Time-Periodic Solutions of the Einstein's Field Equations II

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In this paper, we construct several kinds of new time-periodic solutions of the vacuum Einstein's field equations whose Riemann curvature tensors vanish, keep finite or take the infinity at some points in these space-times, respectively. The singularities of these new time-periodic solutions are investigated and some new physical phenomena are found. The applications of these solutions in modern cosmology and general relativity can be expected.

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1. Introduction. The Einstein's field equations are the fundamental equations in general relativity and play an essential role in cosmology. This paper concerns the time-periodic solutions of the following vacuum Einstein's field equations

$$G_{\mu\nu} \triangleq R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0, \tag{1}$$

or equivalently,

$$R_{\mu\nu} = 0, \tag{2}$$

where $g_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$ is the unknown Lorentzian metric, $R_{\mu\nu}$ is the Ricci curvature tensor, R is the scalar curvature and $G_{\mu\nu}$ is the Einstein tensor.

It is well known that the exact solutions of the Einstein's field equations play a crucial role in general relativity and cosmology. Typical examples are the Schwarzschild solution and Kerr solution. Although many interesting and important solutions have been obtained (see, e.g., [1] and [5]), there are still many fundamental open problems. One such problem is *if there exists a "time-periodic" solution, which contains physical singularities such as black hole, to the Einstein's field equations.* This paper continues the discussion of this problem.

The first time-periodic solution of the vacuum Einstein's field equations was constructed by the first two authors in [3]. The solution presented in [3] is time-periodic, and describes a regular space-time, which has vanishing Riemann curvature tensor but is inhomogenous, anisotropic and not asymptotically flat. In particular, this space-time does not contain any essential singularity, but contains some non-essential singularities which correspond to steady event horizons, time-periodic event horizon and has some interesting new physical phenomena.

In this paper, we focus on finding the time-periodic solutions, which contain physical singularities such as black hole to the vacuum Einstein's field equations (1). We shall construct three kinds of new time-periodic solutions of the vacuum Einstein's field equations (1) whose Riemann curvature tensors vanish, keep finite or go to the infinity at some points in these space-times respectively. The singularities of these new time-periodic solutions are investigated and new physical phenomena are found. Moreover, the applications of these solutions in modern cosmology and general relativity may be expected. In the forthcoming paper [4], we shall construct a time-periodic solution of the Einstein's field equations with black hole, which describes the time-periodic cosmology with many new and interesting physical phenomena.

2. Procedure of finding new solutions.

We consider the metric of the following form

$$(g_{\mu\nu}) = \begin{pmatrix} u & v & p & 0 \\ v & 0 & 0 & 0 \\ p & 0 & f & 0 \\ 0 & 0 & 0 & h \end{pmatrix}, \tag{3}$$

where u, v, p, f and h are smooth functions of the coordinates (t, x, y, z). It is easy to verify that the determinant of $(g_{\mu\nu})$ is given by

$$g \triangleq \det(g_{\mu\nu}) = -v^2 fh. \tag{4}$$

Throughout this paper, we assume that

$$g < 0. \tag{H}$$

Without loss of generality, we may suppose that f and g keep the same sign, for example,

$$f < 0 \ \ (resp.\ f > 0) \quad \text{and} \quad h < 0 \ \ (resp.\ g > 0). \tag{5}$$

In what follows, we solve the Einstein's field equations (2) under the framework of the Lorentzian metric of the form (3).

By a direct calculation, we have the Ricci tensor

$$\begin{aligned} R_{11} \ = \ -\frac{1}{2}\Bigg\{ & \frac{v_x}{v}\left(\frac{f_x}{f}+\frac{h_x}{h}\right)+ \\ & \frac{1}{2}\left[\left(\frac{f_x}{f}\right)^2+\left(\frac{h_x}{h}\right)^2\right]-\left(\frac{f_{xx}}{f}+\frac{h_{xx}}{h}\right)\Bigg\}. \end{aligned} \tag{6}$$

It follows from (2) that

$$\frac{v_x}{v}\left(\frac{f_x}{f}+\frac{h_x}{h}\right)+\frac{1}{2}\left[\left(\frac{f_x}{f}\right)^2+\left(\frac{h_x}{h}\right)^2\right]-\left(\frac{f_{xx}}{f}+\frac{h_{xx}}{h}\right)=0. \tag{7}$$

This is an ordinary differential equation of first order on the unknown function v. Solving (7) gives

$$v = V(t, y, z)\exp\left\{\int \Theta(t, x, y, z)dx\right\}, \tag{8}$$

where

$$\Theta = \left[\frac{f_{xx}}{f}+\frac{h_{xx}}{h}-\frac{1}{2}\left(\frac{f_x}{f}\right)^2-\frac{1}{2}\left(\frac{h_x}{h}\right)^2\right]\frac{fh}{(fh)_x},$$

and $V = V(t, y, z)$ is an integral function depending on t, y and z. Here we assume that

$$(fh)_x \neq 0. \tag{9}$$

In particular, taking the ansatz

$$f = -K(t, x)^2, \quad h = N(t, y, z)K(t, x)^2 \tag{10}$$

and substituting it into (8) yields

$$v = VK_x. \tag{11}$$

By the assumptions (H) and (9), we have

$$V \neq 0, \quad K \neq 0, \quad K_x \neq 0. \tag{12}$$

Noting (10) and (11), by a direct calculation we obtain

$$R_{13} = -\frac{V_z K_x}{KV}. \tag{13}$$

It follows from (2) that

$$R_{13} = 0.$$

Combining (12) and (13) gives

$$V_z = 0. \tag{14}$$

This implies that the function V depends only on t, y but is independent of x and z. Noting (10)-(11) and using (14), we calculate

$$R_{12} = -\frac{1}{2V}\left(\frac{p_{xx}}{K_x}-\frac{K_{xx}p_x}{K_x^2}-\frac{2pK_x}{K^2}+\frac{2K_xV_y}{K}\right). \tag{15}$$

Solving p from the equation $R_{12} = 0$ yields

$$p = AK^2 + V_yK + \frac{B}{K}, \tag{16}$$

where A and B are integral functions depending on t, y and z. Noting (10)-(11) and using (14) and (16), we observe that the equation $R_{23} = 0$ is equivalent to

$$B_z - 2K^3A_z = 0. \tag{17}$$

Since K is a function depending only on t, x, and A, B are functions depending on t, y and z, we can obtain that

$$B = 2K^3A + C(t, x, y), \tag{18}$$

where C is an integral function depending on t, x and y. For simplicity, we take

$$A = B = C = 0. \tag{19}$$

Thus, (16) simplifies to

$$p = V_y K. \tag{20}$$

From now on, we assume that the function N only depends on y, that is to say,

$$N = N(y). \tag{21}$$

Substituting (10)-(11), (14) and (20)-(21) into the equation $R_{02} = 0$ yields

$$u_x V_y + V(u_{yx} - 4V_y K_{xt}) = 0. \tag{22}$$

Solving u from the equation (22) leads to

$$u = 2K_t V. \tag{23}$$

Noting (10)-(11), (14), (20)-(21) and (23), by a direct calculation we obtain

$$R_{03} = 0, \tag{24}$$

$$\begin{cases} R_{22} = (4N^2V^2)^{-1}\left[2NV^2N_{yy} - 4N^2VV_{yy} + 4N^2V_y^2 - 2NVN_yV_y - V^2N_y^2\right], \\ R_{33} = -(4NV^2)^{-1}\left[2NV^2N_{yy} - 4N^2VV_{yy} + 4N^2V_y^2 - 2NVN_yV_y - V^2N_y^2\right] \end{cases} \tag{25}$$

and

$$R_{00} = (2KNV^2)^{-1}\left[4NV_tV_y^2 + 2NV^2V_{tyy} - 2NVV_tV_{yy} - 4NVV_yV_{ty} - VN_yV_tV_y + V^2N_yV_{ty}\right]. \tag{26}$$

Therefore, under the assumptions mentioned above, the Einstein's field equations (2) are reduced to

$$-\frac{N_{yy}}{N} + \frac{1}{2}\left(\frac{N_y}{N}\right)^2 + 2\frac{V_{yy}}{V} + \frac{N_yV_y}{NV} - 2\left(\frac{V_y}{V}\right)^2 = 0 \tag{27}$$

and

$$4V_y^2V_t + 2V^2V_{yyt} - 2VV_{yy}V_t - 4VV_yV_{yt} - \frac{VV_yV_tN_y}{N} + \frac{V^2V_{yt}N_y}{N} = 0. \tag{28}$$

On the other hand, (27) can be rewritten as

$$2\left(\frac{V_y}{V}\right)_y + \frac{V_yN_y}{VN} - \left(\frac{N_y}{N}\right)_y - \frac{1}{2}\left(\frac{N_y}{N}\right)^2 = 0 \tag{29}$$

and (28) is equivalent to

$$2\left(\frac{V_y}{V}\right)_{yt} + \left(\frac{V_y}{V}\right)_t \frac{N_y}{N} = 0. \tag{30}$$

Noting (21) and differentiating (29) with respect to t gives (30) directly. This shows that (29) implies (30). Hence in the present situation, the Einstein's field equations (2) are essentially (29). Solving V from the equation (29) yields

$$V = w(t)|N(y)|^{1/2}\exp\left\{q(t)\int |N(y)|^{-1/2}dy\right\}, \tag{31}$$

where $w = w(t)$ and $q = q(t)$ are two integral functions only depending on t. Thus, we can obtain the following solution of the vacuum Einstein's field equations in the coordinates (t, x, y, z)

$$ds^2 = (dt, dx, dy, dz)(g_{\mu\nu})(dt, dx, dy, dz)^T, \tag{32}$$

where

$$(g_{\mu\nu}) = \begin{pmatrix} 2K_tV & K_xV & KV_y & 0 \\ K_xV & 0 & 0 & 0 \\ KV_y & 0 & -K^2 & 0 \\ 0 & 0 & 0 & NK^2 \end{pmatrix}, \tag{33}$$

in which $N = N(y)$ is an arbitrary function of y, $K = K(t, x)$ is an arbitrary function of t, x, and V is given by (31).

By calculations, the Riemann curvature tensor reads

$$R_{\alpha\beta\mu\nu} = 0, \quad \forall\, \alpha\beta\mu\nu \neq 0202 \text{ or } 0303, \tag{34}$$

while

$$R_{0202} = Kwqq'|N|^{-1/2}\exp\left\{q\int |N|^{-1/2}dy\right\} \tag{35}$$

and

$$R_{0303} = Kwqq'|N|^{1/2}\exp\left\{q\int |N|^{-1/2}dy\right\}. \tag{36}$$

3. Time-periodic solutions. This section is devoted to constructing some new time-periodic solutions of the vacuum Einstein's field equations.

3.1 Regular time-periodic space-times with vanishing Riemann curvature tensor. Take $q = constant$ and let $V = \rho(t)\kappa(y)$, where κ is defined by

$$\kappa(y) = c_1\sqrt{|N|}\exp\left\{c_2\int |N|^{-1/2}dy\right\}, \tag{37}$$

in which c_1 and c_2 are two integrable constants. In this case, the solution to the vacuum Einstein's filed equations in the coordinates (t, x, y, z) reads

$$ds^2 = (dt, dx, dy, dz)(g_{\mu\nu})(dt, dx, dy, dz)^T, \tag{38}$$

where

$$(g_{\mu\nu}) = \begin{pmatrix} 2\rho\kappa\partial_t K & \rho\kappa\partial_x K & \rho K\partial_y\kappa & 0 \\ \rho\kappa\partial_x K & 0 & 0 & 0 \\ \rho K\partial_y\kappa & 0 & -K^2 & 0 \\ 0 & 0 & 0 & NK^2 \end{pmatrix}. \tag{39}$$

Theorem 1 *The vacuum Einstein's filed equations (2) have a solution described by (38) and (39), and the Riemann curvature tensor of this solution vanishes.* ■

As an example, let

$$\begin{cases} w(t) = \cos t, \\ q(t) = 0, \\ K(t, x) = e^x \sin t, \\ N(y) = -(2 + \sin y)^2. \end{cases} \tag{40}$$

In the present situation, we obtain the following solution of the vacuum Einstein's filed equations (2)

$$(\eta_{\mu\nu}) = \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & 0 \\ \eta_{01} & 0 & 0 & 0 \\ \eta_{02} & 0 & \eta_{22} & 0 \\ 0 & 0 & 0 & \eta_{33} \end{pmatrix}, \tag{41}$$

where

$$\begin{cases} \eta_{00} = 2e^x(2 + \sin y)\cos^2 t, \\ \eta_{01} = \dfrac{1}{2}e^x(2 + \sin y)\sin(2t), \\ \eta_{02} = \dfrac{1}{2}e^x \cos y \sin(2t), \\ \eta_{22} = -[e^x \sin t]^2, \\ \eta_{33} = -[e^x(2 + \sin y)\sin t]^2. \end{cases} \tag{42}$$

By (4),

$$\eta \triangleq \det(\eta_{\mu\nu}) = -\frac{1}{4}e^{6x}(2 + \sin y)^4 \sin^4 t \sin^2(2t). \tag{43}$$

Property 1 The solution (41) of the vacuum Einstein's filed equations (2) is time-periodic. ■

Proof. In fact, the first equality in (42) implies that

$$\eta_{00} > 0 \quad \text{for} \ \ t \neq k\pi + \pi/2 \ \ (k \in \mathbb{N}) \ \ \text{and} \ \ x \neq -\infty.$$

On the other hand, by direct calculations,

$$\begin{vmatrix} \eta_{00} & \eta_{01} \\ \eta_{01} & 0 \end{vmatrix} = -\frac{1}{4}e^{2x}(2 + \sin y)^2 \sin^2(2t) < 0,$$

$$\begin{vmatrix} \eta_{00} & \eta_{01} & \eta_{02} \\ \eta_{01} & 0 & 0 \\ \eta_{02} & 0 & \eta_{22} \end{vmatrix} = -\eta_{01}^2\eta_{22} > 0$$

and

$$\begin{vmatrix} \eta_{00} & \eta_{01} & \eta_{02} & 0 \\ \eta_{01} & 0 & 0 & 0 \\ \eta_{02} & 0 & \eta_{22} & 0 \\ 0 & 0 & 0 & \eta_{33} \end{vmatrix} = -\eta_{01}^2\eta_{22}\eta_{33} < 0$$

for $t \neq k\pi, \ k\pi + \pi/2 \ \ (k \in \mathbb{N})$ and $x \neq -\infty$.

In Property 3 below, we will show that $t = k\pi, \ k\pi + \pi/2 \ \ (k \in \mathbb{N})$ are the singularities of the space-time described by (41), but they are not essential (or say, physical) singularities, these non-essential singularities correspond to the event horizons of the space-time described by (41) with (42); while, when $x = -\infty$, the space-time (41) degenerates to a point.

The above discussion implies that the variable t is a time coordinate. Therefore, it follows from (42) that the Lorentzian metric

$$ds^2 = (dt, dx, dy, dz)(\eta_{\mu\nu})(dt, dx, dy, dz)^T \tag{44}$$

is indeed a time-periodic solution of the vacuum Einstein's field equations (2), where $(\eta_{\mu\nu})$ is given by (41). This proves Property 1. □

Noting (34)-(36) and the second equality in (40) gives

Property 2 The Lorentzian metric (44) (in which $(\eta_{\mu\nu})$ is given by (41) and (42)) describes a regular space-time, this space-time is Riemannian flat, that is to say, its Riemann curvature tensor vanishes. ■

Remark 1 *The first time-periodic solution to the Einstein's field equations was constructed by Kong and Liu [3]. The time-periodic solution presented in [3] also has the vanishing Riemann curvature tensor.*

It follows from (43) that the hypersurfaces $t = k\pi$, $k\pi + \pi/2 \ \ (k \in \mathbb{N})$ and $x = \pm\infty$ are singularities of the space-time (44) (in which $(\eta_{\mu\nu})$ is given by (41) and (42)), however, by Property 2, these singularities are not physical (or say, not essential). According to the definition of event horizon (see e.g., Wald [6]), it is easy to show that the hypersurfaces $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$ and $x = +\infty$ are the event horizons of the space-time (44) (in which $(\eta_{\mu\nu})$ is given by (41) and (42)). Therefore, we have

Property 3 The Lorentzian metric (44) (in which $(\eta_{\mu\nu})$ is given by (41) and (42)) does not contain any essential singularity. These non-essential singularities consist of the hypersurfaces $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$ and $x = \pm\infty$. The singularities $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$ and $x = +\infty$ correspond to the event horizons, while, when $x = -\infty$, the space-time (44) degenerates to a point. ■

We now investigate the physical behavior of the space-time (44).

Fixing y and z, we get the induced metric

$$ds^2 = \eta_{00}dt^2 + 2\eta_{01}dtdx. \tag{45}$$

Consider the null curves in the (t, x)-plan, which are defined by

$$\eta_{00}dt^2 + 2\eta_{01}dtdx = 0. \tag{46}$$

Noting (42) gives

$$dt = 0 \quad \text{and} \quad \frac{dt}{dx} = -\tan t. \tag{47}$$

Thus, the null curves and light-cones are shown in Figure 1.

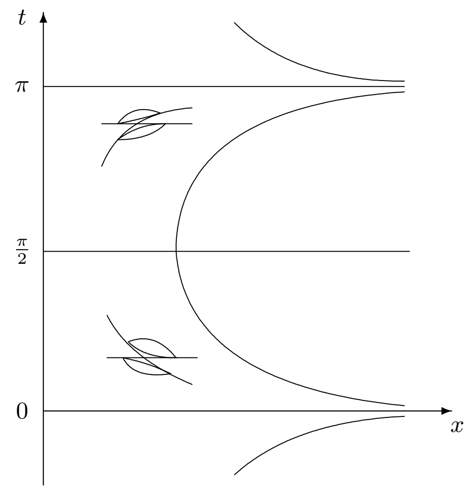

FIG. 1: Null curves and light-cones in the domains $0 < t < \pi/2$ and $\pi/2 < t < \pi$.

We next study the geometric behavior of the t-slices.

For any fixed $t \in \mathbb{R}$, it follows from (44) that the induced metric of the t-slice reads

$$\begin{aligned} ds^2 &= \eta_{22}dy^2 + \eta_{33}dz^2 \\ &= -e^{2x}\sin^2 t[dy^2 + (2 + \sin y)^2 dz^2]. \end{aligned} \tag{48}$$

When $t = k\pi$ $(k \in \mathbb{N})$, the metric (48) becomes

$$ds^2 = 0.$$

This implies that the t-slice reduces to a point. On the other hand, in the present situation, the metric (44) becomes

$$ds^2 = 2e^x(2 + \sin y)dt^2.$$

When $t \neq k\pi$ $(k \in \mathbb{N})$, (48) shows that the t-slice is a three-dimensional cone-like manifold centered at $x = -\infty$.

3.2 Regular time-periodic space-times with non-vanishing Riemann curvature tensor. We next construct the regular time-periodic space-times with non-vanishing Riemann curvature tensor.

To do so, let

$$\begin{cases} w(t) = \cos t, \\ q(t) = \sin t, \\ K(x,t) = e^x \sin t, \\ N = -\dfrac{1}{(2+\sin y)^2}. \end{cases} \tag{49}$$

Then, by (31),

$$V = \frac{\cos t \exp\{(2y - \cos y)\sin t\}}{2+\sin y}.$$

Thus, in the present situation, we have the following solution of the vacuum Einstein's field equations (2)

$$\widetilde{\eta}_{\mu\nu} = \begin{pmatrix} \widetilde{\eta}_{00} & \widetilde{\eta}_{01} & \widetilde{\eta}_{02} & 0 \\ \widetilde{\eta}_{01} & 0 & 0 & 0 \\ \widetilde{\eta}_{02} & 0 & \widetilde{\eta}_{22} & 0 \\ 0 & 0 & 0 & \widetilde{\eta}_{33} \end{pmatrix}, \tag{50}$$

where

$$\begin{cases} \widetilde{\eta}_{00} &= \dfrac{2e^x \cos^2 t \exp\{(2y-\cos y)\sin t\}}{2+\sin y}, \\ \widetilde{\eta}_{01} &= \dfrac{e^x \sin(2t)\exp\{(2y-\cos y)\sin t\}}{2(2+\sin y)}, \\ \widetilde{\eta}_{02} &= e^x \left\{ \sin t \cos t - \dfrac{\cos t \cos y}{(2+\sin y)^2} \right\} \sin t \\ & \quad \times \exp\{(2y-\cos y)\sin t\}, \\ \widetilde{\eta}_{22} &= -e^{2x}\sin^2 t, \\ \widetilde{\eta}_{33} &= -\dfrac{e^{2x}\sin^2 t}{(2+\sin y)^2}. \end{cases} \tag{51}$$

By (4),

$$\begin{aligned} \widetilde{\eta} &\triangleq \det(\widetilde{\eta}_{\mu\nu}) = -(\widetilde{\eta}_{01})^2 \widetilde{\eta}_{22}\widetilde{\eta}_{33} \\ &= -\frac{e^{6x+2(2y-\cos y)\sin t}\sin^2(2t)\sin^4 t}{4(2+\sin y)^4}. \end{aligned} \tag{52}$$

Introduce

$$\triangle(t,x,y) = 6x + 2(2y - \cos y)\sin t.$$

Thus, it follows from (52) that

$$\widetilde{\eta} < 0 \tag{53}$$

for $t \neq k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$ and $\triangle \neq -\infty$. It is obvious that the hypersurfaces $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$ and $\triangle = \pm\infty$ are the singularities of the space-time described by (50) with (51). As in Subsection 3.1, we can prove that the hypersurfaces $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$ are not essential (or say, physical) singularities, these non-essential singularities correspond to the event horizons of the space-time described by (50) with (51).

Similar to Property 1, we have

Property 4 The solution (50) (in which $(\widetilde{\eta}_{\mu\nu})$ is given by (51)) of the vacuum Einstein's filed equations (2) is time-periodic. ■

Similar to Property 2, we have

Property 5 The Lorentzian metric (50) (in which $(\widetilde{\eta}_{\mu\nu})$ is given by (51)) describes a regular space-time, this space-time has a non-vanishing Riemann curvature tensor. ■

Proof. In the present situation, by (34)

$$R_{\alpha\beta\mu\nu} = 0, \quad \forall\ \alpha\beta\mu\nu \neq 0202 \ \text{ or } \ 0303, \tag{54}$$

while

$$\begin{aligned} R_{0202} &= e^x(2+\sin y)\cos^2 t \sin^2 t \\ & \quad \times \exp\{(2y-\cos y)\sin t\}, \end{aligned} \tag{55}$$

and

$$R_{0303} = \frac{e^x \cos^2 t \sin^2 t \exp\{(2y-\cos y)\sin t\}}{2+\sin y}. \tag{56}$$

Property 5 follows from (54)-(56) directly. Thus the proof is completed. □

In particular, when $t \neq k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$, it follows from (55) and (56) that

$$R_{0202},\ R_{0303} \longrightarrow \infty \quad \text{as } \ x + (2y - \cos y)\sin t \to \infty. \tag{57}$$

However, a direct calculation gives

$$\mathbf{R} \triangleq R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} \equiv 0. \tag{58}$$

Thus, we obtain

Property 6 The Lorentzian metric (50) (in which $(\widetilde{\eta}_{\mu\nu})$ is given by (51)) does not contain any essential singularity. These non-essential singularities consist of the hypersurfaces $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$ and $\triangle = \pm\infty$, in which the hypersurfaces $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$ are the event horizons. Moreover, the Riemann curvature tensor satisfies the properties (57) and (58). ■

We next analyze the singularity behavior of $\triangle = \pm\infty$.

Case 1: Fixing $y \in \mathbb{R}$, we observe that

$$\triangle \to \pm\infty \Longleftrightarrow x \to \pm\infty.$$

This situation is similar to the case $x \to \pm\infty$ discussed in Subsection 3.1. That is to say, $x = +\infty$ corresponds to the event horizon, while, when $x \to -\infty$, the space-time (50) with (51) degenerates to a point.

Case 2: Fixing $x \in \mathbb{R}$, we observe that

$$\triangle \to \pm\infty \Longleftrightarrow y \to \pm\infty.$$

In the present situation, it holds that

$$t \neq k\pi\ (k \in \mathbb{N}).$$

Without loss of generality, we may assume that

$$\sin t > 0.$$

For the case that $\sin t < 0$, we have a similar discussion. Thus, noting (57), we have

$$R_{0202},\ R_{0303} \longrightarrow \infty \quad \text{as} \ \ y \to \infty.$$

Moreover, by the definition of the event horizon we can show that $y = +\infty$ is not a event horizon. On the other hand, when $y \to -\infty$, the space-time (50) with (51) degenerates to a point.

Case 3: For the situation that $x \to \pm\infty$ and $y \to \pm\infty$ simultaneously, we have a similar discussion, here we omit the details.

For the space-time (50) with (51), the null curves and light-cones are shown just as in Figure 1. On the other hand, for any fixed $t \in \mathbb{R}$, the induced metric of the t-slice reads

$$\begin{aligned} ds^2 &= \widetilde{\eta}_{22} dy^2 + \widetilde{\eta}_{33} dz^2 \\ &= -e^{2x} \sin^2 t [dy^2 + (2+\sin y)^{-2} dz^2]. \end{aligned} \tag{59}$$

Obviously, in the present situation, the t-slice possesses similar properties shown in the last paragraph in Subsection 3.1.

In particular, if we take (t, x, y, z) as the spherical coordinates (t, r, θ, φ) with $t \in \mathbb{R},\ r \in [0, \infty),\ \theta \in [0, 2\pi),\ \varphi \in [-\pi/2, \pi/2]$, then the metric (50) with (51) describes a regular time-periodic space-time with non-vanishing Riemann curvature tensor. This space-time does not contain any essential singularity, these non-essential singularities consist of the hypersurfaces $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$ which are the event horizons. The Riemann curvature tensor satisfies (58) and

$$R_{0202},\ R_{0303} \longrightarrow \infty \quad \text{as} \ \ r \to \infty.$$

Moreover, when $t \neq k\pi\ (k \in \mathbb{N})$, the t-slice is a three dimensional bugle-like manifold with the base at $x = 0$; while, when $t = k\pi\ (k \in \mathbb{N})$, the t-slice reduces to a point.

3.3 Time-periodic space-times with physical singularities. This subsection is devoted to constructing the time-periodic space-times with physical singularities.

To do so, let

$$\begin{cases} w(t) = \cos t, \\ q(t) = \sin t, \\ K(x,t) = \dfrac{\sin t}{x^2}, \\ N = -\dfrac{1}{(2+\cos y)^2}. \end{cases} \tag{60}$$

Then, by (31) we have

$$V = \frac{\cos t \exp\{(2y + \sin y)\sin t)\}}{2 + \cos y}.$$

Thus, in the present situation, the solution of the vacuum Einstein's field equations (2) in the coordinates (t, x, y, z) reads

$$ds^2 = (dt, dx, dy, dz)(\hat{\eta}_{\mu\nu})(dt, dx, dy, dz)^T, \tag{61}$$

where

$$(\hat{\eta}_{\mu\nu}) = \begin{pmatrix} \hat{\eta}_{00} & \hat{\eta}_{01} & \hat{\eta}_{02} & 0 \\ \hat{\eta}_{01} & 0 & 0 & 0 \\ \hat{\eta}_{02} & 0 & \hat{\eta}_{22} & 0 \\ 0 & 0 & 0 & \hat{\eta}_{33} \end{pmatrix}, \tag{62}$$

in which

$$\begin{cases} \hat{\eta}_{00} = \dfrac{2\cos^2 t \exp\{(\sin y + 2y)\sin t\}}{(2+\cos y)x^2}, \\ \hat{\eta}_{01} = -\dfrac{\sin(2t)\exp\{(\sin y + 2y)\sin t\}}{(2+\cos y)x^3}, \\ \hat{\eta}_{02} = \dfrac{\sin t}{x^2}\left\{\dfrac{\cos t \sin y}{(2+\cos y)^2} + \dfrac{\sin(2t)}{2}\right\}\times \\ \qquad \exp\{(\sin y + 2y)\sin t\}, \\ \hat{\eta}_{22} = -\dfrac{\sin^2 t}{x^4}, \\ \hat{\eta}_{33} = -\dfrac{\sin^2 t}{(2+\cos y)^2 x^4}. \end{cases} \tag{63}$$

By (4), we have

$$\begin{aligned} \hat{\eta} &\triangleq \det(\hat{\eta}_{\mu\nu}) = -(\hat{\eta}_{01})^2 \hat{\eta}_{22}\hat{\eta}_{33} \\ &= -\frac{e^{2(2y+\sin y)\sin t}\sin^2(2t)\sin^4 t}{x^{14}(2+\cos y)^4}. \end{aligned} \tag{64}$$

It follows from (63) that

$$\hat{\eta} < 0 \tag{65}$$

for $t \neq k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$ and $x \neq 0$. Obviously, the hypersurfaces $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$ and $x = 0$ are the singularities of the space-time described by (61) with (62)-(63). As before, we can prove that the hypersurfaces $t = k\pi,\ k\pi+\pi/2\ \ (k \in \mathbb{N})$ are not essential (or, say, physical) singularities, and these non-essential singularities correspond to the event horizons of the space-time described by (61) with (62)-(63), however $x = 0$ is an essential (or, say, physical) singularity (see Property 8 below).

Similar to Property 1, we have

Property 7 The solution (61) (in which $(\hat{\eta}_{\mu\nu})$ is given by (62) and (63)) of the vacuum Einstein's field equations (2) is time-periodic. ■

Proof. In fact, the first equality in (63) implies that

$$\hat{\eta}_{00} > 0 \quad \text{for} \quad t \neq k\pi + \pi/2 \quad (k \in \mathbb{N}) \quad \text{and} \quad x \neq 0. \tag{66}$$

On the other hand, by direct calculations we have

$$\begin{vmatrix} \hat{\eta}_{00} & \hat{\eta}_{01} \\ \hat{\eta}_{01} & 0 \end{vmatrix} = -\hat{\eta}_{01}^2 < 0, \tag{67}$$

$$\begin{vmatrix} \hat{\eta}_{00} & \hat{\eta}_{01} & \hat{\eta}_{02} \\ \hat{\eta}_{01} & 0 & 0 \\ \hat{\eta}_{02} & 0 & \hat{\eta}_{22} \end{vmatrix} = -\hat{\eta}_{01}^2\hat{\eta}_{22} > 0 \tag{68}$$

and

$$\begin{vmatrix} \hat{\eta}_{00} & \hat{\eta}_{01} & \hat{\eta}_{02} & 0 \\ \hat{\eta}_{01} & 0 & 0 & 0 \\ \hat{\eta}_{02} & 0 & \hat{\eta}_{22} & 0 \\ 0 & 0 & 0 & \hat{\eta}_{33} \end{vmatrix} = -\hat{\eta}_{01}^2\hat{\eta}_{22}\hat{\eta}_{33} < 0 \tag{69}$$

for $t \neq k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$ and $x \neq 0$.

The above discussion implies that the variable t is a time coordinate. Therefore, it follows from (63) that the Lorentzian metric (61) is indeed a time-periodic solution of the vacuum Einstein's field equations (2), where $(\hat{\eta}_{\mu\nu})$ is given by (63). This proves Property 7. □

Property 8 When $t \neq k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$, for any fixed $y \in \mathbb{R}$ it holds that

$$R_{0202} \to +\infty \quad \text{and} \quad R_{0303} \to +\infty, \quad \text{as } x \to 0. \tag{70}$$

■

Proof. By direct calculations, we obtain from (35) and (36) that

$$R_{0202} = \frac{(2+\cos y)\sin^2(2t)\exp\{(\sin y + 2y)\sin t\}}{4x^2}, \tag{71}$$

and

$$R_{0303} = \frac{\sin^2(2t)\exp\{(\sin(y) + 2y)\sin t\}}{4x^2(2+\cos y)}. \tag{72}$$

(70) follows from (71) and (72) directly. The proof is finished. □

On the other hand, a direct calculation yields

$$\mathbf{R} \triangleq R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} \equiv 0. \tag{73}$$

Therefore, we have

Property 9 The Lorentzian metric (61) describes a time-periodic space-time, this space-time contains two kinds of singularities: the hypersurfaces $t = k\pi,\ k\pi+\pi/2\ \ (k \in \mathbb{N})$, which are non-essential singularities and correspond to the event horizons, and $x = 0$, which is an essential (or, say, physical) singularity. ■

We now analyze the behavior of the singularities of the space-time characterized by (61) with (63).

By (64), we shall investigate the following cases: (a) $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$; (b) $y \to \pm\infty$; (c) $x \to \pm\infty$; (d) $x \to 0$.

Case a: $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$. According to the definition of the event horizon, the hypersurfaces $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$ are the event horizons of the space-time described by (61) with (63).

Case b: $y \to \pm\infty$. Noting (64), in this case we may assume that $t \neq k\pi$ $(k \in \mathbb{N})$ (if $t = k\pi$, then the situation becomes trivial). Without loss of generality, we may assume that $\sin t > 0$. Therefore, it follows from (71) and (72) that, for any fixed $x \neq 0$ it holds that

$$R_{0202},\ \ R_{0303} \longrightarrow \infty \quad \text{as}\ \ y \to +\infty \tag{74}$$

and

$$R_{0202},\ \ R_{0303} \longrightarrow 0 \quad \text{as}\ \ y \to -\infty. \tag{75}$$

(74) implies that $y = +\infty$ is also a essential singularity, while $y = -\infty$ is not because of (75).

Case c: $x \to \pm\infty$. By (63), in this case the space-time characterized by (61) reduces to a point.

Case d: $x \to 0$. Property 8 shows that $x = 0$ is a physical singularity. This is the biggest difference between the space-times presented in Subsections 3.1-3.2 and the one given this subsection. In order to illustrate its physical meaning, we take (t, x, y, z) as the spherical coordinates (t, r, θ, φ) with $t \in \mathbb{R},\ r \in [0, \infty),\ \theta \in [0, 2\pi),\ \varphi \in [-\pi/2, \pi/2]$. In the coordinates (t, r, θ, φ), the metric (61) with (63) describe a time-periodic space-time which possesses three kind of singularities:

(i) $t \neq k\pi$ $(k \in \mathbb{N})$: they are the event horizons;

(ii) $r \to +\infty$: the space-time degenerates to a point;

(iii) $r \to 0$: it is a physical singularity.

For the case (iii), in fact Property 8 shows that every point in the set

$$\mathfrak{S}_B \triangleq \{(t, r, \theta, \varphi) \,|\, r = 0,\ t \neq k\pi,\ k\pi + \pi/2\ (k \in \mathbb{N})\}$$

is a singular point. Noting (34) and (70), we name the set of singular points $\mathfrak{S}_B$ as a *quasi-black-hole*. Property 8 also shows that the space-time (61) is not homogenous and not asymptotically flat. This space-time perhaps has some new applications in cosmology due to the recent WMAP data, since the recent WMAP data show that our Universe exists anisotropy (see [2]). This inhomogenous property of the new space-time (61) may provide a way to give an explanation of this phenomena.

We next investigate the physical behavior of the space-time (61).

Fixing y and z, we get the induced metric

$$ds^2 = \hat{\eta}_{00} dt^2 + 2\hat{\eta}_{01} dt dx. \tag{76}$$

Consider the null curves in the (t, x)-plan defined by

$$\hat{\eta}_{00} dt^2 + 2\hat{\eta}_{01} dt dx = 0. \tag{77}$$

Noting (63) leads to

$$dt = 0 \quad \text{and} \quad \frac{dt}{dx} = -\frac{2\tan t}{x}. \tag{78}$$

Let

$$\rho = 2 \ln |x|. \tag{79}$$

Then the second equation in (78) becomes

$$\frac{dt}{d\rho} = -\tan t. \tag{80}$$

Thus, in the (t, ρ)-plan the null curves and light-cones are shown in Figure 1 in which x should be replaced by ρ.

We now study the geometric behavior of the t-slices.

For any fixed $t \in \mathbb{R}$, the induced metric of the t-slice reads

$$ds^2 = -\frac{\sin^2 t}{x^4}[dy^2 + (2 + \cos y)^{-2} dz^2]. \tag{81}$$

When $t = k\pi$ $(k \in \mathbb{N})$, the metric (81) becomes

$$ds^2 = 0.$$

This implies that the t-slice reduces to a point. On the other hand, in this case the metric (61) becomes

$$ds^2 = \frac{2}{(2+\cos y)x^2} dt^2.$$

When $t \neq k\pi$ ($k \in \mathbb{N}$), (81) shows that the t-slice is a three-dimensional manifold with cone-like singularities at $x = \infty$ and $x = -\infty$, respectively. In particular, if we take (t, x, y, z) as the spherical coordinates (t, r, θ, φ), then the induced metric (81) becomes

$$ds^2 = -\frac{\sin^2 t}{r^4}[d\theta^2 + (2+\cos\theta)^{-2} d\varphi^2]. \tag{82}$$

In this case the t-slice is a three-dimensional cone-like manifold centered at $r = \infty$.

At the end of this subsection, we would like to emphasize that the space-time (61) possesses a physical singularity, i.e., $x = 0$ which is named as a quasi-black-hole in this paper.

4. Summary and discussion. In this paper we describe a new method to find exact solutions of the Einstein's field equations (1). Using our method, we can construct many interesting exact solutions, in particular, the time-periodic solutions of the vacuum Einstein's field equations. More precisely, we have constructed three kinds of new time-periodic solutions of the vacuum Einstein's field equations: the regular time-periodic solution with vanishing Riemann curvature tensor, the regular time-periodic solution with finite Riemann curvature tensor and the time-periodic solution with physical singularities. We have also analyzed the singularities of these new time-periodic solutions and investigate some new physical phenomena enjoyed by these new space-times.

In particular, in the spherical coordinates (t, r, θ, φ) we construct a time-periodic space-time with essential singularities. This space-time possesses an interesting and important singularity which is named as a *quasi-black-hole*. This space-time is inhomogenous and not asymptotically flat and can perhaps be used to explain the phenomenon that our Universe exists anisotropy from the recent WMAP data (see [2]). We believe some applications of these new space-times in modern cosmology and general relativity can be expected.

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