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de Jong, F.C.J.M.

Publication date:
1997

Citation for published version (APA):
de Jong, F. C. J. M. (1997). *Time-series and cross section information in affine term structure models*. (Discussion papers / CentER for Economic Research; Vol. 1997-86). Econometrics.

CentER for Economic Research

Discussion paper

[redacted]

t time series
t information
t term structure of interest rates
t monetary models
t panel data

V C33
V E43

Tilburg University

Center
for
Economic Research

No. 9786

TIME-SERIES AND CROSS-SECTION INFORMATION IN AFFINE TERM STRUCTURE MODELS

By Frank de Jong

October 1997

ISSN 0924-7815

Time-series and Cross-section Information in Affine Term Structure Models

Frank de Jong*
Tilburg University

September 1997

Abstract

In this paper we provide an empirical analysis of the term structure of interest rates using the affine class of term structure models introduced by Duffie and Kan. We estimate these models by combining time-series and cross-section information in a theoretically consistent way. In the estimation we use an exact discretization of the continuous time factor process and allow for a general measurement error structure. We provide evidence that at least two correlated factors are necessary to describe the term structure. The generalized CIR specification is close to the most general two-factor affine model and is preferred over the Vasicek model. However, there is some evidence that a two factor affine model is misspecified.

Keywords: Term structure, Panel data, Kalman filter.
JEL codes: C33, E43

Please address correspondence to:
Frank de Jong
Department of Econometrics
Tilburg University
P.O. Box 90153
5000 LE, Tilburg, The Netherlands
phone: +31-13-4662911
fax: +31-13-4663280
e-mail: f.dejong@kub.nl

*The author thanks, without implicating, Michael Brennan, Tom Engsted, Jean-Philippe Lesne, Adrian Pagan, Peter Schotman and seminar participants at CREST, UCLA, the CEPR European Summer Symposium in Financial Markets 1996 in Gerzensee, the Econometric Society European Meeting 1996 in Istanbul, and the European Finance Association meeting 1997 in Vienna for useful comments.

1 Introduction

Good models for the term structure of interest rates are essential for the pricing of bonds and other interest rate derivatives, as well as for managing the risk of these financial assets. If we are concerned with derivative pricing, a perfect cross sectional fit of the observed bond prices is essential. This inspired the exogenous term structure models of Hull and White (1990) and Heath, Jarrow and Morton (1992). However, the calibrated coefficients of these models often change rapidly over time. If we are concerned with risk management, stable estimates of model parameters are required. Endogenous term structure models here provide a useful structure. A particularly tractable class of term structure models is proposed by Duffie and Kan (1996). In this class of models the interest rates on bonds of all maturities are linear (affine) functions of a small number of underlying factors. The dynamics of these factors are described by a generalized square root diffusion process. This class of models is able to capture many shapes of the yield curve and, depending on the number of factors, can describe different developments of the yield curve over time. The affine term structure model nests many well-known models, such as the one-factor Vasicek (1977) and Cox, Ingersoll and Ross (CIR, 1985) models, and the two-factor model of Longstaff and Schwartz (1992).

The affine term structure model consists of a dynamic model for the evolution of the factors and a model for bond prices (or yields) as function of the factors and the time to maturity. We will refer to the former as the *time series* dimension, and the latter as the *cross section* dimension of the model. Both dimensions of the model can be analysed separately, but there is a growing literature that estimates term structure models using panel data, i.e. combined cross section and time series data.[1] There are several advantages of using panel data. Firstly, the panel data appoach fully exploits the restrictions imposed by the term structure model is therefore expected to give more accurate estimates of the dynamics of the term structure. Secondly, combined use of time series and cross section data allows for identification of the market price of interest rate risk, which is not identified from each dimension separately. Of course, both points are only valid if the model is correctly specified. The panel data framework provides a natural specification test of the model by testing the restrictions imposed by the model on the parameters of the pricing equations (the cross section dimension) and the dynamic model for the factors (the time series equation).

The contribution of this paper is to analyze a more general model structure than is

[1]An undoubtedly incomplete list is Chen and Scott (1992), Pearson and Sun (1994), Lund (1994,1997), Duan and Simonato (1995), Frachot, Lesne and Renault (1995), Geyer and Pichler (1997), Santa-Clara (1995), Buraschi (1996), Pagan and Martin (1996), Babbs and Nowman (1997), and Bams and Schotman (1997).

employed in most previous papers. Typically, the models anaysed are multi-factor versions of the CIR model with mutually independent factors.[2] We shall allow for feedback among the factors. Moreover, we will allow for a volatility structure that nests the constant volatility (as in the Vasicek model) and the square root volatility model (as in the CIR and Longstaff and Schwartz models) and let the data decide on the best specification. We estimate one and two-factor versions of the affine model on US term structure data. In addition, we provide an extensive specification analysis of the estimated models to assess their ability to describe the cross section of bond prices and the dynamics of the yield curve.

Other contributions of this paper are on the econometric side. The model is specified in continuous time, whereas the data are observed at discrete points in time (monthly in our empirical work). In this paper we provide an exact discretization of the conditional mean and variance of the factors. Estimation is based on a subset of the available yields that covers the maturity spectrum. Typically, the dimension of the observations is higher than the number of factors. Therefore, the factors are treated as latent variables which are integrated out using the Kalman filter. Estimation is by Quasi Maximum Likelihood based on the conditional mean and variance of the process.

The setup of the paper is as follows. Section 2 describes the theoretical model. Section 3 discusses the empirical implementation of the model and gives a brief description of the data. Sections 4 and 5 discuss the empirical results for one-factor models and two-factor models, respectively. Section 6 concludes.

2 The affine class of term structure models

Duffie and Kan (1996) propose a class of affine term structure models in which zero-coupon bond prices are an exponential-affine function of a vector of factors, $F_t \in R^n$:

$$P_t(\tau) = \exp[-A(\tau) - B(\tau)'F_t]. \tag{1}$$

where τ denotes the time to maturity of the bond. Due to this form, the interest rates or yields on zero-coupon bonds are a linear function of the factors, where the intercept and factor loadings are time-invariant functions of the time to maturity

$$Y_t(\tau) \equiv -\ln P_t(\tau)/\tau = A(\tau)/\tau + B(\tau)'/\tau \cdot F_t. \tag{2}$$

[2]A notable exception is the work of Frachot, Lesne and Renault (1995), who estimate a general affine term structure model on French data using indirect inference techniques.

The underlying factors F_t are assumed to follow a diffusion process with a square-root type volatility structure

$$dF_t = \Lambda(F_t - \mu)dt + \Sigma \begin{pmatrix} \sqrt{\alpha_1' F_t + \beta_1}dW_{1t} \\ \cdot \\ \cdot \\ \sqrt{\alpha_n' F_t + \beta_n}dW_{nt} \end{pmatrix} \tag{3}$$

where W_{it} are independent Wiener processes under the 'real world' or empirical probability measure P.

The affine model contains several well-known models as special cases. The model of Langetieg (1980), which generalizes the Vasicek (1977) model to more dimensions, is obtained if $\alpha = 0$.[3] The generalized Cox, Ingersoll and Ross (1985) model is obtained if α is diagonal and $\beta = 0$. In the latter model all yields are guaranteed to be positive, see Pang and Hodges (1996). If, in addition, the mean reversion matrix Λ and the correlation matrix Σ are diagonal, the factors follow mutually independent stochastic processes and we obtain a two-factor CIR model which is observationally equivalent to the Longstaff and Schwartz (1992) model. Jegadeesh and Pennacchi (1996) propose a model where the short rate fluctuates around a stochastic mean. This model is also a special case of the affine class with a particular recursive structure for Λ.

In order to price bonds and other term structure derivatives some assumption about the market price of interest rate risk has to be made. Duffie and Kan (1996) assume that the market price of risk for factor i is proportional to its instantaneous standard deviation, $\psi_i\sqrt{\alpha_i' F_t + \beta_i}$. Under this assumption, an equivalent martingale measure Q can be constructed, under which the transformed innovation process $dW_{it}^* \equiv dW_{it} + \psi_i\sqrt{\alpha_i' F_t + \beta_i}dt$ is a Wiener process. The stochastic process for F_t under Q is given by

$$dF_t = \Lambda^*(F_t - \mu^*)dt + \Sigma \begin{pmatrix} \sqrt{\alpha_1' F_t + \beta_1}dW_{1t}^* \\ \cdot \\ \cdot \\ \sqrt{\alpha_n' F_t + \beta_n}dW_{nt}^* \end{pmatrix} \tag{4}$$

The 'risk-neutral' intercept and mean-reversion parameters are related to the parameters of the real world dynamics through

$$\Lambda^* = \Lambda - \Sigma \begin{pmatrix} \psi_1 \alpha_1' \\ \cdot \\ \cdot \\ \psi_n \alpha_n' \end{pmatrix}, \quad \Lambda^* \mu^* = \Lambda\mu + \Sigma \begin{pmatrix} \psi_1 \beta_1 \\ \cdot \\ \cdot \\ \psi_n \beta_n \end{pmatrix}. \tag{5}$$

[3]For notational convenience, α denotes a matrix with rows α_i' and β a vector with elements β_i.

Using no-arbitrage arguments, Duffie and Kan (1996) show that the coefficients $A(\tau)$ and $B(\tau)$ in the bond pricing equation (1) satisfy the system of ordinary differential equations

$$\frac{dA(\tau)}{d\tau} = A_0 - (\Lambda^* \mu^*)' B(\tau) - 1/2 \sum_i \sum_j B_i(\tau) B_j(\tau) b_{ij} \tag{6a}$$

$$\frac{dB(\tau)}{d\tau} = B_0 + (\Lambda^*)' B(\tau) - 1/2 \sum_i \sum_j B_i(\tau) B_j(\tau) a_{ij} \tag{6b}$$

where the vectors a_{ij} and the scalars b_{ij} are defined by $a_{ij}'x + b_{ij} \equiv [\Sigma(\alpha x + \beta)_d \Sigma']_{ij}$.[4]

The affine model has many parameters which cannot all be identified. Pang and Hodges (1996) show that any invertible rotation of the factors gives the same bond prices. This implies that without loss of generality we may assume that Σ equals the identity matrix. An equivalent, but sometimes more convenient, normalization is that Λ is diagonal and impose n restrictions on Σ. For example, one could assume that Σ, or Σ^{-1}, has diagonal elements equal to one. Since the factors contain an arbitrary scale factor, bond prices are invariant under scale transformations of the factors. Hence, without loss of generality we can normalize $B_0 = \iota$. This normalization also leads to simple numerical solutions of the differential equations (6a)–(6b). The final assumption is $A_0 = 0$, which is typically not restrictive, except in the multivariate CIR model. With these normalizations, the instantaneous interest rate r_t equals the sum of the factors $(r_t = \iota' F_t)$.

Dai and Singleton (1997) discuss some further identfication issues. In particular, they show that in an n factor model, there should be $n-1$ normalizations on the vectors μ and β. This implies that, for example, in the multivariate Vasicek model, where the elements of β are free paramaters, only one element of μ is identified. In the multivariate CIR model, $\beta = 0$ and hence all elements of μ are identified.[5]

A final identification issue arises if the volatility of some factors is constant. Dai and Singleton (1997) show that the model is invariant under certain 'unitary rotations'. For example, in the multivariate Vasicek model this implies that not all elements of Σ can be identified but only $n(n-1)/2$ elements.

3 Empirical implementation of the affine model

In this section we describe the empirical implementation of the affine term structure model. We focus on the the state space formulation, the discretization of the continuous

[4] The notation $(\alpha x + \beta)_d$ denotes a diagonal matrix with elements equal to $\alpha_i' x + \beta_i$.

[5] In the multivariate CIR model the assumption that $A_0 = 0$ is restrictive and could be relaxed. Following Pearson and Sun (1984), an intercept could be added to the bond price equations.

time dynamics, the measurement error structure, and the estimation. We also briefly discuss the data.

3.1 State space formulation

In the panel data framework, the dimension of the vector of observed interest rates is typically higher than the dimension of the factor. Let there be observations for maturities τ_1 through τ_k. Collect the observed yields for period t in the vector

$$y_t \equiv \begin{pmatrix} Y_t(\tau_1) \\ \cdot \\ \cdot \\ Y_t(\tau_k) \end{pmatrix}$$

Also, define the coefficient matrices

$$A \equiv \begin{pmatrix} A(\tau_1)/\tau_1 \\ \cdot \\ \cdot \\ A(\tau_k)/\tau_k \end{pmatrix}, \quad B \equiv \begin{pmatrix} B(\tau_1)'/\tau_1 \\ \cdot \\ \cdot \\ B(\tau_k)'/\tau_k \end{pmatrix}.$$

With these definitions the model is conveniently stated in state space form[6]

$$\begin{aligned} y_t &= A + BF_t && \text{(7a)} \\ F_{t+h} &= \mu + \Phi(F_t - \mu) + \nu_{t+h} && \text{(7b)} \end{aligned}$$

The second equation in this system is called the transition equation of the factors. It is the discrete time equivalent of equation (3). The parameters of the transition follow from the conditional mean and variance of the factors. Appendix A shows that the *exact* conditional mean and variance of F_{t+h} given F_t are given by [7]

$$\begin{aligned} E_t(F_{t+h}) &= \mu + \Phi(F_t - \mu) && \text{(8a)} \\ Var_t(F_{t+h}) &= q(F_t) && \text{(8b)} \end{aligned}$$

with $\Phi = e^{\Lambda h}$ and the elements of $q(F_t)$ given by

$$q_{ij}(F_t) = \frac{1 - e^{-(\kappa_i+\kappa_j)h}}{\kappa_i + \kappa_j}(a_{ij}'\mu + b_{ij}) + \sum_k \frac{e^{-\kappa_k h} - e^{-(\kappa_i+\kappa_j)h}}{\kappa_i + \kappa_j - \kappa_k} a_{ij,k}(F_t - \mu)_k$$

Just like the instantaneous variance, the conditional variance of discrete changes in the factors is an affine function of the current level of the factors.

[6] The data are observed at discrete intervals of length $h > 0$.

[7] The normalization imposed is that Λ is a diagonal matrix, with diagonal elements $(-\kappa_1, .., -\kappa_n)$.

The coefficients A and B of the measurement equation (9a) are functions of the parameters $(\Lambda^*, \mu^*, \Sigma, \alpha, \beta)$ of the risk-neutral factor process (4). In principle, these parameters could be estimated from a *cross-section* of interest rates of different maturity. This is the approach taken by Brown and Dybvig (1986) and Schotman and De Munnik (1994). Note that the market price of risk parameter, ψ, cannot be identified using cross-sectional data only.

The coefficients of the transition equation (7b) are functions of the parameters $(\Lambda, \mu, \Sigma, \alpha, \beta)$ of the real-world process (3). These parameters can be estimated from *time series* data on a particular maturity (or more maturities in a multi-factor model). This is the approach of Aït-Sahalia (1996), Broze, Scaillet and Zakoïan (1995), Chan, Karolyi, Longstaff and Sanders (1992), Conley, Hansen, Luttmer and Scheinkman (1997), Koedijk, Nissen, Schotman and Wolff (1997), and many others. Again, separate identification of the market price of risk is not possible using time-series data only.

The term structure model imposes several restrictions across the parameters of the measurement equation and the transition equation. In particular, (Λ^*, μ^*) are functions of the parameters $\theta = (\Lambda, \mu, \Sigma, \alpha, \beta, \psi)$ through equation (5). Joint estimation, in a panel data framework, of the pricing model and the transition model for the factors allows for identification of the market price of risk parameters, ψ, but also imposes testable restrictions on the parameters Λ, Σ, α and β. These restrictions provide a natural specification test of the affine model, similar to cross equation tests used in models of rational expectations.

3.2 Measurement error structure

The model predicts the exact relation $y_t = A + BF_t$ between the factors and the yields. Obviously, in an n factor model, observations on n maturities could be used to construct the factors by 'inverting' the model. This is the approach of Pearson and Sun (1994). A drawback of this procedure is that the results are potentially very sensitive to the particular choice of maturities. Moreover, the approach neglects useful information in other maturities. When using more maturities than factors the equality $y_t = A + BF_t$ cannot be satisfied by all maturities. Therefore, some form of measurement error is necessary. The important issue is which assumptions to make on the measurement error structure.

Chen and Scott (1992) estimate a model with two factors and four maturities. They assume that two yields are observed without error so that the model for these two maturities can be inverted to obtain the factors. The other yields, or linear combinations thereof, are assumed to be measured with a normally distributed measurement error. The estimation method is Maximum Likelihood.

A number of papers, e.g. Duan and Simonato (1995), Geyer and Pichler (1995), and Jegadeesh and Pennacchi (1996) assume that all interest rates are observed with some measurement error, which is both serially and cross-sectionally uncorrelated. Santa Clara (1995) has a similar econometric approach for estimating a Markovian model of the Heath, Jarrow and Morton (1992) type. Because of the measurement errors, the factors cannot be measured exactly. Instead, the factors are treated as latent variables and integrated out using a Kalman filter. The estimation method in these papers is Quasi Maximum Likelihood based on the prediction errors of the Kalman filter. Bams and Schotman (1997) follow a similar approach but allow for correlation between the errors for different maturities. The strenght of the correlation depends on the difference in maturity.

Frachot, Lesne and Renault (1995) point out that a diagonal error covariance matrix is not robust under linear transformations of the data. They propose to use a more general, non-diagonal, cross sectional correlation matrix for the measurement errors. In their own empirical work, Frachot et al. assume that some yields are observed without error. They also note that QML estimates based on the Kalman filter with an approximation to the exact conditional mean and variance of the factors may suffer from a discretization bias, and propose to use indirect inference methods to correct for this bias. Lund (1994) also argues that the measurement error structure should be unrestricted and estimates a variant of the generalized Vasicek model on Danish term structure data. Buraschi (1996) estimates a multifactor version of the CIR model using indirect inference methods. He allows for measurement errors on all observed variables and uses the Kalman filter to construct the auxiliary model in an indirect inference procedure.

Following these papers we assume that the measurement errors have zero mean, are serially uncorrelated, but may be cross sectionally correlated with time-invariant covariance matrix H. The most convenient way to parameterize H is as LDL', where L is lower triangular with ones on the diagonal and D is the diagonai matrix of eigenvalues. This form makes H positive definite by construction. This parameterization is more general than the assumptions made in many other papers and, more importantly, it makes the estimates of the parameters invariant to linear transformations of the yields. The full state-space form of the model with measurement errors is

$$y_t = A + BF_t + u_t, \qquad Var(u_t) = H \tag{9a}$$

$$F_{t+h} = \mu + \Phi(F_t - \mu) + \nu_{t+h}, \qquad Var(\nu_{t+h}) = q(F_t) \tag{9b}$$

In Appendix B the relation between this state space model and multivariate time series models is explored.

3.3 Estimation

Given the state space setup, the most convenient way to estimate the parameters is by Maximum Likelihood based on the Kalman filter. The relevant equations for the Kalman filter in the affine term structure model are given in Appendix C. The ML estimator is consistent and efficient if the factors and the error terms follow normal distributions, such as in the Vasicek (1977) and Langetieg (1980) models. In most affine term structure models the conditional distribution of the factors is not normal, but if the conditional mean and variance of the factors are correctly specified, one could expect the estimates obtained from the Kalman filter to be constistent by the Quasi Maximum Likelihood principle (see Gouriéroux, Montfort and Renault (1984)). There is one subtle problem with this argument which arises because the conditional variance of the factors depends on the current value of the factors, which are latent variables that can be estimated but not observed exactly. Therefore, the conditional variance used in the likelihood function will not be correct. An additional problem is that the conditional distribution of the factors is not normal, which invalidates the updating rules in the Kalman filter. As a result, the QML estimates obtained from the Kalman filter will be inconsistent, see Lund (1997) for an extensive discussion of this point. The inconsistency could be removed by applying the indirect inference methods of Gouriéroux, Montfort and Renault (1992) or the efficient method of moments of Gallant and Tauchen (1996).[8] On the other hand, the simulation results in Lund (1997) suggest that, for parameters typically found in estimates of term structure models, the bias in the QML estimator is not particularly large. Therefore, we refrain from using simulation-based estimation techniques and report the QML estimates.

Since we have a full error covariance matrix, the number of parameters to be estimated is potentially large. Lund (1994) proposes an EM algorithm for the optimization of the likelihood function that separates the parameters of the measurement error covariance matrix H from the model parameters. This appoach is attractive because it avoids the curse of dimensionality and can therefore deal with a large number of maturities in the vector of observations. However, we experienced problems with convergence of the EM algorithm. Therefore, we decided to limit the number of maturities used in estimation to 4, which leaves 10 free parameters in H to be estimated. Optimization is conducted in two steps by the BHHH algorithm. In the first step, the model is estimated with the error covariance matrix restricted to a constant times the identity matrix. The estimates from this step are used as starting values for the optimization with a full error covariance matrix.

[8]This is the approach followed by Frachot, Lesne and Renault (1995) and Pagan and Martin (1996).

Table 1: Descriptive statistics of the McCullogh and Kwon data

maturity	3 month	1 year	5 year	10 year
average	7.68	8.20	8.75	8.95
standard deviation	2.68	2.58	2.27	2.13
minumum	3.38	3.74	5.15	5.72
maximum	16.00	16.35	15.70	15.07

3.4 Data

Our database is the extended McCulloch dataset (McCulloch and Kwon, 1993). This dataset contains monthly observations on US interest rates with maturities running from 1 month to 30 years. The series starts in 1947 and ends in 1991. The data are zero-coupon rates which were calculated from prices of coupon bonds using McCulloch's interpolation method. From 1985 only bonds which do not have prepayment provisions are used, before that year the data may include such bonds. For the maturities over 10 years, the bond data is quite scarce, so the interpolation is not very accurate. Moreover, there are a lot of missing observations in the early part of the sample. A related problem shows up in the very short term interest rates. The one and two month rate series show some exceptionally large one-period changes. We feel more confident using maturities from 3 months and longer. As for the choice of maturities, to keep the estimation feasible we confine ourselves to four maturities: three months, one year, five years and ten years. We use a subsample of the data that starts in January 1970 and ends in February 1991. In total, there are 254 monthly observations. Figure 1 graphs the data and Table 1 gives some descriptive statistics. The long maturity interest rates are somewhat less variable than the short rates. Moreover, on average the term structure is upward sloping. The large volatilities of the interest rates around 1980 show up clearly. Since this is also a period with high levels of interest rates, the data give some intuitive support for models where the conditional variance depends on the level of the interest rates.

4 Empirical results for one-factor models

In this section we take a first shot at modeling the term structure by one-factor affine term structure models. The one-factor version of the affine model is very tractable because the differential equations (6a)–(6b) have analytical solutions for $A(\tau)$ and $B(\tau)$. In the one-factor model equations (3) and (4) specialize to

$$dF_t = \kappa(\mu - F_t)dt + \sqrt{\alpha F_t + \beta} dW_t \tag{10a}$$

$$dF_t = \kappa^*(\mu^* - F_t)dt + \sqrt{\alpha F_t + \beta}dW_t^* \tag{10b}$$

The functions $A(\tau)$ and $B(\tau)$ can be found from a straightforward generalization of the standard CIR equations which are given e.g. in Hull (1993):

$$\begin{aligned}
A(\tau) &= \frac{2\tilde{\phi}}{\alpha}\ln\left(\frac{2\gamma e^{\frac{(\kappa^*+\gamma)}{2}\tau}}{(\kappa^*+\gamma)(e^{\gamma}-1)+2\gamma}\right) - \frac{\beta}{\alpha}(\tau - B(\tau)) \\
B(\tau) &= \frac{2(e^{\gamma\tau}-1)}{(\kappa^*+\gamma)(e^{\gamma\tau}-1)+2\gamma}
\end{aligned}$$

$$\kappa^* = \kappa + \psi\alpha, \qquad \tilde{\phi} \equiv \kappa\left(\mu + \frac{\beta}{\alpha}\right), \qquad \gamma \equiv \sqrt{(\kappa^*)^2 + 2\alpha}.$$

For the special case of the Vasicek model ($\alpha = 0$) the coefficients are

$$\begin{aligned}
A(\tau) &= \theta(\tau - B(\tau)) + \frac{\beta}{4\kappa}B(\tau)^2 \\
B(\tau) &= \frac{1-\exp(-\kappa\tau)}{\kappa}
\end{aligned}$$

where $\theta \equiv \mu - \frac{\psi\beta}{\kappa} - \frac{\beta}{2\kappa^2}$ is the yield on infinite maturity bonds.

In Table 2 we report estimates of the one-factor affine model. Also, estimates of the one-factor CIR and Vasicek models, which are special cases of the affine yield curve model, are presented. The parameters estimated are the mean reversion coefficient κ, the long run mean of the factor μ, the variance parameters α and β, and the market price of risk parameter ψ.

The estimated mean reversion coefficients under the risk-neutral distribution are very small: κ^* in the affine model is 0.0014, which implies a half life of around 500 years.[9] The result in the CIR model is virtually the same, and in the Vasicek model the estimated half-life is around 30 years. This slow mean reversion implies very flat term structures, as graphed in Figure 2. Although the infinite maturity yield must be constant if κ^* is positive, the mean reversion is slow enough to create considerable movements in, say, 10 year rates.

The estimated intercept of the instantaneous variance is negative, and the 'slope' coefficient α is larger than the comparable estimate for the CIR model. The sensitivity of the conditional variance to the level of the short rate is therefore stronger in the affine model than in the CIR model. Time-series based studies have reported a similar phenomenon, see e.g. Chan et al. (1992). A negative intercept may be somewhat counter-intuitive and may threathen the existence of the model. We also estimated the affine model under the restriction that the intercept β is non-negative. In that case, β was

[9] The half-life of the factor is defined as $\ln(2)/\kappa^*$.

estimated very close to 0; the other parameter estimates were virtually identical the CIR estimates.

The estimates of the market price of risk are significantly negative and of the same order of magnitude in all specifications.[10] This result implies that the risk premium for holding long term bonds is positive. The estimated risk premium for a ten year bond is around 1.65% annually, which corresponds quite well with the observed risk premium.

Given the estimated parameters and factors,[11] we can construct in each month a fitted term structure for all maturities, also for maturities not used in the estimation. The differences between the observed and the fitted term structures (the *residuals*) provide important information on the cross sectional and dymamic fit of the model. Figure 3 graphs the average of the fitted and observed term structures, as well as the root mean squared error (RMSE) of the residuals. The figure shows that the model fits the long end pretty well, but fails to capture the short end of the yield curve: the RMSE is around 150 basis points for the three month rate. There is also substantial serial correlation in the residuals, with first order autocorrelation coefficients around 0.9 or higher.

Another way to judge the quality of the model is by regressing the observed yields on a constant and the estimated factors. The regression can be done in levels or in first differences. Either way, the regression coefficients should be more or less the same as the factor loadings obtained from the term stucture model. Figure 4 shows that this holds for the long maturities (over 5 years) the but for shorter maturities there are large differences between the estimated sensitivities and the model values.

A more formal way to test the specification of the model is by testing the restrictions the model imposes between the pricing equations (the cross section dimension) and the dynamics of the factors (the time series dimension). In the time series dimension the parameters $(\kappa, \mu, \alpha, \beta)$ can be identified. In the cross section dimension, we estimate a separate mean reversion parameter κ_c and variance parameters α_c and β_c. To enhance comparability among estimates, we assume that the long run mean of the factor, μ, in the cross section dimension is the same as in the time series dimension. This leaves the market price of risk ψ as a free parameter. The parameters Λ^* and ϕ^* of the risk-neutral distribution (10b) are calculated from the cross-sectional parameters $(\kappa_c, \mu, \alpha_c, \beta_c, \psi)$. Table 3 reports results for the one-factor model with separate coefficients for the cross-section and the time-series dimension. The restrictions between the time-series and cross section parameters are rejected by the Likelihood Ratio test.[12] The key to this rejection is that the time series estimates show a much stronger mean reversion and

[10]The risk premia are derived in Appendix D.

[11]Estimates of the factors are obtained from the Kalman filter.

[12]The test statistic is 17.76, which is larger than the 5% critical value of a chi-square distribution with 3 degrees of freedom.

higher instantaneous variance than the cross section estimates. To fit the rather flat shape of the observed yield curves a slow mean reversion is necessary, whereas in the time series dimension the mean reversion of interest rates is quite strong. The estimated mean reversion under the risk-neutral distribution is comparable to the previous estimates for the affine and Vasicek model. For the CIR model, the estimate of κ^* is negative, but the shape of the $B(\tau)$ curve is not very different from the other models for the maturities we consider.[13]

All these results point at substantial misspecification of the one-factor affine term structure model. The model fails to give a good fit of the term structure at the short end. Moreover, the dynamics of the yield curve are not well described as evidenced by the strong residual serial correlation and the differences in parameter estimates for the time series and cross section dimensions. We therefore now turn to multifactor models.

5 Empirical results for two-factor models

In this section we present estimates of affine term structure models with two factors. The normalization imposed is that Λ is diagonal and Σ^{-1} has diagonal elements equal to 1. The free parameters are the diagonal elements of Λ, denoted by κ, the off-diagonal elements of Σ^{-1}, denoted by x_{12} and x_{21}, and the elements of μ, α, β and ψ. [14] Table 4 reports the parameter estimates for the two-factor affine model and for two special cases, the generalized CIR and Vasicek models. It turned out that the off-diagonal elements of α were never significantly different from zero. The table therefore only reports results for the affine model with a diagonal α.

The results for the two-factor Vasicek model are not as good as the results of the affine and CIR models. The likelihood for this specification is much lower, and the variance of the measurement errors is typically larger. As a result, and similar to the one-factor case, the Vasicek specification is rejected against the more general affine specification. We therefore confine the discussion of the results to the affine and CIR models.

The estimation results show that there are two factors with very different properties. The mean reversion of the first factor is similar to the mean reversion in the one-factor model, with a half-life over 30 years. The second factor shows a much stronger mean reversion with a half-life of less than one year. There is a significant correlation between

[13]Strictly speaking, a negative mean reversion coefficient implies an explosive process for the instantaneous interest rate under the risk-neutral distribution, but the functions $A(\tau)$ and $B(\tau)$ are well-defined for such values.

[14]In the two-factor Vasicek model, only one element of the mean vector μ is identified, and the restriction $\mu_2 = 0$ is imposed. Similarly, in the affine model only one element of β is identified, and the restriction $\beta_2 = 0$ is imposed.

the factors, which is evident from the non-zero off-diagonal elements of Σ^{-1}. Indeed, the Likelihood Ratio tests reported in Table 6 strongly reject independence of the factors. One way to interpret this result is to write the model in 'feedback' form, with a mean reversion matrix given by

$$\Lambda = \Sigma^{-1} \begin{pmatrix} -\kappa_1 & 0 \\ 0 & -\kappa_2 \end{pmatrix} \Sigma$$

This representation is shown in the second panel of Table 4. It appears that the feedback from the second factor to the first is not very strong. One could think of the first factor as determining the level of the yield curve, whereas the second factor is related to the slope of the yield curve.[15] The graph of the estimated factors in Figure 5 supports this interpretation. The movements in the first factor appear to capture slow movements in inflation or other macro-economic factors. The second factor follows the peaks in the short term interest rates quite closely and appears to capture policy shocks. Figure 6 graphs the implied intercepts $A(\tau)$ and factor loadings $B(\tau)$. The first factor loading, $B_1(\tau)$, is very flat; the impact of a shock in the first factor is around 1 for all maturities considered. Theoretically, $B_1(\tau)$ should converge to zero for large τ but apparently this convergence is so slow that it is hardly detectable at horizons up to ten years. So, although the model implies constant infitite maturity yields, long run yields can vary substantially. The factor loading of the second factor declines much faster, but is not negligible even for the longest maturities we consider.

Turning to the variance parameters, the estimates of α_{11} and α_{22} are strongly significant. In the affine model, β_1 is negative and significant, with a higher estimate for α_{11} than in the CIR model. Therefore, the affine model is slightly preferred to the CIR model, but the results are qualitatively very similar. The market prices of risk for both factors are negative, which implies a positive risk premium for holding long term bonds. With the instantaneous variance evaluated at the long term mean of the factors, the implied risk premium for a ten year bond is 1.32%, split over the first factor (around 0.4%) and the second factor (around 0.9%).

The final estimates concern the variance-covariance matrix of the measurement errors. The 3 month and 10 year maturity have the smallest errors, but their standard deviation is still in the order of 10 to 20 basis points.[16] The measurement error for the middle range maturities is somewhat greater, around 30 to 50 basis points.

Like for the one-factor model, we ran a battery of specification tests on the residuals of the two-factor model. The average fitted term structure and the RMSE of the residuals are graphed in Figure 7. The first thing to notice is that the fit of the two-factor model

[15] Using a pricipal components methodology, Litterman and Scheinkman (1991) document a slope factor. Balduzzi, Das and Foresi (1991) model the second factor as a central tendency.

[16] A Likelihood Ratio test strongly rejects the restriction that the variance of these error terms is zero.

is substantially better than the fit of the one-factor model. Espcially the fit in the long end and in the short end is very good. However, there are still substantial errors for the maturities around 1 year. The explanation for this result is that the term structure is often very steep in the short end. The second factor tracks the three month rate quite closely but the observed term structures are often steeper than the fitted term structures, causing large errors in the 1 year range. The residuals are not free from serial correlation, but the estimated first order serial correlation coefficients are much smaller than in the one-factor model, around 0.6. A brief analysis showed that about half of the variance of the residuals can be explained by lagged residuals. However, the serial covariances do not show a clear pattern and it is hard to detect a common structure which could serve as a third factor in an affine model.[17]

The time series regressions of the observed yields on the estimated factors gives some nice results, see Figure 8. The estimated coefficients are close to the factor loadings implied by the two-factor model. Only in the short end there are some small deviations from the theoretical values.

Finally, we tested the specification of the two-factor affine model by allowing for a different set of parameters for the cross section and the time series dimension of the model. Table 5 reports the results for this less restricted model. The parameter restrictions implied by the theoretical model are rejected on a 5% significance level. The main difference between the two sets of estimates is again the strength of the mean reversion of the factors. The half-life of the first factor in the time series dimension is around 6 years, compared with around 30 years in the cross section dimension. As a result, the conditional variances of the factors are also very different. The cross sectional fit is somewhat better than the fit of the restricted model, but still there isn't enough flexibility to capture the full shape of the yield curve as is clear from the residuals plotted in Figure 9. The hump around the one year maturity is still present, although it is smaller than before.

6 Conclusion

In this paper, we provided an empirical analysis of the affine class of term structure models proposed by Duffie and Kan (1996) on monthly US data. The estimation method combines time series and cross section information in a theoretically consistent way. We estimated one-factor models and models with two factors. The results clearly show that the one-factor models are misspecified: the fit is not very good and there is strong

[17] Geyer and Pichler (1997) estimate CIR models with up to five independent factors. However, their estimates of the third and higher factors are very unstable and mostly seem to fit some outliers.

residual serial correlation. A formal test of equality of the parameters in the bond pricing equations (the cross section dimension of the model) and the factor dynamics (the time series dimension) also rejects the model restrictions.

The two-factor affine model fits the data much better and the estimates of the underlying factors are very intuitive. There is one factor with slow mean reversion, which could proxy a time-varying mean or inflation. The second factor has a much stronger mean reversion and captures short run effects such as policy shocks. The model fits the long end of the term structure quite well, but it has some problems fitting the steep initial part of the yield curve. Perhaps an extension of the model with a stochastic volatility factor as proposed in Andersen and Lund (1997) could give a better fit of the steep short end, but this model is outside the affine class and has no known analytical or simple numerical solutions for bond prices.

A Conditional moments of factors

In this appendix we show how to derive the exact conditional mean and variance of the generalized square root process given in equation (3), which we repeat here for convenience

$$dF_t = \Lambda(F_t - \mu)dt + \Sigma(\alpha F_t + \beta)_d^{1/2} dW_t.$$

The mean reversion coefficient matrix is normalized to be diagonal, $\Lambda = \mathrm{diag}(-\kappa_1, .., -\kappa_n)$. The stochastic differential equation for F_t can be solved using Ito's lemma

$$de^{-\Lambda t}(F_t - \mu) = e^{-\Lambda t}\Sigma(\alpha F_t + \beta)_d^{1/2} dW_t \Rightarrow$$

$$F_{t+h} = \mu + e^{\Lambda h}(F_t - \mu) + \int_0^h e^{\Lambda(h-s)}\Sigma(\alpha F_{t+s} + \beta)_d^{1/2} dW_{t+s}$$

where $e^{\Lambda h} = \mathrm{diag}(\exp(-\kappa_1 h), .., \exp(-\kappa_n h))$. Since the second part of this sum is a martingale, the conditional mean and variance follow immediately as

$$E_t(F_{t+h}) = \mu + e^{\Lambda h}(F_t - \mu)$$

$$Var_t(F_{t+h}) = \int_0^h e^{\Lambda(h-s)}\Sigma(\alpha E_t(F_{t+s}) + \beta)_d \Sigma' e^{\Lambda(h-s)} ds$$

Using that $E_t(F_{t+s}) = \mu + e^{\Lambda t}(F_t - \mu)$ and defining $[\Sigma(\alpha x + \beta)_d \Sigma']_{ij} = a'_{ij}x + b_{ij}$ we obtain

$$\begin{aligned}
Var_t(F_{t+h})_{ij} &= \int_0^h e^{-(\kappa_i+\kappa_j)(h-s)}(a'_{ij}E_t F_{t+s} + b_{ij})ds \\
&= \int_0^h e^{-(\kappa_i+\kappa_j)(h-s)}(a'_{ij}(\mu + e^{\Lambda s}(F_t - \mu)) + b_{ij})ds \\
&= \int_0^h e^{-(\kappa_i+\kappa_j)(h-s)}(a'_{ij}\mu + b_{ij})ds + \int_0^h e^{-(\kappa_i+\kappa_j)(h-s)} a'_{ij} e^{\Lambda s}(F_t - \mu)ds
\end{aligned}$$

Working out the integrals yields the result

$$Var_t(F_{t+h})_{ij} = \frac{1-e^{-(\kappa_i+\kappa_j)h}}{\kappa_i+\kappa_j}(a'_{ij}\mu + b_{ij}) + \sum_k \frac{e^{-\kappa_k h} - e^{-(\kappa_i+\kappa_j)h}}{\kappa_i+\kappa_j-\kappa_k} a_{ij,k}(F_{t,k}-\mu_k)$$

Note that this results implies a very simple form for the unconditional variance of F_t,

$$Var(F_t)_{ij} = \frac{a'_{ij}\mu + b_{ij}}{\kappa_i+\kappa_j}.$$

B Relation to multivariate time series models

In this appendix we discuss the relations between the state space form of the affine yield curve model and multivariate time series models. To simplify the exposition we write the yields in deviation from their long run mean, $a = A + B\mu$. Let k to be the dimension of y_t and n the dimension of F_t. Splitting the vector of observed yields into two parts, with dimensions n and $k-n$, respectively, the system becomes

$$\begin{aligned} y_{1t} - a_1 &= B_1(F_t - \mu) + u_{1t} \\ y_{2t} - a_2 &= B_2(F_t - \mu) + u_{2t} \\ F_{t+h} - \mu &= \Phi(F_t - \mu) + \nu_{t+h} \end{aligned}$$

In general B_1 will be an invertible matrix, so that the factors equal

$$F_t - \mu = B_1^{-1}(y_{1t} - a_1 - u_{1t})$$

Substituting out the factors gives the reduced form of the model

$$\begin{aligned} y_{1,t+h} - a_1 &= T(y_{1t} - a_1) + B_1\nu_{t+h} + u_{1,t+h} - Tu_{1t} \\ y_{2,t+h} - a_2 &= B_2B_1^{-1}T(y_{1t} - a_1) + B_2\nu_{t+h} + u_{2,t+h} - B_2B_1^{-1}Tu_{1t} \end{aligned}$$

with $T = B_1\Phi B_1^{-1}$. This system of equations can be written more compactly as the vector ARMA(1,1) model

$$y_{t+h} = a + \Psi(y_t - a) + B\nu_{t+h} + u_{t+h} - \Psi u_t$$

where $\Psi = B[\Phi B_1^{-1}|0] \equiv BA'$, where A and B are matrices of dimension k by n. The rank of Ψ equals the number of factors n, which may be smaller than the number of yields included in the model. This reduced form contains a moving average term which implies that a first order VAR will be a misspecified model for the term structure. Only if the factors are exact linear combinations of the observed yields, i.e. if $u_{1t} = 0$, the MA terms disappear and the model reduces to a VAR.

C The Kalman filter

The standard state space setup of the affine model is found by defining $\xi_t = F_t - \mu$, $a = A + B\mu$, and $Q_t = Var_{t-h}(F_t)$. All models in this paper are estimated using Quasi Maximum Likelihood based on the following Kalman filter equations[18]

MODEL

$$\begin{aligned} y_t &= a + B\xi_t + e_t, \qquad Var(e_t) = H \\ \xi_t &= \Phi\xi_{t-h} + \eta_t, \qquad Var(\eta_t) = Q_t \end{aligned}$$

INITIAL CONDITIONS

$$\begin{aligned} \hat{\xi}_0 &= E(\xi_t) \\ \hat{P}_0 &= \mathrm{Var}(\xi_t) \end{aligned}$$

PREDICTION

$$\begin{aligned} \xi_{t|t-h} &= \Phi\hat{\xi}_{t-h} \\ P_{t|t-h} &= \Phi\hat{P}_{t-h}\Phi' + Q_t \end{aligned}$$

LIKELIHOOD CONTRIBUTIONS

$$\begin{aligned} v_t &= y_t - a - B\xi_{t|t-h} \\ F_t &= BP_{t|t-h}B' + H \\ -2\ln L_t &= \ln|F_t| + v_t'F_t^{-1}v_t \end{aligned}$$

UPDATING

$$\begin{aligned} K_t &= P_{t|t-h}B'F_t^{-1} \\ L_t &= I - K_tB \\ \hat{\xi}_t &= \xi_{t|t-h} + K_tv_t \\ \hat{P}_t &= L_tP_{t|t-h} \end{aligned}$$

D Risk premia on long bonds

In our empirical work we also want to calculate the risk premium on long maturity bonds. Denote the stochastic process followed by the bond price as

$$dP(\tau) = \mu_{P(\tau)}P(\tau)dt + \sigma_{P(\tau)}P(\tau)dW_t$$

[18] These equations are adapted from Hamilton (1994, Ch.13), where the notation is slightly changed: H' in Hamilton is B in our notation.

where the dependence of coefficients and prices on time is suppressed. The expected instantaneous return on the bond is the risk free rate plus a risk premium, which depends on the market prices of risk and the instantaneous standard deviation of the bond return

$$\mu_{P(\tau)} = r + \lambda' \sigma_{P(\tau)}$$

From Ito's lemma, the standard deviation of the bond return is

$$\sigma_{P(\tau)} = -\sigma_F B(\tau)$$

where $\sigma_F \sigma_F'$ is the instantaneous variance-covariance matrix of the factors. Given the assumed functional forms for σ_F and λ we obtain

$$\mu_{P(\tau)} = r - \sum_i \psi_i (\alpha_i' F + \beta_i) B_i(\tau)$$

This equation shows that the risk premium on each factor is proportional to the instantaneous variance of that factor, multiplied by the factor loading. If all parameters ψ_i are negative, the risk premia are positive. Since the factor loadings are increasing with maturity, longer bonds will typically have a higher expected return than short bonds.

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Table 2: Estimation results one-factor affine models

	Affine	CIR	Vasicek
κ	0.0601	0.0429	0.0222
	(0.0074)	(0.0042)	(0.0028)
μ	6.4642	5.8099	7.3146
	(0.2465)	(0.5397)	(1.4600)
α	0.3961	0.2168	
	(0.0637)	(0.0231)	
β	-1.5137		1.9980
	(0.4369)		(0.2284)
ψ	-0.1481	-0.1446	-0.0928
	(0.0069)	(0.0110)	(0.0178)
κ^*	0.0014	0.0116	0.0222
	[495.10]	[60.00]	[31.19]
$2\ln L$	710.45	702.43	677.60

Measurement error standard deviations (in %) and correlations

model				
affine	1.6414			
	0.92	1.2098		
	0.50	0.77	0.3737	
	-0.18	0.22	0.67	0.1839
CIR	1.6729			
	0.92	1.2434		
	0.54	0.80	0.4052	
	-0.01	0.37	0.75	0.2004
Vasicek	1.7142			
	0.93	1.2946		
	0.59	0.83	0.4505	
	0.18	0.53	0.83	0.2309

Notes: This table reports QML estimates and standard errors of one-factor affine term structure models defined in section 4. For legibility, the estimates of the parameters are scaled as follows: κ, 100μ, $\psi/100$, 100α and 10000β. The number in [] is the half-life of the factors under the risk-neutral distribution in years, calculated as $\ln(2)/\kappa^*$

Table 3: Estimation results one-factor affine models with separate time-series and cross-section parameters

	affine	CIR	Vasicek
κ	0.2183	0.2041	0.2018
	(0.1116)	(0.1242)	(0.1321)
μ	7.4027	7.5788	7.5574
	(0.8220)	(1.0065)	(1.0509)
α	0.4318	0.1988	
	(0.0788)	(0.0232)	
β	-1.7248		1.9220
	(0.4813)		(0.2185)
κ_c	0.0230	0.0307	0.0218
	(0.0034)	(0.0049)	(0.0031)
α_c	0.0088	0.5943	
	(0.0002)	(0.0907)	
β_c	6.0774		6.0548
	(1.1225)		(0.8864)
ψ	-0.0630	-0.0688	-0.0631
	(0.0021)	(0.0068)	(0.0040)
κ^*	0.0225	-0.0102	0.0218
	[31.39]		[31.70]
$2\ln L$	728.21	714.55	693.54

Notes: This table reports QML estimates and standard errors of one-factor affine term structure models with separate parameters for the cross section and time series dimension. Parameters with a subindex c are the cross-section paramaters. For identification the restriction $\mu_c = \mu$ is imposed. See also the notes at Table 2.

Table 4: Estimation results two-factor affine models

	Affine	CIR	Vasicek
κ_1	0.0595 (0.0077)	0.0368 (0.0052)	0.0224 (0.0029)
κ_2	1.2067 (0.1405)	1.2190 (0.1947)	0.8742 (0.0546)
μ_1	3.5547 (0.4527)	3.5600 (0.7596)	6.1900 (1.5609)
μ_2	1.8520 (0.5094)	1.9063 (0.7524)	
α_{11}	0.2017 (0.0777)	0.0982 (0.0370)	
α_{22}	2.4940 (0.8814)	2.3574 (1.2883)	
β_1	-0.5540 (0.2621)		1.4955 (0.2000)
β_2			6.3157 (1.3600)
ψ_1	-0.2230 (0.0132)	-0.1480 (0.0269)	-0.1076 (0.0102)
ψ_2	-0.1568 (0.0205)	-0.1651 (0.0201)	-0.1460 (0.0167)
x_{12}	-0.0372 (0.0229)	-0.0344 (0.0226)	0.0145 (0.0396)
x_{21}	0.4763 (0.3806)	0.4548 (0.3889)	0.0145 (0.0396)
$2\ln L$	1366.35	1356.60	1218.14

Implied parameters of factor process under empirical measure

	Affine		CIR		Vasicek	
Λ	-0.0795	0.0419	-0.0550	0.0400	-0.0222	-0.0124
	0.5369	-1.1867	0.5294	-1.2008	0.0124	-0.8744
κ	0.0595	1.2067	0.0368	1.2190	0.0224	0.8742
	[11.65]	[0.57]	[18.83]	[0.57]	[30.91]	[0.79]

Implied parameters of factor process under risk neutral measure

	Affine		CIR		Vasicek	
Λ^*	-0.0345	0.0419	-0.0405	0.0400	-0.0222	-0.0124
	0.5369	-0.7965	0.5294	-0.8116	0.0124	-0.8744
κ^*	0.0060	0.8214	0.0139	0.8381	0.0224	0.8742
	[115.67]	[0.84]	[49.80]	[0.83]	[30.91]	[0.79]

Measurement error standard deviations (in %) and correlations

model				
affine	0.1145			
	0.38	0.5067		
	0.47	0.92	0.3594	
	-0.01	0.87	0.87	0.2401
CIR	0.1360			
	0.45	0.5025		
	0.54	0.93	0.3575	
	0.08	0.88	0.88	0.2324
Vasicek	0.1132			
	0.66	0.5341		
	0.84	0.94	0.3801	
	0.52	0.94	0.90	0.2488

Notes: This table reports QML estimates and standard errors of two-factor affine term structure models defined in Section 5. The estimates of the parameters are scaled as in Table 2. The parameters x_{12} and x_{21} are the off-diagonal elements of Σ^{-1}.

Table 5: Estimation results two-factor affine models with separate cross-section and time-series parameters

time series parameters

	affine	CIR	Vasicek
κ_1	0.2355	0.1105	0.3352
	(0.1289)	(0.1281)	(0.2407)
κ_2	0.7709	0.7306	0.7960
	(0.2436)	(0.2823)	(0.5256)
μ_1	4.7706	6.5115	6.8300
	(0.8062)	(1.3314)	(0.8546)
μ_2	2.0647	1.6503	
	(0.3876)	(0.6563)	
α_{11}	0.2508	0.1233	
	(0.0527)	(0.0204)	
α_{22}	2.1934	2.1181	
	(0.5078)	(0.5203)	
β_1	-0.6739		1.4958
	(0.2190)		(0.2200)
β_2			6.0639
			(1.4300)
x_{12}	-0.3045	-0.2980	-0.5648
	(0.2241)	(0.2490)	(0.4535)
x_{21}	-0.2027	-0.2943	-0.5648
	(0.3358)	(0.2757)	(0.4535)

cross section parameters

	affine	CIR	Vasicek
κ_1	0.0240	0.0220	0.0239
	(0.0078)	(0.0038)	(0.0031)
κ_2	1.6317	2.2399	0.7569
	(0.2948)	(1.1141)	(0.0411)
α_{11}	0.0008	0.0000	
	(0.0000)	(0.0000)	
α_{22}	55.7775	97.3178	
	(18.9940)	(55.8528)	
β_1	-3.0677		0.6950
	(1.3839)		(0.2200)
β_2			281.16
			(33.67)
x_{12}	-0.0595	-0.0807	0.0501
	(0.0383)	(0.0550)	(0.0445)
x_{21}	0.2565	0.1921	0.0501
	(0.0786)	(0.0918)	(0.0445)
ψ_1	-0.0667	-0.1495	-0.1462
	(0.0216)	(0.0060)	(0.0006)
ψ_2	-0.0201	-0.0180	-0.0098
	(0.0082)	(0.0017)	(0.0013)
$2 \ln L$	1463.02	1447.66	1302.87

Implied parameters of factor process under empirical measure

	Affine		CIR		Vasicek	
Λ	-0.2002	0.1738	-0.0509	0.2025	-0.1193	0.3822
	-0.1157	-0.8061	-0.2000	-0.7902	-0.3822	-1.0118
κ	0.2355	0.7702	0.1105	0.7306	0.3352	0.7960
	[2.94]	[0.90]	[6.27]	[0.95]	[2.07]	[0.87]

Implied parameters of factor process under risk neutral measure

	affine		CIR		Vasicek	
Λ^*	-0.0481	0.0942	-0.0559	0.1762	-0.0220	-0.0368
	0.4062	-0.4879	0.4196	-0.4495	0.0368	-0.7588
κ^*	-0.0263	0.5623	-0.0830	0.5883	0.0239	0.7569
		[1.23]		[1.18]	[29.04]	[0.92]

Measurement error standard deviations (in %) and correlations

affine	0.1518			
	0.02	0.3839		
	0.11	0.91	0.3134	
	-0.47	0.79	0.83	0.2016
CIR	0.1839			
	0.12	0.3994		
	0.21	0.92	0.3201	
	-0.36	0.81	0.84	0.1965
Vasicek	0.0680			
	0.79	0.4572		
	0.96	0.92	0.3415	
	0.73	0.94	0.89	0.2209

Notes: This table reports QML estimates and standard errors of two-factor affine term structure models defined in Section 5, with separate parameters for te cross section and time series dimension of the model. For reasons of identification μ_1 and μ_2 are restricted to be the same for the cross section and the time series dimension. See also notes to Table 4.

Table 6: Likelihood Ratio tests in two-factor models

	affine	CIR	Vasicek
LR1	41.97	38.30	4.39
LR2	89.00	79.70	54.88
LR3	96.67	91.10	84.73

Notes:
LR1 is a test for independence between the factors. This hypothesis imposes the restrictions that α and Σ are diagonal.
LR2 is a test for the absence of measurement errors on the three month and 10 year interest rate. This imposes the restrictions that the rows and columns of H corresponding to these maturities vanish.
LR3 tests the equality of κ, α, β and Σ across the time series and cross section dimension. The test statistic is equal to the difference between the likelihood values reported in Table 4 and Table 5.
* denotes a statistic which is not significant at the 5% level

Figure 1: US term structure data. The figure shows the 3 month, 1, 5 and 10 year zero coupon yields constructed by McCullogh and Kwon (1993) from US treasury coupon bonds.

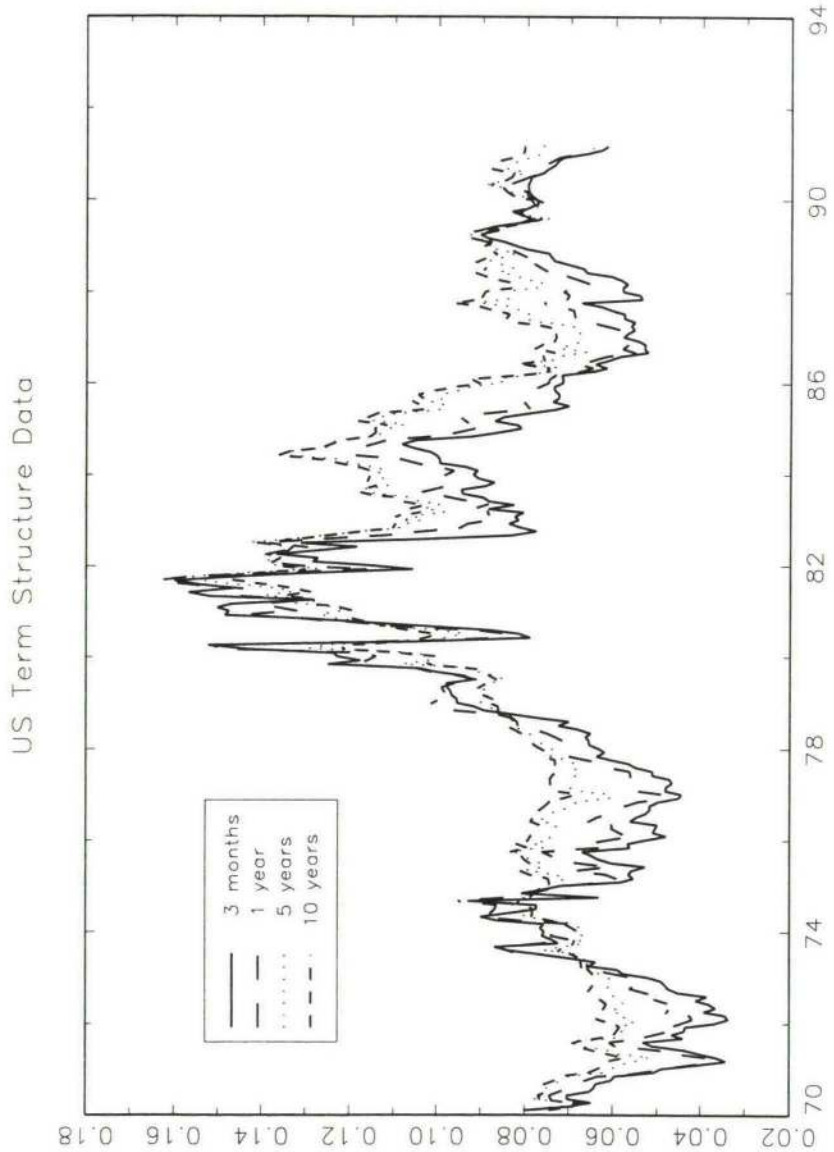


Figure 2: Factor loadings in one-factor affine models. The figure shows the estimated factor loadings $B(\tau)/\tau$ as a function of maturity for the one-factor affine, CIR and Vasicek models.

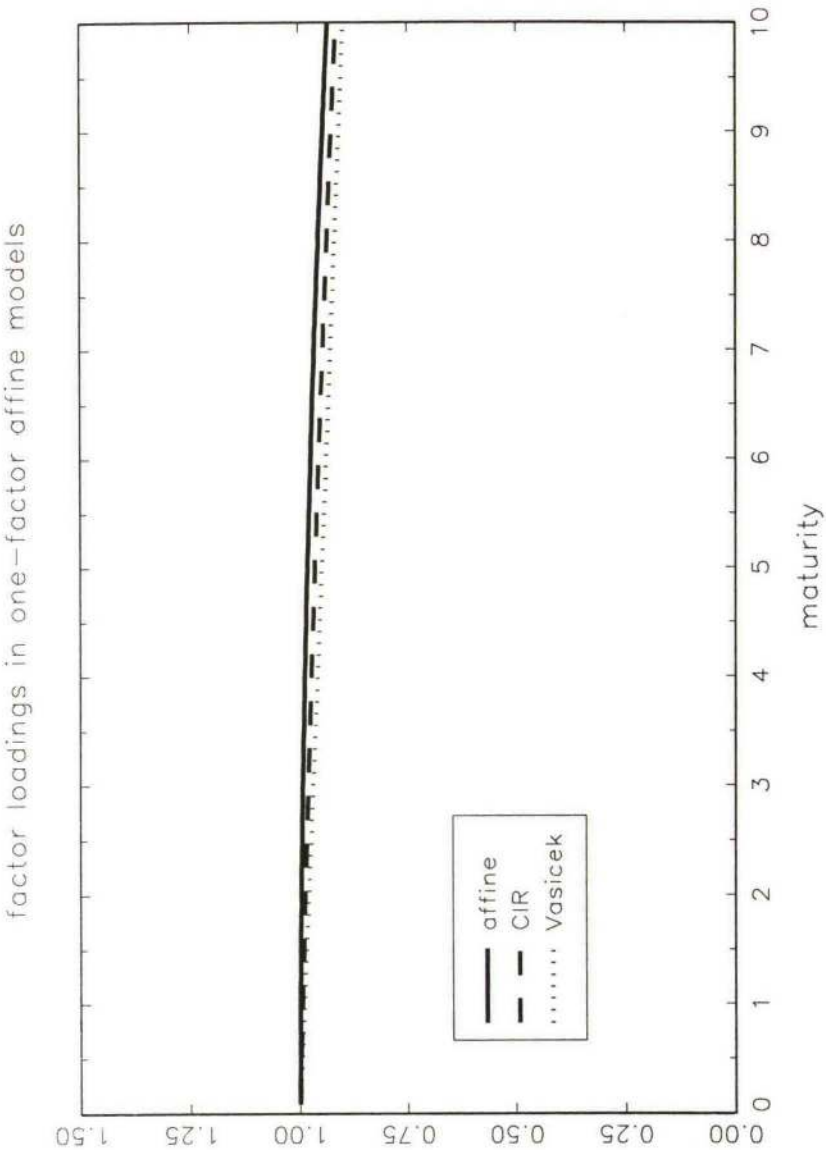


Figure 3: Fit of the one-factor affine model. The figure shows the average of the actual and fitted term structures, as well as the standard deviation of the residuals, in the one-factor affine model.

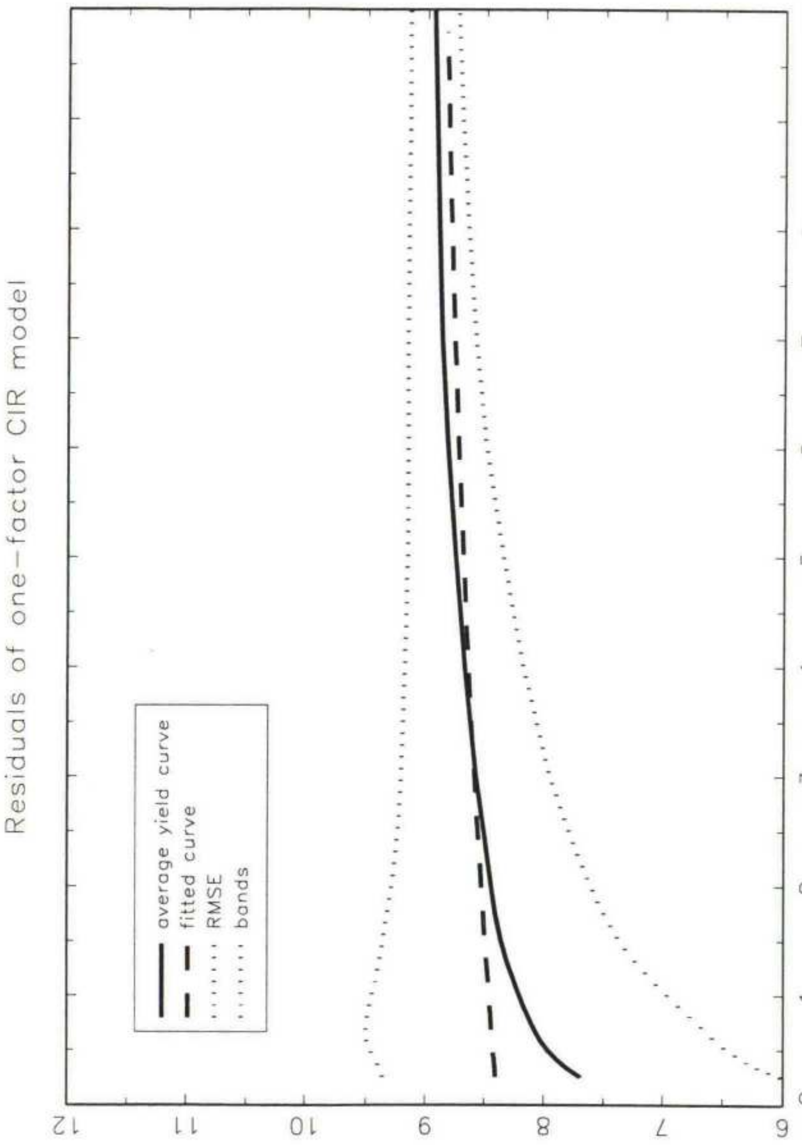


Figure 4: Regression of observed yields on fitted factors. The figure shows the coefficients of a regression (in levels and first differences) of the observed time series of yields on the time series of fitted factors in the one-factor affine model.

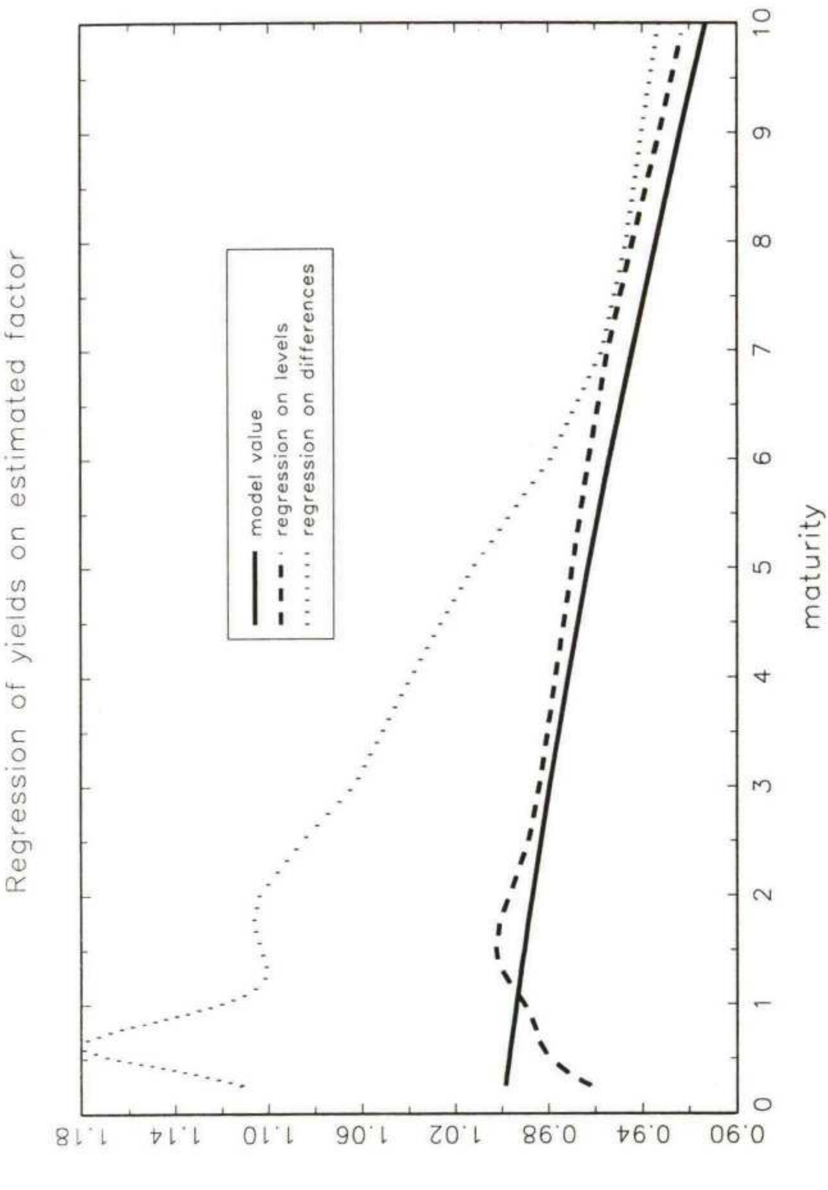


Figure 5: Fitted factors in two-factor affine model. The figure shows the estimated factors in the two-factor affine model.

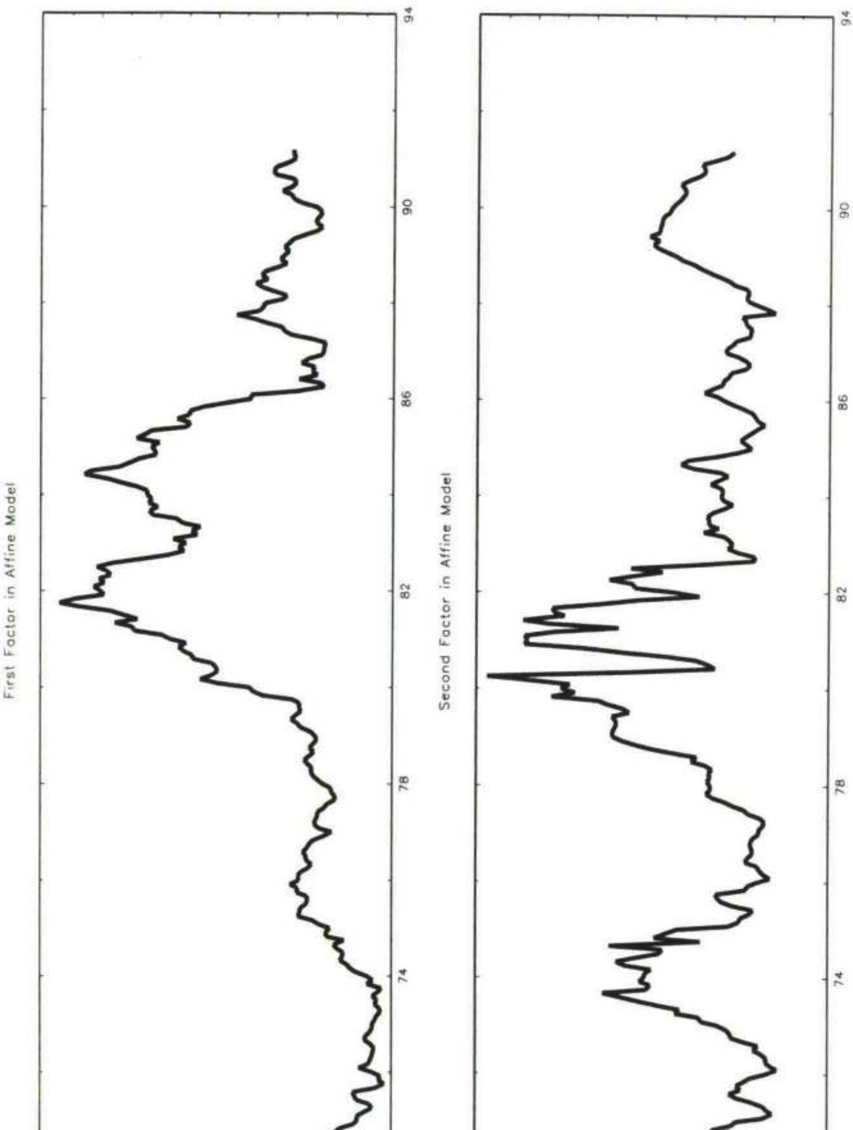


Figure 6: Factor loadings in two-factor affine models. The figure shows the estimated factor loadings $B_i(\tau)/\tau$, $i = 1, 2$ as a function of maturity for the two-factor affine, CIR and Vasicek models.

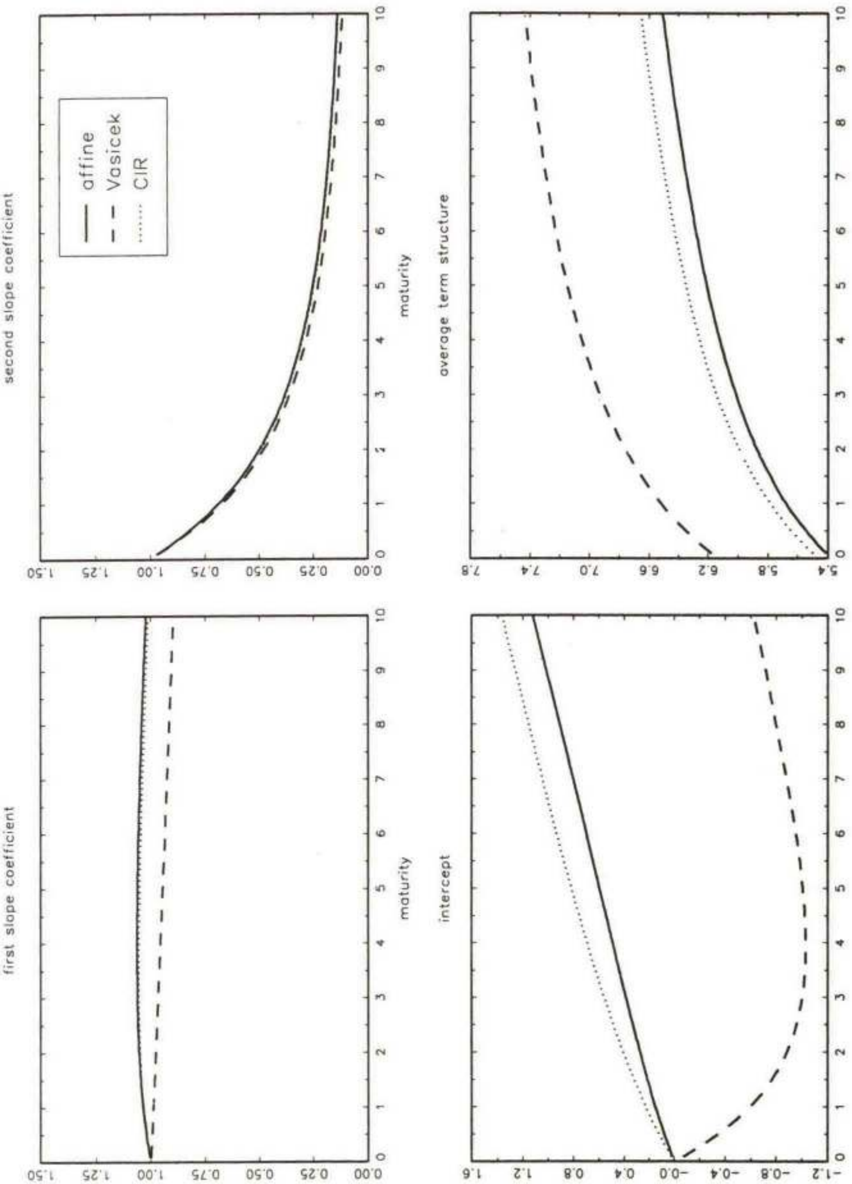


Figure 7: Fit of the two-factor affine model. The figure shows the average of the actual and fitted term structures, as well as the standard deviation of the residuals, in the two-factor affine model.

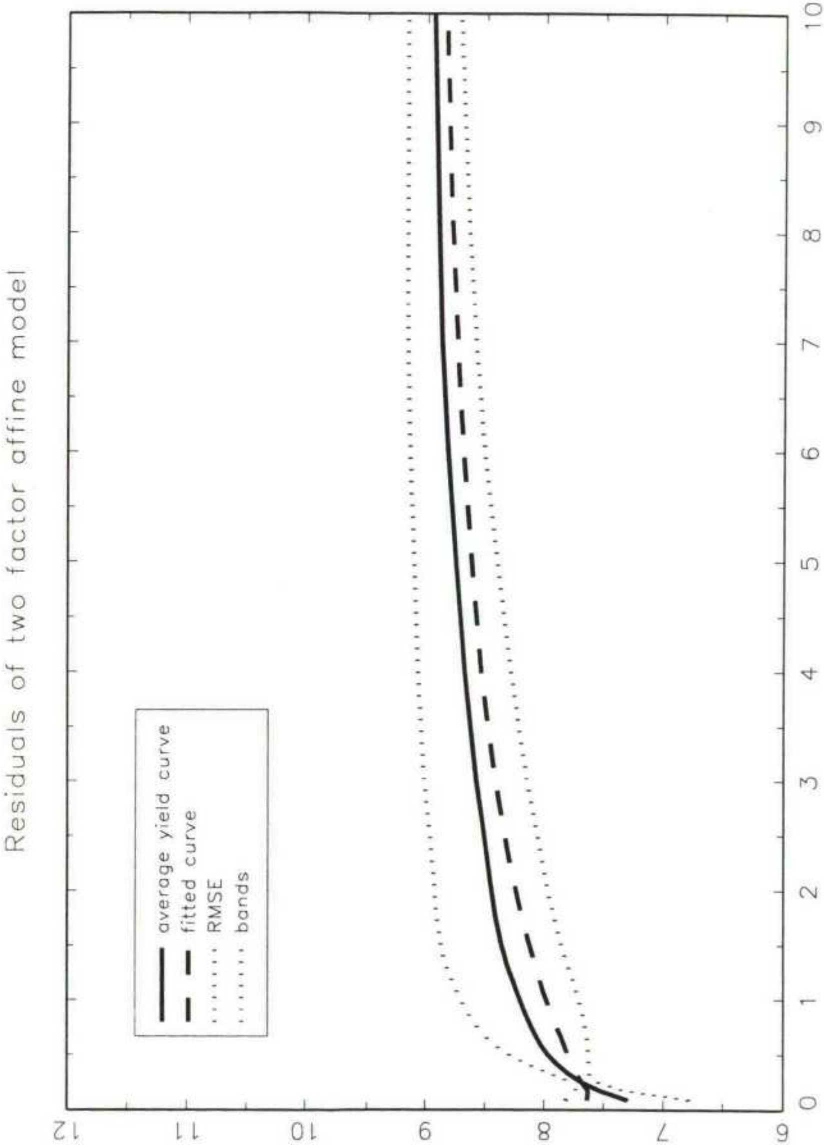
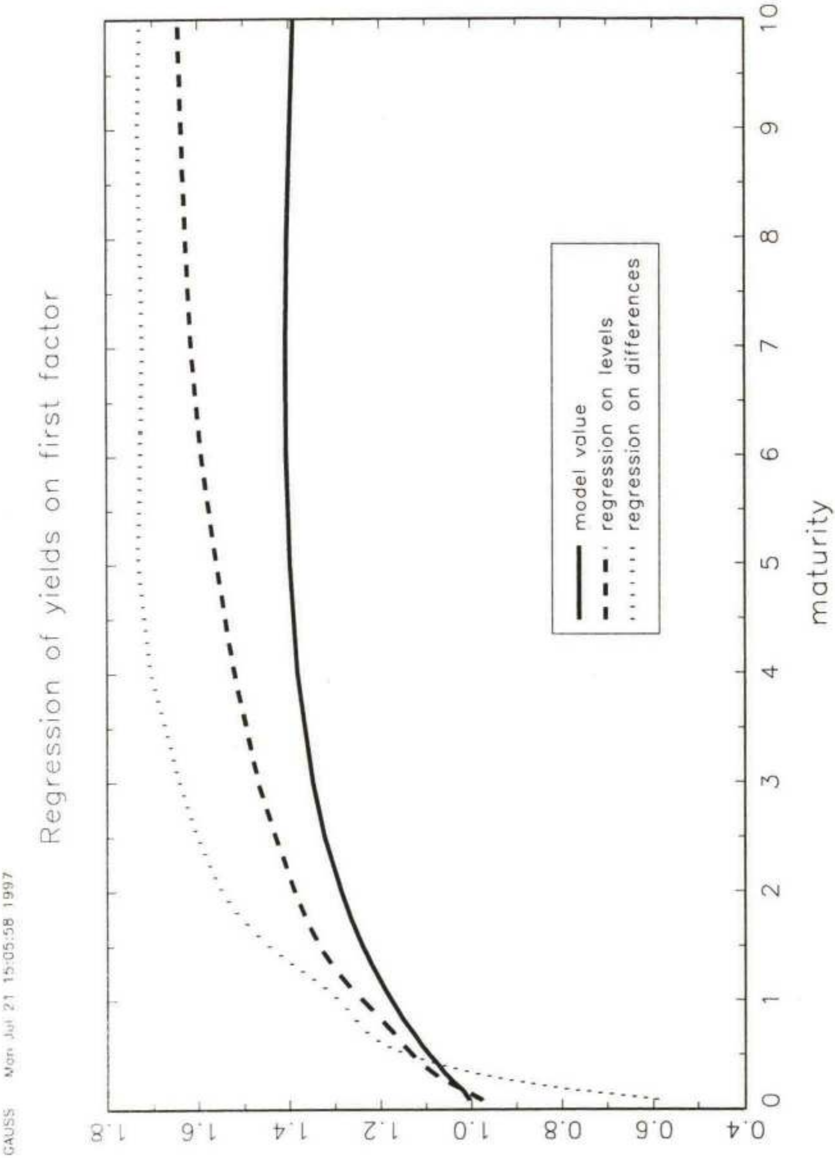


Figure 8: Regression of observed yields on fitted factors. The figure shows the coefficients of a regression (in levels and first differences) of the observed time series of yields on the time series of fitted factors in the two-factor affine model.



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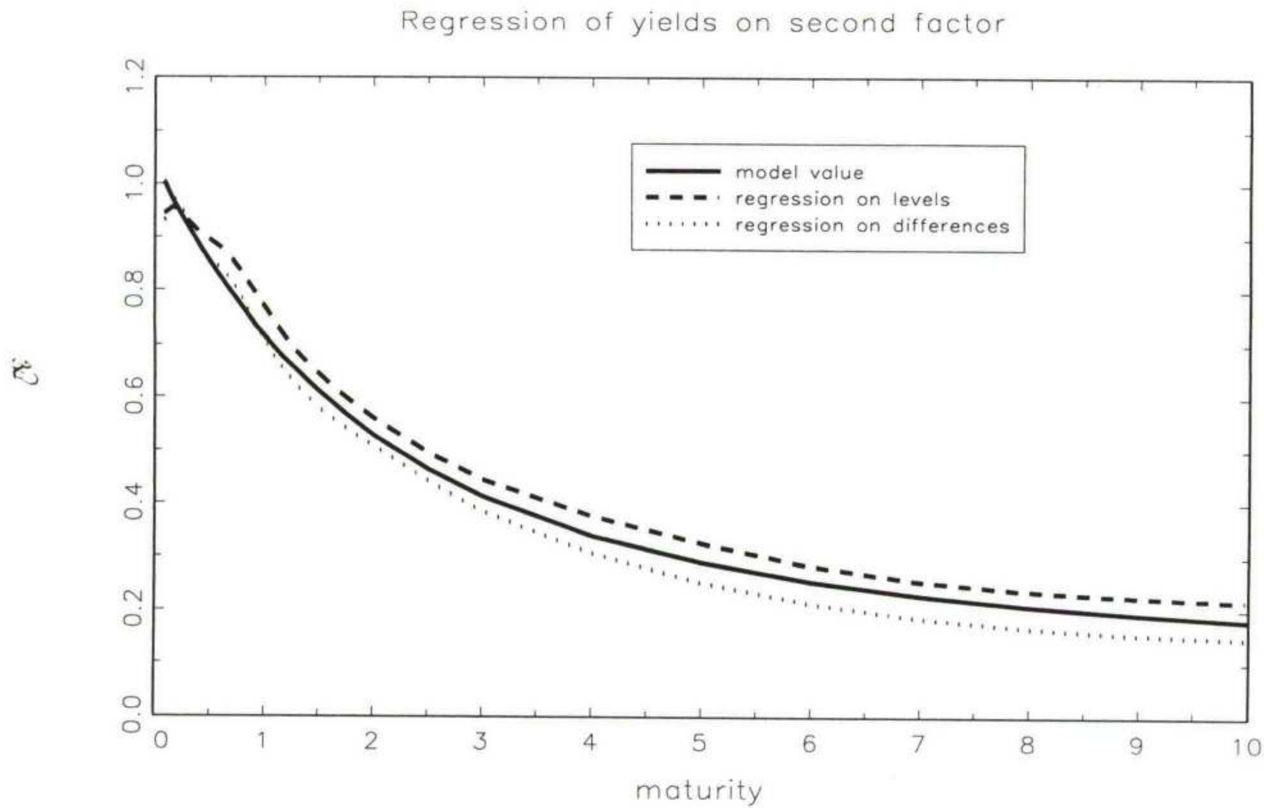
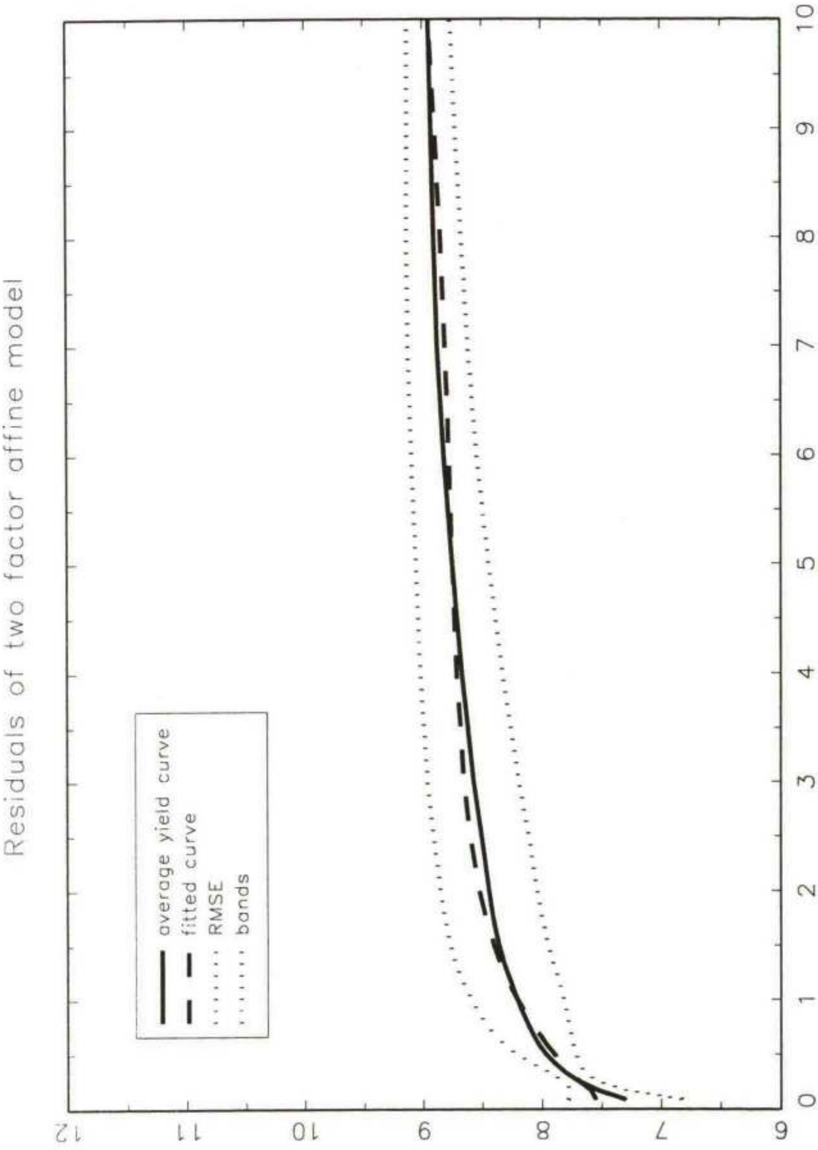


Figure 9: Fit of the unrestricted two-factor affine model. The figure shows the average of the actual and fitted term structures, as well as the standard deviation of the residuals, in the two-factor affine model with separate parameters for the time series and cross section dimension of the model.



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