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TO THE INTEGRABILITY OF THE SYSTEM OF TWO COUPLED NONLINEAR SCHRÖDINGER EQUATIONS

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1. In our previous paper [1] we have introduced the notion of the degenerative dispersion law. Let us imagine a medium with one type of waves with dispersion law $\omega(\boldsymbol{k})$ and consider the process of scattering of n waves into m waves. This process is described by the resonant conditions

$$\begin{aligned}\omega(\boldsymbol{k}_1)+\cdots+\omega(\boldsymbol{k}_n)&=\omega(\boldsymbol{k}_{n+1})+\cdots+\omega(\boldsymbol{k}_{n+m}),\\ \boldsymbol{k}_1+\cdots+\boldsymbol{k}_n&=\boldsymbol{k}_{n+1}+\cdots+\boldsymbol{k}_{n+m}.\end{aligned} \tag{1}$$

Eqs. (1) define some manifold F in the $(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_{n+m})$ space. The dispersion law $\omega(\boldsymbol{k})$ is called degenerative with respect to the process (1) if there is a function $f(\boldsymbol{k})\not\equiv A\omega(\boldsymbol{k})+(\boldsymbol{\alpha},\boldsymbol{k})+B$ which satisfies on the manifold F to the equation

$$f(\boldsymbol{k}_1)+\cdots+f(\boldsymbol{k}_n)=f(\boldsymbol{k}_{n+1})+\cdots+f(\boldsymbol{k}_{n+m}). \tag{2}$$

Here A, B are arbitrary constants, $\boldsymbol{\alpha}$ is an arbitrary constant vector.

The notion of the dispersion laws, degenerative to processes including several types of waves may be defined by analogous way. Let us consider, for example, the process in which one wave with dispersion law $\Omega(\boldsymbol{k})$ is formed as a result of the interaction of two waves with dispersion law $\omega(\boldsymbol{k})$. Such a process is described by the resonant condition

$$\omega(\boldsymbol{k}_1)+\omega(\boldsymbol{k}_2)=\Omega(\boldsymbol{k}_1+\boldsymbol{k}_2). \tag{3}$$

This condition defines the resonant manifold Γ in $(\boldsymbol{k}_1,\boldsymbol{k}_2)$ space.

The dispersion laws $\omega(\boldsymbol{k})$, $\Omega(\boldsymbol{k})$ are degenerative with respect to the process of interaction (3) if there are two functions f and g which satisfy on the manifold Γ to the equations

$$f(\boldsymbol{k}_1)+f(\boldsymbol{k}_2)=g(\boldsymbol{k}_1+\boldsymbol{k}_2) \tag{4}$$

and

$$f(\boldsymbol{k})\not\equiv A\omega(\boldsymbol{k})+(\boldsymbol{\alpha},\boldsymbol{k}),\quad g(\boldsymbol{k})\not\equiv A\Omega(\boldsymbol{k})+(\boldsymbol{\alpha},\boldsymbol{k}).$$

As was demonstrated in [1], the question of the degeneration of the dispersion law of some nonlinear system to appropriate nonlinearity is very important in connection with the problem of integrability of the correspondent nonlinear system or with the problem of existence of the additional motion invariant of that system. In [1] the starting point was that the motion invariants of the dynamic equations must generate motion invariants of the appropriate kinetic equations. This leads to the following important result: if the dynamic motion invariant I has the quadratic part I_q of the form

$$I_q=\int f(\boldsymbol{k})a_k a_k^*\,\mathrm{d}\boldsymbol{k} \tag{5}$$

and waves a_k have the dispersion law $\omega(\boldsymbol{k})$ then there are only two alternative possibilities:

1) the dispersion law $\omega(k)$ is degenerative to the major nonlinear process. This occurs when the matrix element $T_{k_1 \ldots k_n}$ of this n-particle process is nonzero on the resonant manifold.

2) If the dispersion law is nondegenerative, then $T_{k_1 \ldots k_n} = 0$ on the resonant surface. When several types of waves are present the same is true for the set of the dispersion laws and appropriate matrix element.

In this paper we do not need the generalizations of this result, but in fact the kinetic equations in the above argumentation may be replaced by S-f[illegible]tional (the classic analog of the S-matrix):

$$a_k^+ = S[a_k^-], \quad a_k \to a_k^\pm \quad \text{as} \quad k \to \pm\infty.$$

By doing this one can see that the above alternative is true not only for major nonlinear process, but for any process of scattering of n waves into m waves. Of course, the matrix element of the major nonlinear process $T_{k_1 \ldots k_n}$ should be replaced in the above alternative by the amplitude $W_{k_1 \ldots k_n; k_{n+1} \ldots k_{n+m}}$ of the scattering of n waves into m waves. However, this generalization is not the aim of the present paper.

In paper [1] we have proposed to use the alternative mentioned above for systematic checking of the integrability of some nonlinear systems or, in other words, the existence of the additional motion invariants with the quadratic major terms.

In the present paper we consider the system ot two coupled nonlinear Schrödinger equations

$$\begin{aligned} i\psi_{1t} &= C_1\psi_{1xx} + 2\alpha|\psi_1|^2\psi_1 + 2\beta|\psi_2|^2\psi_1, \\ i\psi_{2t} &= C_2\psi_{2xx} + 2\gamma|\psi_2|^2\psi_2 + 2\beta|\psi_1|^2\psi_2, \end{aligned} \tag{6}$$

with the Hamiltonian

$$H = \int_{-\infty}^{\infty} \{c_1|\psi_{1x}|^2 + c_2|\psi_{2x}|^2 + \alpha|\psi_1|^4 + 2\beta|\psi_1|^2|\psi_2|^2 + \gamma|\psi_2|^4\}\, dx. \tag{7}$$

The system (6) arises in nonlinear optics, for example in the problem of interaction of waves with different polarizations considered in [2]. This system is to some extent universal from the point of view of its applications to physics.

System (6) has the following trivial invariants of motion:

$$I_1 = \int |\psi_1|^2\, dx; \quad I_2 = \int |\psi_2|^2\, dx;$$

$$I_3 = \int (\psi_1 \overset{*}{\psi}_{1x} + \psi_2 \overset{*}{\psi}_{2x})\, dx; \quad I_4 = H.$$

In paper [3] it was shown that in the case

$$c_1 = c_2; \quad \alpha = \beta = \gamma \tag{8}$$

the system (6) has a infinite set of motion invariants and may be solved by the inverse scattering method.

The aim of the present paper is to show that besides (8) there is another integrable system of the type (6)

$$c_1 = -c_2; \quad \alpha = -\beta = \gamma. \tag{9}$$

In all other cases except (8), (9) the system (6) does not have any additional motion invariants of the type (5).

2. Let us make some preliminary note. In the one-dimensional case any dispersion law $\omega(k)$ is degenerative to the process of scattering of two waves into two waves

$$\begin{aligned} \omega(k) + \omega(k_1) &= \omega(k_2) + \omega(k_3), \\ k + k_1 &= k_2 + k_3. \end{aligned} \tag{10}$$

Indeed, the system of equations (10) has only two solutions

$$\begin{aligned} k_2 &= k_1; \quad k_3 = k, \\ k_2 &= k; \quad k_3 = k_1 \end{aligned}$$

irrespective to the form of the function $\omega(k)$.

Obviously, the same statement is true for the process of interaction of different types of waves which have coinciding dispersion laws. But if these dispersion laws are different (for example $\omega_1(k)$ and $\omega_2(k)$) then they are in the general case nondegenerative with respect to the process

$$\omega_1(k) + \omega_2(k_1) = \omega_1(k_2) + \omega_2(k_3), \quad k + k_1 = k_2 + k_3. \tag{12}$$

The example we are interested in is

$$\omega_1(k) = c_1 k^2, \quad \omega_2(k) = c_2 k^2, \quad \rho = \frac{c_2}{c_1} \neq \pm 1. \tag{13}$$

Equations (12) describe at any $\omega_1(k)$, $\omega_2(k)$ the two-dimensional manifold Γ_1 in the four-dimensional space (k, k_1, k_2, k_3). One can check directly that in the case (13) the manifold Γ_1 can be parametrized in the following way:

$$k = \frac{\rho - 1}{2} k_1 + \frac{\rho + 1}{2} k_2, \quad k_3 = \frac{\rho + 1}{2} k_1 + \frac{\rho - 1}{2} k_2. \tag{14}$$

Let a_{1k}, a_{2k} be the amplitudes of waves with dispersion laws $\omega_1(k)$, $\omega_2(k)$. If there is an additional motion invariant I with the quadratic part of the form

$$I = \int f_1(k)|a_{1k}|^2 \, dk + \int f_2(k)|a_{2k}|^2 \, dk, \tag{15}$$

then the functions $f_1(k)$, $f_2(k)$ are to satisfy to the equations

$$f_1(k) + f_2(k_1) = f_2(k_2) + f_1(k_3), \quad k + k_1 = k_2 + k_3 \tag{16}$$

on the manifold Γ_1. (Because the matrix element is nonzero on the resonant manifold Γ_1 (12) when $\beta \neq 0$. Using the parametrisation (14) we find that $f_{1,2}(k)$ satisfy to the functional equation

$$f_1\left(\frac{\rho - 1}{2} k_1 + \frac{\rho + 1}{2} k_2\right) + f_2(k_1) = f_2(k_2) + f_1\left(\frac{\rho + 1}{2} k_1 + \frac{\rho - 1}{2} k_2\right). \tag{17}$$

If we differentiate (17) two times by k_1 and one time by k_2 and assume that $k_1 = k_2 = \xi/\rho$, we find:

$$(\rho^2 - 1)(\rho - 1) f_1'''(\xi) = (\rho^2 - 1)(\rho + 1) f_1'''(\xi).$$

Hence if $\rho^2 \neq 1$ we have

$$f_1'''(\xi) = 0, \quad f_1(\xi) = A\xi^2 + B\xi + C. \tag{18}$$

Substituting (18) into (16) we find after some simple calculations:

$$f_2(\xi) = \rho A \xi^2 + B\xi + D. \tag{19}$$

Let us note that the dispersion laws $\omega_1(k)$, $\omega_2(k)$ are defined up to arbitrary constants and equal linear terms. With this accuracy $f_1(k) = \omega_1(k)$, $f_2(k) = \omega_2(k)$. Hence at $\rho^2 \neq 1$ the dispersion laws (3) are nondegenerative.

At $\rho = 1$ $\omega_1 = \omega_2$, and the dispersion law is degenerative. It is degenerative at $\rho = -1$ also.

It is a consequence of a more general fact that dispersion laws $\omega_1(k)$, $\omega_2(k)$ are degenerative with respect to the process (12) if $\omega_2(k) = -\omega_1(k)$, and $\omega_1(k) = \omega_1(-k)$.

Let us return to the system (6). Let $a_{1k}(t)$, $a_{2k}(t)$ be the Fourier components of $\psi_1(x, t)$, $\psi_2(x, t)$. It follows from the fact of nondegenerativeness of the dispersion laws when $c_1^2 \neq c_2^2$ that this system may have an additional motion invariant with the quadratic part of the form (15), provided that the matrix element of the process (12) turns to zero on the manifold Γ_1 only. But this matrix element is the constant value β. Hence the necessary condition of the existence of an additional motion invariant of the system (6) is $\beta = 0$. If $\beta = 0$ the system (6) splits into two independent nonlinear Schrödinger equations whose integrability is well known.

3. As we have seen in section 2, the set of dispersion laws (13) is degenerative to the process (12) when $\rho = \pm 1$. Therefore we can not make any conclusion about the existence of an additional invariant I and have to consider the second order process of scattering of three waves into three waves. Otherwise, in the language of paper [1], the four-particle interaction is trivial and the first nonvanishing terms in the appropriate kinetic equation are six-particle. The kernel of this kinetic equation is the squared second order matrix element in the perturbation theory [3]. It is more convenient for us to deal with the general case $c_1^2 \neq c_2^2$. Let us consider the process described by the resonant conditions of the form:

$$\left.\begin{aligned} &\omega_1(k)+\omega_1(k_1)+\omega_2(k_2)=\omega_2(k_3)+\omega_1(k_4)+\omega_1(k_5),\\ &k+k_1+k_2=k_3+k_4+k_5. \end{aligned}\right\} \tag{20}$$

Eqs. (20) determine a four-dimensional manifold Γ_2 in the six-dimensional space $(k,\dots,k_5)$. In the case under consideration the manifold Γ_2 is rational and may be parametrised by the formulae

$$\begin{aligned} k&=\frac{3P\rho}{1+2\rho}+R\left[u+\frac{1}{u}-\frac{1}{v}+(1+2\rho)v\right],\\ k_1&=\frac{3P\rho}{1+2\rho}+R\left[u+\frac{1}{u}+\frac{1}{v}-(1+2\rho)v\right],\\ k_2&=\frac{3P}{1+2\rho}-\frac{2R}{u}-2Ru,\\ k_3&=\frac{3P}{1+2\rho}+\frac{2R}{u}-2Ru,\\ k_4&=\frac{3P\rho}{1+2\rho}+R\left[u-\frac{1}{u}+(1+2\rho)v+\frac{1}{v}\right],\\ k_5&=\frac{3P\rho}{1+2\rho}+R\left[u-\frac{1}{u}-(1+2)v-\frac{1}{v}\right]. \end{aligned} \tag{21}$$

Here P, R, u, v are independent variables on the manifold Γ_2.

Using parametrisation (21) it is easy to prove that dispersion laws (13) are nondegenerative with respect to the process (20) even in the case $\omega_1 = \pm\omega_2 (\rho = \pm 1)$.

Indeed, if we substitute (21) into the equation

$$f_1(k)+f_1(k_1)+f_2(k_2)=f_2(k_3)+f_1(k_4)+f_1(k_5) \tag{22}$$

and differentiate three times by R and assume that $R=0$ we find $f'''_{1,2}(\xi)=0$.

From the nondegenerativeness of the process (20) it follows that the system (6) may have an additional motion invariant if the matrix element corresponding to the process (20) turns to zero on the manifold (21). If we assume in (20) that $k_2 = k_3$ or $k = k_5$ we obtain the two-dimensional manifolds of the type (12). Those submanifolds of the manifold (21) should be excluded from our consideration, because they correspond to the scattering of two waves into two waves. The six-particle matrix element may be obtained directly by applying the perturbation theory [3] to the system (6) and by summation of the terms of the second order. Those simple but rather extensive calculations result in

$$\begin{aligned} T_{kk_1k_2;k_3k_4k_5} = &-4\alpha\beta\\ &\times\left[\frac{1}{\omega_1(k_1+k_2-k_3)+\omega_2(k_3)-\omega_2(k_2)-\omega_1(k_1)}\right.\\ &+\frac{1}{\omega_1(k+k_2-k_3)+\omega_2(k_3)-\omega_2(k_2)-\omega_1(k)}\\ &+\frac{1}{\omega_1(k+k_1-k_4)+\omega_1(k_4)-\omega_1(k)-\omega_1(k_1)}\\ &\left.+\frac{1}{\omega_1(k+k_1-k_5)+\omega_1(k_5)-\omega_1(k)-\omega_1(k_1)}\right]\\ &-2\beta^2\left[\frac{1}{\omega_2(k+k_2-k_4)+\omega_1(k_4)-\omega_2(k_2)-\omega_1(k)}\right.\\ &+\frac{1}{\omega_2(k+k_2-k_5)+\omega_1(k_5)-\omega_2(k_2)-\omega_1(k)}\\ &+\frac{1}{\omega_2(k_1+k_2-k_4)+\omega_1(k_4)-\omega_2(k_2)-\omega_1(k_1)}\\ &\left.+\frac{1}{\omega_2(k_1+k_2-k_5)+\omega_1(k_5)-\omega_2(k_2)-\omega_1(k_1)}\right] \end{aligned} \tag{23}$$

Substituting (21) into (23) we find after some calculations

$$T_{kk_1k_2;k_3k_4k_5} \sim \Big\{ v^2(1+2\rho)^2\rho[2\alpha\rho - \beta(1+\rho^2)] + \frac{1}{v^2}\rho[2\alpha\rho - \beta(1+\rho^2)] + u^2(1+2\rho)^2(2\beta\rho - 2\alpha) \frac{2}{u^2}\rho^3(\beta - \alpha\rho)\Big\}. \tag{24}$$

One can see from (24) that T turns to zero in two cases only:

1) $\rho = 1$, $\alpha = \beta = \gamma$;
2) $\rho = -1$, $\alpha = -\beta = \gamma$.

The first case in (6) corresponds to the well-known vector nonlinear Schrödinger equation integrated by the inverse scattering method in [4]. In the case when $\rho = -1$, $\alpha = -\beta = \gamma$ we have a rather exotic system of equations nevertheless integrable by the inverse scattering method also. Indeed, the inverse scattering method is applicable to the system [5]

$$\begin{aligned} i\Psi_t &= \Psi_{xx} + \Psi\chi\Psi, \\ -i\chi_t &= \chi_{xx} + \chi\Psi\chi, \end{aligned} \tag{25}$$

where Ψ and χ are matrices. Let us take

$$\Psi = (\psi_1, \ldots, \psi_n); \quad \chi = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_n \end{pmatrix} \tag{26}$$

and consider the reduction

$$\chi = A\overset{*}{\Psi}, \tag{27}$$

where A is the Hermitian matrix. Then we find the system

$$i\psi_{mt} = \psi_{mxx} + U\psi_m, \quad m = 1, \ldots, n, \tag{28}$$

where $U = \Psi A \overset{*}{\Psi}$ is the real function.

By unitary transformation $\Psi \to S\Psi$ A may be reduced to the diagonal form: $A \to \varkappa_i\delta_{ik}$. Therefore after the gauge transformations at $n = 2$ we find that besides the vector Schrödinger equation there is only one integrable system

$$\begin{aligned} i\psi_{1t} &= \psi_{1xx} + (|\psi_1|^2 - |\psi_2|^2)\psi_1, \\ i\psi_{2t} &= \psi_{2xx} + (|\psi_1|^2 - |\psi_2|^2)\psi_2. \end{aligned} \tag{29}$$

If we assume in (6) that $c_1 = c$, $c_2 = -c$, $\alpha = -\beta = \gamma$ and change the variables

$$x \to \sqrt{c}x, \quad \psi_1 \to \frac{1}{\sqrt{2\alpha}}\varphi_1, \quad \psi_2 \to \frac{1}{\sqrt{2\alpha}}\overset{*}{\varphi}_2 \tag{30}$$

we find that φ_1, φ_2 obey the system of equations coinciding with (29).

Let us summarise the results. It is shown that the system (6) does not have any additional motion invariant having the expansion in powers of ψ_1, ψ_2 with the quadratic part of the form (15) and hence cannot be solved by the inverse scattering method. There are two exceptional cases

$$c_1 = c_2, \quad \alpha = \beta = \gamma, \tag{31}$$

$$c_1 = -c_2, \quad \alpha = -\beta = \gamma. \tag{32}$$

From our point of view the above consideration reveals the effectiveness of testing of the integrability of the nonlinear wave systems based on the concept of the degenerative dispersion laws. The results of such type of testing of some nonlinear systems being important from the viewpoint of their applications to physics are to be published elsewhere soon.

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