MATEMATIKA, 2012, Volume 28, Number 2, 127–131

Titchmarsh's Theorem for the Bessel Transform

R. Daher, M. El Hamma and A. El Houasni
Department of Mathematics, Faculty of Sciences Aïn Chock
University of Hassan II, Casablanca, Morocco
e-mail: elmohamed77@yahoo.fr

Abstract In [1] Titchmarsh proved some theorems on the classical Fourier transform of functions satisfying conditions related to the Cauchy-Lipschitz conditions on the Euclidean space $\mathbb{R}$. In this paper we extend one those theorems for the Bessel transform for function on half-line $[0, \infty)$ in a weighted L_p-metric are studied with the use of Bessel generalized translation.

Keywords Bessel operator, Bessel transform, Bessel generalized translation.

2010 Mathematics Subject Classification 42B10, 42A38, 42B37.

1 Introduction and Preliminaries

Integral transforms and their inverses (e.g., the Fourier-Bessel transform) are widely used to solve various problems in calculus, mechanics, mathemAtical physics, and computational mathematics (see, e.g.,[1, 7]).

In this paper, we use the Bessel generalized translation, it is one of the most important generalized translations on the half-line $\mathbb{R}^+ = [0, +\infty)$ [4, 5]. The Bessel generalized translation is used while studying various problems connected with Bessel operators (see [3, 8]).

Titchmarsh ([1], Theorem 84) characterized the set of functions in $\mathrm{L}^p(\mathbb{R})$ satisfying the estimate, namely we have

Theorem 1.1 *Let f(x) belong to* $\mathrm{L}^p(\mathbb{R})$ $(1 < p \leq 2)$*, and let*

$$\int_{-\infty}^{+\infty} |f(x+h) - f(x-h)|^p dx = O(h^{\alpha p}),\ (0 < \alpha \leq 1)$$

as $h \longrightarrow 0$*. Then* $\mathcal{F}(f)(x)$ *belongs to* $\mathrm{L}^{\beta}(\mathbb{R})$ *for*

$$\frac{p}{p + \alpha p - 1} < \beta \leq \frac{p}{p-1}$$

where $\mathcal{F}(f)$ *stands for the Fourier transform of f*

The main of this paper is to establish an analog of Theorem 1.1 in the Bessel operators setting by means of the Bessel generalized translation.

Let

$$\mathrm{B} = \frac{d^2}{dt^2} + \frac{(2\alpha + 1)}{t}\frac{d}{dt},$$

be the Bessel differential operator. By $j_\alpha(t)$ denote the Bessel normed function of the first kind, i.e

$$j_\alpha(t) = \frac{2^\alpha \Gamma(\alpha+1) J_\alpha(t)}{t^\alpha},$$

where $J_\alpha(x)$ is the Bessel function of the first kind and $\Gamma(x)$ is the gamma-function (see[4]). The function $y = j_\alpha(t)$ satisfies the differential equation $\mathrm{B}y + y = 0$ with the initial conditions $y(0) = 1$ and $y'(0) = 0$. The function $j_\alpha(t)$ is infinitely differentiable, entire analytic.

Assume that $\mathrm{L}^p_\alpha(\mathbb{R}_+)$, $\alpha > -\frac{1}{2}$ and $1 < p \leq 2$, is the Banach space of mesurable functions f(t) on $\mathbb{R}_+$ with the finite norm

$$\|f\| = \|f\|_{p,\alpha} = \left(\int_0^\infty |f(t)|^p t^{2\alpha+1} dt\right)^{1/p}.$$

In $\mathrm{L}^p_\alpha(\mathbb{R}_+)$, consider the Bessel generalized translation T_h (see [3, p.121])

$$\mathrm{T}_h f(x) = \frac{\Gamma(\alpha+1)}{\Gamma(1/2)\Gamma(\alpha+\frac{1}{2})}\int_0^\pi f(\sqrt{x^2+h^2-2xh\cos t})\sin^{2\alpha} t dt,\ \alpha > -\frac{1}{2},\ 0 \leq h \leq 1$$

which corresponds to the Bessel operator B.

It is easy to see that

$$\mathrm{T}_0 f(x) = f(x)$$

If $f(x)$ has a continuous first derivative, then

$$\frac{\partial}{\partial h}\mathrm{T}_h f(x)|_{h=0} = 0.$$

If it has a continuous second derivative, then $u(x,h) = \mathrm{T}_h f(x)$ solves the Cauchy problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial h^2} + \frac{2\alpha+1}{h}\frac{\partial u}{\partial h}$$

and

$$u|_{h=0} = f(x),\quad \frac{\partial u}{\partial h}|_{h=0} = 0.$$

The operator T_h is linear, homogeneous, and continuous. Below are some properties of this operator (see [3, pp. 124-125]):

(i) $\mathrm{T}_h j_\alpha(\lambda x) = j_\alpha(\lambda h) j_\alpha(\lambda x)$

(ii) T_h is self-adjoint. If $f(x)$ is continuous function such that $\int_0^\infty x^{2\alpha+1}|f(x)|dx < \infty$ and $g(x)$ is continuous and bounded for all $x \geq 0$, then

$$\int_0^\infty (\mathrm{T}_h f(x))g(x)x^{2\alpha+1}dx = \int_0^\infty f(x)(\mathrm{T}_h g(x))x^{2\alpha+1}dx.$$

(iii) $\mathrm{T}_h f(x) = \mathrm{T}_x f(h)$.

(iv) $\|\mathrm{T}_h f - f\| \longrightarrow 0$ as $h \longrightarrow 0$.

The Bessel transform defined by the formula (see [3, 4, 6])

$$\widehat{f}(\lambda) = \int_0^\infty f(t) j_\alpha(\lambda t) t^{2\alpha+1} dt; \quad \lambda \in \mathbb{R}_+$$

The inverse Bessel transform is given by the formula

$$f(t) = (2^\alpha \Gamma(\alpha+1))^{-2} \int_0^\infty \widehat{f}(\lambda) j_\alpha(\lambda t) \lambda^{2\alpha+1} d\lambda$$

The following relation connect the Bessel generalized translation, and the Bessel transform in [2], we have

$$\widehat{(\mathrm{T}_h f)}(\lambda) = j_\alpha(\lambda h) \widehat{f}(\lambda) \tag{1}$$

For $\alpha > -\frac{1}{2}$, we introduce the Bessel normalized function of the first kind j_α defined by

$$j_\alpha(x) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(n+\alpha+1)} \tag{2}$$

Moreover, from (2) we see that

$$\lim_{x \longrightarrow 0} \frac{(j_\alpha(x) - 1)}{x^2} \neq 0$$

by consequence, there exist $C > 0$ and $\eta > 0$ satisfying

$$|x| \leq \eta \Longrightarrow |j_\alpha(x) - 1| \geq C|x|^2 \tag{3}$$

2 An Analog of Titchmarsh's Theorem

In this section, we give an analog of Titchmarsh's Theorem 84 for the Bessel transform.

Theorem 2.1 *Let $f(x)$ belong to $\mathrm{L}^p_\alpha(\mathbb{R}_+)$, $(1 < p \leq 2)$, and let*

$$\int_0^\infty |\mathrm{T}_h f(x) - f(x)|^p x^{2\alpha+1} dx = O(h^{\gamma p}), \ (0 < \gamma \leq 2)$$

as $h \longrightarrow 0$. Then $\widehat{f}(x)$ belongs to $\mathrm{L}^\beta_\alpha(\mathbb{R}_+)$, for

$$\frac{2p\alpha + 2p}{2p + 2\alpha(p-1) + \gamma p - 2} < \beta \leq \frac{p}{p-1}$$

Proof: For a fixed h the Bessel transform of $\mathrm{T}_h f(x)$ is $j_\alpha(hx)\widehat{f}(x)$. Hence the Bessel transform of $\mathrm{T}_h f(x) - f(x)$, as a function of x, is $(j_\alpha(hx)-1)\widehat{f}(x)$.
Hence

$$\begin{aligned}\int_0^\infty |j_\alpha(hx)-1|^{p'}|\widehat{f}(x)|^{p'}x^{2\alpha+1}dx &< K(p)\left(\int_0^\infty |\mathrm{T}_h f(x)-f(x)|^p x^{2\alpha+1}dx\right)^{1/p-1} \\ &< K(p)h^{\gamma p'}\end{aligned}$$

From formula (3), we have

$$\int_0^{\eta/h} |hx|^{2p'}|\widehat{f}(x)|^{p'}x^{2\alpha+1}dx < K(p)h^{\gamma p'}$$

Then

$$\int_0^{\eta/h} x^{2p'}|\widehat{f}(x)|^{p'}x^{2\alpha+1}dx < K(p)h^{(\gamma-2)p'}$$

let

$$\phi(\xi) = \int_1^\xi |x^2\widehat{f}(x)|^\beta x^{(2\alpha+1)\frac{p'}{\beta}}dx$$

Then, if $\beta < p'$

$$\begin{aligned}\phi(\xi) &\le \left(\int_1^\xi |x^2\widehat{f}(x)|^{p'}x^{2\alpha+1}dx\right)^{\beta/p'}\left(\int_1^\xi dx\right)^{1-\beta/p'} \\ &= O(\xi^{(2-\gamma)p'\frac{\beta}{p'}}\xi^{1-\frac{\beta}{p'}}) \\ &= O(\xi^{2\beta-\gamma\beta+1-\frac{\beta}{p'}})\end{aligned}$$

Hence

$$\begin{aligned}&\int_1^\xi |\widehat{f}(\xi)|^\beta x^{2\alpha+1}dx = \int_1^\xi x^{-2\beta-(2\alpha+1)\frac{\beta}{p'}}\phi'(x)x^{2\alpha+1}dx \\ &= \xi^{-2\beta-(2\alpha+1)\frac{\beta}{p'}}\xi^{2\alpha+1}\phi(\xi) + (2\beta+(2\alpha+1)\frac{\beta}{p'}-(2\alpha+1))\int_1^\xi x^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha}\phi(x)dx \\ &= O(\xi^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+1+1-\gamma\beta+\beta(\frac{p+1}{p})}) + O\left(\int_1^\infty x^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha}x^{1-\gamma\beta+\beta(\frac{p+1}{p})}dx\right) \\ &= O(\xi^{-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+2-\gamma\beta+\beta(\frac{p+1}{p})})\end{aligned}$$

and this bounded as $\xi \longrightarrow \infty$ if $-2\beta-(2\alpha+1)\frac{\beta}{p'}+2\alpha+2-\gamma\beta+\beta(\frac{p+1}{p}) < 0$ i.e, if

$$\beta > \frac{2p\alpha+2p}{2p+2\alpha(p-1)+\gamma p-2}$$

This proves the theorem.

3 Conclusion

There are many theorems known about to classical Fourier transform can be generalized for the Bessel transform, among them Titchmarsh's Theorem. In this work we have succeeded to generalise this theorem for the Bessel transform in the space $\mathrm{L}^p_\alpha(\mathbb{R}_+)$. We proved that if $f(x)$ belongs to $\mathrm{L}^p_\alpha(\mathbb{R}_+)$, and that

$$\int_0^\infty |\mathrm{T}_h f(x) - f(x)|^p x^{2\alpha+1} dx = O(h^{\gamma p}),\ (0 < \gamma \leq 2)$$

as $h \longrightarrow 0$, then its Bessel transform $\widehat{f}$ belongs to $\mathrm{L}^\beta_\alpha(\mathbb{R}_+)$ for

$$\frac{2p\alpha + 2p}{2p + 2\alpha(p-1) + \gamma p - 2} < \beta \leq \frac{p}{p-1},$$

where $0 < \gamma < 1$ and $1 < p \leq 2$.

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