Tits-Kantor-Koecher Superalgebras of Jordan superpairs covered by grids

Dedicated to John Faulkner

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Summary. *In this paper we describe the Tits-Kantor-Koecher superalgebras associated to Jordan superpairs covered by grids, extending results from [**38**] to the supercase. These Lie superalgebras together with their central coverings are precisely the Lie superalgebras graded by a 3-graded root system.*

Introduction

It is well-known that every Jordan superpair and hence also every Jordan superalgebra gives rise to a 3-graded Lie superalgebra, the so-called Tits-Kantor-Koecher superalgebra (=TKK-superalgebra). This connection between Jordan superstructures and Lie superalgebras is the basis of the classifications of simple finite dimensional Jordan superalgebras and Jordan superpairs over algebraically closed fields of characteristic 0 by Kac [**19**] and Krutelevich [**26**] which both are based on Kac's classification of finite-dimensional simple Lie superalgebras [**20**]. This connection is also one of the motivations for classifying simple Jordan superalgebras of growth 1 in the recent memoirs [**22**] by Kac-Martínez-Zelmanov: via the Tits-Kantor-Koecher construction these correspond to superconformal algebras of type K.

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The structure theory of Jordan superpairs covered by grids [**35**] did not make use of this connection to Lie superalgebras. But in light of the fundamental importance of the Tits-Kantor-Koecher construction the following question is of interest: *What are the TKK-superalgebras $\mathfrak{K}(V)$ of Jordan superpairs V covered by a grid $\mathcal{G}$?*

To answer this question is one of the goals of this paper. In general V is a direct sum of ideals V_i covered by a connected grid $\mathcal{G}_i$, and correspondingly $\mathfrak{K}(V)$ is a direct sum of ideals $\mathfrak{K}(V_i)$ (5). It is therefore enough to consider the case of connected grids. In §3 we will describe a model of $\mathfrak{K}(V)$ for each type of connected $\mathcal{G}$, based on the classification of V given in [**35**]. To illustrate our results, we describe here the situation of a rectangular covering grid $\mathcal{R}(J,K) = \{e_{jk} : j \in J, k \in K\}$, i.e., a family of idempotents e_{jk} in $V_{\bar{0}}$ satisfying the multiplication rules of the usual rectangular matrix units E_{jk} such that V is the sum of all Peirce spaces $V_2(e_{jk})$. We note that J and K are arbitrary sets.

(i) $|J|+|K| \geq 4$: In this case V is isomorphic to a rectangular matrix superpair $\mathbb{M}_{JK}(A) = (\mathrm{Mat}(J,K;A), \mathrm{Mat}(K,J;A))$ where A is an associative superalgebra and $\mathrm{Mat}(J,K;A)$ denotes the $J \times K$-matrices over A with only finitely many non-zero entries. Conversely, every $\mathbb{M}_{JK}(A)$ is covered by the rectangular grid $\mathcal{R}(J,K)$ consisting of the idempotents $e_{jk} = (E_{jk}, E_{kj})$. Let $I = J \dot\cup K$ and put $\mathfrak{sl}_I(A) = \{x \in \mathrm{Mat}(I,I;A) : \mathrm{Tr}(x) \in [A,A]\}$ where Tr is the usual trace map and $[A,A]$ is spanned by all supercommutators $[a,b] = ab - (-1)^{|a||b|}ba$. Although $\mathfrak{sl}_I(A)$ is a 3-graded Lie superalgebra, it is in general not the TKK-superalgebra of $\mathbb{M}_{JK}(A)$ since it may have a centre. But we have

$$\mathfrak{K}(\mathbb{M}_{JK}(A)) \cong \mathfrak{psl}_I(A) = \mathfrak{sl}_I(A)/Z(\mathfrak{sl}_I(A)). \tag{1}$$

Of course, $\mathbb{M}_{JK}(A)$ is a Jordan superpair covered by $\mathcal{R}(J,K)$ for any non-empty J,K and (1) holds whenever $|I| \geq 3$. But for $|J|+|K| \leq 3$ more general examples occur.

(ii) A Jordan superpair V is covered by a rectangular grid $\mathcal{R}(J,K)$ with $|J|+|K| = 3$ if and only if $V \cong \mathbb{M}_{JK}(A)$ for an alternative superalgebra A. The TKK-superalgebra $\mathfrak{K}(\mathbb{M}_{JK}(A))$ is no longer given by (1). Rather in 3.2 we provide a model which is the super version of a Lie algebra previously studied by Faulkner in his work on generalizations of projective planes [**13**].

(i) $|J|+|K| = 2$: Here V is covered by a single idempotent which is equivalent to $V \cong (J,J)$ for a unital Jordan superalgebra J. The corresponding TKK-superalgebra is described in 3.1.

Since a TKK-superalgebra $\mathfrak{K}(V)$ is determined by the Jordan superpair V, one can expect that properties of $\mathfrak{K}(V)$ are controlled by V. Indeed, in [**15**] we will show that this is so for the Gelfand-Kirillov dimension of $\mathfrak{K}(V)$ and V. In this paper, we will study the relation between ideals of $\mathfrak{K}(V)$ and of V and this naturally leads us to consider the question of semiprimeness, primeness and simplicity of $\mathfrak{K}(V)$ and V. For example, we prove the following result in 2.6.

Proposition. *Let $\mathfrak{K}(V)$ be the TKK-superalgebra of a Jordan superpair defined over a superring S containing $\frac{1}{2}$. Then $\mathfrak{K}(V)$ is simple if and only if V is simple.*

In [**14**] we have shown that a Jordan superpair V covered by a grid is simple if and only if its supercoordinate system is simple. Combining this with the proposition above we achieve the second goal of this paper, namely *to determine the simple TKK-superalgebras of Jordan superpairs covered by a grid*. The result is given in each case separately, see 3.2, 3.4, 3.6, 3.7, 3.8, 3.9 and 3.10. For example, in the case (i) above the Lie superalgebra $\mathfrak{psl}_I(A)$ is simple if and only if A is a simple associative superalgebra, assuming that A is defined over a superring S containing $\frac{1}{2}$. After completion of this paper we have learned that generalizations of classical Lie superalgebras and questions of simplicity are also studied in a forthcoming paper [**5**] by Benkart, Xu and Zhao.

It is surprising that many classical Lie superalgebras in the sense of [**20**] arise as TKK-superalgebras of simple finite dimensional Jordan superpairs covered by a grid. Besides Lie algebras not of type $\mathrm{E}_8, \mathrm{F}_4$ and G_2 we obtain the following simple Lie superalgebras in the notation of [**20**]:

(i) $\mathbf{A}(m,n)$ for $m+1, n+1$ not relatively prime, 3.4;
(ii) $\mathbf{B}(m,n)$ for $m>0$ and all types $\mathbf{C}(n)$ and $\mathbf{D}(m,n)$, 3.6, 3.7;
(iii) $\mathbf{P}(n)$ for even n, 3.6;
(iv) $\mathbf{Q}(n)$, 3.4;
(v) $\mathbf{F}(4)$ which is the TKK-superalgebra of the Jordan superpair $(\mathrm{K}_{10}, \mathrm{K}_{10})$ where K_{10} is the 10-dimensional Kac superalgebra.

The main result of [**38**] was that Lie algebras graded by 3-graded root systems are exactly the central covers of TKK-algebras of Jordan pairs covered by a grid. The third goal of this paper is *to extend this connection to the setting of superstructures*. A Lie superalgebra L is graded by a root system R if

$$L = \bigoplus\nolimits_{\alpha \in R \cup \{0\}} L_\alpha, \quad [L_\alpha, L_\beta] \subset L_{\alpha+\beta},$$

and this grading is induced by a semisimple split subalgebra $\mathfrak{g} \subset L_{\bar{0}}$ with root system R (see 2.7 for the precise definition). Lie algebras graded by root systems have been studied by several authors, among them Allison-Benkart-Gao [**1**,**2**], Benkart [**4**], Benkart-Zelmanov [**6**] and Berman-Moody [**9**]. One of the reasons for studying these Lie algebras is their connection to extended affine Lie algebras (Allison-Gao [**3**], Berman-Gao-Krylyuk [**7**] and [**8**]). One can expect that, similarly, Lie superalgebras graded by root systems will be of importance for the super version of extended affine Lie algebras. The extension of the aforementioned result to Lie superalgebras is the following result 2.8(d):

Theorem. *A Lie superalgebra over a superring containing $\frac{1}{2}$ and $\frac{1}{3}$ is graded by a 3-graded root system R if and only if it is a perfect central extension of the TKK-superalgebra of a Jordan superpair covered by a grid with associated root system R.*

This theorem together with our description of TKK-superalgebras determines Lie superalgebras graded by a 3-graded root system. A list of all coordinatization results is given in 2.9. We point out that our methods allow us to include root systems of possibly infinite rank, as defined in [**37**], [**29**]. This generality seems to be of interest in view of the importance of the infinite rank affine algebras for Virasoro and Kac-Moody algebras, see e.g. [**21**, 7.11, Ch. 14]. Certain completions of TKK-superalgebras as considered above also arise in the work of Cheng-Wang [**11**] on Lie subalgebras of differential operators on the supercircle.

For example, for the root system R of type A the Jordan superpairs arising in the theorem are covered by a rectangular grid, leading to the following

Corollary. *A Lie superalgebra L is graded by the root system* A *of rank* $|I|-1$, $|I| \geq 4$ *if and only if* $L/Z(L) \cong \mathfrak{psl}_I(A)$ *for some associative superalgebra* A.

1. Review of Jordan superpairs covered by grids.

In this paper we will study Lie superalgebras constructed from Jordan superpairs covered by grids. For the convenience of the reader we give a short review of all the definitions and facts needed from the theory of Jordan superpairs. More details can be found in [**35**] and [**14**].

1.1. Superalgebras. Unless specified otherwise, all algebraic structures are defined over some superextension S of a base ring k, i.e., a unital supercommutative associative k-superalgebra. S-modules, sometimes also called S-supermodules, are assumed to be $\mathbb{Z}_2$-graded, and in particular this is so for subalgebras, ideals, etc. We will write $M = M_{\bar{0}} \oplus M_{\bar{1}}$ for an S-module where $M_{\bar{0}}$ and $M_{\bar{1}}$ denote the even, respectively odd, part of M. For $m \in M_\mu$, $\mu \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ we denote by $|m| = \mu$ the degree (or parity) of m.

Recall that alternative, Jordan and Lie superalgebras can be defined by requiring that the Grassmann envelope is, respectively, an alternative, Jordan or Lie algebra. When considering Lie superalgebras we will often assume $\frac{1}{2} \in k$. All alternative, and hence in particular associative superalgebras, as well as all Jordan superalgebras are assumed to be unital.

1.2. Matrices over associative superalgebras. Let A be an associative superalgebra over S. For two arbitrary sets we denote by $\mathrm{Mat}(J, K; A)$ the A-bimodule of $J \times K$-matrices with entries from A but only finitely many non-zero entries. The S-supermodule $\mathrm{Mat}(J, K; A)$ has a $\mathbb{Z}_2$-grading induced by the grading of A, i.e., $\mathrm{Mat}(J, K; A)_\mu = \mathrm{Mat}(J, K; A_\mu)$ for $\mu = \bar{0}, \bar{1}$. More generally, let $J = M \dot{\cup} N$ and $K = P \dot{\cup} Q$ be two partitions. The S-supermodule $\mathrm{Mat}(M|N, P|Q; A)$ is defined by

$$\mathrm{Mat}(M|N, P|Q; A)_\mu = \begin{pmatrix} \mathrm{Mat}(M, P; A_\mu) & \mathrm{Mat}(M, Q; A_{\mu+\bar{1}}) \\ \mathrm{Mat}(N, P; A_{\mu+\bar{1}}) & \mathrm{Mat}(N, Q; A_\mu) \end{pmatrix} \qquad (1)$$

for $\mu = \bar{0}, \bar{1}$. With the usual matrix product

$$\mathrm{Mat}_{P|Q}(A) := \mathrm{Mat}(P|Q, P|Q; A) \qquad (2)$$

becomes an associative, but in general not unital, superalgebra. As in [**14**, 3.8] one can show that the ideals of the superalgebra $\mathrm{Mat}_{P|Q}(A)$ are given by $\mathrm{Mat}_{P|Q}(B)$ where B is an ideal of A, and hence $\mathrm{Mat}_{P|Q}(A)$ is semiprime, prime or simple if and only if A is semiprime, prime or simple. Following standard practice, we will replace the sets M, N, P, Q in the notations above with their cardinality in case they are all finite. Thus, we will write $\mathrm{Mat}_{p|q}(A) = \mathrm{Mat}_{P|Q}(A)$ if $|P| = p < \infty$ and $q = |Q| < \infty$. Note that in this case $\mathrm{Mat}_{p|q}(A)$ is unital.

1.3. Doubles of associative algebras. Let B be an associative algebra. The double of B is the associative superalgebra $\mathbb{D}(B) = B \oplus Bu$ with $\mathbb{D}(B)_{\bar 0} = B$, $\mathbb{D}(B)_{\bar 1} = uB$ and product $\cdot$ given by $a \cdot b = ab$, $a \cdot bu = (ab)u = au \cdot b$ and $au \cdot bu = ab$ for $a, b \in B$ ([**14**, 2.4]). For $B = \mathrm{Mat}(n, n; k)$ this superalgebra already occurs in [**19**] where it is denoted $Q_n(k)$ because of its relation to the simple Lie superalgebras $\mathbf{Q}_n$. We have the canonical isomorphism of superalgebras ([**14**, 2.4]): $\mathrm{Mat}_{P|Q}(\mathbb{D}(B)) \cong \mathbb{D}(\mathrm{Mat}(P\dot\cup Q, P\dot\cup Q; B))$.

Any ideal of $\mathbb{D}(B)$ has the form $\mathbb{D}(I) = I \oplus Iu$ where I is an ideal of B. Hence $\mathbb{D}(B)$ is semiprime, prime or simple if and only if B is respectively semiprime, prime or simple.

We recall the following characterizations of prime alternative and simple associative superalgebras.

1.4. Theorem. (a) (Shestakov-Zelmanov [**49**]) *A prime alternative superalgebra over a field of characteristic $\neq 2, 3$ either is associative or is a Cayley-Dickson ring.*

(b) (Wall [**45**, Lemma 3]) *An associative superalgebra A is simple as superalgebra if and only if either A is simple as algebra or $A = \mathbb{D}(A_{\bar 0})$ and $A_{\bar 0}$ is simple. In particular, an associative superalgebra A over an algebraically closed field k is finite dimensional and simple if and only if either $A \cong \mathbb{D}(\mathrm{Mat}(m, m; k))$ or $A \cong \mathrm{Mat}_{p|q}(k)$ for finite numbers m, p and q.*

1.5. Jordan superpairs. Jordan superpairs over S are pairs $V = (V^+, V^-)$ of S-modules together with a pair $Q = (Q^+, Q^-)$ of S-quadratic maps $Q^\sigma\colon V^\sigma \to \mathrm{Hom}_S(V^{-\sigma}, V^\sigma)$, $\sigma = \pm$, satisfying certain identities ([**35**, 3.2]). By definition, we therefore have supersymmetric S-bilinear maps $Q^\sigma(.,.)\colon V^\sigma \times V^\sigma \to \mathrm{Hom}_S(V^{-\sigma}, V^\sigma)$ of degree 0 and $S_{\bar 0}$-quadratic maps $Q^\sigma_{\bar 0}\colon V^\sigma_{\bar 0} \to \mathrm{Hom}_S(V^{-\sigma}, V^\sigma)_{\bar 0}$ which are related by $Q^\sigma(u, w) = Q^\sigma_{\bar 0}(u+w) - Q^\sigma_{\bar 0}(u) - Q^\sigma_{\bar 0}(w)$ for $u, w \in V^\sigma_{\bar 0}$. Since $2Q^\sigma_{\bar 0}(u) = Q^\sigma(u, u)$ the maps $Q^\sigma_{\bar 0}$ are determined by Q^σ in case $\frac{1}{2} \in S$. We will follow common practise in Jordan theory and leave out the superscripts σ if no confusion can arise. Also it is sometimes easier to define a Jordan superpair via the (super)triple products which are S-trilinear maps $\{...\}\colon V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma$ related to the maps Q by $\{u\, v\, w\} = (-1)^{|v||w|} Q(u, w)v$. As for superalgebras there exists a Grassmann envelope $G(V)$ for superpairs V, and a pair V is a Jordan superpair over a S if and only if $G(V)$ is a Jordan pair over $G(S)$.

Subpairs and ideals of Jordan superpairs are defined in the obvious way ([**35**, 3.3]). A Jordan superpair is simple if it has non-zero multiplication and if all ideals are trivial. For ideals I, J of a Jordan superpair V their *Jordan product* $I \diamond J = ((I \diamond J)^+, (I \diamond J)^-))$ is given by $(I \diamond J)^\sigma = Q_{\bar 0}(I^\sigma_{\bar 0})J^{-\sigma} + \{I^\sigma\,,\, J^{-\sigma}\,,\, I^\sigma\}$. We note that $I \diamond J$ is in general not an ideal. Then V is called semiprime if $I \diamond I \neq 0$ for any non-zero ideal I of V, and is called prime if $I \diamond J \neq 0$ for any two non-zero ideals I and J of V ([**14**, 3.1]). Primeness can also be defined in terms of the annihilator $\mathrm{Ann}_V(I) = (\mathrm{Ann}(I)^+_{\bar 0} \oplus \mathrm{Ann}(I)^+_{\bar 1}, \mathrm{Ann}(I)^-_{\bar 0} \oplus \mathrm{Ann}(I)^-_{\bar 1})$ of an ideal I, where $z \in \mathrm{Ann}(I)^\sigma_\mu$ for $\sigma = \pm$ and $\mu \in \{\bar 0, \bar 1\}$ if and only if

$$\begin{aligned} 0 &= D(z^\sigma, V^{-\sigma}) = D(V^{-\sigma}, z^\sigma) = Q(z^\sigma, V^\sigma) \\ &= Q_{\bar 0}(I^{-\sigma}_{\bar 0})z = Q_{\bar 0}(I^\sigma_{\bar 0})Q_{\bar 0}(V^{-\sigma}_{\bar 0})z\,, \quad \text{and in addition for } \mu = \bar 0: \\ 0 &= Q_{\bar 0}(z)I^{-\sigma} = Q_{\bar 0}(I^{-\sigma}_{\bar 0})Q_{\bar 0}(z)\,. \end{aligned}$$

The annihilator is an ideal of V, and V is prime if and only if V is semiprime and the annihilator of any non-zero ideal of V vanishes.

1.6. Basic examples of Jordan superpairs. An associative S-superalgebra A gives rise to a Jordan superpair (A, A) by defining $Q_{\bar 0}(u)v = uvu$ and $\{u\,v\,w\} = uvw + (-1)^{|u||v|+|v||w|+|w||u|}\,wvu$. Any Jordan superpair isomorphic to a subpair of some (A, A) is called special.

A second class of examples are the quadratic form superpairs (M, M) determined by a quadratic form q on an S-supermodule M ([**35**, 3.10]). If b denotes the polar of q, the Jordan product is given by $\{m\,n\,p\} = b(m, n)p + mb(n, p) - (-1)^{|n||p|}b(m, p)n$ and $Q_{\bar 0}(m_{\bar 0})n = b(m_{\bar 0}, n)m_{\bar 0} - q_{\bar 0}(m_{\bar 0})n$. Under some mild assumptions, this class is in fact also special ([**18**, 2.2] and [**31**]), at least in case $S = k$.

Let $U = (U^+, U^-)$ be a Jordan pair over k. Then $U_S = (S \otimes_k U^+, S \otimes_k U^-)$ is a Jordan superpair over S, called the S-extension of U ([**35**, 3.8]). Its Jordan triple product satisfies $\{s_u \otimes u, s_v \otimes v, s_w \otimes w\} = (s_u s_v s_w) \otimes \{u\,v\,w\}$. The special case $S = \mathbb{D}(k)$ will be important: the double of U is the superextension $\mathbb{D}(U) := U_{\mathbb{D}(k)} = \mathbb{D}(k) \otimes_k U$. It is known ([**14**, 3.2]) that U is semiprime, prime or simple if $\mathbb{D}(U)$ is so, and the converse is true if $\frac{1}{2} \in k$. We have $\mathbb{D}(B \otimes_k U) = \mathbb{D}(B) \otimes_B (B \otimes_k U) \cong \mathbb{D}(B) \otimes_k U$ for an extension B of k [**14**, 3.2(2)].

1.7. Jordan superpairs covered by grids. We will study Jordan superpairs V covered by a grid $\mathcal{G} = \{g_\alpha : \alpha \in R_1\}$ where (R, R_1) is the associated 3-graded root system (see [**35**, 4] for the precise definition). Thus each $g_\alpha \in \mathcal{G} \subset V_{\bar 0}$ is an idempotent giving rise to a Peirce decomposition $V = V_0(g_\alpha) \oplus V_1(g_\alpha) \oplus V_2(g_\alpha)$, and these Peirce decompositions are pairwise compatible leading to a simultaneous Peirce decomposition $V = \bigoplus_{\alpha \in R_1} V_\alpha$ where $V_\alpha = \bigcap_{\beta \in R_1} V_{i(\beta)}(g_\beta)$ for $i(\beta)$ defined by $g_\alpha \in V_{i(\beta)}(g_\beta)$ (in fact, $i(\beta) = \langle \alpha, \beta^\vee \rangle$ is the Cartan integer of α, β and hence $\langle \alpha, \beta^\vee \rangle = 2(\alpha|\beta)/(\beta|\beta)$ for any invariant inner product $(.|.)$).

If $(R, R_1) = \bigoplus_{i \in I}(R^{(i)}, R^{(i)}_1)$ is a direct sum of 3-graded root systems $(R^{(i)}, R^{(i)}_1)$ we have a corresponding decomposition of $V = \bigoplus_{i \in I} V^{(i)}$ where each $V^{(i)} = \bigoplus_{\alpha \in R^{(i)}_1} V_\alpha$ is an ideal of V covered by the grid $\mathcal{G}^{(i)} = \{g_\alpha : \alpha \in R^{(i)}_1\}$ ([**35**, 4.5]). Since every 3-graded root system is a direct sum of irreducible 3-graded root systems ([**37**] or [**29**]) this allows to reduce questions on Jordan superpairs covered by grids to the case where $\mathcal{G}$ is connected, i.e., the associated 3-graded root system is irreducible. There are the following seven types of connected grids ([**35**, 5]):

(i) rectangular grid $\mathcal{R}(J,K), 1 \le |J| \le |K|$, (R, R_1) is the rectangular grading $\mathrm{A}_I^{J,K}$ where $J \dot{\cup} K = I \dot{\cup} \{0\}$ for some element $0 \notin I$ and R is a root system of type A and rank $|I|$;

(ii) hermitian grid $\mathcal{H}(I)$, $2 \le |I|$, (R, R_1) is the hermitian grading of $R = \mathrm{C}_I$;

(iii) even quadratic form grid $\mathcal{Q}_e(I)$, $3 \le |I|$, (R, R_1) is the even quadratic form grading of $R = \mathrm{D}_{I\dot{\cup}\{0\}}$;

(iv) odd quadratic form grid $\mathcal{Q}_o(I)$, $2 \le |I|$, (R, R_1) is the odd quadratic form grading of $R = \mathrm{B}_{I\dot{\cup}\{0\}}$;

(v) alternating (= symplectic) grid $\mathcal{A}(I)$, $5 \le |I|$, (R, R_1) is the alternating grading of $R = \mathrm{D}_I$;

(vi) Bi-Cayley grid $\mathcal{B}$, $R = \mathrm{E}_6$;

(vii) Albert grid $\mathcal{A}$, $R = \mathrm{E}_7$.

1.8. Standard examples. The coordinatization theorems of [**35**, §5] described in (a) – (i) below can be summarized by saying that a Jordan superpair V is covered by a grid $\mathcal{G}$ if and only if V is isomorphic to a standard example $V(\mathcal{G}, \mathcal{C})$ depending on $\mathcal{G}$ and a supercoordinate system $\mathcal{C}$.

(a) For the rectangular grading of $R = \mathrm{A}_1$ with $|J| = |K| = 1$ we have $|R_1| = 1$ and $\mathcal{G}$ just consists of a single idempotent $\mathcal{G} = \{g\}$ which covers V in the sense that $V = V_2(g)$. Any such Jordan superpair is isomorphic to the superpair $\mathbb{J} = (J, J)$ of a unital Jordan superalgebra J over S. In this case $\mathcal{C} = J$.

(b) The standard examples for the remaining rectangular grids $\mathcal{R}(J,K)$, $|J| + |K| \ge 3$, are the rectangular matrix superpairs $\mathbb{M}_{JK}(A) = (\mathrm{Mat}(J,K;A), \mathrm{Mat}(K, J;A))$ where A is an alternative superalgebra in case $R = \mathrm{A}_2$ and associative otherwise. In the alternative case the product is described in [**35**, 5.4], in the associative case $\mathbb{M}_{JK}(A)$ is a special Jordan superpair canonically imbedded in $(\mathrm{Mat}(J \cup K; A), \mathrm{Mat}(J \cup K; A))$. Here $\mathcal{C} = A$. We have $\mathbb{D}(\mathbb{M}_{JK}(B)) \cong \mathbb{M}_{JK}(\mathbb{D}(B))$ for an associative coordinate algebra B ([**14**, 3.2(3)].

(c) The Jordan superpairs covered by a hermitian grid $\mathcal{H}(I)$, $|I| = 2$ are exactly the $\mathbb{J} = (J, J)$ where J is a Jordan superalgebra with two strongly connected supplementary idempotents giving rise to a Peirce decomposition $\mathfrak{P}$ of J in the form $\mathfrak{P} : J = J_{11} \oplus J_{12} \oplus J_{22}$. In this case, the supercoordinate system of V is $\mathcal{C} = (J, \mathfrak{P})$.

(d) Examples of Jordan superpairs covered by hermitian grids $\mathcal{H}(I)$ are the hermitian Jordan superpairs $\mathbb{H}_I(A, A_0, \pi) = (\mathrm{H}_I(A, A_0, \pi), \mathrm{H}_I(A, A_0, \pi))$, where $\mathrm{H}_I(A, A_0, \pi) = \{x = (x_{ij}) \in \mathrm{Mat}(I, I; A) : x = x^{\pi \mathrm{T}}$, all $x_{ii} \in A_0\}$, A is an alternative superalgebra which is associative for $|I| \ge 4$ and π is a nuclear involution with ample subspace A_0 ([**35**, 5.10]). We have $A_0 \subset \mathrm{H}(A, \pi) = \{a \in A : a^\pi = a\}$ and this is an equality if $\frac{1}{2} \in S$. For an associative A these are special Jordan superpairs, in the alternative case the product is described in [**35**, 5.11]. The $\mathbb{Z}_2$-grading of $\mathrm{H}_I(A, A_0, \pi)$ is induced form the $\mathbb{Z}_2$-grading of A. Conversely, under a weak additional assumption which we will assume to be fulfilled in the following (this is so as soon as $\frac{1}{2} \in S$) any Jordan superpair covered by a hermitian grid $\mathcal{H}(I), |I| \geq 3$ is isomorphic to some hermitian matrix superpair $\mathbb{H}_I(A, A_0, \pi)$. We put $\mathcal{C} = (A, A_0, \pi)$.

(e) For a superextension A of S and a set $I \neq \emptyset$ we denote by $H(I, A)$ the free A-module with even basis $\{h_{\pm i} : i \in I\}$ equipped with the hyperbolic superform q_I satisfying $q_I(h_{+i}, h_{-i}) = 1$ and $q_I(h_{\pm i}, h_{\pm j}) = 0$ for $i \neq j$. The corresponding quadratic form superpair $(H(I, A), H(I, A)) = \mathbb{EQ}_I(A, q_X)$ is covered by an even quadratic form grid $\mathcal{Q}_e(I)$. Conversely, any Jordan superpair covered by an even quadratic form grid $\mathcal{Q}_e(I)$, $|I| \geq 3$ is isomorphic to some $\mathbb{EQ}_I(A, q_X)$ ([**35**, 5.14]). Here $\mathcal{C} = A$.

(f) We let again A be a superextension of S and suppose that X is an A-module with an A-quadratic form q_X with a base point $e \in X_{\bar{0}}$ satisfying $q_X(e) = 1$. For $I \neq \emptyset$ we put $M = H(I, A) \oplus X$, $q = q_I \oplus q_X$. The corresponding quadratic form superpair $(M, M) = \mathbb{OQ}_I(A, q_X)$ is covered by an odd quadratic form grid $\mathcal{Q}_o(I)$. Conversely, any Jordan superpair covered by an odd quadratic form grid $\mathcal{Q}_o(I)$, $|I| \geq 2$ is isomorphic to some $\mathbb{OQ}_I(A, q_X)$ ([**35**, 5.16]). We put $\mathcal{C} = (A, X, q_X)$.

(g) For a superextension A of S we denote by $\mathrm{Alt}(I, A)$ the A-module of all alternating matrices $x \in \mathrm{Mat}(I, I; A)$, i.e., $x^{\mathrm{T}} = -x$ and all diagonal entries $x_{ii} = 0$. The alternating matrix superpair $\mathbb{A}_I(A) = (\mathrm{Alt}(I, A), \mathrm{Alt}(I, A))$ is a subpair of $\mathbb{M}_{II}(A)$; it is covered by an alternating grid $\mathcal{A}(I)$. Conversely, any Jordan superpair covered by an alternating grid $\mathcal{A}(I)$, $|I| \geq 4$ is isomorphic to some $\mathbb{A}_I(A)$ ([**35**, 5.18]. We put $\mathcal{C} = A$.

(h) The examples (e) and (g) are superextensions of the corresponding Jordan pairs which are split of type $\mathcal{G}$ in the sense of [**35**, 4.9]. This is also so for the remaining two standard examples. A Jordan superpair over S is covered by a Bi-Cayley grid $\mathcal{B}$ if and only if it is isomorphic to the Bi-Cayley superpair $\mathbb{B}(A) = A \otimes_k \mathbb{M}_{12}(\mathbb{O}_k)$, the A-extension of the rectangular matrix superpair $\mathbb{B}(k) = \mathbb{M}_{12}(\mathbb{O}_k)$ for $\mathbb{O}_k$ the split Cayley algebra over k ([**35**, 5.20]). Here $\mathcal{C} = A$.

(i) A Jordan superpair V over S is covered by an Albert grid $\mathcal{A}$ if and only if there exists a superextension A of S such that V is isomorphic to the Albert superpair $\mathbb{AB}(A) = A \otimes_k \mathbb{AB}(k)$, the A-extension of the split Jordan pair $\mathbb{AB}(k) = \mathbb{H}_3(\mathbb{O}_k, k \cdot 1, \pi)$ where $\mathbb{O}_k$ is the split Cayley algebra over k with canonical involution π ([**35**, 5.22]). Here again $\mathcal{C} = A$.

1.9. Simple Jordan superpairs covered by grids. We summarize here the main results from [**14**, §3] on simplicity of the standard examples $V(\mathcal{G}, \mathcal{C})$.

(a) $\mathbb{J}$ is simple if and only if J is simple.

(b) $\mathbb{M}_{JK}(A)$, $|J| + |K| \geq 3$ is simple if and only if A is a simple superalgebra. In particular, by 1.4, a simple rectangular matrix superpair over a field of characteristic $\neq 2, 3$ either has a simple associative coordinate superalgebra or is of type $\mathbb{M}_{12}(A)$ for $A = A_{\bar{0}}$ a simple Cayley-Dickson algebra and hence has $\mathbb{M}_{12}(A)_{\bar{1}} = 0$.

(c) Let $V = \mathbb{H}_I(A, A_0, \pi)$ be a hermitian matrix superpair with $|I| \geq 3$. If V is simple, then A is a π-simple superalgebra. Conversely, if A is π-simple and A_0 is the span of all traces and norms, i.e., $A_0 = A_{0,\min}$ as defined in [**35**, 5.10], then V is simple. In particular, if $\frac{1}{2} \in A$ then V is simple if and only if A is π-simple, and if $S = k$ is a field of characteristic $\neq 2, 3$ then there are exactly the following possibilities for a simple V:

  (i) $V \cong \mathbb{M}_{II}(B)$ for a simple associative superalgebra B,
  (ii) A is a simple associative superalgebra, or
  (iii) $A = A_{\bar{0}}$ is a simple Cayley-Dickson algebra and hence $V = V_{\bar{0}}$.

(d) Let $V = \mathbb{OQ}_I(A, q_X)$ be an odd quadratic form superpair with $|I| \geq 2$. If V is semiprime then q_X is nondegenerate, equivalently, $q_I \oplus q_X$ is nondegenerate. Conversely, if q_X is nondegenerate and A is simple then V is simple. In particular, if $\frac{1}{2} \in A$ or $A = A_{\bar{0}}$, then V is simple if and only if q_X is nondegenerate and A is a field.

(e) Let A be a superextension of S. If $V = A \otimes_k U$ is the A-extension of a Jordan pair U over k which is split of type $\mathcal{G}$, then V is simple if and only if A is simple. In particular, V is simple if and only if either $A = A_{\bar{0}}$ is a field (and hence $V = V_{\bar{0}}$ is a simple Jordan pair) or $A_{\bar{0}}$ is a field of characteristic 2 and $V = \mathbb{D}(A_{\bar{0}} \otimes_k U)$.

2. Tits-Kantor-Koecher Superalgebras.

Unless stated otherwise, we retain the setting of the previous sections, in particular k will denote an arbitrary base ring. We will consider superalgebras and superpairs over some superextension S of k. Starting from 2.5 *we will assume $\frac{1}{2} \in k$.*

2.1. Derivations of Jordan superpairs. Recall ([**27**, 1.4]) that a derivation of a Jordan pair U over k is a pair $\Delta = (\Delta^+, \Delta^-) : U \to U$ of k-linear maps which satisfy for all $x \in V^\sigma$ the equation $\Delta^\sigma Q^\sigma(x) = Q^\sigma(\Delta^\sigma(x), x) + Q^\sigma(x)\Delta^{-\sigma}$. Linearization of this identity gives the more familiar condition $\Delta^\sigma\{x\ y\ z\} = \{\Delta^\sigma(x)\ y\ z\} + \{x\ \Delta^{-\sigma}(y)\ z\} + \{x\ y\ \Delta^\sigma(z)\}$ for $x, z \in V^\sigma, y \in V^{-\sigma}$, or equivalently

$$[\Delta^\sigma, D^\sigma(x, y)] = D^\sigma(\Delta^\sigma(x), y) + D^\sigma(x, \Delta^{-\sigma}(y))$$

where $D^\sigma(x, y)$ is defined by $D^\sigma(x, y)z = \{x\, y\, z\}$. Conversely, this last condition is enough for Δ being a derivation if $\frac{1}{2} \in k$.

Let now $V = (V^+, V^-)$ be a Jordan superpair over S. Recall that $\operatorname{End}_S V^+ \times \operatorname{End}_S V^-$ is an S-supermodule whose homogeneous parts are given by $(\operatorname{End}_S V^+)_\alpha \times (\operatorname{End}_S V^-)_\alpha, \alpha \in \mathbb{Z}_2$. A pair $\Delta = (\Delta^+, \Delta^-) : V \to V$ of homogeneous S-linear maps is called a *derivation of* V if the following two conditions are satisfied for all

$x_{\bar 0} \in V^\sigma_{\bar 0}$ and all homogeneous $(x,y) \in V^\sigma \times V^{-\sigma}$:

$$\begin{aligned} \varDelta^\sigma Q^\sigma_{\bar 0}(x_{\bar 0}) &= Q^\sigma_{\bar 0}(\varDelta^\sigma(x_{\bar 0}), x_{\bar 0}) + Q^\sigma_{\bar 0}(x_{\bar 0})\varDelta^{-\sigma} \quad \text{and} \\ [\varDelta^\sigma, D^\sigma(x,y)] &= D^\sigma(\varDelta^\sigma(x), y) + (-1)^{|\varDelta||x|}\, D^\sigma(x, \varDelta^{-\sigma}(y)), \end{aligned} \tag{1}$$

where $[.,.]$ is the supercommutator ($[a,b] = ab - (-1)^{|a||b|}ba$) and $|\varDelta|$ denotes the common degree of $\varDelta^+$ and $\varDelta^-$. The justification for these conditions is the following criterion. If $\varDelta$ is a pair of homogeneous S-linear maps, then for any homogenous $g_\varDelta \in G$ of degree $|\varDelta|$ we have

$$\varDelta \text{ is a derivation of } V \iff g_\varDelta \otimes \varDelta \text{ is a derivation of } G(V). \tag{2}$$

Each $\operatorname{End}_S V^\sigma$ is a Lie superalgebra, denoted $\mathfrak{l}(V^\sigma)$, with respect to the supercommutator, hence so is $\mathfrak{l}(V^+)\times\mathfrak{l}(V^-)$ with respect to the componentwise product. The set $\operatorname{Der} V$ of all derivations of V is a $\mathbb{Z}_2$-graded subalgebra of $\mathfrak{l}(V^+)\times\mathfrak{l}(V^-)$ and hence a Lie superalgebra over S, called the *derivation algebra of* V.

For $(x,y) \in V$ we put

$$\delta(x,y) = (\, D^+(x,y)\,,\, -(-1)^{|x||y|}\, D^-(y,x)\,) \in \operatorname{End}_S V^+ \times \operatorname{End}_S V^-.$$

It follows from the super version of the identities (JP12) and (JP15) of [**27**] that $\delta(x,y)$ is a derivation of V, a so-called *inner derivation.* We denote by $\operatorname{IDer} V$ the $\mathbb{Z}$-span of all $\delta(x,y)$. Because of $s\delta(x,y) = \delta(sx,y)$ this is in fact an S-submodule of $\operatorname{Der} V$. Moreover, (1) implies

$$[\varDelta\,,\, \delta(x,y)] = \delta(\varDelta^+(x), y) + (-1)^{|\varDelta||x|}\, \delta(x, \varDelta^-(y))$$

which says that $\operatorname{IDer} V$ is an ideal of $\operatorname{Der} V$ and hence itself a Lie superalgebra, called the *inner derivation algebra of* V. In the Grassmann envelope we have $g_x g_y \otimes \delta(x,y) = \delta(g_x \otimes x, g_y \otimes y)$ and therefore

$$\operatorname{IDer} G(V) \subset G(\operatorname{IDer} V). \tag{3}$$

We note that $\operatorname{IDer} G(V)$ need not be equal to $G(\operatorname{IDer} V)$ since for odd elements x, y the derivation $1 \otimes \delta(x,y)$ need not be an inner derivation of $G(V)$.

2.2. Tits-Kantor-Koecher superalgebras. Let V be a Jordan superpair over S and let $\mathfrak{D}$ be a subalgebra of the Lie superalgebra $\operatorname{Der} V$ containing $\operatorname{IDer} V$. On the S-supermodule

$$\mathfrak{K}(V,\mathfrak{D}) = V^+ \oplus \mathfrak{D} \oplus V^-$$

we define an S-superalgebra by

$$\begin{aligned} &[x^+ \oplus c \oplus x^-\,,\, y^+ \oplus d \oplus y^-] = (\, c^+y^+ - (-1)^{|d||x^+|}\, d^+x^+\,) \\ &\quad \oplus (\, [cd] + \delta(x^+, y^-) - (-1)^{|x^-||y^+|}\, \delta(y^+, x^-)\,) \oplus (\, c^-y^- - (-1)^{|d||x^-|}\, d^-x^-\,) \end{aligned}$$

where of course $x^\sigma, y^\sigma \in V^\sigma$ and $c, d \in \mathfrak{D}$. We put $\mathfrak{K}(V) = \mathfrak{K}(V, \operatorname{IDer} V)$.

Suppose for a moment that $V_{\bar{1}} = 0$, i.e., V is a Jordan pair. Then $\operatorname{Der} V$ and hence $\mathfrak{D}$ are Lie algebras, and it is well-known that the algebra $\mathfrak{K}(V, \mathfrak{D})$ is a Lie algebra. For example, this is shown in [**33**, §11] (note that the product is sometimes defined slightly different, e.g. in [**28**], but the approach above is more suitable for our purposes). For Jordan pairs associated to Jordan algebras these types of Lie algebras were first considered by Tits [**44**], Kantor [**23**] and Koecher [**24**], [**25**], and they are therefore called the *Tits-Kantor-Koecher algebra of* $(V, \mathfrak{D})$, or the *TKK-algebra of* $(V, \mathfrak{D})$ for short. If $\mathfrak{D} = \operatorname{IDer} V$ we will call $\mathfrak{K}(V)$ the *TKK-algebra of* V.

Coming back to the general situation of a Jordan superpair V we observe that the Grassmann envelope of the algebra $\mathfrak{K}(V, \mathfrak{D})$ is the TKK-algebra of the Grassmann envelope of $(V, \mathfrak{D})$,

$$G(\mathfrak{K}(V, \mathfrak{D})) = \mathfrak{K}(G(V), G(\mathfrak{D})). \tag{1}$$

Hence $\mathfrak{K}(V, \mathfrak{D})$ is a Lie superalgebra over S which we call the *Tits-Kantor-Koecher superalgebra of* $(V, \mathfrak{D})$ or *TKK-superalgebra of* $(V, \mathfrak{D})$ or *TKK-superalgebra* in case $\mathfrak{D} = \operatorname{IDer} V$.

If $V = \oplus_{\gamma \in \Gamma} V[\gamma]$ is a Γ-grading, [**35**, 3.3], define for $\gamma \in \Gamma$

$$\operatorname{IDer}_\gamma V := \textstyle\sum_{\gamma = \alpha + \beta} \delta(V^+[\alpha], V^-[\beta]) \tag{2}$$

$$\mathfrak{K}(V)_\gamma := V^+[\gamma] \oplus \operatorname{IDer}_\gamma V \oplus V^-[\gamma]. \tag{3}$$

Then $\mathfrak{K}(V)_\gamma$ is a $\mathbb{Z}_2$-graded S-submodule of $\mathfrak{K}(V)$. Moreover, $\operatorname{IDer}_\gamma V$ consists of the inner derivations of degree γ, hence $\operatorname{IDer} V = \oplus_{\gamma \in \Gamma} \operatorname{IDer}_\gamma V$ by [**10**, §11.6] and therefore

$$\mathfrak{K}(V) = \oplus_{\gamma \in \Gamma} \mathfrak{K}(V)_\gamma \,. \tag{4}$$

It is an immediate consequence of the axioms for a grading that we also have $[\mathfrak{K}(V)_\gamma \,, \mathfrak{K}(V)_\delta] \subset \mathfrak{K}(V)_{\gamma+\delta}$. Thus, (4) is a Γ-grading of the Lie superalgebra $\mathfrak{K}(V)$.

If U is a subpair of V it is in general not true that $\operatorname{IDer} U$ imbeds in $\operatorname{IDer} V$, and hence $\mathfrak{K}(U)$ is not necessarily a subalgebra of $\mathfrak{K}(V)$, see however [**36**, 3.2] where imbedding is shown in a special case. Nevertheless, we have

$$\begin{aligned} &V = \textstyle\bigoplus_{i \in I} V^{(i)} \quad \text{(direct sum of ideals)} \\ &\Rightarrow \mathfrak{K}(V) = \textstyle\bigoplus_{i \in I} \mathfrak{K}(V^{(i)}) \quad \text{(direct sum of ideals)} \end{aligned} \tag{5}$$

Indeed, in this case $\operatorname{IDer} V = \bigoplus_{i \in I} \operatorname{IDer} V^{(i)}$ is a direct sum of ideals. If $f : V \to W$ is an isomorphism of Jordan superpairs [**35**, 3.3] then

$$\mathfrak{K}(f)\colon \mathfrak{K}(V) \to \mathfrak{K}(W) : x^+ \oplus d \oplus y^- \mapsto f^+(x^+) \oplus f d f^{-1} \oplus f^-(y^-) \tag{6}$$

is an isomorphism of Lie superalgebras. Conversely, any isomorphism $\mathfrak{K}(V) \to \mathfrak{K}(W)$ respecting the 3-gradings in the sense of 2.3 arises in this way.

2.3. 3-graded Lie superalgebras. A *3-grading* of a Lie superalgebra L over S is a decomposition $L = L_1 \oplus L_0 \oplus L_{-1}$ where each L_i is a submodule, hence $L_i = L_{i\bar{0}} \oplus L_{i\bar{1}}$ for $i = 0, \pm 1$, satisfying

$$[L_i, L_j] \subset L_{i+j} \qquad (1)$$

where $L_{i+j} = 0$ if $i + j \neq 0, \pm 1$. In other words, $L = L_1 \oplus L_0 \oplus L_{-1}$ is a $\mathbb{Z}$-grading with at most 3 non-zero homogeneous spaces. Because of this, 3-gradings are sometimes also called *short $\mathbb{Z}$-gradings*, like in [**48**]. A Lie superalgebra is called *3-graded* if it has a 3-grading. If L is a 3-graded Lie superalgebra its Grassmann envelope is a 3-graded Lie algebra in the sense of [**38**, 1.5].

A 3-graded Lie superalgebra $L = L_1 \oplus L_0 \oplus L_{-1}$ will be called *Jordan 3-graded* if

(i) $[L_1, L_{-1}] = L_0$, and
(ii) there exists a Jordan superpair structure on (L_1, L_{-1}) whose Jordan triple product is related to the Lie product by

$$\{x\,y\,z\} = [[xy]z] \text{ for all } x, z \in L_{\sigma 1}, y \in L_{-\sigma 1}, \sigma = \pm. \qquad (2)$$

In this case, $V = (L_1, L_{-1})$ will be called the *associated Jordan superpair.*

The prototype of a Jordan 3-graded Lie algebra is the TKK-superalgebra of a Jordan superpair V. The relation between general Jordan 3-graded Lie algebras and TKK-superalgebras is described below. If L is Jordan 3-graded, the subalgebra $G(L_1) \oplus [G(L_1), G(L_{-1})] \oplus G(L_{-1})$ of the 3-graded Lie algebra $G(L)$ is a Jordan 3-graded Lie algebra in the sense of [**38**, 1.5].

If $\frac{1}{2} \in S$ the associated Jordan superpair is unique: its product is given by (2) and by $Q_{\bar{0}}(x_{\bar{0}})y = \frac{1}{2}[[x_{\bar{0}}, y]x_{\bar{0}}]$. Conversely, these two formulas define a pair structure on (L_1, L_{-1}) which will be a Jordan superpair in any situation where Jordan superpairs are defined by linear identities. Hence, by [**35**, (3.2.1)], a 3-graded Lie superalgebra L over S with $[L_1, L_{-1}] = L_0$ is Jordan 3-graded as soon as 2 and 3 are invertible in S.

2.4. Lemma. *Let L be a Jordan 3-graded Lie superalgebra with associated Jordan superpair V.*

(a) *$\mathfrak{K}(V) \cong L/C$, where $C = \{x \in L_0 : [x, L_1] = 0 = [x, L_{-1}]\}$ is the 0-part of the centre of L. Moreover,*

$$G(C) = C_{G(L)} := \{x \in G(L)_0 : [x, G(L)_1] = 0 = [x, G(L)_{-1}]\}. \qquad (1)$$

(b) *V is finitely generated if and only if L is so.*

Proof. (a) Let $f\colon L \to \mathfrak{K}(V)$ be the surjective linear map defined by $f|L_{\pm 1} = \mathrm{Id}$ and $f(x) = (ad\ x|_{V^+}, ad\ x|_{V^-}) = \sum_i \delta(x_i, y_i)$ for $x = \sum_i [x_i, y_i] \in L_0$. Its kernel is C. Moreover, it is easy to check, using 2.3.2, that f is a homomorphism. (1) is straightforward using that G is a free k-module.

(b) If $\mathcal{B}$ is a finite generating set of V, it also generates L in view of 2.3.2 and $L_0 = [V^+, V^-]$. Conversely, let $\mathcal{B}$ be a finite generating set for L. We can assume that the elements in $\mathcal{B}$ are homogeneous with respect to the 3-grading, i.e., $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_0 \cup \mathcal{B}_{-1}$ where $\mathcal{B}_i \subset L_i$. The elements in $\mathcal{B}_0$ are sums of commutators of V^+ and V^-, so all of them are generated by a finite number of elements in V. These elements together with $\mathcal{B}_1$ and $\mathcal{B}_{-1}$ form a finite generating set for V.

In the following we will study the interplay of ideals of a Jordan 3-graded Lie algebra L and ideals in the associated Jordan superpair V. For TKK-superalgebras this will naturally lead to results relating (semi)primeness and simplicity of V and $\mathfrak{K}(V)$. In order to express the quadratic maps $Q_{\bar{0}}(.)$ of V by the Lie algebra product of L *we assume in the following that* $\frac{1}{2} \in S$. Then the triple product suffices to define ideals for a Jordan superpair V over S. Indeed, specializing the definitions of [**35**, 3.3], an *ideal* U of V is a pair of S-submodules satisfying $\{U^\sigma\, V^{-\sigma}\, V^\sigma\} + \{V^\sigma\, U^{-\sigma}\, V^\sigma\} \subset U^\sigma$, and V is semiprime if and only if $U^3 = (\{U^+\, U^-\, U^+\}, \{U^-\, U^+\, U^-\}) \neq 0$ for any non-zero ideal U of V.

2.5. Lemma. *Let $L = L_1 \oplus L_0 \oplus L_{-1}$ be a Jordan 3-graded Lie superalgebra with associated superpair $V = (L_1, L_{-1})$ and assume that $\frac{1}{2} \in S$.*

(a) *We denote by π_σ, $\sigma = \pm 1$, the canonical projections of L onto L_σ. If $I \lhd L$ is an ideal of L, then $I \cap V = (I \cap L_1, I \cap L_{-1})$ and $\pi(I) = (\pi_+(I), \pi_-(I))$ are ideals of V with $\pi(I)^3 \subset I \cap V \subset \pi(I)$.*

(b) *Any ideal $U = (U_1, U_{-1})$ of V generates a* split ideal *$\mathfrak{I}(U)$ of L, given by*

$$\mathfrak{I}(U) = U_1 \oplus ([U_1, L_{-1}] + [L_1, U_{-1}]) \oplus U_{-1}\,.$$

If $U = (U_1, U_{-1})$ and $W = (W_1, W_{-1})$ are two ideals of V satisfying for $\sigma = \pm 1$

$$[U_\sigma, W_{-\sigma}] = 0 \quad \textit{and} \quad \{U_\sigma\, L_{-\sigma}\, W_\sigma\} = 0 \tag{1}$$

then $[\mathfrak{I}(U), \mathfrak{I}(W)] = 0$. In particular, if L is a TKK-superalgebra then

$$[\mathfrak{I}(U), \mathfrak{I}(\mathrm{Ann}_V(U))] = 0. \tag{2}$$

Proof. (a) For $x_\sigma, z_\sigma \in L_\sigma$, $y_{-\sigma} \in L_{-\sigma}$ and arbitrary $l \in L$ we have

$$\{x_\sigma\, y_{-\sigma}\, \pi_\sigma(l)\} = \pi_\sigma([[x_\sigma, y_{-\sigma}], l]) \quad \text{and} \quad \{x_\sigma\, \pi_{-\sigma}(l)\, z_\sigma\} = [[x_\sigma, l], z_\sigma] \in L_\sigma\,.$$

These formulas imply that $I \cap V$ and $\pi(I)$ are ideals of V. The second formula also proves $\{\pi_\sigma(I)\, \pi_{-\sigma}(I)\, \pi_\sigma(I)\} \subset I \cap L_\sigma$.

(b) That $\mathfrak{I}(U) \lhd L$ is a straightforward verification. For easier notation let $W_0 = [W_1, L_{-1}] + [L_1, W_{-1}]$ and similarly for U_0. Then

$$[\mathfrak{I}(U), \mathfrak{I}(W)] = ([U_1, W_0] + [U_0, W_1]) \oplus ([U_1, W_{-1}] + [U_{-1}, W_1] + \\ + [U_0, W_0]) \oplus ([U_0, W_{-1}] + [U_{-1}, W_0])$$

Here $[U_1, W_0] = [U_1, [W_1, L_{-1}] + [L_1, W_{-1}]] = \{U_1\, L_{-1}\, W_1\} + [[L_1, W_{-1}]U_1] = 0$ by (1). Since our assumptions are symmetric in U and W and in $\sigma = \pm 1$, it now follows from (1) that $[\mathfrak{I}(U), \mathfrak{I}(W)] = [U_0, W_0]$. But by the Jacobi identity,

$$\begin{aligned} [U_0, W_0] &= [U_0, [W_1, L_{-1}] + [L_1, W_{-1}]] \\ &\subset [[U_0, W_1], L_{-1}] + [W_1, [U_0, L_{-1}]] + [[U_0, L_1], W_{-1}] + [L_1, [U_0, W_{-1}]] \\ &= 0 \end{aligned}$$

since $[U_0, W_\sigma] = 0$ by the first part of the proof, $[W_1, [U_0, L_{-1}]] \subset [W_1, U_{-1}] = 0$ by (1) and similarly for the term $[[U_0, L_1], W_{-1}]$. Finally, $W = \mathrm{Ann}_V(U)$ satisfies the assumption (1), whence (2).

2.6. Proposition. *Let V be a Jordan superpair over S with $\frac{1}{2} \in S$.*
(a) *If V is semiprime then so is $\mathfrak{K}(V)$.*
(b) *V is prime if and only if $\mathfrak{K}(V)$ is prime and V is semiprime.*
(c) *V is simple if and only if $\mathfrak{K}(V)$ is simple.*

For linear Jordan pairs, i.e., $\frac{1}{2}$ and $\frac{1}{3} \in S$, (c) is proven in [**26**, Lemma 6]. We will include a short proof here for sake of completeness.

Proof. (a) Let V be semiprime, and let $0 \neq I \triangleleft \mathfrak{K}(V)$. If $I \cap V = 0$ the ideal $\pi(I)$ has $\pi(I)^3 = 0$ by 2.5(a). Therefore $\pi(I) = 0$ by semiprimeness of V, which means that $I \subset \mathfrak{K}(V)_0 = \mathrm{IDer}\, V$. But then $[I, \mathfrak{K}(V)_{\pm 1}] \subset I \cap \mathfrak{K}(V)_{\pm 1} = 0$ and hence $I = 0$ since the representation of $\mathrm{IDer}\, V$ on V is faithful. This proves

$$V \text{ semiprime, } 0 \neq I \triangleleft \mathfrak{K}(V) \quad \Rightarrow \quad I \cap V \neq 0. \tag{1}$$

Using again the semiprimeness of V, we have $0 \neq \{I \cap V^\sigma\,, I \cap V^{-\sigma}\,, I \cap V^\sigma\} \subset [[I, I], I] \subset [I, I]$, i.e., $\mathfrak{K}(V)$ is semiprime.

(b) Suppose V is prime and let I, J be two non-zero ideals of $\mathfrak{K}(V)$. By (1), both $I \cap V$ and $J \cap V$ are non-zero ideals of V. By primeness of V, we have $0 \neq \{I \cap V^\sigma\,, J \cap V^{-\sigma}\,, I \cap V^\sigma\} \subset [[I, J], I] \subset [J, I]$, proving primeness of $\mathfrak{K}(V)$. Conversely, let $\mathfrak{K}(V)$ be prime and let $0 \neq U \triangleleft V$. Then $\mathfrak{I}(\mathrm{Ann}_V(U)) = 0$ by 2.5.2 and therefore $\mathrm{Ann}_V(U) = 0$. This and the semiprimeness of V imply that V is prime ([**14**, 3.1]).

(c) Suppose V is simple and let again $0 \neq I \triangleleft \mathfrak{K}(V)$. Since V is in particular semiprime we have $I \cap V \neq 0$ by (1) above. But then $I \cap V = V$, so I contains the generating set $V^+ \oplus V^-$ of $\mathfrak{K}(V)$ and hence $I = \mathfrak{K}(V)$. Conversely, if $\mathfrak{K}(V)$ is simple and $0 \neq U \triangleleft V$ then the ideal $\mathfrak{I}(U)$ of $\mathfrak{K}(V)$ is non-zero, hence $\mathfrak{I}(U) = \mathfrak{K}(V)$ and $U = V$ follows.

Remark. In the following section §3 we will determine the TKK-superalgebras of Jordan superpairs covered by a grid. They not only provide examples for the general theory of TKK-superalgebras but, by (c) above, also examples of simple Lie superalgebras by making use of the classification of simple Jordan superpairs covered by a grid, see 1.9. Moreover, these TKK-superalgebras also are examples of root-graded Lie superalgebras. In the remainder of this section we will explain this connection.

2.7. Root-graded Lie superalgebras. Let R be a reduced, possibly infinite root system as defined in [**37**] and let $\mathcal{Q}(R) = \mathbb{Z}[R]$ be the $\mathbb{Z}$-span of R. A *Lie superalgebra graded by* R, also called a *R-graded Lie superalgebra*, is a Lie superalgebra L over S together with a decomposition

$$L = \bigoplus\nolimits_{\alpha \in R \cup \{0\}} L_\alpha\,, \tag{1}$$

where the $L_\alpha = L_{\alpha\bar{0}} \oplus L_{\alpha\bar{1}}$ are S-submodules, and k-subalgebras $\mathfrak{h} \subset \mathfrak{g} \subset L$ such that the following conditions are satisfied:

(i) the decomposition (1) is a $\mathcal{Q}(R)$-grading in the sense that $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$, with the understanding that $L_{\alpha+\beta} = \{0\}$ if $\alpha+\beta \notin R \cup \{0\}$;

(ii) $L_0 = \sum_{\alpha\in R} [L_\alpha, L_{-\alpha}]$;

(iii) there exists a family $(x_\alpha : \alpha \in R)$ of non-zero elements $x_\alpha \in L_{\alpha\bar{0}}$ such that, putting $h_\alpha = -[x_\alpha, x_{-\alpha}]$, we have

$$\mathfrak{h} = \sum\nolimits_{\alpha\in R} k \cdot h_\alpha\,, \tag{2}$$

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus\nolimits_{\alpha\in R} k\cdot x_\alpha\,, \text{ and} \tag{3}$$

$$[h_\alpha, y_\beta] = \langle \beta, \alpha^\vee\rangle y_\beta \tag{4}$$

for all $\alpha \in R$ and $y_\beta \in L_\beta$, $\beta \in R \cup \{0\}$ (here $\langle\beta,\alpha^\vee\rangle$ are the Cartan integers).

A Lie superalgebra will be called *root-graded* if it is graded by some reduced root system R. The subalgebra $\mathfrak{g}$ is called the *grading subalgebra*.

Remarks. (1) Suppose $\frac{1}{2}, \frac{1}{3} \in k$. It is shown in [**29**] that for Lie algebras (instead of superalgebras) the definition above is equivalent to the one in [**38**, 2.1]. The paper [**38**] contains a description of Lie algebras graded by 3-graded root systems. The super version of this result is described in 2.8 and 2.9.

Split semisimple locally finite Lie algebras are examples of root-graded Lie algebras. Their classification has been published in [**34**]. Lie algebras over fields of characteristic 0 graded by finite root systems have been introduced and described in [**9**] for simply-laced root systems and in [**6**] for the others. The papers [**9**], [**6**] and [**38**] describe root-graded Lie algebras up to central extensions. The central extensions of Lie algebras graded by finite reduced root systems were described in [**1**] for Lie algebras over fields of characteristic 0. The root gradings of finite-dimensional semisimple complex Lie algebras are determined in [**40**]. Lie algebras graded by non-reduced root systems are studied in [**2**].

(2) The Grassmann envelope of a root-graded Lie superalgebra is in general not root-graded since condition (ii) need not be true. However, the subalgebra generated by the union $\bigcup_{\alpha\in R} G(L_\alpha)$ of the Grassmann envelopes of each L_α is root-graded.

(3) Suppose k is a field of characteristic 0. As shown in [**29**], R can be identified with a set of k-linear forms on $\mathfrak{h}$ such that

$$L_\alpha = \{x \in L : [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\} \tag{5}$$

for all $\alpha \in R \cup \{0\}$. Moreover, $\mathfrak{h}$ is a maximal abelian subalgebra and $\{h_\alpha : \alpha \in R\}$ is isomorphic to the dual root system of R.

In particular, if R is finite $\mathfrak{g}$ is a finite-dimensional split semisimple Lie algebra with splitting Cartan subalgebra $\mathfrak{h}$ and root system R. Thus, in this case L is a Lie superalgebra with the following properties:

(a) $L_{\bar{0}}$ contains a subalgebra $\mathfrak{g}$ which is a finite-dimensional split semisimple Lie algebra whose root system relative to a splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is $R \subset \mathfrak{h}^*$,

(b) L has a decomposition (1) where the L_α are given by (5), and

(c) $L_0 = \sum_{\alpha \in R} [L_\alpha, L_{-\alpha}]$.

The conditions (a), (b) and (c)=(ii) are the axioms used by Berman-Moody [**9**] and Benkart-Zelmanov [**6**] for root-graded Lie algebras. It is obvious that, conversely, (a), (b) and (c) imply that L is a R-graded Lie superalgebra as defined above. (This characterization indicates an interesting generalization of the concept of a root-graded Lie superalgebra: replace $\mathfrak{g}$ by a classical simple Lie superalgebra and R by its "super root system", i.e., the set of weights of the $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}$ with respect to a splitting Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\bar{0}}$.)

The results in [**38**, §2] on root-graded Lie algebras easily generalize to root-graded Lie superalgebras as defined above. Since the proofs remain the same, we only mention the following results.

2.8. Theorem. (a) ([**38**, 2.2]) *Let L be a R-graded Lie superalgebra and let $C \subset L_0$ be a central ideal (that $C \subset L_0$ is automatic if $\frac{1}{2} \in k$). Then L/C is R-graded. Conversely, if $\tilde{L}$ is a perfect central extension of a R-graded Lie superalgebra L and if $\frac{1}{2}, \frac{1}{3}, \frac{1}{5} \in S$ then $\tilde{L}$ is also R-graded.*

(b) ([**38**, 2.3]) *Let $R = R_1 \dot\cup R_0 \dot\cup R_{-1}$ be a 3-graded root system and let L be a R-graded Lie superalgebra over S containing $\frac{1}{2}$ and $\frac{1}{3}$. Put*

$$L_{(\pm 1)} = \bigoplus_{\alpha \in R_{\pm 1}} L_\alpha \quad \textit{and} \quad L_{(0)} = \bigoplus_{\alpha \in R_0 \cup \{0\}} L_\alpha.$$

Then $L = L_{(1)} \oplus L_{(0)} \oplus L_{(-1)}$ is a Jordan 3-graded Lie superalgebra. Its associated Jordan superpair V is covered by a standard grid $\mathcal{G}$ whose root system is isomorphic to R. In particular, L is a central extension of $\mathfrak{K}(V)$.

(c) ([**38**, 2.6]) *Let $V = \bigoplus_{\alpha \in R_1} V_\alpha$ be a Jordan superpair covered by a standard grid $\mathcal{G} = \{g_\alpha : \alpha \in R_1\}$ with associated 3-graded root system $R = R_1 \dot\cup R_0 \dot\cup R_{-1}$. Then $\mathfrak{K}(V)$ is R-graded with respect to*

$$\mathfrak{K}(V)_\alpha = \begin{cases} V_\alpha^+ & \alpha \in R_1, \\ \sum_{\alpha = \beta - \gamma} \delta(V_\beta^+, V_\gamma^-) & \alpha \in R_0 \cup \{0\}, \\ V_{-\alpha}^- & \alpha \in R_{-1}, \end{cases} \tag{1}$$

and $\mathfrak{g}$ the k-subalgebra generated by $\mathcal{G}$.

(d) ([**38**, 2.7]) *Let (R, R_1) be a 3-graded root system and assume $\frac{1}{2}, \frac{1}{3} \in S$. Then a Lie superalgebra L over S is R-graded if and only if L is a perfect central extension of the TKK-superalgebra $\mathfrak{K}(V)$ of a Jordan superpair V covered by a grid whose associated 3-graded root system is R.*

2.9. Classification of R-graded Lie superalgebras for a 3-graded R. By Theorem 2.8(d) above, the classification up to central extensions of Lie superalgebras graded by a 3-graded root system is reduced to determining TKK-superalgebras $\mathfrak{K}(V)$ of Jordan superpairs V covered by a standard grid. For the convenience of the reader we list below where one can find these TKK-superalgebras for all irreducible 3-graded root systems.

(1) $R = \mathrm{A}_1$: These are the TKK-superalgebras of unital Jordan superalgebras, see 3.1.
(2) $R = \mathrm{A}_2$: $V = \mathbb{M}_{12}(A)$, two models for $\mathfrak{K}(V)$ are given in 3.2 and 3.3.
(3) $R = \mathrm{A}_I, |I| \geq 3$: $V = \mathbb{M}_{JK}(A)$ for A associative and $I\dot\cup\{\infty\} = J\dot\cup K$, $\infty \notin I$, $\mathfrak{K}(V) = \mathfrak{psl}_{J\dot\cup K}(A)$, 3.4.
(4) $R = \mathrm{B}_2$ and $R = \mathrm{C}_3$: $\mathfrak{K}(V) = \mathfrak{K}(\mathbb{J})$ for $\mathbb{J}$ described in 3.5.
(5) $R = \mathrm{B}_I, |I| \geq 3$: $V = \mathbb{OQ}_{I'}(A, q_X)$ where $I' = I \setminus \{\infty\}$ for some $\infty \in I$, $\mathfrak{K}(V) = \mathfrak{eosp}(q_I \oplus q_X)$, 3.7.
(6) $R = \mathrm{C}_I, |I| \geq 4$: $V = \mathbb{H}_I(A, A_0, \pi)$ for A associative, $\mathfrak{K}(V) = \mathfrak{psu}_I(A, A_0, \pi)$, 3.6.
(7) $R = \mathrm{D}_I, |I| \geq 4$: there are two 3-gradings for these root systems, the even quadratic form grading and the alternating grading which are non-isomorphic for $|I| \geq 5$. For the first, $V = \mathbb{EQ}_{I'}(A)$ where I' is defined as in (5), and for the second $V = \mathbb{A}_I(A)$. Both superpairs have isomorphic TKK-superalgebras, namely $\mathfrak{eosp}(q_I)$, 3.7 and 3.8.
(8) $R = \mathrm{E}_6$ and E_7: $V = \mathbb{B}(A)$ and $V = \mathbb{AB}(A)$ respectively, $\mathfrak{K}(V)$ is described in 3.9 and 3.10.
(9) $R = \mathrm{E}_8, \mathrm{F}_4, \mathrm{G}_2$: these root systems are not 3-graded.

2.10. Refined root gradings. Let Λ be an abelian group and let $L = \bigoplus_{\alpha \in R\cup\{0\}} L_\alpha$ be a R-graded Lie superalgebra with respect to the grading subalgebra $\mathfrak{g}$. A *refined root grading of L of type Λ* is a Λ-grading $L = \bigoplus_{\lambda\in\Lambda} L^\lambda$ of L which is compatible with the root grading in the following sense:

(i) $L^\lambda = \bigoplus_{\alpha\in R\cup\{0\}} (L^\lambda \cap L_\alpha)$ for every $\lambda \in \Lambda$, or equivalently $L_\alpha = \bigoplus_{\lambda\in\Lambda} (L_\alpha \cap L^\lambda)$ for all $\alpha \in R \cup \{0\}$, and
(ii) $\mathfrak{g} \subset L^0$.

Refined root gradings of Lie algebras have been introduced in [**46**, 2]. The paper [**46**] contains a classification of special types of refined root gradings, so-called (pre)division gradings, for $\Lambda = \mathbb{Z}^n$ and R simply-laced of rank ≥ 3. The case $R = \mathrm{A}_2$ is treated in [**47**].

Let L be a R-graded Lie superalgebra with a refined root grading of type Λ. We put $L_\alpha^\lambda = L^\lambda \cap L_\alpha$. Then

$$L_0^\lambda = \textstyle\sum_{\alpha\in R, \kappa\in\Lambda} [L_\alpha^\kappa\,,\, L_{-\alpha}^{\lambda-\kappa}]. \qquad (1)$$

If $C = \bigoplus_{\lambda\in\Lambda, \alpha\in R\cup\{0\}}(C \cap L_\alpha^\lambda)$ is a central ideal, then L/C has a canonical refined root grading of type Λ. In particular, this is so for $L/Z(L)$.

It is therefore natural to classify refined root gradings of $L/Z(L)$ only. For 3-graded root systems this means describing the refined root gradings of TKK-superalgebras. The following result which is immediate from the definitions translates this problem into a problem for Jordan superpairs.

2.11. Proposition. *Let V be a Jordan superpair covered by a standard grid $\mathcal{G}$, and denote by $\mathfrak{g}$ the k-subalgebra of $\mathfrak{K}(V)$ generated by $\mathcal{G} \subset \mathfrak{K}(V)$.*

(a) *Suppose that $(V, \mathcal{G})$ has a refined root grading of type Λ with homogenous spaces $(V^{\pm}\langle\lambda\rangle : \lambda \in \Lambda)$, as defined in [**35**, 4.8]. Then so has $(\mathfrak{K}(V), \mathfrak{g})$. The homogeneous spaces $\mathfrak{K}(V)^{\lambda}$ are given by* 2.2.2 *and* 2.2.3*; in particular*

$$\mathfrak{K}(V)^{\lambda}_{\pm\alpha} = V^{\pm}\langle\lambda\rangle \cap V^{\pm}_{\alpha} \quad \textit{for } \alpha \in R_1. \tag{1}$$

(b) *Conversely, suppose $\frac{1}{2} \in S$ and that $\mathfrak{K}(V)$ has a refined root grading of type Λ. Then* (1) *defines a refined root grading of $(V, \mathfrak{g})$.*

Refined root gradings for Jordan superpairs covered by a grid have been classified in [**35**, §5]. Hence, the proposition together with the results of §3 gives the corresponding description of refined root gradings of Lie superalgebras. Details will be left to the reader.

3. TKK-superalgebras of Jordan superpairs covered by grids.

By 1.7 and 2.2.5, the TKK-superalgebra of a Jordan superpair V over S covered by a grid is a direct sum of ideals, namely the TKK-superalgebras of ideals of V covered by a connected grid. For the description of TKK-superalgebras we can therefore assume that the covering grid is connected. Hence V is one of the standard examples of 1.8. In this section we will give concrete models for their TKK-superalgebras, generalizing the results of [**38**] in the non-supersetting. We will also describe which of these Lie superalgebras are simple, using the simplicity criterion 2.6(c) and results from 1.9.

3.1. TKK-superalgebras of unital Jordan superalgebras. Let $V = \mathbb{J} = (J, J)$ be the Jordan superpair of a unital Jordan superalgebra J over S, and denote by 1_J its identity element and by U its quadratic representation [**35**, 3.11]. Following the classical theory, see e.g. [**18**, 1.8], the *structure algebra* of J is the $\mathbb{Z}_2$-graded algebra $\operatorname{Str} J = (\operatorname{Str} J)_{\bar{0}} \oplus (\operatorname{Str} J)_{\bar{1}}$ where a homogeneous $s \in (\operatorname{End}_S J)_{|s|}$ lies in $(\operatorname{Str} J)_{|s|}$ if and only if s and $s' = s - U_{s1_J, 1_J}$ satisfy

$$\begin{aligned} sU_{\bar{0}}(x_{\bar{0}}) &= U(sx_{\bar{0}}, x_{\bar{0}}) + U_{\bar{0}}(x_{\bar{0}})s' \quad \text{and} \\ [s, U(x,y)] &= U(sx, y) + (-1)^{|s||x|}\, U(x, s'y) \end{aligned} \tag{1}$$

for all $x_{\bar{0}} \in J_{\bar{0}}$ and homogenous $x, y \in J$. The product on $\operatorname{Str} J$ is the supercommutator. $\operatorname{Str} J$ is a Lie superalgebra since $G(\operatorname{Str} J) = \operatorname{Str} G(J)$ is a Lie algebra. The map

$$\flat\colon \operatorname{Der} \mathbb{J} \xrightarrow{\approx} \operatorname{Str} J : d = (d_+, d_-) \mapsto d_+ \tag{2}$$

is an isomorphism of Lie superalgebras with inverse given by $\sharp\colon s \mapsto (s, s')$. The *derivation algebra* $\operatorname{Der} J$ of J is the subalgebra of $\operatorname{Str} J$ consisting of all $s \in \operatorname{Str} J$ with $s(1_J) = 0$. If we let $\operatorname{Der}(\mathbb{J}, 1_J)$ be the subalgebra of $\operatorname{Der} \mathbb{J}$ whose elements annihilate $(1_J, 1_J)$, the restriction map $\flat$ induces an isomorphism

$$\flat\colon \operatorname{Der}(\mathbb{J}, 1_J) \xrightarrow{\approx} \operatorname{Der} J$$

with inverse $\sharp\colon d \mapsto (d, d)$. Under the isomorphism $\flat$, the inner derivation algebra $\operatorname{IDer} \mathbb{J}$ maps onto the *inner structure algebra* $\operatorname{IStr} J$, the span of all maps $D(x, y), x, y \in J$ for $D(x, y)z = \{xyz\}$. The *inner derivation algebra* $\operatorname{IDer} J$ of J is the annihilator of 1_J in $\operatorname{IStr} J$.

For the remainder of this subsection assume $\frac{1}{2} \in S$. Then J can be viewed as a linear Jordan superalgebra with a bilinear supercommutative product xy, [**35**, (3.11.3) and (3.11.4)]. The Jordan triple product and the bilinear product are related by

$$\{xyz\} = 2(xy)z + 2x(yz) - (-1)^{|x||y|} 2y(xz), \quad xy = \tfrac{1}{2}\{x\,,\, 1_J\,,\, y\}.$$

In this case a homogeneous endomorphism d of J is a derivation if and only if $d(xy) = d(x)y + (-1)^{|d||x|} x d(y)$ for all homogeneous $x, y \in J$. Denote by L_x the left multiplication of J, and let $L_J = \{L_x : x \in J\}$. We then have

$$\operatorname{Str} J = L_J \oplus \operatorname{Der} J \quad \textit{if } \tfrac{1}{2} \in S. \tag{3}$$

Indeed, any $s \in \operatorname{Str} J$ can be uniquely decomposed as $s = L_x + (s - L_x)$ where $x = s1_J$ and $s - L_x \in \operatorname{Der} J$. (We note that this decomposition is no longer true in characteristic 2, even in the classical case. For example, let A be a finite-dimensional central-simple associative algebra over a field of characteristic 2 and let $J = A^+$ be the corresponding Jordan algebra, so $U_x y = xyx$. By [**17**, Thm. 14] one knows that $\operatorname{Str} J$ consists of all maps of the form $s\colon x \mapsto ax + xb$ for arbitrary $a, b \in A$. Hence $\operatorname{Der} J = \operatorname{IDer} J = V_J$ for $V_x = xy + yx$ in this case.)

All maps $\Delta_{x,y} = [L_x, L_y]$ are derivations of J. Since $\{xyz\} = 2L_{xy}z + 2\Delta_{x,y}z$ we have

$$\operatorname{IDer} J = \operatorname{span}\{\Delta_{x,y} : x, y \in J\} \quad \text{if } \tfrac{1}{2} \in S. \tag{4}$$

The TKK-superalgebra $\mathfrak{K}(\mathbb{J})$ is therefore defined on the S-module

$$\mathfrak{K}(\mathbb{J}) = J^+ \oplus (L_J \oplus \operatorname{IDer} J) \oplus J^-$$

where $J^+ = J^- = J$ as S-modules and the superscript indicates the position, and has product

$$\begin{aligned}
[x^+ \oplus L_z \oplus c \oplus u^-\,,\, y^+ \oplus L_w \oplus d \oplus v^-] &= (\, zy - xw + c(y) - (-1)^{|d||x|} d(x)\,)^+ \\
&\oplus (2L_{xv-uy} + L_{c(w)} - (-1)^{|z||d|} L_{d(z)}\,) \oplus (\,[cd] + \Delta_{z,w} + 2\Delta_{x,v} + 2\Delta_{u,y}\,) \\
&\oplus (\, uw - zv + c(v) - (-1)^{|d||u|}\, d(u)\,)^-
\end{aligned}$$

where $u, v, w, x, y, z \in J$ and $c, d \in \operatorname{IDer} J$. An equivalent version of this Lie superalgebra was given by Kac in [**19**, §3].

By 2.6(c) and 1.9(a), $\mathfrak{K}(J)$ is simple if and only if J is simple. The simple finite-dimensional Jordan superalgebras over algebraically-closed fields of characteristic $\neq 2$ have recently been determined ([**42**], [**43**], [**30**]) extending the earlier papers [**19**] and [**16**] for characteristic zero. These papers deal with Jordan superalgebras with non-zero odd part. For the classification of simple Jordan algebras see [**32**]. Recall however 2.2.6 which implies that isotopic, but not necessarily isomorphic Jordan superalgebras give rise to isomorphic TKK-superalgebras.

3.2. TKK-superalgebra of a rectangular matrix superpair $\mathbb{M}_{12}(A)$ for a unital alternative superalgebra A. A triple $t = (t_1, t_2, t_3)$ of S-linear homogeneous endomorphisms of A is called a *triality of A* if $|t_1| = |t_2| = |t_3| =: |t|$ and

$$t_1(ab) = t_2(a)b + (-1)^{|a||t|}at_3(b) \tag{1}$$

for homogeneous elements a and b in A. We denote by $\mathcal{T}_{\bar{n}}$ the set of all trialities of degree $\bar{n} \in \{\bar{1}, \bar{0}\}$ and let $\mathcal{T} = \mathcal{T}_{\bar{0}} \oplus \mathcal{T}_{\bar{1}}$. Our concept of triality in an alternative superalgebra is compatible with that of triality in an alternative algebra in the following way: given homogeneous S-linear endomorphisms (t_1, t_2, t_3) of A of the same degree we have

$$\begin{aligned} &(t_1, t_2, t_3) \textit{ is a triality of } A \quad \Longleftrightarrow \\ &(g_1 \otimes t_1, g_2 \otimes t_2, g_3 \otimes t_3) \textit{ is a triality of } G(A), \end{aligned} \tag{2}$$

where g_1, g_2 and g_3 are elements of degree $|t_i|$ in the Grassmann algebra G. Because of (2) we have that the Grassmann envelope of $\mathcal{T}$ coincides with the Lie algebra of trialities of the alternative algebra $G(A)$, whence $\mathcal{T}$ is a subalgebra of the Lie superalgebra $\mathfrak{l}_S(A) \times \mathfrak{l}_S(A) \times \mathfrak{l}_S(A)$. We put

$$\begin{aligned} M &= \oplus_{1 \le i \neq j \le 3} AE_{ij} \ (\subset \mathrm{Mat}(3,3;A)), \\ \mathrm{Der}\, M_{\bar{\sigma}} &= \{h = (h_{ij}) \in \oplus_{1 \le i \neq j \le 3} \mathfrak{l}_S(A) : (h_{ij}, h_{il}, h_{lj}) \in \mathcal{T}_{\bar{n}} \text{ for all } i,j,k \neq\}, \\ \mathrm{Der}\, M &= \mathrm{Der}\, M_{\bar{0}} \oplus \mathrm{Der}\, M_{\bar{1}}\,. \end{aligned}$$

By (2) and [**38**, 3.3] we have that $\mathrm{Der}\, M$ is a Lie superalgebra with componentwise multiplication. Define a linear map

$$\varphi : \mathrm{Der}\, M \to \mathcal{T} : (h_{ij}) \mapsto (h_{12}, h_{13}, h_{32}).$$

Since the Grassmann envelope $G(\varphi)$ is the isomorphism of Lie algebras from $\mathrm{Der}\, G(M)$ to the set of trialities of $G(A)$ defined in [**38**, 3.3(2)], we have that

$$\varphi\colon \mathrm{Der}\, M \to \mathcal{T} \textit{ is an isomorphism of Lie superalgebras.} \tag{3}$$

For homogeneous elements $a, b \in A$ let $L_a b = ab$, $R_a b = (-1)^{|a||b|}ba$ and define

$$\begin{aligned} X(a,b) &:= (L_aL_b + (-1)^{|a||b|}R_{ba}\,,\ L_aL_b\,,\ R_aR_b) \text{ and} \\ Y(a,b) &:= (L_aL_b\,,\ L_aL_b + (-1)^{|a||b|}R_{ba}\,,\ -(-1)^{|a||b|}L_bL_a) \end{aligned}$$

A direct calculation, using (2), shows that both $X(a,b), Y(a,b) \in \mathcal{T}_{|a|+|b|}$, for homogeneous elements $a, b \in A$. We denote by $\mathfrak{D}^0$ the span of all $X(a,b)$ and $Y(c,d)$ where a, b, c, d are homogeneous elements in A. On

$$\mathfrak{F}(A) := \mathfrak{D}^0 \oplus M$$

we define a superanticommutative product $[\cdot,\cdot]$ by:

(i) for $t = (t_1, t_2, t_3)$, $t' = (t'_1, t'_2, t'_3) \in \mathfrak{D}^0$:

$$[t, t'] = ([t_1, t'_1], [t_2, t'_2], [t_3, t'_3])$$

(ii) for $t \in \mathfrak{D}^0$ with $\varphi^{-1}(t) = (t_{ij}) \in \operatorname{Der} M$ and $m = \Sigma_{i \neq j} m_{ij} E_{ij} \in M$:

$$[t, m] = \Sigma_{ij} t_{ij}(m_{ij}) E_{ij} \in M.$$

(iii) for homogeneous elements $aE_{ij}, bE_{pq} \in M$:

$$\begin{aligned}
[aE_{ij}, bE_{pq}] &= \delta_{jp} ab E_{iq} - (-1)^{|a||b|} \delta_{iq} ba E_{pj} \in M, \\
&\quad \text{if } |\{i,j\} \cap \{p,q\}| = 1, \\
[aE_{ij}, bE_{ij}] &= 0, \\
[aE_{12}, bE_{21}] &= X(a,b), \\
[aE_{13}, bE_{31}] &= Y(a,b), \\
[aE_{23}, bE_{32}] &= Y(a,b) - X(1, ab).
\end{aligned}$$

With the product defined above, $\mathfrak{F}(A)$ is a Lie superalgebra since its Grassmann envelope $G(\mathfrak{F}(A))$ is the Lie algebra defined in [**38**, 3.3(4)]. Moreover, $\mathfrak{F}(A)$ is Jordan 3-graded by

$$\begin{aligned}
\mathfrak{F}(A)_1 &= AE_{12} \oplus AE_{13} \cong \operatorname{Mat}(1,2;A), \\
\mathfrak{F}(A)_0 &= \mathfrak{D}^0 \oplus AE_{23} \oplus AE_{32} \text{ and} \\
\mathfrak{F}(A)_{-1} &= AE_{21} \oplus AE_{31} \cong \operatorname{Mat}(2,1;A).
\end{aligned}$$

with associated Jordan superpair $\mathbb{M}_{12}(A)$ and has 0 centre. Therefore, by 2.4(a)

$$\mathfrak{F}(A) \cong \mathfrak{K}(\mathbb{M}_{12}(A)). \tag{4}$$

Since in the non-supercase $\mathfrak{F}(A)$ is a subalgebra of a Lie algebra first studied by J. Faulkner in [**13**], we call $\mathfrak{F}(A)$ the *Faulkner algebra of A*.

Assume $\frac{1}{2} \in S$. By 2.6(c) and 1.9(b) $\mathfrak{F}(A)$ is simple if and only if A is simple. Therefore, by 1.9(b) again, if $S = k$ is a field of characteristic $\neq 2, 3$, either A is a simple associative superalgebra and hence $\mathfrak{F}(A) \cong \mathfrak{psl}_3(A)$, see 3.4, or $A = A_{\bar{0}}$ is a simple Cayley-Dickson algebra and hence $\mathfrak{F}(A)$ is a simple A_2-graded Lie algebra.

3.3. Another model for $\mathfrak{K}(\mathbb{M}_{12}(A))$. We keep the setting and notations of 3.2. Under some assumptions on S we will present another model for $\mathfrak{K}(\mathbb{M}_{12}(A))$ which is the super version of a Lie algebra recently studied in [**8**] and [**47**]. In order to do so, we start by decomposing the Lie superalgebras $\mathfrak{T}$ and $\mathfrak{D}^0$, analogous to [**38**, 3.3]. For homogeneous a, b in A define

$$\Delta(a,b) := L_{[a,b]} - 3[L_a, R_b] - R_{[a,b]}.$$

Then $\xi_a\xi_b \otimes \Delta(a,b) = \Delta_{G(A)}(\xi_a \otimes a, \xi_b \otimes b)$, where ξ_a and ξ_b are elements of G of degree $|a|$ and $|b|$ respectively and where $\Delta_{G(A)}(\cdot,\cdot)$ denotes the (standard) inner derivation of the alternative algebra $G(A)$. Therefore, $\Delta(a,b)$ is a derivation of A (of degree $|a|+|b|$) which we will call a *(standard) inner derivation.* Let $\mathrm{Der}(A)$ denote the Lie superalgebra of all derivations of A and let $\mathrm{IDer}(A)$ be the span of all inner derivations $\Delta(a,b)$ for homogeneous elements a and b in A. For $d \in \mathrm{Der}\, A$ we have

$$[d, \Delta(a,b)] = \Delta(d(a), b) + (-1)^{|d||a|}\Delta(a, d(b)) \qquad (1)$$

whence $\mathrm{IDer}\, A$ is an ideal of $\mathrm{Der}\, A$. For every $a \in A$,

$$\lambda(a) := (L_a, L_a + R_a, -L_a) \quad \text{and} \quad \varrho(a) := (R_a, -R_a, L_a + R_a)$$

are trialities and the same holds for

$$\Delta_{\mathfrak{T}} = (\Delta, \Delta, \Delta) \qquad \text{for } \Delta \in \mathrm{Der}\, A.$$

Let

$$\begin{aligned} \mathfrak{T}^\lambda &= \{\lambda(a) \in \mathfrak{T} : a \in A\} \cong A, \quad \mathfrak{T}^\varrho = \{\varrho(a) \in \mathfrak{T} : a \in A\} \cong A, \\ \mathfrak{T}^0 &= \{\Delta_{\mathfrak{T}} \in \mathfrak{T} : \Delta \in \mathrm{Der}(A)\} \text{ and} \\ \mathfrak{D}^{00} &= \{\Delta_{\mathfrak{T}} : \Delta \in \mathrm{IDer}(A)\} \cong \mathrm{IDer}(A). \end{aligned}$$

Then, repeating the proof of [**38**, (3.3.5)] shows

$$\mathfrak{T} = \mathfrak{T}^\lambda \oplus \mathfrak{T}^0 \oplus \mathfrak{T}^\varrho \quad \text{and} \quad \mathfrak{D}^0 = \mathfrak{T}^\lambda \oplus \mathfrak{D}^{00} \oplus \mathfrak{T}^\varrho \quad \text{if } \tfrac{1}{3} \in S.$$

For the remainder of this subsection suppose that 2 *and* 3 *are invertible in* S. On

$$\mathfrak{psl}_3(A) = \mathrm{IDer}\, A \oplus (A \otimes_k \mathfrak{sl}_3(k))$$

we define a superanticommutative product by

(i) $\mathrm{IDer}\, A$ is a subalgebra;
(ii) for homogeneous $d \in \mathrm{IDer}\, A$, $a \in A$ and $x \in \mathfrak{sl}_3(k)$: $[d, a \otimes x] = d(a) \otimes x$;
(iii) for $a, b \in A$ and $x, y \in \mathfrak{sl}_3(k)$:

$$\begin{aligned} [a \otimes x, b \otimes y] = \tfrac{1}{3}\mathrm{Tr}(xy)\Delta(a,b) \oplus \big(\tfrac{1}{2}(ab + (-1)^{|a||b|}ba) \otimes [x,y] + \\ + \tfrac{1}{2}(ab - (-1)^{|a||b|}ba) \otimes (xy + yx - \tfrac{2}{3}\mathrm{Tr}(xy)E_3) \end{aligned}$$

where Tr is the trace and E_3 the 3×3-identity matrix.

The Grassmann envelope of this superalgebra is the Lie algebra $\mathfrak{psl}_3(G(A))$ of [**8**, 2.5], hence $\mathfrak{psl}_3(A)$ is a Lie superalgebra. It is Jordan 3-graded by

$$\begin{aligned}
\mathfrak{psl}_3(A)_1 &= AE_{12} \oplus AE_{13} \cong \mathrm{Mat}(1,2;A),\\
\mathfrak{psl}_3(A)_0 &= \mathrm{IDer}\, A \oplus (A(E_{11}-E_{22}) \oplus A(E_{22}-E_{33}) \oplus AE_{23} \oplus AE_{32}) \text{ and}\\
\mathfrak{psl}_3(A)_{-1} &= AE_{21} \oplus AE_{31} \cong \mathrm{Mat}(2,1;A)
\end{aligned}$$

with associated Jordan superpair $\mathbb{M}_{12}(A)$ and has trivial centre, hence by 2.4(a)

$$\mathfrak{F}(A) \cong \mathfrak{K}(\mathbb{M}_{12}(A)) \cong \mathfrak{psl}_3(A). \tag{2}$$

This can also be seen directly. Indeed, it is easy to check that the linear map $\varphi : \mathfrak{F}(A) \to \mathfrak{psl}_3(A)$ defined by

$$\begin{aligned}
aE_{ij} \in M &\mapsto aE_{ij} \in A \otimes \mathfrak{sl}_3(k),\\
X(a,b) \in \mathfrak{D}^0 &\mapsto \frac{1}{3}\Delta(a,b) \oplus ab \otimes (E_{11} - \frac{1}{3}E_3) - (-1)^{|a||b|} ba \otimes (E_{22} - \frac{1}{3}E_3),\\
Y(a,b) \in \mathfrak{D}^0 &\mapsto \frac{1}{3}\Delta(a,b) \oplus ab \otimes (E_{11} - \frac{1}{3}E_3) - (-1)^{|a||b|} ba \otimes (E_{33} - \frac{1}{3}E_3),
\end{aligned}$$

provides an isomorphism from $\mathfrak{F}(A)$ to $\mathfrak{psl}_3(A)$. Notice that $\varphi(X(a,b)) = [a \otimes E_{12}, b \otimes E_{21}]$ and that $\varphi(Y(a,b)) = [a \otimes E_{13}, b \otimes E_{31}]$.

3.4. TKK-superalgebra of a rectangular matrix superpair $\mathbb{M}_{JK}(A)$ for a unital associative superalgebra A. Put $I := J \dot\cup K$. The associative superalgebra $\mathrm{Mat}(I,I;A)$ becomes a Lie superalgebra, denoted $\mathfrak{gl}_I(A)$, with the usual supercommutator product. Put

$$\begin{aligned}
\mathfrak{gl}_I(A)_1 &= \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathrm{Mat}(J,K;A) \right\},\\
\mathfrak{gl}_I(A)_0 &= \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a \in \mathrm{Mat}(J,J;A), d \in \mathrm{Mat}(K,K;A) \right\},\\
\mathfrak{gl}_I(A)_{-1} &= \left\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} : c \in \mathrm{Mat}(K,J;A) \right\}
\end{aligned}$$

Then $\mathfrak{gl}_I(A) = \mathfrak{gl}_I(A)_1 \oplus \mathfrak{gl}_I(A)_0 \oplus \mathfrak{gl}_I(A)_{-1}$ is a 3-grading for $\mathfrak{gl}_I(A)$. Moreover,

$$\mathfrak{sl}_I(A) := \mathfrak{gl}_I(A)_1 \oplus [\mathfrak{gl}_I(A)_1, \mathfrak{gl}_I(A)_{-1}] \oplus \mathfrak{gl}_I(A)_{-1}$$

is Jordan 3-graded with associated Jordan superpair $\mathbb{M}_{JK}(A)$. By 2.4(a) we therefore have

$$\mathfrak{K}(\mathbb{M}_{JK}(A)) \cong \mathfrak{sl}_I(A)/C =: \mathfrak{psl}_I(A), \tag{1}$$

where $C = \{x \in [\mathfrak{gl}_I(A)_1, \mathfrak{gl}_I(A)_{-1}] : [x, \mathfrak{gl}_I(A)_{\pm 1}] = 0\}$. Arguing as in [**38**, 3.4(2),(3)], we get the following descriptions of $\mathfrak{sl}_I(A)$ and C:

$$\mathfrak{sl}_I(A) = \{x \in \mathfrak{gl}_I(A) : \mathrm{Tr}(x) \in [A,A]\} \quad \text{if } |I| \geq 3,$$

where Tr is the usual trace map, $[A,A]$ is spanned by all supercommutators $[a,b] = ab - (-1)^{|a||b|}ba$, and

$$C = \begin{cases} Z(A)E_n \cap \mathfrak{sl}_n(A) & \text{if } n = |I| < \infty, \\ \{0\} & \text{if } |I| = \infty \end{cases} \tag{2}$$

where $Z(A) = \{z \in A : [z,A] = 0\}$ denotes the centre of the superalgebra A, E_n is the $n \times n$-identity matrix and $\mathfrak{sl}_n(A) := \mathfrak{sl}_I(A)$ for $|I| = n$. We also put $\mathfrak{psl}_n(A) = \mathfrak{psl}_I(A)$ for $|I| = n$. Suppose $\frac{1}{2} \in S$. It then follows from 2.6(c) and 1.9(b) that

$$\mathfrak{psl}_I(A) \textit{ is simple if and only if } A \textit{ is a simple superalgebra } (\tfrac{1}{2} \in S). \tag{3}$$

Examples: (1) $\mathfrak{psl}_3(A)$ is isomorphic to the Lie superalgebras constructed in 3.2 and 3.3 for an associative A.

(2) $\mathfrak{sl}_n(A) = \mathfrak{psl}_n(A)$ if $\frac{1}{n} \in S$ and $Z(A) \cap [A,A] = 0$. For algebras the condition $Z(A) \cap [A,A] = 0$ is for example fulfilled if A is a quantum torus ([**7**]) or, more generally, a so-called G-tori [**39**]. In the first case the Lie algebras $\mathfrak{sl}_n(A)$ appear as centre-less core of extended affine Lie algebras of type A_{n-1}.

(3) Suppose $\frac{1}{2} \in S$ and $A = \mathbb{D}(A_{\bar{0}})$ as in 1.3. Then $[A,A] = A_{\bar{0}} \oplus u[A_{\bar{0}}, A_{\bar{0}}]$, $Z(A) = Z(A_{\bar{0}}) \subset A_{\bar{0}}$ and hence with obvious notation

$$\begin{aligned} \mathfrak{sl}_I(A) &= \mathfrak{gl}_I(A_{\bar{0}}) \oplus u\,\mathfrak{sl}_I(A_{\bar{0}}) \\ \mathfrak{psl}_I(A) &= \mathfrak{pgl}_I(A_{\bar{0}}) \oplus u\,\mathfrak{sl}_I(A_{\bar{0}}) \quad \text{where} \\ \mathfrak{pgl}_I(A_{\bar{0}}) &= \begin{cases} \mathfrak{gl}_I(A_{\bar{0}})/Z(A_{\bar{0}})E_n & \text{if } |I| = n < \infty \\ \mathfrak{gl}_I(A_{\bar{0}}) & \text{if } |I| = \infty. \end{cases} \end{aligned}$$

We have the following matrix realization of $\mathfrak{sl}_I(A)$ in $2I \times 2I$ matrices:

$$\mathfrak{sl}_I(A) \cong \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a \in \mathfrak{gl}_I(A_{\bar{0}}),\, b \in \mathfrak{sl}_I(A_{\bar{0}}) \right\}.$$

It follows from (3) and 1.4(b) that $\mathfrak{psl}_I(\mathbb{D}(A_{\bar{0}}))$ *is simple if and only if* $A_{\bar{0}}$ *is a simple (associative) algebra.*

An interesting special case is $A_{\bar{0}} = \mathrm{Mat}(M,M;B)$ for a unital associative algebra B. Then $A = \mathrm{Mat}(M,M;\mathbb{D}(B))$ by 1.3 and

$$\mathfrak{sl}_I(\mathrm{Mat}(M,M;\mathbb{D}(B))) \cong \mathfrak{sl}_{I\times M}(\mathbb{D}(B)), \tag{4}$$

$$\mathfrak{pgl}_I(\mathrm{Mat}(M,M;\mathbb{D}(B))) \cong \mathfrak{pgl}_{I\times M}(\mathbb{D}(B)). \tag{5}$$

In particular, for $B = k$ an algebraically closed field of characteristic 0 we obtain the Lie superalgebras $\mathfrak{sl}_n(\mathbb{D}(k)) = \widetilde{\mathbf{Q}}_{n-1}$ and its simple quotient $\mathfrak{psl}_n(\mathbb{D}(k)) = \mathbf{Q}_{n-1}$ in the notation of [**20**, 2.1.4].

(4) In the notation of 1.2.2, suppose $A = \mathrm{Mat}_{P|Q}(B)$ for a unital *commutative* associative algebra. To have a proper superalgebra we assume $|P|, |Q| \geq 1$ in the following. Then by straightforward matrix multiplication

$$[A, A] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{P|Q}(B) : \mathrm{Tr}(a) = \mathrm{Tr}(d) \right\} \quad \text{and}$$
$$Z(A) = \begin{pmatrix} Z(\mathrm{Mat}(P, P; B)) & 0 \\ 0 & Z(\mathrm{Mat}(Q, Q; B)) \end{pmatrix} = Z(A_{\bar{0}})$$

where $Z(\mathrm{Mat}(P, P; B))$ has the same description as C above in (2). We have a canonical isomorphism

$$\mathfrak{gl}_I(\mathrm{Mat}_{P|Q}(B)) = (\mathrm{Mat}_{I\times P|I\times Q}(B))^{(-)}$$

where $^{(-)}$ indicates that the associative superalgebra on the right hand side is considered as Lie superalgebra with respect to the supercommutator. Under this isomorphism

$$\mathfrak{sl}_I(\mathrm{Mat}_{P|Q}(B)) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{I\times P|I\times Q}(B) : \mathrm{Tr}(a) = \mathrm{Tr}(d) \right\}$$

The description of C in this case depends on the cardinalities of I, P, Q and also on the "characteristic" of B. It is straightforward and will be left to the reader.

The Lie superalgebra $\mathfrak{psl}_I(\mathrm{Mat}_{P|Q}(B))$ is simple if and only if B is simple. In particular, if $B = k$ is a field of characteristic 0 and $|I| = n$, $|P| = p$ and $|Q| = q$ are finite, then $\mathfrak{psl}_n(\mathrm{Mat}_{p|q}(k))$ is the simple Lie superalgebra $\mathbf{A}(np, nq)$ in the notation of [**20**, 2.1.1].

We point out that by 1.4(b) any finite-dimensional simple associative superalgebra over an algebraically closed field k is isomorphic to $\mathrm{Mat}_{p|q}(k)$ as above or to $\mathrm{Mat}(m, m; \mathbb{D}(k))$ as in (3).

3.5. TKK-superalgebras of Jordan superpairs covered by a hermitian grid of rank 2 or 3. The Jordan superpairs in question are all of the form $V = \mathbb{J}$ for a specific unital Jordan superalgebra J (see 1.8(c) and (d)) and hence 3.1 provides a description of $\mathfrak{K}(V)$. Note however that in case $J = \mathbb{H}_I(A, A_0, \pi)$ for $|I| = 2, 3$ and A associative the corresponding TKK-superalgebra is described in 3.6 below.

3.6. TKK-superalgebra of a hermitian matrix superpair $\mathbb{H}_I(A, A_0, \pi)$ for an associative superalgebra A and $|I| \geq 3$. We will use the notation of 1.8(d), denote by A' the S-span of $\{ab + a^\pi b^\pi : a, b \in A\} \cup \{a_0 b_0 : a_0, b_0 \in A_0\}$ and define the Lie superalgebra

$$\mathfrak{su}_I(A, A_0, \pi) = \left\{ \begin{pmatrix} a & b \\ c & -(a^\pi)^T \end{pmatrix} : a \in \mathrm{Mat}(I, I; A) \text{ with } \mathrm{Tr}(a) \in A', \right.$$
$$\left. b, c \in H_I(A, A_0, \pi) \right\}.$$

It has a natural Jordan 3-grading for which $\mathbb{H}_I(A, A_0, \pi)$ is the associated Jordan superpair, whence

$$\mathfrak{K}(\mathbb{H}_I(A, A_0, \pi)) \cong \mathfrak{su}_I(A, A_0, \pi)/C := \mathfrak{psu}_I(A, A_0, \pi).$$

Here $C = \{x \in \mathfrak{su}_I(A, A_0, \pi)_0 : [x, \mathfrak{su}_I(A, A_0, \pi)_{\pm 1}] = 0\}$ satisfies

$$C = \begin{cases} \{zE_{2n} : z \in Z(A), z^\pi = -z, nz \in A'\} & \text{if } n = |I| < \infty, \\ \{0\} & \text{if } |I| = \infty \end{cases}$$

where E_{2n} is the $2n \times 2n$-identity matrix and $Z(A)$ denotes the centre of A.

For the remainder of this subsection we assume $\frac{1}{2} \in A$. Then $A_0 = \mathrm{H}(A, \pi)$ and $A' = \mathrm{H}(A, \pi) \oplus ([A, A] \cap \mathrm{S}(A, \pi))$ where $\mathrm{S}(A, \pi) = \{a \in A : a^\pi = -a\}$. Therefore in this case we have for $\mathfrak{su}_I(A, \pi) := \mathfrak{su}_I(A, \mathrm{H}(A, \pi), \pi)$ that

$$\mathfrak{su}_I(A, \pi) = \Big\{ x = \begin{pmatrix} a & b \\ c & -(a^\pi)^{\mathrm{T}} \end{pmatrix} : \ \mathrm{Tr}(x) \in [A, A]$$

$$a \in \mathrm{Mat}(I, I; A), b, c \in H_I(A, \pi), \Big\} \quad \text{if } \tfrac{1}{2} \in A$$

$$C = \{zE_{2n} : z \in Z(A) \cap \mathrm{S}(A, \pi), nz \in [A, A]\}$$
$$\text{if } \tfrac{1}{2} \in A \text{ and } |I| = n < \infty$$

Moreover, it follows from 2.6(c) and 1.9(c) that

$$\mathfrak{psu}_I(A, \pi) \textit{ is simple if and only if } (A, \pi) \textit{ is a simple superalgebra,} \tag{1}$$

Examples: We will consider two classes of examples of associative superalgebras with involutions. By [**41**, Prop. 13, 14] they include finite-dimensional superalgebras A with involutions over an algebraically closed field k of characteristic 0.

(a) First, for B an associative algebra and $A = \mathrm{Mat}_{Q|Q}(B)$ we have the supertranspose involution

$$A \ni \begin{pmatrix} w & x \\ y & z \end{pmatrix} \quad \overset{\tau}{\longmapsto} \quad \begin{pmatrix} z^{\mathrm{T}} & -x^{\mathrm{T}} \\ y^{\mathrm{T}} & w^{\mathrm{T}} \end{pmatrix} \in A, \tag{2}$$

where z^{T} is the usual transpose of z ([**41**, Prop.13]). As fixed point set of an involution,

$$\mathrm{H}(\mathrm{Mat}_{Q|Q}(B), \tau) = \left\{ \begin{pmatrix} a & b \\ c & a^{\mathrm{T}} \end{pmatrix} \in \mathrm{Mat}_{Q|Q}(B) : b \text{ skew}, c \text{ symmetric} \right\}$$
$$=: \mathrm{P}_Q(B) \tag{3}$$

is a Jordan superalgebra. We have a canonical isomorphism $\mathrm{H}_I(\mathrm{Mat}_{Q|Q}(B), \tau) \cong \mathrm{P}_{I \times Q}(B)$. As is already proven in [**19**, Thm. 2] for finite Q, the TKK-superalgebra of the Jordan superpair associated to $\mathrm{H}(\mathrm{Mat}_{Q|Q}(k), \tau) = \mathrm{P}_Q(k)$ is

$$\mathfrak{K}(\mathrm{P}_Q(k)) \cong \left\{ \begin{pmatrix} x & y \\ z & -x^{\mathrm{T}} \end{pmatrix} \in \mathrm{Mat}_{2Q|2Q}(k) : \mathrm{Tr}(x) = 0, y = y^{\mathrm{T}}, z = -z^{\mathrm{T}} \right\}$$
$$=: \mathbf{P}_{2Q}(k) \tag{4}$$

where we put $2Q = Q \times \{1,2\}$. Replacing $\mathrm{H}(A,\tau)$ by $\mathrm{H}_I(A,\tau)$ then leads to

$$\mathfrak{K}(\mathrm{P}_{I\times Q}(k)) \cong \mathbf{P}_{2(I\times Q)}(k) \tag{5}$$

(b) Similarly, let s be the block diagonal matrix inducing the symplectic involution $d \mapsto sd^{\mathrm{T}}s^{\mathrm{T}}$, i.e.,

$$s = \operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \dots\right), \tag{6}$$

For B an associative algebra and $A = \mathrm{Mat}_{P|Q}(B)$ the orthosymplectic involution is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{\sigma}{\longmapsto} \begin{pmatrix} a^{\mathrm{T}} & c^{\mathrm{T}}s^{\mathrm{T}} \\ sb^{\mathrm{T}} & sd^{\mathrm{T}}s^{\mathrm{T}} \end{pmatrix}, \tag{7}$$

where Q is either of finite even order or infinite. The fixed point superalgebra

$$\begin{aligned}\mathrm{H}(\mathrm{Mat}_{P|Q}(B),\sigma) &= \left\{\begin{pmatrix} a & b \\ sb^{\mathrm{T}} & d \end{pmatrix} \in \mathrm{Mat}_{P|Q}(B) : a = a^{\mathrm{T}}, d = sd^{\mathrm{T}}s^{\mathrm{T}}\right\} \\ &=: \mathrm{OSP}_{P|Q}(B)\end{aligned} \tag{8}$$

is called the *orthosymplectic (Jordan) superalgebra* and denoted BC in [**19**]. We have

$$\mathrm{H}_I(\mathrm{Mat}_{P|Q}(B)) \cong \mathrm{OSP}_{I\times P|I\times Q}(B) \tag{9}$$

which is a simple Jordan superalgebra if $|I| \geq 3$ and B is simple over a ring k containing $\frac{1}{2}$. For the TKK-superalgebra of the Jordan superpair associated to the orthosymplectic superalgebra $\mathrm{OSP}_{P|Q}(k)$ we have (see the proof of [**19**, Thm. 2])

$$\mathfrak{K}(\mathrm{OSP}_{P|Q}(k)) \cong \mathfrak{eosp}(q_Q^+ \oplus q_P^-) \tag{10}$$

where the Lie superalgebra on the right is the elementary orthosymplectic Lie superalgebra (3.7) of the quadratic form which is the orthogonal sum of the standard symmetric form q_Q^+ on the even space $k^{(P)}$ and the standard skew-symmetric form on the odd space $(\Pi k)^{(P)}$.

In terms of the canonical matrix representation, $\mathfrak{eosp}(q_Q^+ \oplus q_P^-)$ consists of matrices

$$\begin{pmatrix} a & b & w & x \\ c & -a^{\mathrm{T}} & y & z \\ z^{\mathrm{T}} & x^{\mathrm{T}} & d & e \\ -y^{\mathrm{T}} & -w^{\mathrm{T}} & f & -d^{\mathrm{T}} \end{pmatrix} \in \mathrm{Mat}_{2Q|2P}(k), \qquad \begin{array}{l} b, c \text{ skew-symmetric} \\ e, f \text{ symmetric} \end{array} \tag{11}$$

where each matrix a, b, c is $Q \times Q$, each w, x, y, z is $Q \times P$ and each d, e, f is $P \times P$ ([**20**, 2.1.2]). The 3-grading realizing $\mathfrak{eosp}(q_Q^+ \oplus q_P^-)$ as TKK-superalgebra of $\mathrm{OSP}_{P|Q}(k)$ is given by

$$\begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

where the matrix indicates membership in the corresponding homogeneous parts for a matrix $X \in \mathfrak{eosp}(q_Q^+ \oplus q_P^-)$ written as in (11). Thus for finite P, Q the simple Lie superalgebra $\mathfrak{K}(\mathrm{OSP}_{P|Q})$ is of type $\mathbf{D}(q,p)$ if $q > 1$ or of type $\mathbf{C}(p+1)$ if $q = 1$ in the notation of [**20**, 2.1.2].

3.7. TKK-superalgebras of quadratic form superpairs. For the description of the TKK-superalgebras of even and odd quadratic form superpairs, we first need to establish some general notions regarding quadratic forms. For a quadratic form $q : M \to A$, A a superextension of S and M an A-module, we define the *orthosymplectic Lie superalgebra of* q as

$$\mathfrak{osp}(q) = \{x \in \mathrm{End}_A M : q(xm,n) + (-1)^{|m||n|} q(xn,m) = 0, \text{ for all } m,n \in M\}.$$

Its Grassmann envelope coincides with the orthogonal Lie algebra of the quadratic form $G(q)$. For $n \in M$ let n^* be the A-linear form on M given by $n^*(p) = q(n,p)$. Moreover, for $m, n \in M$ we define an endomorphism mn^* of M by $mn^*(p) = mq(n,p)$. Finally, let

$$\mathfrak{eosp}(q) = \mathrm{span}_A\{mn^* - (-1)^{|m||n|} nm^* : m,n \in M \text{ homogeneous}\}.$$

This is an ideal of $\mathfrak{osp}(q)$ and will be called the *elementary orthosymplectic Lie superalgebra of* q since $G(\mathfrak{eosp}(q)) = \mathfrak{eo}(G(q))$ is the elementary orthogonal Lie algebra of $G(q)$. More information on the superalgebras $\mathfrak{eosp}(q)$ is given in [**12**]. In particular, we mention: *if A is a field of characteristic $\neq 2$, $\dim_A M < \infty$ and q is nondegenerate then $\mathfrak{osp}(q) = \mathfrak{eosp}(q)$.*

We need $\mathfrak{eosp}(.)$ for the special case of the orthogonal sum of (M, q) and the hyperbolic superplane $(H(A), q_1)$ over A, i.e. the hyperbolic superspace of rank 2 over A in the sense of [**35**, 5.13]:

$$M_\infty := H(A) \oplus M = Ah_\infty \oplus M \oplus Ah_{-\infty}, \quad q_\infty := q_1 \oplus q. \tag{1}$$

With respect to the decomposition (1), every element of $\mathfrak{eosp}(q_\infty)$ can be represented by a matrix B, and it is straightforward from the results in [**38**, 5.1] that for a homogeneous B we have

$$B \in \mathfrak{eosp}(q_\infty) \iff B = \begin{pmatrix} \alpha & -m^* & 0 \\ n & x & m \\ 0 & -n^* & -\alpha \end{pmatrix} =: M(\alpha, m, n, x),$$

where $\alpha \in A$, $m, n \in X$ and $x \in \mathfrak{eosp}(q)$ all are homogenous of the same degree. The description of the Lie algebra product in the non-supercase, given in [**38**, (5.1.4)], also holds in the super setting. In particular, $\mathfrak{eosp}(q_\infty)$ is 3-graded by

$$\begin{aligned} \mathfrak{eosp}(q_\infty)_1 &= \{M(0,m,0,0) : m \in M\} \cong M, \\ \mathfrak{eosp}(q_\infty)_0 &= \{M(\alpha,0,0,x) : \alpha \in A, x \in \mathfrak{eosp}(q_X)\} \\ \mathfrak{eosp}(q_\infty)_{-1} &= \{M(0,0,n,0) : n \in M\} \cong M \end{aligned}$$

and $\mathfrak{eosp}(q_\infty)_1 \oplus [\mathfrak{eosp}(q_\infty)_1\,,\,\mathfrak{eosp}(q_\infty)_{-1}] \oplus \mathfrak{eosp}(q_\infty)_{-1}$ is a Jordan 3-graded Lie superalgebra whose associated Jordan superpair is the quadratic form superpair $V = (M, M)$ of q as defined in [**35**, 3.10]. By [**38**, (5.1.6)] we have

$$\text{if there exists } m_{\bar 0} \in M_{\bar 0} \text{ such that } q(m_{\bar 0}, m_{\bar 0}) \text{ is invertible in } S \\ \text{then } \mathfrak{K}((M, M)) = \mathfrak{eosp}(q_\infty). \tag{2}$$

Since $q_I(h_{+i}+h_{-i}, h_{+i}+h_{-i}) = 2$ we can use (2) to describe the TKK-superalgebras of the even and odd quadratic form superpairs $\mathbb{EQ}_I(A)$ and $\mathbb{OQ}_I(A, q_X)$ as described in 1.8:

$$\text{if } \tfrac{1}{2} \in S \text{ then } \mathfrak{K}(\mathbb{EQ}_I(A)) = \mathfrak{eosp}((q_I)_\infty) = \mathfrak{eosp}(q_{I\dot\cup\{\infty\}}) \quad \text{and} \tag{3}$$

$$\mathfrak{K}(\mathbb{OQ}_I(A, q_X)) = \mathfrak{eosp}((q_{I\dot\cup\{\infty\}} \oplus q_X)). \tag{4}$$

Examples: (a) If $A = A_{\bar 0}$ then $\mathbb{EQ}_I(A)$ is a Jordan pair and the TKK-superalgebra $\mathfrak{K}(\mathbb{EQ}_I(A))$ is a Lie algebra. In particular, for $\frac{1}{2} \in S$ we have $\mathfrak{K}(\mathbb{EQ}_I(A)) = \mathfrak{eosp}(q_{I\dot\cup\{\infty\}})$ is prime or simple if and only if $A = A_{\bar 0}$ is an integral domain or a field.

(b) Suppose again $\frac{1}{2} \in S$. Then $\mathfrak{K}(\mathbb{OQ}_I(A, q_X))$ is simple if and only if $A = A_{\bar 0}$ is a field and q_X is nondegenerate. In this case, $\mathfrak{K}(\mathbb{OQ}_I(A, q_X))$ is the elementary orthosymplectic Lie superalgebra of a quadratic form on the space $M = M_{\bar 0} \oplus M_{\bar 1}$ where $M_{\bar 0}$ is the orthogonal sum of the hyperbolic space $\mathrm{H}_{I\dot\cup\{\infty\}}(A)$ and the quadratic space $(X_{\bar 0}, q|X_{\bar 0})$ and where $M_{\bar 1} = X_{\bar 1}$ with the skew-symmetric form $q|X_{\bar 1} \times X_{\bar 1}$. In particular, for finite I and finite-dimensional X we obtain the orthosymplectic Lie superalgebras $\mathfrak{osp}(2+2|I|+\dim X_{\bar 0}, \dim X_{\bar 1})$ of type $\mathbf{B}(m, n) = \mathfrak{osp}(2m+1, 2n)$, $m > 0, n \geq 0$, $\mathbf{C}(n) = \mathfrak{osp}(2, 2n-2)$, $n \geq 2$ and $\mathbf{D}(m, n) = \mathfrak{osp}(2m, 2n)$, $m \geq 2, n > 0$ of [**20**, 2.1.2]. Thus, among all orthosymplectic Lie superalgebars only $\mathbf{B}(0, n) = \mathfrak{osp}(1, 2n)$ is missing.

3.8. TKK-superalgebra of an alternating matrix superpair. To describe the TKK-superalgebra of $\mathbb{A}_I(A) = (\mathrm{Alt}(I; A), \mathrm{Alt}(I; A))$ for a super extension A of S (see 1.8) we will use $\mathfrak{eosp}(q_I)$ where q_I is the hyperbolic form on the hyperbolic superspace over A of rank $2|I|$. Recall from [**35**, 5.13] that $H(I, A)$ has a decomposition $H(I, A) = H_+(I, A) \oplus H_-(I, A)$ where $H_\pm(I, A) = \bigoplus_{i\in I} Ah_{\pm i}$. With respect to this decomposition every $B \in \mathfrak{eosp}(q_I)$ can be represented as a matrix such that $\mathfrak{eosp}(q_I)$ can be identified with

$$\mathfrak{eosp}(q_I) = \left\{ \begin{pmatrix} a & b \\ c & -a^{\mathrm{T}} \end{pmatrix} : a \in \mathrm{Mat}(I, I; A), b, c \in \mathrm{Alt}(I, A) \right\}$$

with $\mathbb{Z}_2$-grading given by the $\mathbb{Z}_2$-grading of A. This Lie superalgebra has a canonical Jordan 3-grading with associated Jordan superpair $\mathbb{A}_I(A)$. An application of 2.4.1 then shows:

$$\text{if } \tfrac{1}{2} \in S \text{ then } \mathfrak{K}(\mathbb{A}_I(A)) = \mathfrak{eosp}(q_I). \tag{1}$$

Assuming $\frac{1}{2} \in S$, we conclude from 1.9(e) and 2.6(c) that $\mathfrak{eosp}(q_I)$ is simple if and only $A = A_{\bar 0}$ is a field, in which case $\mathfrak{eosp}(q_I)$ is a simple Lie algebra.

If $\frac{1}{2} \notin S$, the description of $\mathfrak{K}(\mathbb{A}_I(A))$ is more complicated. The results of [**38**, 6.2] for the non-supercase also hold in the super setting with obvious modifications.

3.9. TKK-superalgebra of a Bi-Cayley superpair. The TKK-superalgebra of the Bi-Cayley superpair $\mathbb{B}(A)$ (1.8) is a special case of 3.2 and 3.3: $\mathfrak{K}(\mathbb{B}(A)) = \mathfrak{F}(\mathbb{M}_{12}(\mathbb{O}_A))$. As in [**35**, (7.2.2)] one can show that in case $\frac{1}{3} \in S$ we have $\mathfrak{K}(\mathbb{B}(A)) \cong A \otimes_{\mathbb{Z}[\frac{1}{3}]} \mathfrak{F}(\mathbb{O}_{\mathbb{Z}[\frac{1}{3}]})$. It follows as in 3.8 above that in case $\frac{1}{2} \in S$ the Lie superalgebra $\mathfrak{K}(\mathbb{B}(A))$ is simple if and only if $A = A_{\bar{0}}$ is a field.

3.10. TKK-superalgebras for an Albert superpair. Since [**35**, Lemma 7.1] also holds in the supersetting, the TKK-algebra of an Albert superpair $\mathbb{AB}(A)$ (see 1.8) is $\mathfrak{K}(\mathbb{AB}(A)) \cong A \otimes_{\mathbb{Z}[\frac{1}{2}]} \mathfrak{K}(\mathbb{AB}(\mathbb{Z}[\frac{1}{2}]))$ if $\frac{1}{2} \in S$. In this case, $\mathfrak{K}(\mathbb{AB}(A))$ is simple if and only if $A = A_{\bar{0}}$ is a field.

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