To filter or not to filter? Impact on stability of delay-difference and neutral equations with multiple delays

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***Abstract*— For control systems where the closed-loop system description is governed by linear delay-differential equations of neutral type, it is known that stability may be fragile, in the sense of sensitive to infinitesimal perturbations to parameters in the system model or arbitrarily small errors in the implementation of the controller. A natural approach to resolve this problem of ill-posedness and to break down the underlying instability mechanisms, rooted in characteristic roots moving from the left plane to the right one via the point at infinity, consists of including a low-pass filter in the control loop, provided the inclusion preserves stability. Independently of the particular control problem, the addition of a low-pass filter essentially boils down to a "regularization" of delay-difference equations and delay equations of neutral type in terms of parametrized delay equations of retarded type, where the parameter can be interpreted as the inverse of the filter's cut-off frequency. In this paper, the stability properties of these parametrized delay equations are analyzed in a general, multi-delay setting, with focus on the transition to the original delay-difference or neutral equations. It is illustrated that the spectral abscissa may not be continuous at the transition, which may impact stability. Hence, conditions for preservation of stability in terms of a robustified stability indicator called filtered spectral abscissa are presented, for which mathematical characterizations and a computationally tractable expression are provided. The application of a PD controller to a time-delay system with relative degree one is used to motivate the structure of the equations studied throughout the paper, and to explicate the implications of the presented results on control design, discussed in the last section.**

***Index Terms*— Delay-difference and neutral equations, stability robustness, well-posedness, low-pass filtering**

I. Introduction

In the context of PID control of time-delay systems (see, e.g., [1]–[8] for some recent results) the application of a PI or PID regulator to systems with relative degree one gives rise to a closed-loop system description by delay-differential equations of neutral type. Neutral equations also naturally appear in the context of vibration control applications if the output of accelerometers is directly used for feedback [9] or if inverse delay-based input shapers are deployed [10].

In general, a linear time-invariant system of neutral type exhibits characteristic root chains, along which the imaginary parts tend to infinity, yet the real parts have a finite limit, and whose asymptotes are determined by the spectrum of an associated delay-difference equation [11]–[14]. Along such infinite root chains, the sensitivity of individual characteristic roots with respect to parameters may grow unbounded. This phenomenon may make the closed-loop system fragile and induce, for instance, a sensitivity of stability with respect to infinitesimal parametric perturbations. Such perturbations may be inherent to the system model (such as perturbations on the corresponding delay parameters [15] that led to the notion of strong stability [16]–[19]) or they may correspond to small errors in the implementation of the controller, such as a small feedback delay [20], a finite-difference approximation of a derivative to any precision [21], [22] or any other approximation that gives rise to multiplicative uncertainty in the form of *approximate identities* in the sense of [23].

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An intuitive solution to damp the highly sensitive, high-frequency characteristic roots and to block the underlying instability mechanisms, grounded in characteristic roots crossing the imaginary axis via "the point at infinity", consists of adding a low-pass filter to the control loop, as already successfully done in [24] to remove instability problems inferred from the implementation of integrals in predictive controllers, and suggested by the well-posedness conditions in [23], [25]. The addition of the filter will give rise to the "regularization" of neutral equations and delay-difference equations in terms of delay-differential equations of retarded type that will be investigated in Sections II-III. This approach is successful under one very simple but fundamental condition, namely that the filter itself is not destabilizing. As we shall see, the phenomena induced by including the filter are, however, non-intuitive. The inclusion may for instance result in a strictly positive jump of the spectral abscissa and induce instability, even if the cut-off frequency of the filter is arbitrarily large, which will lead to additional constraints for a robust control design.

Before we turn ourselves in the following sections to analyzing the regularization of classes of delay-difference equations and neutral equations, in the above sense, we make the previous statements more concrete and motivate the structure of the considered equations by means of an application. We consider the feedback interconnection of a system described by

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A_0 x(t) + \sum_{i=1}^{m} A_i x(t-\tau_i) + Bu(t), \\ y(t) & = & C_0 x(t) + \sum_{i=1}^{m} C_i x(t-\tau_i) \end{array} \right. \tag{1}$$

and a PD control law

$$u(t) = K_P y(t) + K_D \dot{y}(t), \tag{2}$$

such that $\det(I - BK_D C_0) \neq 0$. It leads to a closed-loop system of neutral type,

$$\dot{x}(t) = BK_D C_0 \dot{x}(t) + \sum_{i=1}^{m} BK_D C_i \dot{x}(t-\tau_i) + \hat{A}_0 x(t) + \sum_{i=1}^{m} \hat{A}_i x(t-\tau_i), \tag{3}$$

where $\hat{A}_i = A_i + BK_P C_i$ for $i = 0, \ldots, m$, and to ease a comparison later on, we deliberately do not collect the two terms corresponding to $\dot{x}(t)$ into one term. The delay-difference equation associated to (3) is given by [16], [26]

$$x(t) = BK_D C_0 x(t) + \sum_{i=1}^{m} BK_D C_i x(t-\tau_i). \tag{4}$$

In this context, we note that neutral equations also arise if the relative degree ρ of (1) satisfies $\rho > 1$ and an extended PID controller (in the sense of [27]), relying the ρ-th derivative of the output is used, and if nontrivial feedthrough terms are added to the plant model and a PI controller is used.

Even if the zero solution of closed-loop system (1)-(2) is exponentially stable, infinitesimal variations on the delays in the system model may induce instability (see the examples in [11, Chapter 1]), and to ensure stability robustness against small delay perturbations, i.e., to guarantee so-called strong stability [16], an additional condition is needed, expressed in terms of the coefficients matrices of (4). Fragility problems related to the implementation of the controller rather than to parameter variations in the model may already occur if the plant model is delay-free, as illustrated with the following example from [21].

Example 1: For the plant model

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t),$$

which is a special instance of (1), the control law (2) with $K_D = -2$ and $K_P = -1$ is stabilizing, leading to characteristic equation $s^2 + s + 1 = 0$. However, adding feedback delay $r > 0$ leads to a neutral system with characteristic equation

$$(1 - 2e^{-sr})s^2 - s - 1 = 0,$$

which is exponentially unstable for any $r > 0$ as the associated delay-difference equation $z(t) - 2z(t-r) = 0$ is exponentially unstable. At the same time, in [21, Example] it is demonstrated that the control law

$$u(t) = K_P y(t) + K_D \frac{y(t) - y(t-r)}{r},$$

approximating the derivative action by a finite difference, is also destabilizing for any $r > 0$.

With $G(s)$ and $C(s)$ denoting the transfer function of plant (1) and controller (2), respectively, and with $\Re(s)$, $\Im(s)$ and $|s|$ standing for real part, imaginary part and modulus of complex number s, the instability problems are characterized by

$$\lim_{|s|\to\infty,\ \Re(s)\geq 0} G(s)C(s) \neq 0 \tag{5}$$

and induced by the behavior of characteristic roots with a large modulus. Adding a first-order low-pass filter to the second term in the right-hand side of (2), results in control law

$$\begin{cases} u(t) = K_P y(t) + K_D z(t), \\ T\dot{z}(t) + z(t) = \dot{y}(t), \end{cases} \tag{6}$$

where $1/T$ is the cut-off frequency of the filter. With this modification, which corresponds to replacing differentiator s by $\frac{1}{1+Ts}$ in the Laplace domain, the controller becomes proper, and the modified closed-loop system is described by delay equations of retarded type, accordingly.

The characteristic equation of closed-loop system (1) and (6) can be expressed in the form

$$\det\left(s\left(I - \frac{1}{1+sT}\left[BK_DC_0 + \sum_{i=1}^{m} BK_DC_i e^{-s\tau_i}\right]\right) - \hat{A}_0 - \sum_{i=1}^{m} \hat{A}_i e^{-s\tau_i},\right) = 0 \tag{7}$$

whose structure suggests to analyze in the first step the effect of the filtering on delay-difference equation (4), leading to differential equation

$$T\dot{x}(t) + x(t) = BK_DC_0x(t) + \sum_{i=1}^{m} BK_DC_i x(t-\tau_i) \tag{8}$$

with T small, and in the second step address the spectrum of (1) and (6) in comparison to the one of neutral equation (3) that constitutes the limit case $T = 0$. This will be precisely the approach followed in the remainder of the paper, albeit in a more general setting. The special case where $C_i = 0,\ i = 1, \ldots, m$, such that delay-difference equation (4) reduces to an algebraic equation, is addressed in [21] and will be discussed in Remark 1.

On the one hand, in (8) the addition of the filter can be interpreted in terms of adding fast dynamics, yielding a connection with singularly perturbed systems (see [28] and the references therein). On the other hand, a re-scaling of time to normalize the cut-off frequency of the filter results in a delay equation with delays τ_i/T. This is the reason why, from a methodological point of view, a key reference will be article [29] on the stability analysis of delay linear equations with large delays in an asymptotic sense (see also [30] for related results on stability of periodic orbits).

The structure of the paper is as follows. In Section II we study the effect of low-pass filtering on delay-difference equations, inspired by the structure of (8), leading to the definition and characterization of a novel robustified stability notion. In Section III we address neutral equations. Finally, in Section IV we discuss the implications of the result on control design, along with some concluding remarks.

II. Stability of Filtered Delay-Difference Equations

We consider the delay-difference equation

$$x(t) = H_0x(t) + \sum_{i=1}^{m} H_i x(t-\tau_i), \tag{9}$$

where $x(t) \in \mathbb{R}^n$ is the state-variable at time t, numbers $\tau_i,\ i = 1, \ldots, m$, satisfying $0 < \tau_1 < \cdots < \tau_m$, represent the time-delays, and matrices $H_i,\ i = 0, \ldots, m$, are assumed to be real valued. We aim to relate the stability properties of (9) with these of equation

$$T\dot{x}(t) + x(t) = H_0x(t) + \textstyle\sum_{i=1}^{m} H_i x(t-\tau_i) \tag{10}$$

for small $T \geq 0$. Note that (10) can be interpreted as being obtained from (9) by applying a low-pass filter with cut-off frequency $\frac{1}{T}$ to the right-hand side of (9). In the other way, (9) is obtained from (10) by setting $T = 0$. The specific decomposition of the contribution from $x(t)$ into two terms (9) stems from the motivating problem in the introduction that led to (4) and (8).

For reasons of well-posedness and to exclude a trivial, degenerate case, we make the following assumptions throughout the paper.

Assumption 1: $\det(I - H_0) \neq 0$.

Assumption 2: The set of characteristic roots of (9) is non-empty.

We also assume that the delays in (9) are commensurate.

Assumption 3: There exist numbers $k_i \in \mathbb{N},\ 1 \leq i \leq m$, and basis delay $\tau \in \mathbb{R}_{>0}$ such that $\tau_i = k_i\tau,\ i = 1, \ldots, m$.

A necessary and sufficient condition for the exponential stability of (9) is a strictly negative spectral abscissa, defined as

$$c_0 := \sup_{s\in\mathbb{C}} \left\{ \Re(s) :\ \det\left(I - H_0 - \sum_{i=1}^{m} H_i e^{-sk_i\tau}\right) = 0 \right\}.$$

By observing that the characteristic function is a polynomial in $e^{-s\tau}$, we can express

$$c_0 = -\frac{1}{\tau} \log\left(\min_{1\leq i\leq \ell} |z_i|\right),$$

where $z_i,\ i = 1, \ldots, \ell$, are the roots of equation

$$\det\left(I - H_0 - \sum_{i=1}^{m} H_i z^{k_i}\right) = 0,\ z \in \mathbb{C}.$$

Note further that $c_0 = \max(\mathcal{I}_0)$, where

$$\mathcal{I}_0 := \left\{ c \in \mathbb{R} : 0 \in \left(\bigcup_{\theta\in[0,\ 2\pi]} \sigma\left(-I + H_0 + \sum_{i=1}^{m} H_i e^{-ck_i\tau} e^{-\imath k_i\theta}\right)\right)\right\},$$

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notation $\sigma(\cdot)$ stands for the spectrum of the matrix, and $\imath = \sqrt{-1}$ is the imaginary unit.

The exponential stability of system (10), parameterized by T, is also determined by its corresponding spectral abscissa (function) $c : \mathbb{R}_{\geq 0} \to \mathbb{R}$, defined by

$$c(T) := \sup_{s \in \mathbb{C}} \{\Re(s) : \ \Delta(s; \ T) = 0\},$$

with

$$\Delta(s; \ T) := \det\left((1+sT)I - H_0 - \sum_{i=1}^{m} H_i e^{-s\tau_i}\right).$$

The characteristic shape of the spectrum of linear delay-difference equations features infinite characteristic root chains along which the imaginary parts tend to infinity, yet the real parts have a finite limit [11], [12]. Intuitively, one might expect that the inclusion of a low-pass filter as in (10), which results in a delay differential equation model of retarded type, induces a bending of these characteristic root chains towards the left, due to the addition of damping at high frequencies. However, more complex behavior may appear, as the following example illustrates.

Example 2: We consider scalar equation

$$x(t) = \frac{7}{5}x(t-1) - \frac{4}{5}x(t-2), \tag{11}$$

which is exponentially stable with

$$c_0 = c(0) = -0.11,$$

as well as its filtered counter part

$$T\dot{x}(t) + x(t) = \frac{7}{5}x(t-1) - \frac{4}{5}x(t-2). \tag{12}$$

In Figures 1-2 we plot the rightmost characteristic roots of (12) for different values of T, where the difference between the figures lies in a different scaling of the vertical axis. For $T = 0$, the characteristic roots lie on a vertical line in the complex plain. All individual characteristic roots along the chain continuously depend on T, but their sensitivity increases if the imaginary part increases, as seen in Figure 1. Zooming out in vertical direction (Figure 2), one observes that the perturbed root chain bends to the right (and back), inducing a strictly positive spectral abscissa. By reducing the value of $T > 0$, we see smaller shifts of roots in Figure 1, as expected, but zooming out, the unstable roots shift to higher frequencies, as indicated by the arrows in Figure 2, also if T is further reduced. In fact, it holds that

$$\lim_{T \to 0+} c(T) = 0.071.$$

Hence, despite the continuous dependence of individual characteristic roots on parameter T, the spectral abscissa function has a discontinuous at $T = 0$. Moreover, system (12) is unstable for any sufficiently small $T > 0$, contrasting the exponential stability at $T = 0$.

The possible lack of continuity of the spectral abscissa function of (10) at $T = 0$ motivates us to introduce a robustified notion of spectral abscissa of (9). Conceptually, it bares similarities to the notion of robust spectral abscissa for delay-differential algebraic systems [31], where small delay perturbations are considered.

Definition 1: The *filtered spectral abscissa* of (9), c_F, is defined as

$$c_F := \limsup_{T \to 0+} c(T),$$

with $c(T)$ the spectral abscissa of (10).

Note that c_F depends on the formulation of relation (9), more precisely on the decomposition of the contributions from $x(t)$. It does not depend on the choice of h in Assumption 3. Clearly we

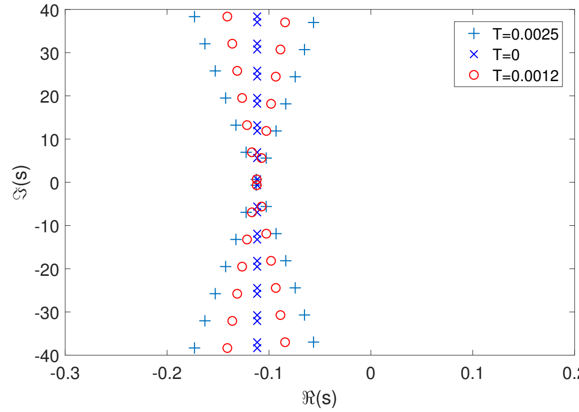

Fig. 1. Characteristic roots of (12) in the rectangular set described by $\Im(s) \in [-40, \ 40]$ and $\Re(s) \in [-0.3, \ 0.2]$, for $T \in \{0, \ 0.0012, \ 0.0025\}$.

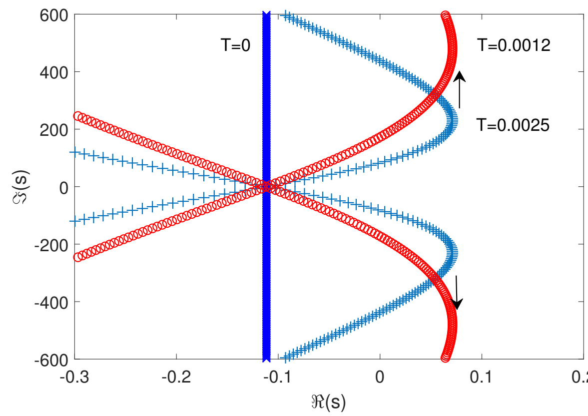

Fig. 2. Characteristic roots of (12) for $\Im(s) \in [-600, \ 600]$ and $\Re(s) \geq -0.3$, for $T \in \{0, \ 0.0012, \ 0.0025\}$.

have $c_F \geq c_0$, and as Example 2 illustrates, the inequality can be strict. The following proposition directly follows.

Proposition 1: There exist a number $\hat{T} > 0$ such that (10) is uniformly exponentially stable for $T \in [0, \ \hat{T}]$ if and only if $c_F < 0$.

In order to provide a mathematical characterization of the filtered spectral abscissa, we need the following set that contains $\mathcal{I}_0$,

$$\mathcal{I}_F := \left\{ c \in \mathbb{R} : \left(\bigcup_{\theta \in [0, \ 2\pi]} \sigma\left(-I + H_0 + \sum_{i=1}^{m} H_i e^{-ck_i\tau} e^{-\imath k_i \theta}\right)\right) \bigcap \imath\mathbb{R} \neq \phi \right\}. \tag{13}$$

The following lemma, whose proof is strongly influenced by the proof of Theorem 3 of [29], characterizes the set $\mathcal{I}_F$.

Lemma 1: If real number c satisfies $c \in \mathcal{I}_F$, then for all $\epsilon > 0$, there is a $\hat{T} > 0$, such that

$$\forall T \in (0, \hat{T}), \exists s \in \mathbb{C} : \ |\Re(s) - c| < \epsilon, \ \text{and} \ \Delta(s; \ T) = 0.$$

If $c \notin \mathcal{I}_F$, then there are constants $\epsilon > 0$ and $\hat{T} > 0$ such that $\Delta(s; \ T) \neq 0$ whenever $T \in [0, \ \hat{T})$ and $\Re(s) \in (c - \epsilon, \ c + \epsilon)$.

Proof. If $c \in \mathcal{I}_F$, then there is a $\theta \in [0, \ 2\pi]$ and $\gamma \geq 0$ such that

$$\det\left(-(1+\imath\gamma)I + H_0 + \sum_{i=1}^{m} H_i \left(e^{-\left(c+\imath\frac{\theta}{\tau}\right)\tau}\right)^{k_i}\right) = 0. \tag{14}$$

Consider the entire function

$$f(s;\ T) = \det\Big(-TsI - \imath\frac{2\pi T}{\tau}\left\lceil\frac{\gamma\tau}{2\pi T}\right\rceil I \\ -I + H_0 + \sum_{i=1}^{m} H_i \left(e^{-s\tau}\right)^{k_i}\Big),$$

parametrized by $T > 0$, where the operation $\lceil\cdot\rceil$ corresponds to the ceiling function. On compact subsets of $\mathbb{C}$, function $f(\cdot;\ T)$ uniformly converges as $T \to 0+$ to function f_0, given by

$$f_0(s) := \det\left(-(1+\imath\gamma)I + H_0 + \sum_{i=1}^{m} H_i \left(e^{-s\tau}\right)^{k_i}\right).$$

From (14) it follows that f_0 has a zero $c + \imath\frac{\theta}{\tau}$. By an application of Rouché's theorem, it follows that for sufficiently small T, function $f(\cdot;\ T)$ has a zero $c_T + \imath\frac{\theta_T}{\tau}$ such that

$$\lim_{T\to 0+} c_T = c, \quad \lim_{T\to 0+} \theta_T = \theta. \tag{15}$$

Thus we have

$$\begin{aligned} 0 &= f\left(c_T + \imath\tfrac{\theta_T}{\tau}\right) \\ &= \det\Big(-T\left(c_T + \imath\left(\tfrac{\theta_T}{\tau} + \tfrac{2\pi}{\tau}\left\lceil\tfrac{\gamma\tau}{2\pi T}\right\rceil\right)\right) I \\ &\quad -I + H_0 + \textstyle\sum_{i=1}^{m} H_i \left(e^{-c_T\tau}e^{-\imath\theta_T}e^{-\imath 2\pi\left\lceil\frac{\gamma\tau}{2\pi T}\right\rceil}\right)^{k_i}\Big), \end{aligned}$$

which implies that $c_T + \imath\left(\frac{\theta_T}{\tau} + \frac{2\pi}{\tau}\left\lceil\frac{\gamma\tau}{2\pi T}\right\rceil\right)$ is a characteristic root of (10) for small T. The assertion follows by combining this results with (15).

Now consider the case where $c \neq \mathcal{I}_F$. As in the definition of this set, the union is taken over a compact interval, there exists an $\epsilon > 0$ such that

$$\left(\bigcup_{\theta\in[0,\ 2\pi]} \sigma\left(-I + H_0 + \sum_{i=1}^{m} H_i e^{-\alpha k_i\tau}e^{-\imath k_i\theta}\right)\right)\bigcap \imath\mathbb{R} \neq \phi, \\ \forall\alpha\in[c-\epsilon,\ c+\epsilon]. \tag{16}$$

Suppose there exist $\alpha \in [c-\epsilon,\ c+\epsilon]$ and $\beta\in\mathbb{R}$ such that $\Delta(\alpha + \imath\beta;\ T) = 0$. The latter implies that

$$(\alpha+\imath\beta)T \in \sigma\left(-I + H_0 + \sum_{i=1}^{m} H_i e^{-\alpha k_i\tau}e^{-\imath\beta k_i\tau}\right)$$

and for small T, we arrive at a contradiction with (16). The proof is completed. □

We can now state the main theoretical result of the paper.

Theorem 1: If matrix $-I + H_0$ has all its eigenvalues in the open left half plane, then it holds that

$$c_F = \max(\mathcal{I}_F), \tag{17}$$

with $\mathcal{I}_F$ defined by (13). If matrix $-I + H_0$ has an eigenvalue in the open right half plane, then $c_F = +\infty$.

Proof. First, we assume that $\sigma(-I + H_0) \subset \mathbb{C}_-$. If s is a characteristic root of (10), then

$$Ts \in \sigma\left(-I + H_0 + \sum_{i=1}^{m} H_i e^{-sk_i\tau}\right).$$

For large $\Re(s)$, the spectrum of $-I + H_0 + \sum_{i=1}^{m} H_i e^{-sk_i\tau}$ lies in the left half plane, while Ts lies in the right half plane. Hence, there exist a number $\hat{c}$, which does not depend on T, such that all characteristic roots of (10) have real part smaller than $\hat{c}$. The assertion then follows from Lemma 1.

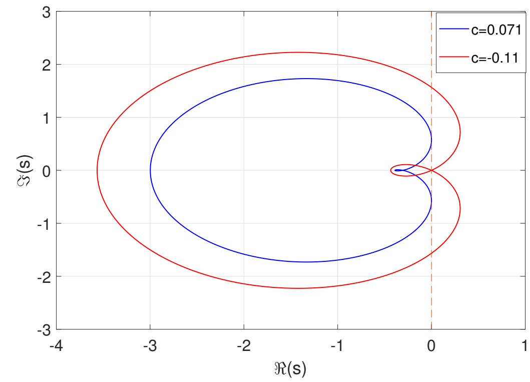

Fig. 3. Curve (18 in the complex plane.

Next we consider the case where $-I + H_0$ has an eigenvalue s_0 with $\Re(s_0) > 0$. With the substitution $\tilde{s} = sT$, Equation $\Delta(s;\ T) = 0$ can be written as $g(\tilde{s};\ T) = 0$, where

$$g(\tilde{s};\ T) := \det\left(-\tilde{s}I - I + H_0 + \sum_{i=1}^{m} H_i e^{-\frac{\tilde{s}k_i\tau}{T}}\right).$$

On any compact subset of $\mathbb{C}_+$, function $g(\cdot;\ T)$ uniformly converges to function g_0 as $T \to 0+$, where $g_0(s) := \det(-\tilde{s}I - I + H_0)$. Because g_0 has right-half plane zero s_0, it follows that for sufficiently small T, $g(\cdot;\ T)$ has a zero s_T such that $\lim_{T\to 0+} s_T = s_0$. Since we have $\Delta\left(\frac{s_T}{T};\ T\right) = 0$, we arrive at $c_F = +\infty$. □

Example 3: We reconsider Example 2, for which $-I + H_0 = -1 < 0$ and $\mathcal{I}_0 = \{-0.11\}$, $\mathcal{I}_F = (-\infty,\ 0.071]$. In Figure 3 we plot the following curve that plays a role in the definition of $\mathcal{I}_0$ and $\mathcal{I}_F$,

$$\theta\in[0,\ 2\pi] \mapsto -1 + \frac{7}{5}e^{-c}e^{-\imath\theta} - \frac{4}{5}e^{-2c}e^{-\imath 2\theta} \tag{18}$$

for $c = c_0 = \max(\mathcal{I}_0)$, respectively $c = c_F = \max(\mathcal{I}_F)$. In the former case, the curve passes through zero, in accordance with $c_0 = \max(\mathcal{I}_0)$. In the latter cases it touches the imaginary axis, in accordance with Theorem 1.

Example 4: We rewrite delay-difference equation (11) in the form

$$x(t) = 2x(t) - \frac{7}{5}x(t-1) + \frac{4}{5}x(t-2),$$

and apply a filtering of the right-hand size, resulting in

$$T\dot{x}(t) + x(t) = 2x(t) - \frac{7}{5}x(t-1) + \frac{4}{5}x(t-2). \tag{19}$$

We are now in the situation where $-I + H_0 = 1 > 0$, and Theorem 1 we have $c_F = +\infty$, whereas the set $\mathcal{I}_F$ is the same as for the previous example.

We conclude the section by another characterization of the filtered spectral abscissa, amenable from a computational point of view.

Theorem 2: If matrix $-I + H_0$ has all its eigenvalues in the open left half plane, then it holds that

$$c_F = \frac{1}{\tau}\log\left(\max_{\omega\geq 0} r_\sigma\left(M(\imath\omega)\right)\right), \tag{20}$$

where $r_\sigma(\cdot)$ denotes the spectral radius and

$$M(s) = \begin{bmatrix} 0 & I & & & & & \\ & & \ddots & & & & \\ & & & & & I & \\ (1+s)I - H_0 & \cdots & -H_1 & \cdots & -H_{m-1} & \cdots \end{bmatrix}^{-1} \times \begin{bmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & H_m \end{bmatrix} \in \mathbb{R}^{k_m n \times k_m n},$$

such that $-H_i$ appears in the $(k_i + 1)$-th block column, $i = 1, \ldots, m-1$.

Proof. Matrix $M(\imath\omega)$ exists for $\omega \geq 0$, since $M(s)$ exists if and only if $\det((1+s)I - H_0) \neq 0$. Substituting $e^{-c\tau}e^{-\imath\vartheta}$ in (13) by z, equality (17) can be recast as

$$e^{-c_F\tau} = \min_{z\in\mathbb{C}} \left\{ |z| : \det\Big(-(1+\imath\omega)I + H_0 + \sum_{i=1}^{m} H_i z^{k_i} \Big) = 0 \ \text{ for some } \omega \geq 0 \right\}.$$

In this expression, the zero determinant condition can be rewritten as

$$\det\left(I - M(\imath\omega) z I_{k_m n}\right) = 0,$$

resulting in $e^{-c_F\tau} = \min_{\omega\geq 0} \ (r_\sigma(M(\imath\omega)))^{-1}$. The assertion directly follows. □.

Example 5: For equation (11) the function $\omega \mapsto r_\sigma(M(\imath\omega))$ is unimodal. A unique maximum 1.074 is reached at $\omega = 0.58$, leading to $c_F = 0.071$.

To conclude the section, let's briefly discuss (9) as limit of (10), interpreted as a singularly perturbed system with $\epsilon = T$ as perturbation parameter. In view of Tikhonov's theorem for delay-free singularly perturbed systems, the discontinuity of the spectral abscissa at $T = 0$, which motivated the introduction of the filtered spectral abscissa, may be considered highly non-intuitive. The rich and surprising dynamics are in line with the following observations. In the time-domain, reducing parameter T not just introduces faster dynamics but at the same time it increases the effective delays compared to their characteristic time-scale, since the delays in (10) do not scale with perturbation parameter T. Furthermore, the descriptions of fast and slow/limit dynamics are interwoven in one equation. In the frequency domain, analyzing the sensitivity of stability around $T = 0$ boils down to analyzing characteristic roots in the neighborhood of infinity, which is an essential singularity of the exponential function.

III. Stability of Neutral Equations

In this section, we make the leap from (filtered) delay-difference equations to neutral equations. The starting point is the neutral equation

$$\dot{x}(t) = H_0\dot{x}(t) + \sum_{i=1}^{m} H_i \dot{x}(t - k_i\tau) + A_0 x(t) + \sum_{i=1}^{m} A_i x(t - k_i\tau), \quad (21)$$

where the additional matrices A_i, $i = 0, \ldots, m$, are real valued and the associated delay-difference equation is given by (9). The application of a filter to the derivatives in the right-hand side leads to

$$\begin{cases} \dot{x}(t) &= z(t) + A_0 x(t) + \sum_{i=1}^{m} A_i x(t - k_i\tau), \\ T\dot{z}(t) + z(t) &= H_0 \dot{x}(t) + \sum_{i=1}^{m} H_i \dot{x}(t - k_i\tau). \end{cases} \quad (22)$$

Note that for $s \neq 0$, the characteristic equation of (22) can be written in the form

$$\det\left[\left(I - \frac{1}{Ts+1}\left(H_0 + \sum_{i=1}^{m} H_i e^{-sk_i\tau} \right) \right) - \frac{1}{s}\left(A_0 + \sum_{i=1}^{m} A_i e^{-sk_i\tau} \right) \right] = 0, \quad (23)$$

whose structure is consistent with characteristic equation (7) of motivating example (1) and (6). The following result concerns the preservation of stability for small T.

Theorem 3: If system (21) is exponentially stable and the associated delay-difference equation (9) satisfies $c_F < 0$, then there is a $\hat{T} > 0$ such that (22) is uniformly exponentially stable for all $T \in [0,\ \hat{T}]$.

Proof. Since $c_F < 0$, there exist numbers $\epsilon > 0$ and $\hat{T} > 0$ such that $\Delta(s;\ T) \neq 0$ for all $T \in [0,\ \hat{T}]$ and $s \in \mathbb{C}$ satisfying $\Re(s) \geq -\epsilon$. Hence, for $T \leq \hat{T}$ and $\Re(s) \geq -\epsilon$, the characteristic equation can be expressed as

$$s \in \sigma\left(\left(I - \frac{1}{Ts+1}\left(H_0 + \sum_{i=1}^{m} H_i e^{-sk_i\tau} \right) \right)^{-1} \times \left(A_0 + \sum_{i=1}^{m} A_i e^{-sk_i\tau} \right) \right),$$

which implies

$$|s| \leq \sup_{\Re(s)\geq -\epsilon, T\in[0,\ \hat{T}]} \left(\|A_0\|_2 + \sum_{i=1}^{m} \|A_i\|_2 e^{-\epsilon k_i\tau} \right) \times \left\| \left(I - \frac{1}{Ts+1}\left(H_0 + \sum_{i=1}^{m} H_i e^{-sk_i\tau} \right) \right)^{-1} \right\|_2 < +\infty. \quad (24)$$

Let us describe arguments why the second factor in the right-hand side of the first inequality can be uniformly bounded. We have

$$\begin{aligned} &\det\left(I - \tfrac{1}{Ts+1}\left(H_0 + \textstyle\sum_{i=1}^{m} H_i e^{-sk_i\tau} \right) \right) = \\ &\det\left(\tfrac{Ts}{1+Ts} I - \tfrac{1}{1+Ts}\left(-I + H_0 + \textstyle\sum_{i=1}^{m} H_i e^{-sk_i\tau} \right) \right). \end{aligned} \quad (25)$$

For the considered range of T and s, we have $\sigma(-I + H_0 + \sum_{i=1}^{m} H_i e^{-sk_i\tau}) \in \mathcal{D}$, where $\mathcal{D} := \{s \in \mathbb{C} :\ -a_1 < \Re(s) < -a_2$ and $|\Im(s)| < b\}$ for some $a_1, a_2 > 0$ and $b > 0$, inferred from Lemma 1. For any $\nu \in \mathcal{D}$, the expression

$$\left| \det\left(\frac{Ts}{1+Ts} I - \frac{1}{1+Ts}\nu \right) \right|$$

is bounded away from zero for $\Re(s) \geq -\epsilon$ and $T \in [0,\ \hat{T}]$ provided $-\epsilon\hat{T} > -a_2$ (otherwise one simply has to repeat the arguments listed above with a smaller choice of $\hat{T}$). It follows that also the modulus of the determinant (25) is uniformly bounded away from zero, which implies the bound on the inverse.

Finally, based on (24), in searching for characteristic roots in the right-half plane $\Re(s) \geq -\epsilon$, we can restrict ourselves to a compact set that does not depend on $T \in [0,\ \hat{T}]$. On this compact set, the characteristic equation of (22) uniformly converges to the one of (21). The assertion then follows once again from Rouché's theorem. □

Consider the situation where system (21) is exponentially stable but $c_F \geq 0$. Following from the proof of Lemma 1 and taking into

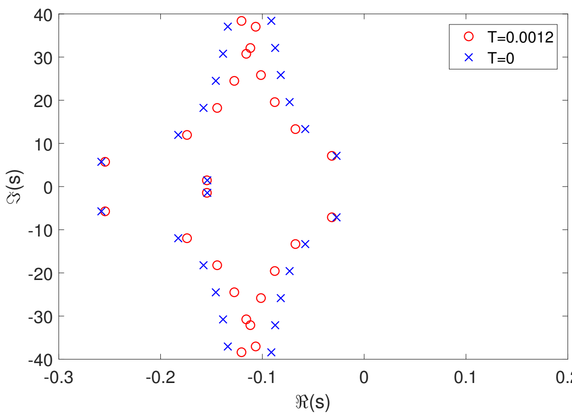

Fig. 4. Characteristic roots of (27) in the rectangular set described by $\Im(s) \in [-40,\ 40]$ and $\Re(s) \in [-0.3,\ 0.2]$, for $T \in \{0,\ 0.0012\}$.

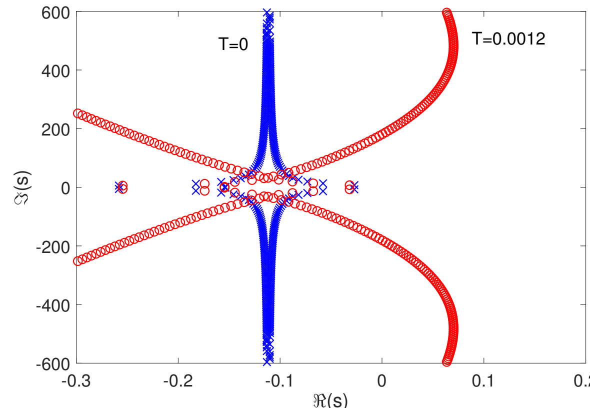

Fig. 5. Characteristic roots of (27) for $\Im(s) \in [-600,\ 600]$ and $\Re(s) \geq -0.3$, for $T \in \{0,\ 0.0012\}$.

account that $c_F > c_0$ (since $c_0 < 0$ is implied by the exponential stability of (21)), there exists a sequence of real positive numbers $\{T_n\}_{n\geq 0}$ converging to zero, such that for $T = T_n$, delay-difference equation (10) has a characteristic root s_n such that

$$\lim_{n\to\infty} \Re(s_n) = c_F, \quad \lim_{n\geq\infty} \Im(s_n) = +\infty,$$

At the same time, the second term within the determinant in (23) is bounded in any right half plane, where it tends to zero if $|s| \to \infty$. These observations suggest an asymptotic matching of the high-frequency characteristic roots of (22) and (10), and preservation of exponential instability for small $T > 0$ if $c_F > 0$. This is confirmed by our numerical experiments and illustrated by the following example.

Example 6: Assume that (21) has the form

$$\dot{x}(t) = \frac{7}{5}\dot{x}(t-1) - \frac{4}{5}\dot{x}(t-2) - 2x(t) + \frac{1}{2}x(t-1), \tag{26}$$

whose associated delay-difference equation was analyzed in Example 2, yielding $c_F = 0.071$. In Figures 4-5 we depict the rightmost characteristic roots of the filtered equation

$$\begin{cases} \dot{x}(t) = z(t) - 2x(t) + \frac{1}{2}x(t-1), \\ T\dot{z}(t) + z(t) = \frac{7}{5}\dot{x}(t-1) - \frac{4}{5}\dot{x}(t-2). \end{cases} \tag{27}$$

A comparison between Figure 5 and Figure 2 makes the matching of high frequency characteristic roots apparent, which causes instability of (27) for any sufficiently small $T > 0$, even though (26) is exponentially stable.

Remark 1: Assumption 2 excludes the special situation where $H_i = 0$, $i = 1, \ldots, m$. In this situation we can rely on [21, Proposition 4.5] and conclude about uniform exponential stability of (22) over an interval $T \in [0,\ \hat{T}]$ with $\hat{T} > 0$, if (21) is exponentially stable and $\sigma(-I + H_0) \in \mathbb{C}_-$.

IV. Implications on Control Design and Concluding Remarks

We studied the stability properties of equations (10) and (22) for small values of T, in relation to delay-difference equation (9) and neutral equation (21). As was illustrated, the spectral abscissa may not be continuous at $T = 0$. This led to the notion of filtered spectral abscissa of a delay-difference equation, which played a major role in the derived conditions for preservation of stability.

The addressed problems were inspired by the use of a low-pass filter in order to resolve well documented fragility problems for controlled time-delay and other infinite-dimensional systems, which are all related to a feedthrough at high frequencies, in the sense of (5), and characteristic roots moving from one half plane to the other one via the point at infinity. Coming back to (1)-(2), a robustified design procedure for determining stabilizing gain values, inferred from Theorem 3, consists, for instance, of solving the constrained optimization problem

$$\begin{array}{l} \min_{(K_P, K_D)} c_N(K_P, K_D), \\ \text{subject to } c_F(K_D) < 0, \end{array}$$

where c_N is the spectral abscissa of closed-loop system (1)-(2) and c_F is the filtered spectral abscissa of delay-difference equation (4). To solve this optimization problem, the characterization (20) of c_F can be employed, and the constraint can be handled by a penalty method as in [21]. A feasible starting point, if it exists, can be found by solving $\min_{K_D} c_F(K_D)$ first. Note that a feasible point characterized by a negative value of the objective function ensures a low-pass filter with sufficiently high cut-off frequency can be added to the control loop without affecting stability (recall that the filtered spectral abscissa is a characteristic of well-posedness and robustness of the system without filter, i.e. $T = 0$). Hence, the control law can be implemented in the form (6) with small T, which implies that the closed-loop system corresponds to an exponentially stable time-delay system of retarded type, satisfying

$$\lim_{|s|\to\infty,\ \Re(s)\geq 0} G(s)C_F(s) = 0,$$

where C_F is the transfer function of the controller (6). Then, the fragility and instability problems outlined in the introduction and illustrated with Example 1, can not occur anymore. Note that the parameters of control law (6), (K_P, K_D, K_I, T), can then be further optimized using the approach of [32], see Section III of this reference.

Finally, we comment on Assumption 3, which was employed in the technical proof of Lemma 1 in order to render Rouché's theorem applicable. If this assumption is dropped and we define

$$\hat{\mathcal{I}}_F =: \Bigg\{ c \in \mathbb{R} : \left(\bigcup_{\vec{\theta} \in [0,\ 2\pi]^m} \sigma\left(-I + H_0 + \sum_{i=1}^{m} H_i e^{-c\tau_i} e^{-\imath\theta_i} \right) \right) \bigcap \imath\mathbb{R} \neq \phi \Bigg\}$$

with $\vec{\theta} = (\theta_1, \ldots, \theta_m)$, then it can be easily shown that the statement of Theorem 1 holds provided (17) is replaced by

$$c_F \leq \max(\hat{\mathcal{I}}_F). \tag{28}$$

Hence, if $\sigma(-I + H_0) \subset \mathbb{C}_-$, then $\max(\hat{\mathcal{I}}_F) < 0$ is a sufficient condition for stability of (10) for small T. We conjecture that for rationally independent delays the condition is actually necessary and sufficient and the inequality in (28) can be replaced by an equality. To support this claim, let $\{\vec{\tau}_n\}_{n \geq 1}$ be a sequence of commensurate delays converging to rationally independent delays $\vec{r}$. Then, relying on Kronecker's theorem [33, Theorem 444], we have $\lim_{n\to\infty} \max(\mathcal{I}_F(\vec{\tau}_n)) = \max(\hat{\mathcal{I}}_F(\vec{r}))$. This result also ensures that by a sufficiently good approximation of rationally independent delays by commensurate delays, the stability conditions from Theorems 1 and 3 are still reliable.

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