots\int d^nx\,x_j|\phi^0_{p,q}(x)|^2=q_j$$

$$\int\cdots\int d^nk\,k_j|(\phi^0_{p,q})^\wedge(k)|^2=p_j,\qquad j=1,\cdots,n$$

where $\hat{f}$ denotes the Fourier transform of f,

$$\hat{f}(k)=(2\pi)^{-n/2}\int\cdots\int d^nx\,e^{-ik\cdot x}f(x).$$

Except when specified otherwise, all the integrals below will be n-dimensional; we shall replace $\int\cdots\int d^nx$ by $\int dx$ for brevity.

The $\phi^0_{p,q}$ also minimize the uncertainty relation inequality:

$$\left[\int dx|\phi^0_{p,q}(x)|^2(x-q)^2\right]\cdot\left[\int dk|(\phi^0_{p,q})^\wedge(k)|^2(k-p)^2\right]=n^2/4.$$

In this sense the function $\phi^0_{p,q}$ is the L^2 function which achieves, of all the L^2 functions, the best phase space localization around the phase space point (p,q).

Perhaps the most important property of the coherent states is the "resolution of identity" [15]. This states that any function $f\in L^2(\mathbb{R}^n)$ can be reconstructed easily from the scalar products

$$(\phi_{p,q},f)=\int dx\,\overline{\phi_{p,q}(x)}f(x).$$

One has indeed

$$\begin{aligned}(2\pi)^{-n}&\int dp\int dq\,\phi_{p,q}(x)(\phi_{p,q},f)\\&=(2\pi)^{-n}\int dp\int dq\int dy\,e^{ip\cdot(x-y)}\cdot\phi(x-q)\overline{\phi(y-q)}f(y)\\&=\int dq\int dy\,\delta(x-y)\phi(x-q)\overline{\phi(y-q)}f(y)\\&=f(x)\int dq|\phi(x-q)|^2=f(x)\int dq|\phi(q)|^2.\end{aligned}$$

If therefore ϕ is normalized, i.e., if $\int dq|\phi(q)|^2=1$, then for all $f\in L^2(\mathbb{R}^n)$ one finds

$$f=(2\pi)^{-n}\int dp\int dq\,\phi_{p,q}(\phi_{p,q},f). \tag{5}$$

The resolution of the identity operator as given by (5) is valid for any choice of $\phi\in L^2(\mathbb{R}^n)$. However, if one makes the "canonical choice" $\phi=\phi^0$, then (5) has the following nice physical interpretation. For all phase space points (p, q), one first projects f onto the best localized function around (p, q), by means of the operation

$$\phi^0_{p,q}\left(\phi^0_{p,q}, f\right);$$

integrating over all of phase space then regenerates f.

Remarks

1) Note that the map $f \to \Phi(p, q) = (\phi_{p,q}, f)$ sends a function of one variable into a function of two. This new function is square integrable,

$$\int dp \int dq |\Phi(p, q)|^2 = (2\pi)^n \int dx |f(x)|^2 < \infty$$

(this immediately follows from (5)). Because there is redundancy in Φ, the range of this map is a subspace much smaller than $L^2(\mathbb{R}^2)$. For special choices of ϕ, this subspace has been explicitly characterized. For $\phi = \phi^0$, for instance, one finds that any such Φ can be written in the form

$$\Phi(p, q) = \exp\left[-\frac{1}{4}(p^2 + q^2)\right]\psi(p + iq),$$

where ψ is an entire analytic function on $\mathbb{C}^n$. Conversely, any square integrable Φ of this form lies in the range of the map $f \to \Phi$ (see [15] for more details, and for the original references).

2) The choice $\phi = \phi^0$ is special in more than one respect. Since $\widehat{(\phi^0)} = \phi^0$, we have

$$\widehat{(\phi_{p,q})}(k) = e^{ip\cdot q}\phi_{-q,p}(k),$$

and

$$(\phi_{p,q}, f) = e^{-ip\cdot q}\left(\phi_{-q,p}, \hat{f}\right).$$

This means that, except for an unimportant phase factor, the phase space pictures associated with f and $\hat{f}$ can be obtained from each other by a simple 90° rotation in phase space.

III. Constructing Phase Space Localization Operators Using the Coherent States

From the preceding interpretation of the "resolution of identity" for canonical coherent states, it is clear how one can build phase space localization operators using these canonical coherent states. Restricting the integral in (5) to a subset S of the phase space $\mathbb{R}^n \times \mathbb{R}^n$ results in a function $P_s f$ reconstructed from only those phase space projections of f corresponding to points in S, i.e.,

$$P_s f = (2\pi)^{-n} \int_{(p,q)\in S} dp \int dq\, \phi^0_{p,q}\left(\phi^0_{p,q}, f\right). \tag{6}$$

The operator P_S, defined in (6), is positive and bounded by one. There is no restriction on the shape of S, apart from the fact that S should be measurable. This contrasts with the operators $L_{T,W}$ discussed before, which focus on rectangles $[-T, T] \times [-W, W]$.

Note that the phase space cutoff defined by P_S is not "sharp," in the sense that the function $P_S f$ will have some phase space content outside the set S. This is illustrated by the fact that at least for some $(p', q') \notin S$, we have $(\phi^0_{p',q'}, P_S f) \neq 0$. The following lemma provides us with an upper bound on these inner products.

Lemma: For any ϵ between 0 and 1, one has

$$\left|\left(\phi^0_{p,q}, P_S f\right)\right| \leq \epsilon^{-n/2}\|f\| \exp\left[-\frac{1-\epsilon}{4}\,\mathrm{dist}((p, q), S)^2\right], \tag{7}$$

where $\mathrm{dist}((p, q), S)$ is the Euclidean distance, in phase space, between (p, q) and S:

$$\mathrm{dist}((p, q), S)^2 = \inf_{(p',q')\in S}\left[(p - p')^2 + (q - q')^2\right].$$

Proof: An easy calculation, using the definitions (3) and (4), leads to the following expression for the inner product between canonical coherent states,

$$\left(\phi^0_{p,q}, \phi^0_{p',q'}\right) = \exp\left[\frac{i}{2}(p' - p)\cdot(q' + q) - \frac{1}{4}(p - p')^2 - \frac{1}{4}(q - q')^2\right].$$

Thus, for $0 < \epsilon < 1$,

$$\begin{aligned}
\left|\left(\phi^0_{p,q}, P_S f\right)\right| &\leq (2\pi)^{-n} \int_{(p',q')\in S} dp' \int dq' \left|\left(\phi^0_{p,q}, \phi^0_{p',q'}\right)\left(\phi^0_{p',q'}, f\right)\right| \\
&\leq (2\pi)^{-n} \exp\left[-\frac{1-\epsilon}{4} \inf_{(p',q')\in S}\left((p - p')^2 + (q - q')^2\right)\right] \\
&\quad \cdot \int dp' \int dq' \exp\left[-\frac{\epsilon}{4}\left((p - p')^2 + (q - q')^2\right)\right]\left|\left(\phi^0_{p',q'}, f\right)\right| \\
&\leq (2\pi)^{-n/2}\epsilon^{-n/2} \exp\left[-\frac{1-\epsilon}{4}\,\mathrm{dist}((p, q), S)^2\right] \\
&\quad \times \left[\int dp' \int dq' \left|\left(\phi^0_{p',q'}, f\right)\right|^2\right]^{1/2} \qquad \text{(by Cauchy–Schwarz)} \\
&\leq \epsilon^{-n/2} \exp\left[-\frac{1-\epsilon}{4}\,\mathrm{dist}((p, q), S)^2\right]\|f\| \qquad \text{(by the resolution of identity (5))},
\end{aligned}$$

which concludes the proof.

In particular, (7) implies that for $\operatorname{dist}((p,q),S) \geq \sqrt{n/2}$,

$$\left|\left(\phi^0_{p,q}, \boldsymbol{P}_S f\right)\right| \leq \left(\frac{2e}{n}\right)^{n/2} \left[\operatorname{dist}((p,q),S)\right]^n \cdot \exp\left[-\frac{1}{4}\operatorname{dist}((p,q),S)^2\right].$$

This shows that the "tail" of $\boldsymbol{P}_S f$ outside S decays very fast. Note that such tails are unavoidable. It is well-known that for a bounded subset S of phase space, no "sharp" phase space filter for S exists. In the case of the prolate spheroidal wave functions, for instance, the operator $L^*_{T,W}, L_{T,W} = P_W Q_T P_W$ is sharp in the frequency variable but has a tail in the time variable behaving as t^{-1}. The converse is true, of course, for $L_{T,W} L^*_{T,W} = Q_T P_W Q_T$. Our operators $\boldsymbol{P}_S$ are not sharp in any phase space direction, but on the other hand, their tails decay as a Gaussian rather than as a negative power. By using a dilation argument (see also Section V) one can choose the tails to decay faster in one direction (at the expense of other directions), but they cannot be made to disappear altogether.

All this can be generalized to operators which use a weight function $F(p,q)$ on phase space rather than a cutoff. Thus one defines

$$\boldsymbol{P}_F f = (2\pi)^{-n} \int dp \int dq F(p,q) \phi^0_{p,q} \left(\phi^0_{p,q}, f\right).$$

If F is positive, and bounded by one, then so is $\boldsymbol{P}_F$.

The operators $\boldsymbol{P}_F$, $\boldsymbol{P}_S$ are in fact integral operators,

$$(\boldsymbol{P}_F f)(x) = \int dy K_F(x,y) f(y)$$

with

$$K_F(x,y) = 2^{-n}\pi^{-3n/2} \exp\left[-\frac{1}{4}(x-y)^2\right] \cdot \int dp \int dq \exp\left[ip\cdot(x-y) - \left(q - \frac{x+y}{2}\right)^2\right] F(p,q). \quad (8)$$

(For $\boldsymbol{P}_S$ it suffices to replace $F(p,q)$ by the function $F_S(p,q) = 1$ if $(p,q) \in S$, 0 otherwise).

If S is bounded, or if F is in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$, then the operator $\boldsymbol{P}_S$, respectively $\boldsymbol{P}_F$, is trace-class, i.e., its spectrum is then purely discrete, and the sum of the eigenvalues is finite. One finds indeed, for any orthonormal basis $\{\psi_l\}$ of $L^2(\mathbb{R}^n)$, that

$$\sum_l (\psi_l, \boldsymbol{P}_F \psi_l) = (2\pi)^{-n} \int dp \int dq F(p,q) \int dx \, |\phi_{p,q}(x)|^2 = (2\pi)^{-n} \int dp \int dq F(p,q).$$

Since $\boldsymbol{P}_F$ is positive, this implies that $\boldsymbol{P}_F$ is trace-class if F is integrable. This means, in particular, that the eigenvalues λ_k tend to zero for $k \to \infty$. It follows that all functions f which are essentially concentrated in phase space on S, i.e., $f \approx \boldsymbol{P}_S f$, can be represented up to an error ϵ as linear combinations of a finite number of eigenfunctions of $\boldsymbol{P}_S$; the "effective dimension," i.e., the number of eigenfunctions needed, here only depends on ϵ.

In general, the eigenfunctions of $\boldsymbol{P}_S$, $\boldsymbol{P}_F$ will not be easy to construct, however. In the next section we shall restrict ourselves to an important family of special cases in which this problem does not occur.

IV. Spherical Symmetry in Phase Space

Throughout this section we shall restrict ourselves to functions $F(p,q)$ (or subsets S of phase space) which only depend on the n variables $r_j^2 = p_j^2 + q_j^2$, $j = 1, \cdots, n$, i.e.,

$$F(p,q) = \mathscr{F}\left(r_1^2, \cdots, r_n^2\right). \quad (9)$$

For such functions F the eigenvectors and eigenvalues are given explicitly by the following theorem.

Theorem: Let F be a function of phase space which depends only on $r_j^2 = p_j^2 + q_j^2$, as in (9). Then 1) the eigenfunctions of $\boldsymbol{P}_F$ are the n-dimensional Hermite functions

$$H_{[k]}(x) = \prod_{j=1}^n H_{k_j}(x_j)$$

with $[k] = (k_1, \cdots, k_n) \in \mathbb{N}^n$ and

$$H_l(t) = \pi^{-1/4}(2^l l!)^{-1/2}\left(t - \frac{d}{dt}\right)^l e^{-t^2/2},$$

and 2) the corresponding eigenvalues

$$\boldsymbol{P}_F H_{[k]} = \lambda_{[k]} H_{[k]} \quad (10)$$

are given by

$$\lambda_{[k]} = \left(\prod_{j=1}^n k_j!\right)^{-1} \int_0^\infty ds_1 \cdots \int_0^\infty ds_n \mathscr{F}(2s_1, \cdots, 2s_n) \cdot \left(\prod_{j=1}^n s_j^{k_j}\right) \exp\left(-\sum_{j=1}^n s_j\right). \quad (11)$$

Proof: We shall prove (10) and (11) by showing that

$$\left(H_{[k]}, \boldsymbol{P}_F H_{[k']}\right) = \lambda_{[k]} \prod_{j=1}^n \delta_{k_j, k'_j} \quad (12)$$

where δ_{kl} is the Kronecker-delta ($\delta_{kl} = 1$ if $k = l$, $\delta_{kl} = 0$ otherwise). Since the Hermite functions $H_{[k]}$ are an orthonormal basis in $L^2(\mathbb{R}^n)$, this proves all our assertions. The only ingredient needed to prove (12) is the inner product $(H_{[k]}, \phi^0_{p,q})$. This can be calculated in many ways (e.g., by using the generating function for the Hermite polynomials). The result is

$$\left(H_{[k]}, \phi^0_{p,q}\right) = \left[\prod_{j=1}^n \left(k_j! 2^{k_j}\right)\right]^{-1/2} \cdot e^{-(r_1^2 + \cdots + r_n^2)/4} \prod_{j=1}^n \left(q_j - ip_j\right)^{k_j}$$

where $r_j^2 = p_j^2 + q_j^2$. Introducing the angles θ_j defined by

$q_j = r_j \cos\theta_j$, $p_j = r_j \sin\theta_j$, we can rewrite this as

$$\left(H_{[k]}, \phi^0_{p,q}\right) = \left[\prod_{j=1}^{n}\left(k_j! 2^{k_j}\right)\right]^{-1/2} \cdot e^{-(r_1^2+\cdots+r_n^2)/4} \prod_{j=1}^{n}\left(r_j^{k_j} e^{-ik_j\theta_j}\right).$$

We can now use this to calculate $(H_{[k]}, \boldsymbol{P}_F H_{[k']})$. Since F depends only on $r_1^2, \cdots, r_n^2$, the θ_j integrations can be carried out immediately, and we find

$$\begin{aligned}\left(H_{[k]}, \boldsymbol{P}_F H_{[k']}\right) &= \left[\prod_{j=1}^{n} \delta_{k_j,k'_j}\left(k_j! 2^{k_j}\right)^{-1}\right] \\ &\quad \times \int_0^\infty dr_1 r_1 \cdots \int_0^\infty dr_n r_n e^{-(r_1^2+\cdots+r_n^2)/2} \\ &\quad \cdot \mathscr{F}\left(r_1^2, \cdots, r_n^2\right) \prod_{j=1}^{n}\left(r_j^{2k_j}\right) \\ &= \lambda_{[k]} \prod_{j=1}^{n} \delta_{kj,k'_j},\end{aligned}$$

which concludes the proof.

The above proof has the virtue of leading directly to an explicit expression, namely (11), for the eigenvalues $\lambda_{[k]}$. The very special role of the Hermite functions here may, however, seem magical to readers less familiar with coherent states. The key to this magic is simple: we are dealing with operators commuting with a family of second-order partial differential operators. This is similar to the case of the prolate spheroidal wave functions. There it turned out to be crucial that the operator $L_{T,W}$ commutes with a second-order differential operator; the discovery of this fact [1] marked a breakthrough in the analysis of band-limited signals. Several generalizations of the original work on the prolate spheroidal wave functions have concentrated on this commutation aspect; see, e.g., [4], [17], [18]. All these generalizations have, however, concerned themselves only with operators of the type $L_{T,W}$ or $\tilde{L}_{T,W}$ (as defined by (1), (2)); in the present situation, as explained above, our construction is based more on a geometric picture of phase space *as a whole* rather than on a picture of switching back and forth between two complementary variables (e.g., time and frequency). The commuting second-order partial differential operators in our present situation are suggested naturally by this geometric phase space picture. The following argument shows how. For the sake of simplicity we restrict ourselves to $n = 1$. (In more than one dimension the argument is entirely similar; it then has to be carried out in each of the n variables $x_1, \cdots, x_n$). In classical mechanics, the harmonic oscillator Hamiltonian leads to a time evolution which is represented by circles on phase space. More explicitly, the solution of the Hamilton–Jacobi equations associated with the Hamiltonian $h(p,q) = \frac{1}{2}(p^2+q^2)$, with initial conditions $p(0) = p_0, q(0) = q_0$, is given by

$$(p(t), q(t)) = (p_0 \cos t - q_0 \sin t, q_0 \cos t + p_0 \sin t) \quad (13)$$

where we have taken units such that the harmonic oscillator frequency $\omega = 1$. These phase space rotations occur in quantum mechanics as well. Expressed in differential equations, one finds that the solution to the equation

$$i\frac{\partial}{\partial t}\psi(x,t) = \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2} + x^2 - 1\right)\psi(x,t)$$

(i.e., the quantum mechanics time evolution equation for the harmonic oscillator Hamiltonian; readers familiar with quantum mechanics will recognize that the ground state energy has been subtracted from the Hamiltonian, which leads to a simpler expression for formula (14) below), with initial condition

$$\psi(x,0) = e^{-ipq/2}\phi^0_{p,q}(x)$$

is given by

$$\psi(x,t) = e^{-ip(t)q(t)/2}\phi^0_{p(t),q(t)}(x), \quad (14)$$

where $p(t), q(t)$ are given by (13). This means that, in a coherent state picture, the time evolution generated by $\frac{1}{2}(-(\partial^2/\partial x^2) + x^2 - 1)$ corresponds to rotation in phase space. Given this geometric picture, it is natural that a construction using coherent states and following a rotationally symmetric procedure (that is essentially what (9) means) leads to operators which are invariant under the time evolution generated by the harmonic oscillator Hamiltonian $\frac{1}{2}(-(\partial^2/\partial x^2) + x^2 - 1)$. The invariance of $\boldsymbol{P}_F$ under this time evolution is but another way of saying that $\boldsymbol{P}_F$ and $\frac{1}{2}(-(\partial^2/\partial x^2) + x^2 - 1)$ commute.

In n dimensions ($n \neq 1$) one argues similarly that $\boldsymbol{P}_F$ and $\frac{1}{2}((\partial^2/\partial x_j^2) + x_j^2 - 1)$ commute, for all $j = 1, \cdots, n$. Since the common set of eigenvectors of these n second-order partial differential operators are the Hermite functions $H_{[k]}$, this explains the role of the Hermite functions as the eigenvectors of $\boldsymbol{P}_F$. One can of course also check these commutations by direct computation, using (8) and substituting (9) for F. In what follows, we shall have a closer look at two examples. For the sake of simplicity of notation, we again restrict ourselves to the case $n = 1$.

A. Gaussian Weights in Time and Frequency

In this case, we take

$$F(p,q) = \exp\left[-\alpha(p^2+q^2)\right].$$

This is clearly a special case of (9). Hence

$$\boldsymbol{P}_F H_k = \lambda_k H_k$$

with

$$\lambda_k = \frac{1}{k!}\int_0^\infty ds\, s^k e^{-(1+2\alpha)s} = (1+2\alpha)^{-(k+1)}. \quad (15)$$

The integral kernel of $\boldsymbol{P}_F$ given by (8) can be calculated explicitly. One finds

$$K_F(x,y) = \frac{1}{2}[\pi\alpha(\alpha+1)]^{-1/2} \cdot \exp\left[-\frac{\alpha+1}{4\alpha}(x-y)^2 - \frac{\alpha}{4(1+\alpha)}(x+y)^2\right]$$
$$= \frac{1}{2}[\pi\alpha(\alpha+1)]^{-1/2} \cdot \exp\left[-\frac{2\alpha+1}{4\alpha(\alpha+1)}(x-y)^2 - \frac{\alpha}{2(1+\alpha)}(x^2+y^2)\right].$$

From this expression for the integral kernel K_F it is easy to check that the operator P_F can also be written as

$$P_F = (2\alpha+1)^{-1/2} L_\alpha^* L_\alpha$$

with

$$(L_\alpha f)(x) = \frac{1}{\sqrt{2\pi}\,\gamma(\alpha)} \cdot \int dy \exp\left[-\frac{(x-y)^2}{2\gamma(\alpha)^2}\right] \exp\left[-\frac{y^2}{2\rho(\alpha)^2}\right] f(y) \quad (16)$$

where

$$\gamma(\alpha) = \sqrt{\alpha(\alpha+1)/(2\alpha+1)} \quad (17)$$

$$\rho(\alpha) = \sqrt{(1+\alpha)/\alpha}. \quad (18)$$

In the form (16) the operator L_α has been studied previously in [13]. The singular functions for L_α are indeed the Hermite functions, and the expression for the singular values of L_α given in [13],

$$\mu_k = \left(\rho/\left(\gamma + \sqrt{\rho^2+\gamma^2}\right)\right)^{2k+1},$$

does indeed lead to the eigenvalues (15) for P_F, when γ, ρ are replaced by their expressions (17), (18) in function of α.

Note that L_α, and hence P_F, might seem slightly less general than the Gaussian operators in [13], in which the parameters γ and ρ are independent. The present approach, with only one parameter (i.e., α), can, however, easily be extended to a two-parameter situation by the use of a dilation argument (see also Section V). The fact that the Hermite functions are the singular functions of L_α, as already pointed out in [13], thus finds a natural geometric explanation in our present approach.

B. Localization on Disks in the Time–Frequency Plane

In this case we revert to our original interpretation of the operator P_S, corresponding to a projection onto the subset S of phase space. To fit into the rotational symmetry of the time–frequency plane, the function F_S (defined by $F_S(p,q)=1$ if $(p,q)\in S$, 0 if $(p,q)\notin S$) must satisfy (9). This condition holds in the special case where S is a disk in the time–frequency plane, centered around the origin,

$$S_R = \{(p,q)\in\mathbb{R}^2,\ p^2+q^2\le R^2\}.$$

The operator $P_R = P_{S_R}$, corresponding to the projection onto the disk S_R in phase space, satisfies all the conditions of the theorem above. Its eigenfunctions are therefore the Hermite functions, independently of R,

$$P_R H_k = \lambda_k(R) H_k.$$

Applying formula (11) we find that the eigenvalues of P_R are given by incomplete gamma functions [19]

$$\lambda_k(R) = \frac{1}{k!}\gamma(k+1, R^2/2) \quad (19a)$$

$$= \frac{1}{k!}\int_0^{R^2/2} ds\, s^k e^{-s} \quad (19b)$$

$$= 1 - e^{-R^2/2}\sum_{j=0}^{k}\frac{1}{j!}2^{-j}R^{2j}. \quad (19c)$$

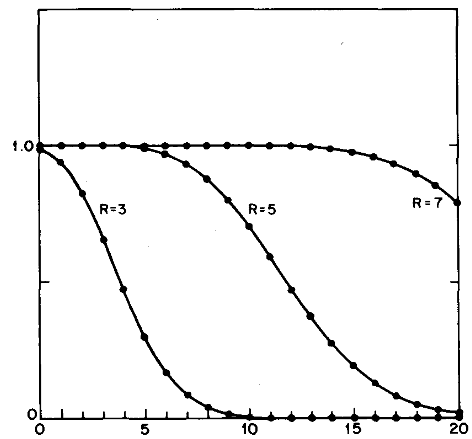

Fig. 1. Eigenvalues $\lambda_n(R)$, $n = 0, 1, \cdots, 30$ for $R=3$, $R=5$, and $R=7$.

Fig. 1 shows a plot of $\lambda_k(R)$, in function of k, for different values of R. For low values of R, the eigenvalues $\lambda_k(R)$ are close to one; near an R-dependent threshold value for k, they plunge to zero, and for higher values of k they stay close to zero. Using various asymptotic expansions for incomplete γ-functions [19], [20] one can make these statements more precise. One finds the following.

1) *Asymptotic behavior of the* $\lambda_k(R)$ *(*k *fixed):*

$$\lambda_k(R) \underset{R\to 0}{\sim} \frac{1}{(k+1)!}2^{-(k+1)}R^{2(k+1)}(1+O(R^2))$$

$$\lambda_k(R) \underset{R\to\infty}{\sim} 1 - \frac{1}{k!}R^{2k}2^{-k}e^{-R^2/2}(1+O(R^{-2})).$$

(these can easily be deduced from (19b), (19c)).

2) *Threshold value for* k*, for large* R*:*

$$\lambda_k(R) \ge 1/2 \leftrightarrow k \le R^2/2 + O(1). \quad (20)$$

3) *Width of the "plunge" region:* For all $\epsilon \in [0,1/2)$, define y_ϵ by

$$\operatorname{erf} y_\epsilon = \int_0^{y_\epsilon} dt\, e^{-t^2} = \sqrt{\pi}\left(\frac{1}{2} - \epsilon\right).$$

Then

$$\#\{k; \epsilon_1 \le \lambda_k(R) < 1 - \epsilon_2\} = (y_{\epsilon_1} + y_{\epsilon_2})(R + y_{\epsilon_1} - y_{\epsilon_2}) + O(1) \quad (21)$$

(where $\#V$ denotes the number of elements of the set V). Formulas (20) and (21) follow from the following asymptotic formula due to Tricomi [20]:

$$\gamma(\alpha+1, \alpha+\sqrt{2\alpha}\,y) = \Gamma(1+\alpha)\left[\frac{1}{2} + \frac{1}{\sqrt{\pi}} \operatorname{erf} y + O(\alpha^{-1/2})\right].$$

The behavior of the eigenvalues $\lambda_k(R)$, as described, is qualitatively very similar to that of the singular values of $L_{T,W}$ which correspond to the prolate spheroidal wave functions. Note also that the number of eigenvalues larger than 1/2, viz., $R^2/2$, is exactly equal to the area of S_R, πR^2, multiplied by the Nyquist density $(2\pi)^{-1}$. This is again similar to what happens for $L_{T,W}$ where the number of singular values larger than 1/2 is given by $2TW/\pi = (2\pi)^{-1}$ area of $[-T,T]\times[-W,W]$. All these similarities are hardly surprising since the operators $\boldsymbol{P}_R$ and $L_{T,W}$ both prescribe a "projection" onto a subset of phase space, in one case the disk S, in the other the rectangle $[-T,T] \times [-W,W]$. However, because of the distinctly different construction, there are a few significant differences, and we end this section with a short discussion of them.

First, note that the eigenfunctions of $\boldsymbol{P}_R$ do not depend on R; the R-dependence is completely contained in the eigenvalues. This is not the case for the prolate spheroidal wave functions $\psi_{T,W;k}$ associated with $L_{T,W}$. It is well-known that, apart from the scaling factor $(T/W)^{1/2}$, the $\psi_{T,W,k}$ depend on T, W via the product TW,

$$\psi_{T,W,k}(x) = (T/W)^{1/4}\psi_{\sqrt{TW},\sqrt{TW},k}(\sqrt{TW}\,x).$$

To make the fairest possible comparison with $\boldsymbol{P}_R$, one should choose units such that $T = W$. (If $T \ne W$, one should compare $L_{T,W}$ to a localization operator singling out an elliptical rather than a circular subset of phase space; see Section V.) In this case the scale factor $(T/W)^{1/2} = 1$, but the singular functions $\psi_{T,T,k}$ still depend on T.

A comparison between $L_{T,T}$ and $\boldsymbol{P}_R$ leads to the following heuristic asymptotic argument. Since the differences between the two operators $L_{T,T}$ and $\boldsymbol{P}_R$ concern mainly boundary effects (difference in the tails, and the difference between a square and a circular boundary), one expects that the effect of these differences on the kth singular function, k fixed, would tend to disappear as R, T tend to infinity. Since the eigenfunctions of $\boldsymbol{P}_R$ do not depend on R, this means that one expects

$$\psi_{T,T,k} \xrightarrow[T\to\infty]{} H_k.$$

It is well-known that this is indeed the case [21].

Apart from being independent of R, the eigenfunctions H_k of $\boldsymbol{P}_R$ also have the virtue of being extremely easy to calculate. As for any family of orthogonal polynomials, there exists a simple recursion formula, in this case a three-term recursion, which can be used to calculate the $H_k(x)$ even for very large values of k. Similarly, the eigenvalues $\lambda_k(R)$ are given by a very simple formula. In both theory and practice, the algorithms for calculating prolate spheroidal wave functions and their associated singular values are much more complicated. On the other hand, as has been proved in [2], [3], the prolate spheroidal wave functions are optimal for the analysis of bandlimited signals. However, for some applications the greater simplicity of calculation offered by $\boldsymbol{P}_R$ (as contrasted with $L_{T,W}$) and the Hermite functions and associated eigenvalues might outweigh the loss of optimality suffered from choosing a different set of basic analyzing functions.

V. Some Generalizations

A. Domains Not Centered Around the Origin

If the function F has spherical symmetry around a point (p^0, q^0) in phase space other than the origin, say

$$F(p,q) = \mathscr{F}\left((p_1 - p_1^0)^2 + (q_1 - q_1^0)^2, \cdots, (p_n - p_n^0)^2 + (q_n - q_n^0)^2\right),$$

then the spectral analysis of $\boldsymbol{P}_F$ is essentially the same, up to a shift in phase. The eigenvalues of $\boldsymbol{P}_F$ are still given by formula (11) and are thus independent of p^0, q^0. The eigenvectors of $\boldsymbol{P}_F$ are phase space shifted Hermite functions

$$H_{[k]}^{p_0,q_0}(x) = e^{ip_0\cdot x} H_{[k]}(x - q_0).$$

B. Elliptical Rather than Spherical Symmetry

The analysis of Section IV can be adapted to situations where ellipses rather than disks in time–frequency plane are singled out, as in chirped radar. This is done by using a simple dilation argument. Let us illustrate this for the case $n = 1$, and for the case where the set S is the ellipse

$$S_{\alpha,R} = \{(p,q) \in \mathbb{R}^2; \alpha^2 q^2 + \alpha^{-2}p^2 \le R^2\}$$

where $\alpha \ne 0$.

In that case we introduce the dilation operator B_α,

$$(B_\alpha f)(x) = \sqrt{\alpha}\, f(\alpha x).$$

Define also

$$\phi^{(0,\alpha)}(x) = (B_\alpha \phi^0)(x) = \sqrt{\alpha}\,\pi^{-1/4}\exp(-\alpha^2 x^2/2)$$

$$\phi_{p,q}^{(0,\alpha)}(x) = e^{ipx}\phi^{(0,\alpha)}(x - q).$$

One easily checks that

$$\phi_{p,q}^{(0,\alpha)} = B_\alpha \phi_{p/\alpha,\alpha q}^{0}. \tag{22}$$

Let us now define the phase space localization operator "projecting" onto $S_{\alpha,R}$ by

$$P_R^\alpha f = \frac{1}{2\pi} \int_{(p,q)\in S_{\alpha,R}} dp \int dq\ \phi_{p,q}^{(0,\alpha)} \left(\phi_{p,q}^{(0,\alpha)}, f\right).$$

Since $\phi_{p,q}^{(0,\alpha)}$ is centered, in phase space, around (p,q), the heuristical arguments of Section II still apply; P_R^α does describe a phase space localization operator singling out the phase space subset $S_{\alpha,R}$. The "tails" associated to P_R^α (in the sense given to them in Section II) are not isotropic in phase space, however.

From (22) one finds that

$$\begin{aligned} B_\alpha^{-1} P_R^\alpha B_\alpha f &= \frac{1}{2\pi} \int_{(p,q)\in S_{\alpha,R}} dp \int dq\ \phi_{p/\alpha,\alpha q}^{0} \left(\phi_{p/\alpha,\alpha q}^{0}, f\right) \\ &= \frac{1}{2\pi} \int_{p'^2+q'^2\le R^2} dp' \int dq'\ \phi_{p',q'}^{0} \left(\phi_{p',q'}^{0}, f\right) \\ &= P_R f. \end{aligned}$$

Hence the eigenvalues of P_R^α are the same as those of P_R, i.e., the $\lambda_k(R)$ given by (19), and the eigenvectors of P_R^α are dilated Hermite functions,

$$\begin{aligned} H_k^\alpha(x) &= \left(B_\alpha^{-1} H_k\right)(x), \qquad B_\alpha^{-1} = B_{1/\alpha} \\ &= \frac{1}{\sqrt{\alpha}} H_k\left(\frac{x}{\alpha}\right). \end{aligned}$$

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