$$V(x) = x_1'Q_1x_1 + \sum_{i=1}^{n-n_1} z_i^2 - \alpha \int_0^{c'x} f_1(\theta)\, d\theta \tag{10}$$

where $z_i|_{x_1=0}$ are linear independent combinations in the components of x_2.

Finally select $\zeta_0 > 0$ sufficiently small such that, for all $|x| \leqslant \zeta_0$,

$$\alpha \int_0^{c'x} f_1(\theta)\, d\theta \leqslant \tfrac{1}{2}\left(\sum_{i=1}^{n-n_1} z_i^2 + x_1'x_1\right). \tag{11}$$

This is possible since $f_1^{(1)}(0) = 0$. Combining (10) and (11) with the fact that $\dot{V}(x) \leqslant 0$ for all $t \geqslant 0$ shows that, for some Q_2,

$$\sum_{i=1}^{n-n_1} z_i^2 \leqslant x_1'Q_2x_1 \tag{12}$$

if $|x| \leqslant \zeta_0$.

Define $N(x) = \{x;\ |x| \leqslant \zeta_0;\ C(x) \leqslant \epsilon_0^2\}$, $C(x) = x_1'\hat{C}x_1$, where $\hat{C} = \hat{C}' > 0$ is the solution of $\hat{C}A_{11} + A_{11}'\hat{C} = I$.

1) An initial state x_0 can be selected arbitrarily close to the origin, such that the corresponding trajectory leaves $N(x)$ after a finite-time interval. Indeed, suppose that $|x| \leqslant \zeta_0$ for all $t \geqslant 0$. Then

$$\dot{C}(x) = x_1'x_1 - 2b_1'x_1f_1(c'x)$$

where for $C(x) = \epsilon^2$ the first term in the right-hand side is infinitely small of second order and positive, while the second term is infinitely small of third order, because of (12) and $f_1^{(1)}(0) = 0$. Hence there exist $\epsilon_0^2 > 0$ and $\delta > 0$, such that $\dot{C}(x) - \delta C(x) \geqslant 0$ for $C(x) \leqslant \epsilon_0^2$, which proves the existence of a finite t_0 such that $C[x(t_0)] > \epsilon_0^2$.

2) This trajectory cannot reenter $N(x)$ for $t > t_0$. Indeed, if it does so as time t, there are two possibilities. Either $C[x(t)] \leqslant \epsilon_0^2$, $|x(t)| = \zeta_0$, which is impossible by (12) for ϵ_0^2 sufficiently small, or $C[x(t)] = \epsilon_0^2$, $|x(t)| \leqslant \zeta_0$. Now the trajectory cannot enter $N(x)$ as $\dot{C}(x) > 0$. It follows that

$$C(x) + x'x \geqslant \min(\zeta_0^2, \epsilon_0^2), \quad \forall\ t \geqslant t_0,$$

which implies (3).

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Time-Varying System Stability–Interchangeability of the Bounds on the Logarithmic Variation of Gain

M. K. SUNDARESHAN AND M. A. L. THATHACHAR

***Abstract*—A frequency-domain criterion for the L_2-stability of systems containing a single time-varying gain in an otherwise time-invariant linear feedback loop is given. This is an improvement upon the earlier criteria presented by the authors in permitting an interchangeability of the allowable bounds on the logarithmic variation of the gain.**

I. Introduction

The analysis of the L_2-stability of a feedback system consisting of a cascade of a linear time-invariant causal operator G in L_2 and a time-varying gain $k(t)$ was the subject of a recent publication [1], in which certain frequency-domain criteria permitting the use of noncausal multipliers were presented. The principal factor of these results is the employment of an upper and a lower bound on the rate of variation of the gain $(1/k(t)\, dk(t)/dt)$, these bounds being determined from certain allowable "shifts" in the causal and the anticausal parts of the multiplier. However, an examination of the most general result of [1] will give one the impression that the shift in the causal part of the multiplier is associated with the upper bound, while the shift in the anticausal part is associated with the lower bound. The main purpose of the present correspondence is to emphasize the fact that this is not mandatory and, in fact, for linear systems, the bounds on $(1/k(t))(dk(t)/dt)$ are interchangeable.[1]

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II. Problem Formulation

Notations and Definitions

While the notations used in the earlier paper [1] will be followed, certain additional notations will be introduced now. Let R, R^+, and J^+ denote, respectively, the real numbers, the nonnegative real numbers, and the nonnegative integers. An operator H in $L_2(L_{2e})$ is a single-valued mapping of $L_2(L_{2e})$ into itself. H is a time-invariant convolution operator in $L_2(L_{2e})$ if

$$Hx(t) = \{h_i, h(t)\} \circledast x(t), \quad (\circledast \text{ denotes convolution})$$

$$= \sum_{i \in J^+} h_i x(t - \tau_i) + \int_{-\infty}^{+\infty} h(\tau)\, x(t - \tau)\, d\tau \qquad \forall\ x(\cdot) \in L_2(L_{2e}),$$

where $\{\tau_i\}$, $i \in J^+$, is a sequence in R^+ and $\{h_i\}$, $i \in J^+$, is a sequence in R. $H(j\omega)$, the Fourier-transform of the kernel $\{h_i, h(\cdot)\}$ of H is given by

$$H(j\omega) = \sum_{i \in J^+} h_i \exp(-j\omega\tau_i) + \int_{-\infty}^{+\infty} h(t) \exp(-j\omega t)\, dt.$$

Let $\mathcal{B}$ denote the Banach algebra of linear bounded time-invariant convolution operators H in L_2, with an identity E. An operator $H \in \mathcal{B}$ is said to be regular in $\mathcal{B}$ if $H^{-1} \in \mathcal{B}$. Let $\mathcal{B}_c$ and $\mathcal{B}_{ac}$ denote, respectively, the subalgebras of $\mathcal{B}$ of causal and anticausal operators (for the definition of causality, see [1]).

Let $\mathcal{K}$ be the class of memoryless time-varying operators K in L_{2e}, defined by $Kx(t) = k(t)\, x(t)\ \forall\ x(\cdot) \in L_{2e}$, $0 < \inf k(t) \leqslant k(t) \leqslant \sup k(t) < \infty\ \forall\ t \in R^+$. Let $\mathcal{K}^\beta \subset \mathcal{K} \ni K \in \mathcal{K}^\beta \Longrightarrow dk(t)/dt \leqslant 2\beta\, k(t)\ \forall\ t \in R^+$ and some $\beta \in R^+$, and let $\mathcal{K}_\alpha \subset \mathcal{K} \ni K \in \mathcal{K}_\alpha \Longrightarrow dk(t)/dt \geqslant -2\alpha k(t)\ \forall\ t \in R^+$ and some $\alpha \in R^+$. Let $\mathcal{K}_\alpha{}^\beta = \mathcal{K}_\alpha \cap \mathcal{K}^\beta$. It is simple to note that $K \in \mathcal{K} \Longrightarrow K^{-1} \in \mathcal{K}$ and $K \in \mathcal{K}_\alpha{}^\beta \Longrightarrow K^{-1} \in \mathcal{K}_\beta{}^\alpha$.

System: The system (Fig. 1) is described by the input–output relations $e_1(\cdot) = u_1(\cdot) - w_2(\cdot)$, $e_2(\cdot) = u_2 + w_1(\cdot)$, with $w_1(\cdot) = Ge_1(\cdot)$, $G \in \mathcal{B}_c$ and $w_2(\cdot) = Ke_2(\cdot)$, $K \in \mathcal{K}$.

Problem: Given that $u_1(\cdot)$, $u_2(\cdot) \in L_2$, and $e_1(\cdot)$, $e_2(\cdot) \in L_{2e}$, find conditions on G and K which ensure that $e_1(\cdot)$, $e_2(\cdot) \in L_2$.

III. Main Result

Theorem

If there exists an operator $M \in \mathcal{B}$ such that

$$M \text{ is regular in } \mathcal{B} \tag{1}$$

$$M = M_1 + M_2 \ni M_1 \in \mathcal{B}_c \text{ and } M_2 \in \mathcal{B}_{ac} \tag{2}$$

$$\operatorname{Re} M(j\omega)\, G(j\omega) \geqslant \delta > 0\ \forall\ \omega \in R \tag{3}$$

$$\operatorname{Re} M_1(j\omega - \beta) \geqslant 0\ \forall\ \omega \in R \text{ and some } \beta \in R^+ \tag{4}$$

and

$$\operatorname{Re} M_2(j\omega + \alpha) \geqslant 0\ \forall\ \omega \in R \text{ and some } \alpha \in R^+, \tag{5}$$

then the system under consideration is L_2-stable (i.e., $u_1(\cdot)$, $u_2(\cdot) \in L_2 \Longrightarrow e_1(\cdot)$, $e_2(\cdot) \in L_2$) for all $K \in \mathcal{K}_\alpha{}^\beta \cup \mathcal{K}_\beta{}^\alpha$.

[1] It appears that this property does not hold in the case of systems containing an additional nonlinear operator in the loop.

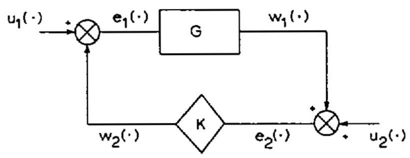

Fig. 1. The feedback system under consideration.

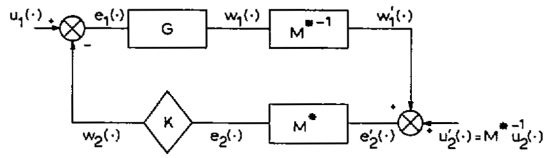

Fig. 2. System transformed with the introduction of multipliers.

Proof: A proof of the theorem will be given only for the case $K \in \mathcal{K}_\beta{}^\alpha$, since it has been proved for the other case of $K \in \mathcal{K}_\alpha{}^\beta$ in [1]. (See [1, theorem 3]. It must be noted that this theorem is proved in [1] under a more restrictive condition of $\text{Re } M_1(j\omega - \beta) + \text{Re } M_2(j\omega + \alpha) \geqslant \epsilon > 0 \ \forall\ \omega \in R$, than is implied by the present conditions (4) and (5). However, the proof with the presently employed relaxed conditions follows as will be detailed below for the case $K \in \mathcal{K}_\beta{}^\alpha$.)

We will follow the same method of proof as was employed in [1], viz., the application of the positivity theorem after the introduction of multipliers into the loop [2],[3]. Let us transform the system, as shown in Fig. 2, by introducing the operators M^* and M^{*-1} (M^* denotes the "adjoint" of M). Note that these are well-defined operators in L_2 since[2] $M \in \mathcal{B} \Longrightarrow M^* \in \mathcal{B}$ and M regular in $\mathcal{B} \Longrightarrow M^{-1} \in \mathcal{B} \Longrightarrow (M^{-1})^* \in \mathcal{B}$ and, further, $(M^{-1})^* = M^{*-1}$. It is now sufficient to prove, in view of the positivity theorem [2],[3], that: 1) M^{*-1} admits a factorization $M^{*-1} = M_{ac}M_c \ni M_c \in \mathcal{B}_c$, $M_{ac} \in \mathcal{B}_{ac}$, and M_c and M_{ac} are regular in $\mathcal{B}_c$ and $\mathcal{B}_{ac}$, respectively; 2) $M^{*-1}G$ is strongly positive with finite gain; and 3) KM^* is positive (for the definitions of positivity, strong positivity, and finiteness of gain of operators, see [1]).

1) *Factorization of M^{*-1}:* From (2),

$$\text{Re } M(j\omega) = \text{Re } M_1(j\omega) + \text{Re } M_2(j\omega) \geqslant 0 \ \forall\ \omega \in R,$$

because of (4) and (5), which implies M is positive.

Now,

$$\langle x(\cdot), M^{*-1}x(\cdot)\rangle = \langle M^*y(\cdot), y(\cdot)\rangle,\ y(\cdot) = M^{*-1}x(\cdot) = \langle y(\cdot), My(\cdot)\rangle,$$

and hence M positive $\Longleftrightarrow M^{*-1}$ positive.

Further, as mentioned earlier, M^*, $M^{*-1} \in \mathcal{B}$. Thus, M^{*-1} is positive and is regular in $\mathcal{B}$ and hence admits a factorization of the desired form, invoking the lemma in [5].

2) *Strong Positivity of $M^{*-1}G$:*

$$\begin{aligned}\langle x(\cdot), M^{*-1}Gx(\cdot)\rangle &= \langle x(\cdot), (M^{-1})^* Gx(\cdot)\rangle \\ &= \langle M^{-1}x(\cdot), Gx(\cdot)\rangle \\ &= \langle y(\cdot), GMy(\cdot)\rangle,\ y(\cdot) = M^{-1}x(\cdot) \\ &\geqslant \delta\|y(\cdot)\|^2,\end{aligned}$$

because of (3) and Parseval's theorem (the norm indicated being the L_2-norm).

Now, $x(\cdot) = My(\cdot)$ and hence,

$$\|x(\cdot)\| \leqslant \gamma(M)\|y(\cdot)\|$$

where $\gamma(M)$ is the gain of M (note that this follows from the definition of the gain,

$$\left.\gamma(M) = \sup_{\substack{y(\cdot)\in L_2\\ y(\cdot)\neq 0}} \frac{\|My(\cdot)\|}{\|y(\cdot)\|}\right).$$

Thus,

$$\begin{aligned}\langle x(\cdot), M^{*-1}Gx(\cdot)\rangle &\geqslant \frac{\delta}{[\gamma(M)]^2}\|x(\cdot)\|^2 \\ &= \boldsymbol{\delta}\langle x(\cdot), x(\cdot)\rangle \ \forall\ x(\cdot) \in L_2,\end{aligned}$$

where $\boldsymbol{\delta} = \delta/[\gamma(M)]^2 > 0$ (note that $M \in \mathcal{B} \Longrightarrow \gamma(M) < \infty$) and hence $M^{*-1}G$ is strongly positive.

Further, $M^{*-1}G$ has finite gain since $M^{*-1} \in \mathcal{B} \Longrightarrow \gamma(M^{*-1}) < \infty$, $G \in \mathcal{B}_c \Longrightarrow \gamma(G) < \infty$, and $\gamma(M^{*-1}G) \leqslant \gamma(M^{*-1})\gamma(G)$.

3) *Positivity of KM^*:*

$$\begin{aligned}\langle x(\cdot), KM^*x(\cdot)\rangle &= \langle MKx(\cdot), x(\cdot)\rangle, \text{ since } K \in \mathcal{K} \text{ is self adjoint} \\ &= \langle y(\cdot), K^{-1}M^{-1}y(\cdot)\rangle,\ y(\cdot) = MKx(\cdot) \\ &\geqslant 0 \ \forall\ y(\cdot) \in L_2,\end{aligned}$$

working as in the proof of [1, theorem 3] (note that $K \in \mathcal{K}_\beta{}^\alpha \Longrightarrow K^{-1} \in \mathcal{K}_\alpha{}^\beta$).

Thus all the requirements of the positivity theorem are fulfilled and hence the system is L_2-stable. Q.E.D.

A Few Remarks

Remark 1: The present result generalizes the stability criteria of [1] in the following aspects:

1) The bounds on the rate of variation of the gain $k(t)$ are more relaxed.

2) Less stringent conditions are imposed on the shifted-imaginary-axis behavior of the causal and the anticausal parts of the multiplier.

Remark 2: For the example considered in [1], of the system with

$$G(s) = \frac{(s^2 + 4.22s + 10.6)(s^2 + 200.1s + 20)}{(s^2 + 2s + 10)(s^2 + s + 16)},$$

the choice of a multiplier

$$M(s) = \frac{(s + 3.22)(s^2 - 4.22s + 10.6)}{(s^2 - 2s + 10)(s + 4)},$$

and an application of the stability theorem (for details, see [1]), proves the L_2-stability of the system for all time-varying gains $k(t)$ satisfying either of the following restrictions:

1) $$-k(t) \leqslant \frac{dk(t)}{dt} \leqslant 6k(t)$$

or

2) $$-6k(t) \leqslant \frac{dk(t)}{dt} \leqslant k(t).$$

It may be noted that [1] proved stability only in the case when the restrictions on $k(t)$ are given by 1).

Remark 3: A comparison of the present result with the L_2-stability criterion of Freedman and Zames [6] is interesting. While [6] imposes average variation constraints on $k(t)$ that are less stringent (note that $k(t)$ need not be differentiable everywhere), it is also less general than the present result in permitting causal multipliers only. The derivation of an average variation result permitting noncausal multipliers is a potentially useful problem for future investigation.

[2] Since $M \in \mathcal{B} \Longrightarrow M$ is a bounded linear operator, the Riesz representation theorem guarantees the existence of M^*; in fact, M^* is also linear and bounded (see Hille and Philips [4, p. 43]).

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On the Existence of Solutions to a Class of Optimization Problems

ANTONIO SALLES CAMPOS

Abstract—**A sufficient condition for the existence and uniqueness of solutions to a class of optimization problems in nonlinear programming form, with strictly convex cost functions, convex inequality and linear equality side constraints, and closed convex constraint sets is studied.**

I. Introduction

The study of sufficient conditions for the existence of solutions to problems of mathematical programming, calculus of variations, and optimal control [1]–[6] is of great interest. This work presents such a sufficient condition for a nonlinear programming problem with strictly convex inequality and linear equality side constraints, and a closed convex constraint set. The main advantage of this sufficient condition rests on its high practicability due to its extreme simplicity.

II. The Basic Problem

Let C be a closed convex subset of E^n, let $f:E^n \to E^1$ be a continuous strictly convex function on C, and let $g:E^n \to E^m$ be a continuous convex function on C. Let $h:E^n \to E^k$ be an affine linear function on C such that $\nabla_x h_i(x)$, where ∇_x denotes gradient in x, $x \in C$, $i = 1,\cdots,k$, are linearly independent vectors. Also, let it be assumed that there exists a vector x^* such that $x^* \in C$, $g(x^*) \leq 0$, and $h(x^*) = 0$ (for $v \in E^n$, the notations $v \leq 0$ and $v = 0$ mean, respectively, $v_i \leq 0$ and $v_i = 0$, $i = 1,\cdots,n$). Find a vector $\hat{x}$ in E^n such that $\hat{x} \in C$, $g(\hat{x}) \leq 0$, $h(\hat{x}) = 0$, and $f(\hat{x}) \leq f(x)$, for all x in C with $g(x) \leq 0$ and $h(x) = 0$.

III. Existence of an Optimum Solution

Consider the following lemma:

Lemma: Assume that there exists a vector $\bar{x}$ that minimizes $f(x)$ on C. Then, if the set $S_\alpha = \{x:x \in C, f(x) \leq \alpha\}$, where α is a real number, is not empty, it is a compact convex set.

Proof: Since f is strictly convex on the convex set C, $\bar{x}$ is unique. Taking $\alpha \geq f(\bar{x})$, S_α is a nonempty set. By the convexity of f on C, S_α is a convex set. Since f is continuous on C, S_α is a closed set. Assume, without loss of generality, that $\bar{x} = 0$ and $f(\bar{x}) = 0$. Hence, by the strict convexity of f on C,

$$f(\beta x) < \beta f(x), \qquad 0 < \beta < 1, \qquad \text{for all } x \in C \text{ such that } x \neq 0.$$

Assume that S_α is not a bounded set. Then, there exists a sequence $\{x^j\}$, $x^j \in S_\alpha$, $j = 1,2,\cdots$, such that $\|x^j\| \to \infty$ with $j \to \infty$, where ($\|\ \|$) denotes the usual Euclidean norm. Pick up only those values of j for which $\|x^j\| > 1$. Then $f(x^j/\|x^j\|) < f(x^j)/\|x^j\| \leq \alpha/\|x^j\|$ for all j such that $\|x^j\| > 1$. But $\lambda = \min_{\|x\|=1, x\in C} f(x)$ exists, for the set $\{x:x \in C, \|x\| = 1\}$ is nonempty and compact, and, moreover, $\lambda > 0$, for $\bar{x} \notin \{x:x \in C, \|x\| = 1\}$ and $\bar{x}$ is unique. Therefore, $0 < \lambda \leq f(x^j/\|x^j\|) < \alpha/\|x^j\|$ for every j such that $\|x^j\| > 1$. But this is a contradiction, for, if $\|x^j\| \to \infty$ with $j \to \infty$, $\alpha/\|x^j\| \to 0$ with $j \to \infty$. Therefore, such a sequence $\{x^j\}$ cannot exist, i.e., S_α must be a bounded set.

Now it is possible to formulate the basic theorem of this work:

Theorem: If there exists a vector $\bar{x}$ that minimizes $f(x)$ on C, then there exists a unique solution $\hat{x}$ to the problem stated in Section II.

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Proof: Since there exists a vector $x^* \in C$ such that $g(x^*) \leq 0$, $h(x^*) = 0$, and since g is a continuous convex function and h is an affine linear function on the closed convex set C,

$$X = \{x:x \in C, g(x) \leq 0, h(x) = 0\}$$

is a nonempty closed convex set. Consider the nontrivial case $\bar{x} \notin X$ and assume, without loss of generality, that $\bar{x} = 0$ and $f(\bar{x}) = 0$. Take the sets

$$S_j = \{x:x \in C, f(x) \leq j\}, \qquad j = 1,2,\cdots.$$

As proved in the preceding lemma, these sets are nonempty compact convex sets. Since there exists some j' such that $S^j \cap X$ is a nonempty set for every $j \geq j'$, let it be assumed, for simplicity of notation, that $j' = 1$. Then $S_1 \cap X$ is a nonempty compact convex set and, as f is continuous and strictly convex on $S_1 \cap X$, there exists a unique point $\hat{x}^1$ that minimizes $f(x)$ on $S_1 \cap X$, with $0 < f(\hat{x}^1) \leq 1$. Moreover, by the same reasons, for any $j \geq 1$, there can be found a unique $\hat{x}^j$ that minimizes $f(x)$ on $S_j \cap X$. Since S_1 is contained in S_j, $j = 1,2,\cdots$, then $S_1 \cap X$ is contained in $S_j \cap X$, $j = 1,2,\cdots$, and hence $0 < f(\hat{x}^j) \leq f(\hat{x}^1) \leq 1$, $j = 1,2,\cdots$. So, $\hat{x}^j \in S_1 \cap X$, $j = 1,2,\cdots$, i.e., $\hat{x}^j = \hat{x}^1$ for every j. Therefore, there exists $\hat{x} = \hat{x}^1$ which is the unique solution to the problem stated in Section II.

IV. Conclusions

The solutions to a class of nonlinear programming problems were shown to exist and to be unique under certain conditions. These conditions are very simple for they limit themselves to the existence of the minimum of the cost function on a particular closed convex set that coincides, in many practical cases, with all the Euclidean space under consideration.

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Partially Singular Linear–Quadratic Control Problems

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Abstract—**Necessary and sufficient conditions are given for the nonnegativity of a partially singular quadratic functional associated with a linear system. The conditions parallel known conditions for the totally singular problem, and a known sufficiency condition for the partially singular problem can be derived from them.**

Introduction

Consider the following linear optimal control problem. Minimize

$$J[u(\cdot)] = \int_{t_0}^{t_f} [\tfrac{1}{2}x'Qx + \tfrac{1}{2}u'Ru + u'Cx] + \tfrac{1}{2}x'(t_f)S_f x(t_f) \quad (1)$$

subject to

$$\dot{x} = Ax + Bu \quad x(t_0) = 0 \quad Dx(t_f) = 0. \quad (2)$$

Here, the state vector x is n-dimensional, and the control vector u is m-dimensional. The matrices A, B, C, Q, and R are time varying

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