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Time-Varying International Diversification and the Forward Premium

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While standard theory predicts a slope of one, the empirically observed slope of the regression of currency returns on the interest rate differential between two countries is negative for most country pairs. This empirical fact that, on average, investors require higher returns on bonds denominated in a currency expected to appreciate, poses a strong challenge for economic models. In this paper, we propose a potential explanation within an infinite horizon dynamic stochastic general equilibrium model with incomplete markets. Heterogenous investors experience varying risk aversion as a result of habit formation.

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1 Introduction

The forward premium anomaly, discovered by Fama (1984), is one of the most prevalent puzzles in international finance. The uncovered interest parity (UIP) implies that high interest currencies should depreciate. However, a large body of empirical literature finds exactly the opposite — high interest currencies tend to appreciate. Investors in high yield currencies benefit twice, once from the interest rate spread and once from the expected appreciation.

Given the complexity and resilience of the puzzle, economists have been searching for a potential explanation ever since its discovery. Approaches towards solving the puzzle may be organized into three categories: irrational expectations, market frictions and risk premia. This paper develops a two-country model under rational expectations without market frictions, and thus attributes the anomaly to large and time-varying risk premia.

This is achieved through two major model ingredients. First, markets are assumed to be incomplete. There is no asset that directly allows individuals to insure their income risk, preventing them from completely aggregating their individual risk. The extent to which insurance is possible depends on the correlation of income with existing financial assets. Second, we assume that consumers form habits according to their consumption history. The predominant effect of habit formation is that it changes the price of risk over time. When consumption drops close to the habit level, marginal utility increases and the implied risk aversion rises. Contrarily, a large wedge between consumption and habit implies small risk aversion.

We attribute the anomaly to time-varying international diversification decisions that arise endogenously due to the varying risk aversion. In the absence of habit expected exchange rates directly affect interest rates and thus the correlation between the two would be always plus one. Now, time varying risk aversion introduces time-varying international investment decisions, which are negatively correlated with the expected exchange rate[1]. This allows, for sufficiently large habit levels, to overcome the primary effect and reproduce the negative correlation.

With habit levels common in the literature[2], we are able to reproduce the forward premium anomaly for six different country pairs, which are formed by combining Germany, Japan, United Kingdom and United States. We distinguish between two different calibrations. In a first attempt, we assume the same habit level for each country. Although this implies an almost complete lack of flexibility in the model calibration the UIP slope is still almost within the 95% confidence interval of the empirical estimation for each country pair. In a second attempt, we allow for different habit levels for each country pair. Then the slope of each country pair lies within one standard deviation of the empirical observation.

Furthermore our model is capable of matching many empirical moments that are crucial in such a setting. The introduction of tradables and non-tradable goods allows to match the first moments of inflation and exchange rate processes simultaneously. In addition, the calibration matches the first moments of the income growth and most notably the correlation between income growth and exchange rates for every country.

Few other papers are successful in reproducing negative slope coefficients. Two notable exceptions under rational expectations and without market frictions are Bansal and

[1]E.g. when a currency is expected to appreciate more, the model dynamics imply that the investor has a preference for the other currency at the same time.

[2]Campbell and Cochrane (1999), Verdelhan (2010)

Shaliastovich (2009) and Verdelhan (2010).

Bansal and Shaliastovich extend the long run risk framework proposed in Bansal and Yaron (2004) to two countries. Time-varying volatility of the long run risk factor in combination with Epstein and Zin (1989) preferences allow the authors to achieve a negative UIP slope coefficient.

Verdelhan provides an explanation to the anomaly in the Campbell and Cochrane (1999) habit framework. He assumes pro-cyclical interest rates, which, combined with habit driven counter-cyclical risk aversions, replicate the anomaly.

Bansal and Shaliastovich assume fully country specific goods[3], while Verdelhan assumes large transportation costs. In both cases this implies the absence of trade and allows to model consumption as an exogenous process.

This paper is similar to Verdelhan in that we attribute the anomaly to risk premia, which vary due to habit formation. In our model, however, we allow for trade and international investment decisions by determining consumption endogenously. Theoretically, this allows for feedback effects between the two countries and generally for richer dynamics within the model. Empirically, it allows to replicate more and different moments. Most notably, as these are risk premium stories, we are capable of reproducing the negative UIP slope, while matching the correlation between income and exchange rates.

The paper is organized in the following way. In section two we present our model followed by a description of our numerical solution method in section three. In section four we describe our calibration. Section five discusses our model results and section six concludes.

2 The model

2.1 General setup

2.1.1 Real Economy

This model describes an exchange economy of two infinitely lived countries, in which each country is endowed with two types of nondurable consumption goods, one tradable, one nontradable. Each country is represented by one agent[4]. In each period, agents receive a share α of their endowments in the nontradable good $y_{NG,t} = \alpha y_t$ and a share $1-\alpha$ in the tradable good $y_{TG,t} = (1-\alpha)y_t$. The agents consider the foreign tradable good as a perfect substitute for the domestic tradable good and possess Cobb-Douglas preferences over the two consumption goods,

$$u(c_{TG,t}, c_{NG,t}) = \frac{2}{1-\gamma}\left(c_{TG,t}^{\frac{1}{2}} c_{NG,t}^{\frac{1}{2}}\right)^{1-\gamma}.$$

For ease of notation we refer to the vector of consumption $c_t = (c_{TG,t}, c_{NG,t})$, whenever an explicit distinction between tradables and non-tradables is not necessary.

[3]Also referred to as full home bias. Each country produces a different good and consumers only have preferences over the good produced in their own country. This assumption is crucial because in a two-country and complete markets setup, the representative investors can fully share risk and the real exchange rate is constant, defeating any potential explanation of the anomaly.

[4]We name them agent H and agent F and they reside in country 1 and country 2.

2.1.2 Financial Economy

Each country has separate exogenous price levels, determining the relative value of the currency. We measure the price level in terms of the nominal price of the tradable good in each country.[5] In addition, we assume that goods and assets can only be traded in the currency of the home country. Furthermore, we assume that Purchasing Power Parity (PPP) holds for tradable goods, thus determining the nominal exchange rate as

$$S_t = \frac{p_{1,t}}{p_{2,t}},$$

where $p_{1,t}$ and $p_{2,t}$ refer to the price levels (e.g. prices of tradables) in the two countries.[6]

Each country issues a one-period bond with no possibility to default on its debt. Denoting prices and nominal holdings of bonds, issued by country i by $q_{i,t}$ and $B_{i,t}$ respectively, and introducing a superscript to identify the country that chooses the economic variable,[7] the home country's budget constraint in nominal terms can be written as

$$C^H_{TG,t} \leq W^H_t + Y^H_{TG,t} - q_{1,t}B^H_{1,t} - q_{2,t}B^H_{2,t},$$

where $W^H_t = B^H_{1,t-1} + S_t B^H_{2,t-1}$ represents nominal wealth of country H. For this wealth, we assume there is some boundary on real debt

$$\frac{W^H_{1,t}}{p_{1,t}} \geq \bar{w}^H.$$

2.1.3 Uncertainty

Uncertainty enters the model through real and monetary shocks, z_t will denote the vector of all such shocks. It is assumed to follow a first order Markov Process with transition function $\Pi(z_{t+1}|z_t)$. Real shocks change the endowment of consumption good $(y_{NG,t}(z_t), y_{TG,t}(z_t))$ available to each country, whereas monetary shocks change the inflation in each country. This has two important implications. Firstly, these shocks determine the exchange rate by PPP. Secondly, although the countries cannot default and the bond appears as riskless, the introduction of shocks to price levels imply a consumption risk of holding bonds.

Note that, since the financial economy consists of only two bonds and there are in general many shocks to income and inflation, markets are incomplete.

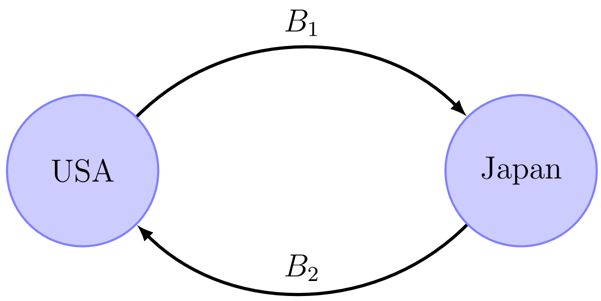
2.1.4 Summary

Figure 1 summarizes our model setup. We assume, there are two countries, for example United States and Japan. Each country has a distinct currency and issues a nominally riskless zero bond. Risk enters the model through stochastic inflation rates in each country, affecting the real payouts of the bonds.

[5] Prices of non-tradables have no impact on agents' decisions in our model.

[6] Given the empirical evidence on absolute and relative PPP it cannot be claimed that PPP holds for a general basket of goods. Burstein, Eichenbaum, and Rebelo (2005), however, show that PPP holds approximately for tradables, if one chooses the definition of tradable good carefully. In particular, they distinguish between production and distribution of tradable goods. They argue that distribution is essentially nontradable. Based on this distinction they show empirically that even in times of extreme exchange rate fluctuations PPP holds approximately for tradables.

[7] This notation will be used throughout the paper.



Currency: \$	Currency: ¥
Stochastic inflation rate: π_t^H	Stochastic inflation rate: π_t^F
Riskless zero bond (\$): B_1	Riskless zero bond (¥): B_2
Bond price: $q_{1,t}$	Bond price : $q_{2,t}$

Figure 1: Market Setup

2.2 Habit utility

Agents in our model form habits according to the history of aggregate consumption. Abel (1990) first proposed this external habit, commonly referred to as "catching up with the Joneses." Agents' effective consumption is reduced by their habit levels, resulting in

$$u(c_t, h_t) = u(c_t - h_t) = u(c_{TG,t} - h_{TG,t}, c_{NG,t} - h_{NG,t}).$$

Concerning the habit process, we follow the standard specification in the literature[8], denoting habit as a weighted average of past consumption.[9] In recursive form this process can be written as

$$h_{t+1} = \rho h_t + \eta c_{t+1}.$$

For simplicity we consider the same habit level for tradables and nontradable goods, where we take nontradable consumption as a proxy for aggregate consumption. The good-specific habit levels are then formed as fractions of the aggregate habit level proportionally to the amount of tradables and nontradables in the economy.

This specification of habit increases the local curvature of the utility function, and thus, increases the risk aversion of agents. Moreover risk aversion changes as agents experience different shocks to endowment. In times of consumption levels close to habit levels, marginal utilities are large and agents very risk averse. Contrarily, in times, when consumption is much higher than habit, marginal utilities are relatively small and the price of risk is low. Thus habit formation allows for large, time varying risk premia.

For the purpose of the calibration, ρ and η are hard to interprete. Therefore we use and report the first two moments of the habit process. It can be shown that there exists the following mapping from the moments to ρ and η,

$$\rho = \frac{\frac{\mathbb{E}[h]^2}{\mathbb{E}[c]^2}\mathbb{V}[c] - \mathbb{V}[h]}{\frac{\mathbb{E}[h]^2}{\mathbb{E}[c]^2}\mathbb{V}[c] + \mathbb{V}[h]},$$

[8] See among many others Constantinides (1990), Ferson and Constantinides (1991), Heaton (1995).

[9] Campbell and Cochrane (1999) and Verdelhan (2010) use a nonlinear habit process, which is set on the surplus consumption ratio $\frac{c-h}{c}$. Working with the surplus consumption ratio is very convenient in their models and the nonlinearity of the process has interesting implications in their framework. However, in our opinion, the approach by Constantinides is the economically more intuitive choice.

$$\eta = (1-\rho)\frac{\mathbb{E}[h]}{\mathbb{E}[c]},$$

where $\mathbb{E}$ refers to the unconditional expectation and $\mathbb{V}$ to the unconditional variance of the consumption and reference process.

2.3 Optimization problem

The optimization problem for each agent is

$$\max_{C_t, B_{1,t}, B_{2,t}} \sum_{t=0}^{\infty} u(c_t, h_t), \tag{1}$$

subject to the budget constraint, the law of motion of wealth and the borrowing constraint. We seek a *competitive equilibrium*, that is a sequence of asset prices $q_t = (q_{1,t}, q_{2,t})$ and portfolio holdings $B_t = (B_{1,t}, B_{2,t})$[10], such that given q_t, the choice of B_t solves (1), subject to the constraints and market clearing. For each agent we can rewrite the sequence problem into the corresponding recursive problem. Define $z_t = (\pi_t^H, \pi_t^F, Y_t^H, Y_t^F)$, $\Psi_t = (W_t, z_t, h_t)$, then

$$V_t(\Psi_t) = \max_{C_t, B_{1,t}, B_{2,t}} u(c_t - h_t) + \delta\, \mathbb{E}_t[V_{t+1}(\Psi_{t+1})], \tag{2}$$

subject to

$$\begin{aligned}
C_{NG,t} &\leq Y_{NG,t},\\
C_{TG,t} &\leq W_t + Y_{TG,t} - q_{1,t}B_{1,t} - q_{2,t}B_{2,t},\\
W_{t+1} &= B_{1,t} + S_{t+1}B_{2,t},\\
B_{1,t} &\geq \underline{B},\\
B_{2,t} &\geq 0.
\end{aligned}$$

Defining real bond holdings $b_{1,t} = \frac{B_{1,t}}{p_{1,t}}$, we impose the following market clearing conditions:

- Bonds are in zero net supply

$$\begin{aligned}
b_{1,t}^H + b_{1,t}^F &= 0,\\
b_{2,t}^H + b_{2,t}^F &= 0.
\end{aligned}$$

- Nontradable goods cannot be traded

$$\begin{aligned}
c_{NG,t}^H - y_{NG,t}^H &= 0,\\
c_{NG,t}^F - y_{NG,t}^F &= 0.
\end{aligned}$$

- Aggregate consumption in tradables is equal to aggregate endowments

$$c_{TG,t}^H + c_{TG,t}^F - y_{TG,t}^H - y_{TG,t}^F = 0.$$

[10] Through the budget constraint, portfolio holdings imply a consumption path.

$$E[\Delta s] \downarrow \qquad \Leftrightarrow \qquad i^H - i^F \uparrow$$

(a) Empirical Observation

$$\begin{tikzcd}
& y^H \uparrow \arrow[r, "corr(y^H, s) < 0"] & E[\Delta s] \downarrow \arrow[r, "payoff \uparrow"] \arrow[drr, dashed, leftrightarrow, "\text{Puzzle}"] & i^H \downarrow \arrow[r] & i^H - i^F \downarrow \\
y^H \uparrow \arrow[r] & c \uparrow \arrow[r] & H \uparrow \arrow[r] & RP \uparrow \arrow[r, "i^H \text{ riskier}"] & i^H - i^F \uparrow
\end{tikzcd}$$

(b) Model Dynamics (negative correlation)

$$\begin{tikzcd}
& y^H \uparrow \arrow[r, "corr(y^H, s) > 0"] & E[\Delta s] \uparrow \arrow[r, "payoff \uparrow"] \arrow[drr, dashed, leftrightarrow, "\text{Puzzle}"] & i^H \uparrow \arrow[r] & i^H - i^F \uparrow \\
y^H \uparrow \arrow[r] & c \uparrow \arrow[r] & H \uparrow \arrow[r] & RP \uparrow \arrow[r, "i^F \text{ riskier}"] & i^H - i^F \downarrow
\end{tikzcd}$$

(c) Model Dynamics (positive correlation)

Figure 2: Illustration of the central Mechanism

2.4 Model Dynamics

2.4.1 Incomplete markets

The only financial assets in the model are the two bonds. As they fall short of spanning the state space, markets are incomplete. In complete markets, agents would be able to fully share their risk. They could trade the aggregate income stream so that in equilibrium each agent received a constant share of aggregate income in each state. In our incomplete market setup agents do not have this possibility. Completing markets in our model would require introducing an asset whose payout depends on the change in GDP. We generally do not observe such an asset in the real world, which justifies the assumption of incomplete markets. Because, in such a setup, agents are only partly able to insure their income risk, we obtain higher and more-volatile risk premia, both necessary to explain the forward premium anomaly[11].

2.4.2 Reproducing the negative slope coefficient

Figure 2 displays the underlying mechanism in our model. The objective is to reproduce the empirical fact that $E[\Delta s]$ is negatively correlated with $i^H - i^F$. There are two major channels in the model linking exchange rates and interest rate differentials. Empirically we observe positive as well as negative correlations between exchange rates and income growth. The model dynamics are slightly different in the two cases, therefore we will now describe the dynamics for each case separetely.

[11]Engel (1996).

Negative Correlation

Firstly, we will elaborate the dynamics, when income growth is negatively correlated with the exchange rate. Please refer to Figure 2(b). The first channel is the standard UIP effect. By assumption, a positive shock to income in the home country leads to an expected appreciation ($s_t \downarrow$) of the currency. Given an appreciation of the home currency ($E[\Delta s] \uparrow$), investors will demand a lower return on the home bond ($i^H \downarrow$) according to the uncovered interest parity.

The second channel is novel and provides the explanation for the existence of the forward premium. In addition to the direct effect of a positive income shock on expected exchange rates, a positive income shock also induces more consumption and thus a higher habit level and thus risk aversion. This stimulates the home countrys demand for foreign bonds, reduces the foreign interest rate, leading to a larger interest rate differential.

If the second effect quantitatively outweighs the first effect, this could provide a possible explanation for the forward premium as the home currency appreciates, while the interest rate differential increases.

Positive Correlation

Secondly, consider the case, where income growth is positivelf correlated with the exchange rate, displayed in Figure 2(c). The underlying idea is the same as for negative correlation. Only two crucial steps change their sign. First, obviously as correlation is positive, a positive income shock leads to a depreciation ($s_t \uparrow$). However, now the investor in the home country, considers the foreign asset as more risky, therefore the second channel changes signs as well and the negative slope coefficient can still be reproduced.

Overall international diversification enables agents to hedge some of their income risk. As a result of income fluctuation and peoples' habit formation, the desire for international diversification fluctuates over time. This leads to time-varying preferences over the home versus the foreign bond, which are negatively correlated with the expected change in the exchange rate. Thus, possibly resulting in a negative correlation between interest rate differential and expected exchang rates, depending on the relative magnitude of the UIP-payoff channel, versus the habit channel.

3 Computation

In order to solve the dynamic programming problem (2) computationally, it is necessary as a preliminary step to discretize the problem to a finite number of shocks. In practice this translates into approximating the estimated processes (i.e. income and exchange rate process for each country) by a discrete shock vector and an associated transition matrix. We simply follow the standard choice in the literature and use an implementation of Tauchens algorithm (Tauchen (1986), Tauchen and Hussey (1991)).

Furthermore, we discretize the habit process into a discrete number of habit states. As the process for habits takes in theory a continuum of possible values, we always set habit at the beginning of each period to the closest level within the discrete space of possible habit values.

Equipped with shock and transition matrices, net wealth of agent A fully describes the endogenous state space, as it summarizes the past actions of agent A. Wealth of agent B can simply be deduced through market clearing. Given the relevant state space of the economy, we use standard dynamic programming techniques to solve for the competitive equilibrium.

In particular we iterate over the consumption policy. The initial policy for each agent is mostly complete debt roll-over, i.e. indebted agents pay back only a small amount of their loan in a two-period model.[12] Then, in each step of the time iteration, we solve the nonlinear system of equations (see Appendix A.3, page 18) on a finite grid over net wealth and subsequently approximate the new consumption policy with cubic splines.

There is no theorem guaranteeing the convergence to or even the existence of a policy function satisfying the dynamic programming problem.[13] However, as long as we observe convergence towards a policy function, we know that it is a solution to the infinite horizon dynamic programming problem within the computational margin of error[14].

Finally, we simulate a large number of exogenous shocks for income and exchange rates and compute many possible outcomes of the economy given the optimal policy functions. Then we perform the interest parity regression on the simulated data to test for the slope and observe additional implications of our model on various economic and financial variables.

4 Calibration

To assess our model's power to explain the forward premium anomaly, we calibrate the model to observed data for various countries. The set of countries, picked by economic significance, comprises Canada (CA), Germany (DE), Japan (JP), United Kingdom (UK) and the United States (US). The analysis puts special emphasis on the country pair United States and Japan, since they are the two largest economies, represent two dominant currencies; and most importantly as the anomaly is particularly robust for this country pair[15].

4.1 Currency baskets

To calibrate the Markov Chain, we need inflation and income data. While income data is readily available, tradable good inflation is not. Broad price indices, such as the Consumer Price Index (CPI), are not suitable since they incorporate both tradable and nontradable prices. More seriously, the usage of these indices would result in a model-implied exchange rate process that is completely different from the one observed in the data. This stands in sharp contrast to the paper's main goal of explaining the relationship between exchange rates and interest rates.

To avoid the above issues connected with price indices, we exploit the fact that the relation between tradable good prices in two countries is given as the exchange rate under PPP. More precisely, tradable good inflation in one country is is measured as the valution

[12] In some cases of relatively high habit levels, this turned out to be not fully robust. In these cases, we first solve for lower levels of habit and then use the solutions as starting policies for higher levels of habit.

[13] For a discussion see Duffie, Geanakoplos, Mas-Colell, and McLennan (1994) and Kubler and Schmedders (2005).

[14] The maximum deviations we allowed for were 10^{-10} for each individual FOC and 10^{-7} for the maximum change in consumption policies.

[15] Han (2004) performs a large cross-country, cross-period comparison to test whether the anomaly is universal. Performing regressions for varying time horizons in the range 1979 to 1998, he finds the percentage of observed negative beta coefficients is 96% for the US and Japan.

Country	AU	CA	CH	DE	DK	FR	NL	NO	SE	SG	UK
Share	3.3	11.9	4.6	24.8	2.6	14.5	9.8	2.5	4.0	6.2	15.7

Table 1: Composition of the reference currency basket for the country-pair Japan-US

of that country's currency against a broad index of other countrys' currencies. For each currency pair, we construct a currency basket of all remaining countries.[16] Tradable good inflation for one country is then derived as a weighted average of exchange rates of this country to all other countries in the basket. More formally, for a given country pair a, b, tradable good inflation is given as

$$\Pi_j = \sum_{\forall i \neq a,b} w_i S^{i,j} \qquad j = a, b,$$

where w_i is the weight of currency i in the basket and $S^{i,a}$ is the price of currency a in terms of currency i.

The weights reflect the importance of country i for tradable good inflation in country j. Therefore, we choose the weights as the shares on world trade. More precisely, the relative vlaue of the sum of each country's aggregate imports and exports with country i. As an example the currency basket for the country-pair US - JP is displayed in Table 1. From here on, we will refer to the tradable good inflation process of a country simply as the country's (basket) exchange rate.

4.2 Estimation of exogeneous state variables

Equipped with the exchange rate process for each country, we can estimate the majority of the model parameters from data. In particular we estimate the exogenous shocks to income and inflation with a Vector Autoregressive Regression (VaR) of order one as

$$\begin{pmatrix} \Delta y_t \\ \Delta s_t \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} \theta_y & \theta_{y,s} \\ \theta_{s,y} & \theta_s \end{pmatrix} \begin{pmatrix} \Delta y_{t-1} \\ \Delta s_{t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{t,y} \\ \epsilon_{t,s} \end{pmatrix}, \tag{3}$$

where Δy and Δs refer to the change in logs of income and exchange rates against the currency basket[17], α and θ are the estimated coefficients and ϵ residuals.

4.2.1 Inflation of Tradables — Persistence

It turns out empirically that in the VaR regression (3) the coefficients $\theta_{y,s}$ and θ_s are universally insignificant. Thus, none of the analyzed countries show signs of significant persistence in exchange rates. Table 2 displays the persistence estimates and p-values for 21 different currencies. Only two currencies even come close to the significance treshold. Therefore we simply set these values for all countries to zero. Arguably this assumption makes a difference for our model economy. Without it, a second channel for a direct payoff effect opens up, working against the above proposed habit effect. However, in our opinion, the empirical evidence legitimates the assumption of zero exchange rate persistence.

4.2.2 Markov Chain Approximation

[16] It is convenient to exclude both countries in the basket in order to still obtain an exact match of the exchange rate when applying PPP.

[17] Exchange Rates are denoted as reference currency over home currency.

Ctry	Parameter				Data	[s.e.]	Model
JP	FX growth	Mean	$\mathbb{E}[\Delta s]$		1.009	0.005	1.009
		Std.	$\sigma[\Delta s]$		0.055	0.004	0.044
	Income growth	Mean	$\mathbb{E}[\Delta y]$		1.005	0.001	1.005
		Std.	$\sigma[\Delta y]$		0.011	0.001	0.009
		Pers.	θ_y		0.233	0.093	0.166
		Corr.	$\rho_{\Delta s,\Delta y}$		-0.208	0.089	-0.207
US	FX growth	Mean	$\mathbb{E}[\Delta s]$		0.998	0.004	0.998
		Std.	$\sigma[\Delta s]$		0.045	0.003	0.035
	Income growth	Mean	$\mathbb{E}[\Delta y]$		1.007	0.001	1.007
		Std.	$\sigma[\Delta y]$		0.009	0.000	0.007
		Pers.	θ_y		0.350	0.074	0.254
		Corr.	$\rho_{\Delta s,\Delta y}$		-0.030	0.079	-0.030

Table 3: Empirical and Model Moments

The remaining results of the VaR regression (3) need to be discretized to accomodate our model. Therefore we discretize the process for each country into a Markov Chain with 9 states. Table 3 displays various statistics describing the result of the empirical estimation for the country pair US - JP, showing the high quality of the Markov Chain Approximation. There are some minor deviations in standard deviations and persistences due to some bias in the discretization, but correlation, clearly a crucial parameter, is matched precisely. Similar accuracy is achieved for the remaining country pairs.

Ctry1	Ctry2	Pers.	Pval
AU	CH	0.01	0.90
AU	DE	-0.01	0.89
AU	FR	0.02	0.86
AU	JP	0.07	0.38
AU	UK	-0.05	0.52
AU	US	0.07	0.38
CH	DE	-0.08	0.37
CH	FR	0.01	0.94
CH	JP	0.10	0.21
CH	UK	0.02	0.81
CH	US	0.02	0.84
DE	FR	-0.02	0.82
DE	JP	0.11	0.20
DE	UK	0.09	0.34
DE	US	0.09	0.35
FR	JP	0.17	0.05
FR	UK	0.10	0.31
FR	US	0.14	0.15
JP	UK	0.13	0.11
JP	US	0.06	0.49
UK	US	0.16	0.05

Table 2: Persistence estimates and P-Values of Log Returns on Exchange Rates

4.3 Remaining parameters

Some parameters, especially preference parameters, cannot easily be estimated from data. Our calibration for these parameters is summarized in table 4. The share of nontradables in each country is set to 0.5[18]. The discount factor is set to 0.99 which is standard in the literature for quarterly horizons. Relative risk aversion (γ) is set to 2.00, which is consistent with economic experiments and is also standard in the literature.

The habit volatility is chosen as approximately half the level of income volatility. This reflects the

[18]Burstein, Eichenbaum, and Rebelo (2005) estimated the share of nontradables as 0.43 for US and 0.48 for Korea.

fact that habit is implicitly driven by changes in in-
come, yet varies less than income. Furthermore the average level of habit is chosen such that it matches the model implied β with the empirically observed β. The resulting average habit level ($\bar{h}$) amounts to 0.95.[19]

Habit for each country is discretized on a linear grid, resulting in two additional parameters to be picked in our parameterization. Firstly, we set the grid's range. We calculate the upper bound as the average habit plus some scaling factor times the volatility of habit ($\mathbb{E}[H] + \zeta\sigma[H]$) and the lower bound accordingly. Secondly, we set the scaling factor to 1 and choose three points for the size of the grid.

Finally, we also need to choose the discretization of wealth. We assume that countries cannot accumulate debt beyond 10% of their net wealth. We choose 11 gridpoints for the approximation of the wealth state space.[20]

Parameter (Quarterly)		Value
Share of Nontradables	α	0.50
Discount factor	δ	0.99
Risk aversion	γ	2.00
Average Habit Level	$\mathbb{E}[H]$	0.91
Habit volatitility	$\sigma[H]$	0.005
Habit scaling factor	ζ	1
Habit grid size		3
wealth boundaries	$\bar{w}^H = \bar{w}^F$	-0.1
wealth grid size		11

Table 4: Calibrated Parameters

5 Results

5.1 Simulation

Given the optimal policies, we simulate the model economy. Table 5 displays the intercept and slope coefficient of a UIP regression using our model economy's data and corresponding actual data for the country pair US - JP. Our model matches both the slope and the intercept reasonably well.

5.2 Impact of habit

Figure 3 displays the impact of the average implied risk aversion on the slope coefficient for three different levels of habit volatility. The implied risk aversion, displayed on the abcisse is strictly increasing in habit and is locally interpretable as the relative risk aversion of an

[19]Due to the discretization of the habit grid, the actual empirical moments might vary slightly from these values. Therefore we report in table ? also the empirical moments.

[20]Arguably the choices for the discretization of habit and wealth are arbitrary. We are currently working on a robustness section, which will show that the calibration of habit (e.g. the choice of $\mathbb{E}[H]$ and $\sigma[H]$) might vary in these parameters, yet the overall results remain unchanged for values within a reasonable range.

Parameter (Quarterly)		Data	[s.e.]	Model	[s.e.]
FP - intercept	α	0.03	[0.01]	0.02	[< 0.001]
FP - slope	β	-0.65	[0.25]	-0.44	[0.058]

Table 5: Exemplary simulation of Japan versus United States

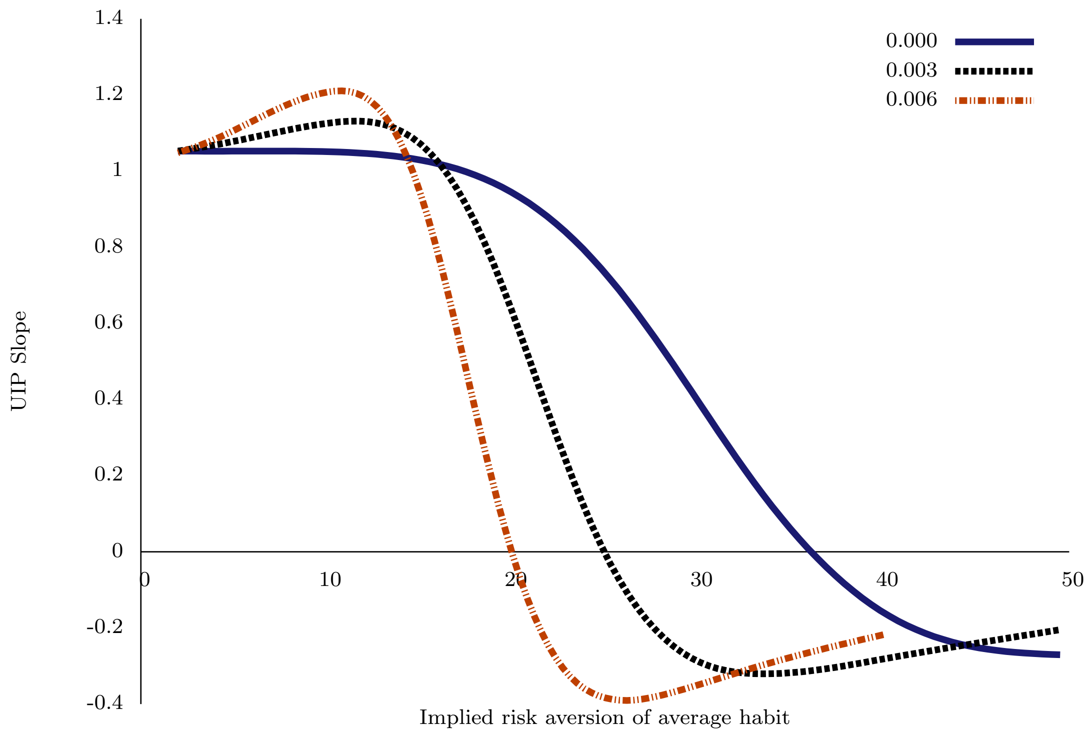


Figure 3: UIP slope for the country pair Japan - US over average habit. The three lines represent different levels of habit volatility.

agent with power utility. It is a common measure used to assess the risk aversion implied by different habit levels and thus allows for comparison across models. In the case of relatively small habit levels the model simply reproduces the uncovered interest parity, $\beta \approx 1$. Habit is too small to create large enough risk premia, which could break the direct link between interest rate differentials and subsequent exchange rate fluctuations.

As average habit levels and thus risk aversion rise, the habit induced international diversification effect becomes more and more relevant and encompasses the primary payoff effect.

For a constant level of habit [21] (the solid line), an implied risk aversion of 35 is needed to create sufficiently high risk premia for a negative UIP slope. As we increase the volatility of habit (the two dashed lines), lower and lower levels of average habit are necessary.

Figure 4 shows the variation in the UIP slope, when changing the volatility of habit. Zero volatility results in the standard prediction of a slope coefficient close to one. When

[21] Please not that this does not imply constant risk aversion, since consumption varies and risk aversion is given as $\gamma/(c-h)$

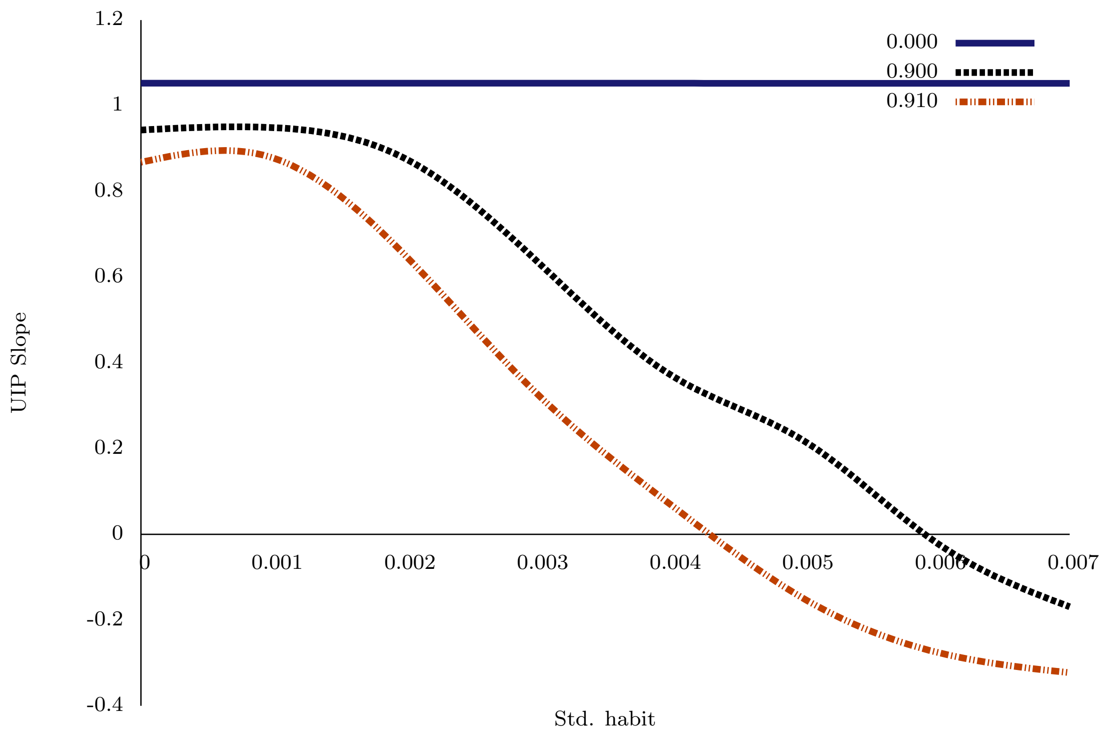


Figure 4: UIP slope for the country pair Japan - US for over standard deviation (volatility) of habit. The three lines represent different levels of average habit.

the average habit level is zero, e.g. risk aversion is around two, an increase in the volatility of habit has no impact on the slope coefficient.

Habit volatility starts to impact the slope coefficient, only when average habit levels are higher (e.g. 0.9 and 0.91 which roughly amounts to a risk aversion of around 20). As we increase habit volatility the slope coefficient decreases almost monotonously.

5.3 Multiple countries

In addition to the detailed analysis for the country-pair US-Japan, we conduct our analysis for five additional country pairs. These are formed by pairwise combination of Germany, Japan, United Kingdom and the United States. Initially, we estimate a Markov Chain for each country pair individually as described in the calibration section, then we solve for optimal policies and finally compute the model implied UIP slope coefficient, which we compare to the data.

For this purpose, we look at two different calibration scenarios. Initilaly, we assume that every country has the same habit level. Then, we allow for a different habit level for each country pair.[22]

[22] It is somehow inconsistent to assume different habit levels for the same country, when compared to different other countries. Ideally, we would like to have a country-specific habit level, instead of a country-pair specific country level. Due to the introduced interdependencies this becomes computationally very intense and constitutes current work in progress.

Ctry1	Ctry2	Emp. β	[s.e.]	Model β
JP	DE	0.116	[0.296]	-0.196
JP	US	-0.628	[0.251]	-0.218
UK	DE	0.274	[0.186]	-0.034
UK	JP	-1.049	[0.386]	-0.205
UK	US	-0.043	[0.198]	0.069
US	DE	-0.026	[0.220]	0.374

Table 6: Empirical versus Model implied UIP slope coefficient for a common habit calibration ($E[H] = 0.95, \sigma[H] = 0.006$)

Ctry1	Ctry2	EA	sigmaA	Emp. β	[s.e.]	Model β
JP	DE	0.880	0.006	0.116	[0.296]	0.127
JP	US	0.930	0.005	-0.628	[0.251]	-0.416
UK	DE	0.800	0.003	0.274	[0.186]	0.342
UK	JP	0.900	0.007	-1.049	[0.386]	-0.682
UK	US	0.950	0.006	-0.043	[0.198]	0.069
US	DE	0.950	0.007	-0.026	[0.220]	-0.036

Table 7: Empirical versus Model implied UIP slope coefficient for varying levels of habit

5.3.1 Common habit level

Table 6 shows the result for one common habit calibration. Here we want to almost completely tie our hands in terms of the parameter choice. Most parameters are either determined by empirical data (e.g. the calibration of income and exchange rate processes) or have almost no impact on the results (e.g. the technical parameters for the discretization).[23]. The only parameters we vary to match the empirical slope are $E[H] = 0.9$ and $\sigma[H] = 0.006$. Despite the strongly reduced degrees of freedom, the model remains capable of explaining the empirically observed slope coefficient within two standard deviations for each country pair.

5.3.2 Varying habit level

Finally, we vary the two parameters of the habit process freely across country pairs. Table 7 displays the resulting slope coefficients against the empirical counterparts. In this setup the model is able to reproduce the empirical slope coefficient within one standard deviation.

6 Conclusion

The present paper proposes an explanation for the forward premium anomaly in a rational expectation framework. The habit-based model with endogenous saving decisions is able to reproduce the empirical anomaly for six different country pairs.

[23]We are currently working on a robustness section, demonstrating the claim of no impact on the results.

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A First order conditions

A.1 Normalization

In order to solve our model we first rewrite the nominal problem (eq. 2) into the according real problem. For this purpose we set price level of nontradables to 1 and the price level of tradables as p_1 respectively p_2 for each country.

Let us denote $R_{1,t} = \frac{1}{1+\pi_t^H}$ and $R_{2,t} = \frac{1}{1+\pi_t^F}$ as the real returns of each bond. Furthermore we redefine the shock vector and state space in real terms as follows: $z_t = (R_{1,t}, R_{2,t}, y_t^H, y_t^F)$ and $\Psi_t = (w_t, z_t, h_t)$. Then the dynamic programming problem transforms into

$$V_t(\Psi_t) = \underset{c_t, b_{1,t}, b_{2,t}}{max} u(c_t, h_t) + \beta\, \mathbb{E}_t[V_{t+1}(\Psi_{t+1})],$$

subject to

$$\begin{aligned}
c_{TG,t} &\leq w_t + y_{TG,t} - q_{1,t}b_{1,t} - q_{2,t}b_{2,t},\\
c_{NG,t} &\leq y_{NG,t},\\
w_{t+1} &= R_{1,t+1}b_{1,t} + R_{2,t+1}b_{2,t},\\
b_{1,t} &\geq \frac{\bar{b_1}}{E[R_{1,t+1}]},\\
b_{2,t} &\geq \frac{\bar{b_2}}{E[R_{2,t+1}]},\\
\bar{w} &\leq b_{1,t}q_{1,t} + b_{2,t}q_{2,t},\\
c_{TG,t} &\geq h_{TG,t}.
\end{aligned}$$

The last equation is unnecessary in theory. The utility function is simply not defined for values smaller than 0. However, it is necessary to enforce the condition for computational reasons, as a solver might try to step outside and evaluate the function for $c_{TG,t} < (1-\eta)h_t$. Depending on the choice of the risk aversion, this could either result in complex numbers or even lead to a potential solution of the equation system, with no economic meaning.

To facilitate computation, we additionally normalize each agent's problem with factors κ^A and κ^B respectively. Given homothetic preferences the individual's policies are simply scaled by the normalization factor. Thus, the equilibrium remains unchanged under the appropriate adjustment of market clearing conditions (see next section).
Define $\tilde{z}_t = (R_{1,t+1}, R_{2,t+1}, \tilde{y}_t^H, \tilde{y}_t^F)$ and $\tilde{\Psi}_t = (\tilde{w}_t, \tilde{z}_t, \tilde{h}_t)$ where $\kappa\tilde{y}_t = y_t$, then

$$V_t(\tilde{\Psi}_t) = \underset{\tilde{c}_t, \tilde{b}_{1,t}, \tilde{b}_{2,t}}{max} u(\tilde{c}_t, \tilde{h}_t) + \beta\, \mathbb{E}_t[V_{t+1}(\tilde{\Psi}_{t+1})],$$

subject to

$$
\begin{aligned}
\tilde{c}_{TG,t} &\leq \tilde{w}_t + \tilde{y}_{TG,t} - q_{1,t}\tilde{b}_{1,t} - q_{2,t}\tilde{b}_{2,t},\\
\tilde{c}_{NG,t} &\leq \tilde{y}_{NG,t},\\
\tilde{w}_{t+1} &= R_{1,t+1}\tilde{b}_{1,t} + R_{2,t+1}\tilde{b}_{2,t},\\
\tilde{b}_{1,t} &\geq \frac{\tilde{\bar{b}}}{E[R_{1,t+1}]},\\
\tilde{b}_{1,t} &\geq \frac{\tilde{\bar{b}}_1}{E[R_{1,t+1}]},\\
\tilde{b}_{2,t} &\geq \frac{\tilde{\bar{b}}_2}{E[R_{2,t+1}]},\\
\tilde{\bar{w}} &\leq \tilde{b}_{1,t}q_{1,t} + \tilde{b}_{2,t}q_{2,t},\\
\tilde{c}_{TG,t} &\geq \tilde{h}_{TG,t}
\end{aligned}
$$

A.2 Kuhn-Tucker Conditions

Concavity of the utility function allows us to impose equality for the first two conditions. Inserting conditions two and three and denoting for simplicity $u(c_t, h_t) = u(c_{TG,t})$ we can write the Lagrangian as

$$
\begin{aligned}
\mathcal{L} = u(\tilde{c}_{TG,t}) + \beta\,\mathbb{E}_t[V_{t+1}(\tilde{\Psi}_{t+1})] \;&+\; \mu\,(\tilde{y}_{TG,t} + \tilde{w}_t - q_{1,t}\tilde{b}_{1,t} - q_{2,t}\tilde{b}_{2,t} - \tilde{c}_{TG,t})\\
&+\; \lambda_1\,(\tilde{b}_{1,t}E_t[R_{1,t+1}] - \tilde{\bar{b}}_1)\\
&+\; \lambda_2\,(\tilde{b}_{2,t}E_t[R_{2,t+1}] - \tilde{\bar{b}}_2)\\
&+\; \lambda_3\,(\tilde{b}_{1,t}q_{1,t} + \tilde{b}_{2,t}q_{2,t} - \tilde{\bar{w}})\\
&+\; \lambda_{nn}\,(\tilde{c}_{TG,t} - \tilde{h}_{TG,t}).
\end{aligned}
$$

Deriving the Lagrangian with respect to each choice variable, adding the conditions and restrictions on the Lagrange multipliers provides us with the following system of first order conditions:

$$\frac{\partial \mathcal{L}}{\partial \tilde{c}_{TG,t}} = u'(\tilde{c}_{TG,t}) - \mu + \lambda_{nn} \overset{!}{=} 0,$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\tilde{b}_{1,t}} &= \beta \mathbb{E}_t \left[\frac{\partial V_{t+1}}{\partial \tilde{b}_{1,t}} \right] - \mu q_{1,t} + \lambda_1 E[R_{1,t+1}] + \lambda_3 q_{1,t} \\
&= \beta \sum_{z_{t+1} \in \Gamma(z_t)} \left[\pi(z_{t+1}|z_t) \frac{\partial u}{\partial \tilde{c}_{TG,t+1}} \frac{\partial \tilde{c}_{TG,t+1}}{\partial \tilde{w}_{t+1}} \frac{\partial \tilde{w}_{t+1}}{\partial \tilde{b}_{1,t}} \right] - \mu q_{1,t} + \lambda_1 E[R_{1,t+1}] + \lambda_3 q_{1,t} \\
&= \beta \sum_{z_{t+1} \in \Gamma(z_t)} \left[\pi(z_{t+1}|z_t) \frac{\partial u}{\partial \tilde{c}_{TG,t+1}} \frac{\partial \tilde{c}_{TG,t+1}}{\partial \tilde{w}_{t+1}} R_{1,t+1} \right] - \mu q_{1,t} + \lambda_1 E[R_{1,t+1}] + \lambda_3 q_{1,t} \overset{!}{=} 0, \\
\frac{\partial \mathcal{L}}{\tilde{b}_{2,t}} &= \beta \mathbb{E}_t \left[\frac{\partial V_{t+1}}{\partial \tilde{b}_{2,t}} \right] - \mu q_{2,t} + \lambda_2 + \lambda_3 q_{2,t} \\
&= \beta \sum_{z_{t+1} \in \Gamma(z_t)} \left[\pi(z_{t+1}|z_t) \frac{\partial u}{\partial \tilde{c}_{TG,t+1}} \frac{\partial \tilde{c}_{TG,t+1}}{\partial \tilde{w}_{t+1}} \frac{\partial \tilde{w}_{t+1}}{\partial \tilde{b}_{2,t}} \right] - \mu q_{2,t} + \lambda_2 E[R_{2,t+1}] + \lambda_3 q_{2,t} \\
&= \beta \sum_{z_{t+1} \in \Gamma(z_t)} \left[\pi(z_{t+1}|z_t) \frac{\partial u}{\partial \tilde{c}_{TG,t+1}} \frac{\partial \tilde{c}_{TG,t+1}}{\partial \tilde{w}_{t+1}} R_{2,t+1} \right] - \mu q_{2,t} + \lambda_2 E[R_{2,t+1}] + \lambda_3 q_{2,t} \overset{!}{=} 0,
\end{aligned}$$

$$\begin{aligned}
\lambda_1(\tilde{b}_{1,t} E[R_{1,t+1}] - \tilde{\bar{b}}_1) &= 0, \\
\lambda_2(\tilde{b}_{2,t} E_t[R_{2,t+1}] - \tilde{\bar{b}}_2) &= 0, \\
\lambda_3(\tilde{b}_{1,t} q_{1,t} + \tilde{b}_{2,t} q_{2,t} - \bar{w}) &= 0 \\
\lambda_{nn}(\tilde{c}_{TG,t} - (1-\eta) h_t) &= 0,
\end{aligned}$$

$$\lambda_1 \geq 0, \quad \lambda_2 \geq 0, \quad \lambda_3 \geq 0, \quad \lambda_{nn} \geq 0$$

where $\Gamma(z_t)$ denotes all states possibly following z_t and $\pi(z_{t+1}|z_t)$ are the transition probabilities.

The same set of equations exists for the second agent and is completed by the market clearing conditions

$$\begin{aligned}
\kappa^A \tilde{b}^A_{1,t} + \kappa^B \tilde{b}^B_{1,t} &= 0, \\
\kappa^A \tilde{b}^A_{2,t} + \kappa^B \tilde{b}^B_{2,t} &= 0, \\
\kappa^A \tilde{c}^A_{TG,t} + \kappa^B \tilde{c}^B_{TG,t} &= \kappa^A \tilde{y}^A_{TG,t} + \kappa^B \tilde{y}^B_{TG,t}.
\end{aligned}$$

The market clearing conditions apply to the unnormalized economy. Thus, terms are unnormalized with the agent specific normalization coefficient.

A.3 Alternative Conditions

It's computationally inconvenient to work with the inequality constraints for the lagrange-multiplier. Therefore we use the following reformulation as described in Zangwill and Garcia (1981).

The key is to replace the Lagrange multipliers by slacks, which are decomposed into a positive and negative part

$$\alpha^+ = [\ max(0, \alpha)\ ]^k,$$
$$\alpha^- = [\ max(0, -\alpha)\ ]^k.$$

One would expect a k of 2 or 3 to work best to avoid any kinks in the nonlinear system of equations. However, surprisingly, we find that $k = 1$ outperforms any other choice.

This allows us to rewrite the first order conditions into the following equivalent system

$$\frac{\partial \mathcal{L}}{\partial \tilde{c}_{TG,t}} = u'(\tilde{c}_{TG,t}) - \mu + \alpha_{nn} + \quad \stackrel{!}{=} 0,$$
$$\frac{\partial \mathcal{L}}{\tilde{b}_{1,t}} = \beta \sum_{z_{t+1} \in \Gamma(z_t)} \left[\pi(z_{t+1}|z_t) \frac{\partial u}{\partial \tilde{c}_{TG,t+1}} R_{1,t+1} \right] - \mu q_{1,t} + \alpha_1 E_t[R_{1,t+1}] + \alpha_3 q_{1,t} \quad \stackrel{!}{=} 0,$$
$$\frac{\partial \mathcal{L}}{\tilde{b}_{2,t}} = \beta \sum_{z_{t+1} \in \Gamma(z_t)} \left[\pi(z_{t+1}|z_t) \frac{\partial u}{\partial \tilde{c}_{TG,t+1}} R_{2,t+1} \right] - \mu q_{2,t} + \alpha_2 E_t[R_{2,t+1}] + \alpha_3 q_{2,t} \quad \stackrel{!}{=} 0$$

$$\alpha_1 - (\tilde{b}_{1,t} E[R_{1,t+1}] - \tilde{\tilde{b}}_1) = 0,$$
$$\alpha_2 - (\tilde{b}_{2,t} E_t[R_{2,t+1}] - \tilde{\tilde{b}}_2) = 0,$$
$$\alpha_3 - (\tilde{b}_{1,t} q_{1,t} + \tilde{b}_{2,t} q_{2,t} - \bar{w}) = 0,$$
$$\alpha_{nn} - (\tilde{c}_{TG,t} - (1 - \eta) h_t) = 0,$$

α can be interpreted as the shadow price of the borrowing constraint. If the constraint does not bind then α is negative and α^- positive which equalizes the $\geq$ constraint. So essentially the borrowing constraint does not have a shadow price. If α is positive α^+ is positive showing up in the FOCs while the borrowing constraint exactly binds. The higher α the more costly is the constraint.