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TO THE THEORY OF INFINITELY DIFFERENTIABLE SEMIGROUPS OF OPERATORS

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Abstract. Given a linear relation (multivalued linear operator) with certain growth restrictions on the resolvent, an infinitely differentiable semigroup of operators is constructed. It is shown that the initial linear relation is a generator of this semigroup. The results obtained are intimately related to certain results in the monograph "Functional analysis and semi-groups" by Hille and Phillips.

§1. Introduction

Let X be a complex Banach space and $\mathcal{B}(X)$ the Banach algebra of bounded linear operators on X. By a semigroup of operators, we mean a strongly continuous operator-valued function $T : \mathbb{R}_+ = (0,\infty) \to \mathcal{B}(X)$ such that $T(t+s) = T(t)T(s)$ for all $t, s \in \mathbb{R}_+$. A semigroup T is said to be *degenerate* if its kernel $\operatorname{Ker} T = \bigcap_{t>0} \operatorname{Ker} T(t)$ is a nonzero subspace of X. For such semigroups, a generator can be introduced as a linear relation (multivalued linear operator). In §2, a summary of the theory of linear relations is presented. The main results of this paper are about the construction of infinitely differentiable semigroups generated by a given linear relation. These results are closely linked with similar statements in [1, §12.2].

Infinitely differentiable semigroups of operators are used in the study of linear differential inclusions of the form

$$\dot{x}(t) \in \mathcal{A}x(t), \quad t \geq 0,$$

where $\mathcal{A}$ is a linear relation on a Banach space X, i.e., a linear subspace of the Cartesian product $X \times X$. Differential inclusions arise naturally in the study of differential equations with a noninvertible operator at the derivative in Banach spaces. The techniques of passage from such differential equations to differential inclusions was widely used in the monograph [2] containing numerous examples (see also the monograph [3]). Differential inclusions often lead to degenerate semigroups of operators, and the problem of defining their generators arises. In this paper, we use the following definitions of a generator of an operator semigroup $T : \mathbb{R}_+ \to \mathcal{B}(X)$ (these definitions were introduced in [4]). Below, A_0 stands for the infinitesimal generating operator for T (see [1]) and, after identification with its graph, A_0 is viewed as a linear relation on X.

Definition 1.1. *The senior generator of a semigroup* T is a relation $\mathbb{A} \in LR(X)$ consisting of the pairs $(x,y) \in X \times X$ that satisfy the following conditions:

1) $x \in \overline{\operatorname{Im} T}$;
2) $T(t)x - T(s)x = \int_s^t T(\tau)y \, d\tau$ for all $0 < s \leq t < \infty$.

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Definition 1.2. An arbitrary relation $\mathcal{A} \in LR(X)$ satisfying the conditions
1) $A_0 \subset \mathcal{A} \subset \mathbb{A}$,
2) $\mathcal{A}$ commutes with $T(t)$, $t > 0$ (see §2)
is called a generator of the semigroup $T : \mathbb{R}_+ \to \mathcal{B}(X)$. A generator $\mathcal{A}$ is said to be *basic* if the resolvent set $\rho(\mathcal{A})$ includes the half-plane $\mathbb{C}_\omega = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq \omega\}$ for some $\omega \in \mathbb{R}$.

This definition of a generator makes it possible to avoid additional restrictions on the behavior of a semigroup near zero (for instance, in [1, Chapter 12] semigroups of class A were considered). Therefore, the main result of this paper (see Theorem 3.1) is obtained under much looser conditions than the corresponding results in [1] (see Theorem 12.7.1 therein).

§2. Some information about linear relations

We present the most widely used definitions and results of the theory of linear relations. They can be found in the monographs [2, 3, 5] and in the paper [6].

Definition 2.1. A linear subspace $\mathcal{A}$ of the Cartesian product $X \times X$ is called a *linear relation* on a Banach space X. If $\mathcal{A}$ is closed, it is called a *closed linear relation*.

The set of all linear relations on X is denoted by $LR(X)$, and the set of all closed linear relations is denoted by $LCR(X)$. The set $LO(X)$ of linear operators acting in X is regarded as a subset of $LR(X)$ by identification of an operator with its graph. Thus, $\mathcal{B}(X) \subset LO(X) \subset LR(X)$.

The subspace $D(\mathcal{A}) = \{x \in X : \exists y \in X \text{ with } (x,y) \in \mathcal{A}\}$ is called the *domain* of $\mathcal{A} \in LR(X)$. For $x \in D(\mathcal{A})$, we denote by $\mathcal{A}x$ the set $\{y \in X : (x,y) \in \mathcal{A}\}$. Next, $\operatorname{Ker}\mathcal{A} = \{x \in D(\mathcal{A}) : (x,0) \in \mathcal{A}\}$ is the *kernel* of $\mathcal{A}$, and $\operatorname{Im}\mathcal{A} = \{y \in X : \exists x \in D(\mathcal{A})$ with $(x,y) \in \mathcal{A}\} = \bigcup_{x \in D(\mathcal{A})} \mathcal{A}x$ is the *range* of $\mathcal{A}$.

For $\mathcal{A} \in LR(X)$, the set $\mathcal{A}0$ is a linear subspace of X, and for all $x \in D(\mathcal{A})$ and $y \in \mathcal{A}x$ we have $\mathcal{A}x = y + \mathcal{A}0$.

The *sum* $\mathcal{A} + \mathcal{B}$ of two relations $\mathcal{A}, \mathcal{B} \in LR(X)$ is defined by $\mathcal{A} + \mathcal{B} = \{(x,y) \in X \times X : x \in D(\mathcal{A}) \cap D(\mathcal{B}),\ y \in \mathcal{A}x + \mathcal{B}x\}$, where $\mathcal{A}x + \mathcal{B}x$ is the algebraic sum of the sets $\mathcal{A}x$ and $\mathcal{B}x$.

The *inverse* to a linear relation $\mathcal{A} \subset X \times X$ is defined by $\mathcal{A}^{-1} = \{(y,x) \in X \times X : (x,y) \in \mathcal{A}\}$.

A relation $\mathcal{A} \in LR(X)$ is said to be *injective* if $\operatorname{Ker}\mathcal{A} = \{0\}$, and *surjective* if $\operatorname{Im}\mathcal{A} = X$.

Definition 2.2. A relation $\mathcal{A} \in LR(X)$ is said to be *continuously invertible* if it is injective and surjective; then $\mathcal{A}^{-1} \in \mathcal{B}(X)$ provided $\mathcal{A}$ is closed.

Definition 2.3. The *resolvent set* of a relation $\mathcal{A} \in LR(X)$ is the set $\rho(\mathcal{A}) = \{\lambda \in \mathbb{C} : (\mathcal{A} - \lambda I)^{-1} \in \mathcal{B}(X)\}$. The *spectrum* of $\mathcal{A} \in LR(X)$ is the set $\sigma(\mathcal{A}) = \mathbb{C} \setminus \rho(\mathcal{A})$.

For $\mathcal{A} \in LR(X)$, the resolvent set $\rho(\mathcal{A})$ is open and the spectrum $\sigma(\mathcal{A})$ is closed.

Definition 2.4. The mapping

$$R(\cdot, \mathcal{A}) : \rho(\mathcal{A}) \to \mathcal{B}(X), \quad R(\lambda, \mathcal{A}) = (\mathcal{A} - \lambda I)^{-1}, \quad \lambda \in \rho(\mathcal{A}),$$

is called the *resolvent* of the relation $\mathcal{A} \in LR(X)$.

It should be noted that the resolvent of an arbitrary relation $\mathcal{A} \in LR(X)$ is a pseudoresolvent in a usual sense (see [1, §4.8]), and therefore it satisfies the Hilbert identity

$$R(\lambda, \mathcal{A}) - R(\mu, \mathcal{A}) = (\lambda - \mu) R(\lambda, \mathcal{A}) R(\mu, \mathcal{A}).$$

Definition 2.5. The *extended spectrum* $\widetilde{\sigma}(\mathcal{A})$ of a linear relation $\mathcal{A} \in LR(X)$ is a subset of the extended complex plane $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$; this subset coincides with $\sigma(\mathcal{A})$ if $\mathcal{A} \in \mathcal{B}(X)$ and with $\widetilde{\sigma}(\mathcal{A}) \cup \{\infty\}$ otherwise.

Theorem 2.1 (see [6])**.** *For a relation $\mathcal{A} \in LR(X)$, the extended spectrum of $\mathcal{A}^{-1}$ is representable in the form*

$$\widetilde{\sigma}(\mathcal{A}^{-1}) = \{\lambda^{-1} : \lambda \in \widetilde{\sigma}(\mathcal{A})\}.$$

Corollary 2.1. *If $\mathcal{A} \in LR(X)$ and $\mu \in \rho(\mathcal{A})$, then*

$$\sigma(R(\mu, \mathcal{A})) = \{(\mu - \lambda)^{-1} : \lambda \in \widetilde{\sigma}(\mathcal{A})\}.$$

The *adjoint relation* $\mathcal{A}^* \in LCR(X)$ consists of all pairs $(\xi, \eta) \in X^* \times X^*$ (X^* is the conjugate of X) such that $\eta(x) = \zeta(y)$ for all $(x, y) \in \mathcal{A}$. Clearly, $\mathcal{A}^* 0 = \{\eta \in X^* : \eta(x) = 0 \text{ for all } x \in D(\mathcal{A})\}$.

Consider a relation $\mathcal{A} \in LR(X)$ with $\rho(\mathcal{A})$ nonempty. A closed subspace $X_0 \subset X$ is said to be *invariant* for $\mathcal{A}$ if X_0 is invariant for all operators $R(\lambda, \mathcal{A})$ with $\lambda \in \rho(\mathcal{A})$. The *restriction* of a relation $\mathcal{A}$ to its invariant subspace X_0 is the relation $\mathcal{A}_0 \in LR(X)$, the resolvent of which is the restriction to X_0 of the resolvent $R(\cdot, \mathcal{A}) : \rho(\mathcal{A}) \to \mathcal{B}(X)$, that is, the mapping $R_0 : \rho(A) \to \mathcal{B}(X_0)$ defined by $R_0(\lambda) = R(\lambda, \mathcal{A})|X_0$, $\lambda = \rho(\mathcal{A})$. We use the notation $\mathcal{A}_0 = \mathcal{A}|X_0$.

An operator $B \in \mathcal{B}(X)$ is said to *commute* with a relation $\mathcal{A} \in LR(X)$ if $(Bx, By) \in \mathcal{A}$ whenever $(x, y) \in \mathcal{A}$.

§3. Construction of infinitely differentiable semigroups

In this section, we present the main results of the paper. We consider the class of linear relations in $LCR(X)$ that have a resolvent whose behavior is controlled by functions in the following class.

Definition 3.1 (see [1])**.** A function $\psi : \mathbb{R} \to \mathbb{R}$ is attributed to the class Ψ if it satisfies the following conditions:

(i) ψ is positive, continuously differentiable, and monotone nondecreasing as $|\tau|$ grows;
(ii) $\psi(\tau) \to \infty$ as $|\tau| \to \infty$;
(iii) $\psi'(\tau)$ is bounded;
(iv) $\int_{-\infty}^{\infty} e^{-t\psi(\tau)}\, d\tau < \infty$ for every $t > 0$.

In particular, (iv) is fulfilled if $\lim_{|\tau| \to \infty} \frac{\psi(\tau)}{\ln |\tau|} = \infty$.

Lemma 3.1. *Suppose $\mathcal{A} \in LR(X)$ is such that $\rho(\mathcal{A})$ includes the half-plane $\mathbb{C}_{\omega_0} = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \omega_0\}$ and*

$$\|R(\lambda, \mathcal{A})\| \le (1 + |\lambda|)^{\alpha}, \quad \lambda \in \mathbb{C}_{\omega_0}, \tag{3.1}$$

for some $\alpha > 0$. If $\gamma > \omega_0$ and $x \in D(\mathcal{A}^{[\alpha]+2})$ ($[\alpha]$ is the integral part of α), then the formula

$$y(t, x) = -\lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda t} R(\lambda, \mathcal{A}) x \, d\lambda \tag{3.2}$$

defines a function continuous for $t \ge 0$; moreover, $y(0, x) = x$ and

$$R(\lambda, \mathcal{A}) x = \int_0^{\infty} e^{-\lambda t} y(t, x) \, dt, \quad \operatorname{Re} \lambda > \gamma. \tag{3.3}$$

Proof. Since $x \in D(\mathcal{A}^m)$, $m = [\alpha]+2$, we have $x = R(\lambda_0, \mathcal{A})^m x_0$ for some $x_0 \in X$, where λ_0 is a point in $\mathbb{C}_{\omega_0}$ with $\operatorname{Re}\lambda < \gamma$. The Hilbert identity implies the relation

$$R(\lambda, \mathcal{A})x = -\frac{x}{\lambda - \lambda_0} - \sum_{n=1}^{m-1} \frac{R(\lambda_0, \mathcal{A})^{m-n} x_0}{(\lambda - \lambda_0)^{n+1}} + \frac{R(\lambda, \mathcal{A})x_0}{(\lambda - \lambda_0)^m}.$$

It follows that

$$-\frac{1}{2\pi i}\int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda t} R(\lambda, \mathcal{A})x\, d\lambda = e^{\lambda_0 t}\left(x + \sum_{n=1}^{m-1} \frac{t^n}{n!} R(\lambda_0, \mathcal{A})^{m-n} x_0\right) - \frac{1}{2\pi i}\int_{\gamma - i\omega}^{\gamma + i\omega} e^{\lambda t} \frac{R(\lambda, \mathcal{A})^m x_0}{(\lambda - \lambda_0)^m}\, d\lambda.$$

Passing to the limit as $\omega \to \infty$ and observing that the integral on the right in the last formula converges absolutely by (3.1), we obtain

$$y(t, x) = e^{\lambda_0 t}\left(x + \sum_{n=1}^{m-1} \frac{t^n}{n!} R(\lambda_0, \mathcal{A})^{m-n} x_0\right) - \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} \frac{R(\lambda, \mathcal{A})x_0}{(\lambda - \lambda_0)^m}\, d\lambda.$$

Consequently, the function $y(t, x)$ is continuous for $t \geq 0$, and we have $y(0, x) = x$. The above formula for $y(t, x)$ implies that

$$\begin{aligned} \int_0^\infty e^{-\lambda t} y(t, x)\, dt = &-\frac{x}{\lambda - \lambda_0} - \sum_{n=1}^{m-1} \frac{R(\lambda, \mathcal{A})^{m-n} x_0}{(\lambda - \lambda_0)^{n+1}} \\ &+ \frac{1}{2\pi i}\int_0^\infty e^{-\lambda t}\left(\int_{\gamma - i\infty}^{\gamma + i\infty} e^{\mu t} \frac{R(\mu, \mathcal{A})x_0}{(\lambda - \lambda_0)^m}\, d\mu\right) dt. \end{aligned} \tag{3.4}$$

The double integral converges absolutely; therefore, changing the order of integration and using the residue calculus, we arrive at the identity

$$\frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{(\mu - \lambda)t} \frac{R(\mu, \mathcal{A})x_0}{(\mu - \lambda)(\lambda - \lambda_0)^m}\, d\mu = \frac{R(\lambda, \mathcal{A})x_0}{(\lambda - \lambda_0)^m}.$$

Thus, the right-hand side of (3.4) coincides with $R(\lambda, \mathcal{A})$. □

Lemma 3.2. *If the resolvent set $\rho(\mathcal{A})$ of a relation $\mathcal{A} \in LR(X)$ includes a sequence (λ_n) with $\lim_{n\to\infty} \|R(\lambda_n, \mathcal{A})\| = 0$, then $\overline{D(\mathcal{A}^m)} = \overline{D(\mathcal{A})}$ for $m \geq 2$.*

Proof. We show that $\overline{D(\mathcal{A}^n)} = \overline{D(\mathcal{A}^{n+1})}$ for $n \geq 2$. For any $x \in D(\mathcal{A}^{n-1})$, the Hilbert identity implies that

$$R(\lambda_0, \mathcal{A})x - (\lambda_k - \lambda_0)R(\lambda_k, \mathcal{A})R(\lambda_0, \mathcal{A})x = R(\lambda_k, \mathcal{A})x.$$

Passing to the limit as $k \to \infty$, we see that $D(\mathcal{A}^n) \subset \overline{D(\mathcal{A}^{n+1})}$. Consequently, $\overline{D(\mathcal{A}^n)} \subset \overline{D(\mathcal{A}^{n+1})}$ for every n; since $D(\mathcal{A}^n) \supset D(\mathcal{A}^{n+1})$ for every n, we arrive at the claimed identity.

Now, let x be an arbitrary vector in $D(\mathcal{A})$, and let $m > 2$. By the above, there exists a sequence $(x_n^{(1)})$ in $D(\mathcal{A}^2)$ with $\lim_{n\to\infty} x_n^{(1)} = x$. Next, there exists a sequence $(x_n^{(2)})$ in $D(\mathcal{A}^3)$ with $\lim_{n\to\infty} \|x_n^{(2)} - x_n^{(1)}\| = 0$, and so on. Continuing in this way, we arrive at a sequence (y_n) in $D(\mathcal{A}^m)$ with $\lim_{n\to\infty} \|y_n - x_n^{(1)}\| = 0$. Therefore, $\lim_{n\to\infty} y_n = x$, whence we see that $D(\mathcal{A}) \subset \overline{D(\mathcal{A}^m)}$; consequently, $\overline{D(\mathcal{A})} = \overline{D(\mathcal{A}^m)}$. □

Theorem 3.1. *Suppose that a relation $\mathcal{A} \in LR(X)$ satisfies the following conditions*:

1) $\rho(\mathcal{A}) \supset \mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0\}$;

2) *there exists a function* ψ *in* Ψ *and a constant* $M > 0$ *such that*

$$\|R(i\tau, \mathcal{A})\| \leq \frac{M}{\psi(\tau)}, \quad \tau \in \mathbb{R};$$

3) *the resolvent of* $\mathcal{A}$ *satisfies*

$$\|R(\lambda, \mathcal{A})\| \leq M_1(1 + |\lambda|)^{\alpha}, \quad \lambda \in \mathbb{C}_+,$$

for some $\alpha \geq -1$ *and* $M_1 > 0$.

Then $\mathcal{A}$ *is the basic generator of the infinitely differentiable semigroup* $T : \mathbb{R}_+ \to \mathcal{B}(X)$ *defined by*

$$T(t) = -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\lambda t} R(\lambda, \mathcal{A})\, d\lambda, \quad t > 0, \tag{3.5}$$

where $a > -a_0$, $a_0 = \inf_{\tau \in \mathbb{R}} \frac{M}{\psi(\tau)}$. *The integral in* (3.5) *converges in the principal value sense. Moreover, we have*

$$\|T(t)\| \leq M_2 e^{at}, \quad t > 0, \tag{3.6}$$

with $M_2 > 0$.

Proof. Put

$$T(t, a) = -\frac{1}{2\pi i} \int_{a-i\omega}^{a+i\omega} e^{\lambda t} R(\lambda, \mathcal{A})\, d\lambda, \tag{3.7}$$

where $a > -a_0$. The required semigroup T will be constructed as the limit as $\omega \to \infty$ of a family of operators of the form (3.7) in the uniform operator topology.

We use the identity

$$\sigma(R(i\tau, \mathcal{A})) = \Big\{ \frac{1}{i\tau - \lambda} : \lambda \in \widetilde{\sigma}(\mathcal{A}) \Big\}, \quad \tau \in \mathbb{R}$$

(see Corollary 2.1) and the estimate

$$\|R(i\tau, \mathcal{A})\| \geq r(R(i\tau, \mathcal{A})) = \sup_{\lambda \in \widetilde{\sigma}(\mathcal{A})} \frac{1}{|i\tau - \lambda|}.$$

Taking condition 2) into account, we obtain

$$\frac{M}{\psi(\tau)} \geq \sup_{\lambda \in \widetilde{\sigma}(\mathcal{A})} \frac{1}{|i\tau - \lambda|} \geq \frac{1}{|i\tau - (\xi + i\tau)|} = \frac{1}{|\xi|}, \quad \xi = \operatorname{Re} \lambda,$$

where $\lambda = \xi + i\tau$ and $r(R(i\tau, \mathcal{A}))$ is the spectral radius of $R(i\tau, \mathcal{A})$. So, $|\xi| = |\mathrm{Re}\lambda| \geq \frac{\psi(\tau)}{M}$ for every $\lambda \in \sigma(\mathcal{A}) \subset \operatorname{Im} \lambda = \tau$. Thus, the domain $\mathbb{C}_{\psi,M} = \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq -\frac{\psi(\operatorname{Im} \lambda)}{M}\}$ bounded by the curve $\Gamma_{\psi,M} = \{\lambda \in \mathbb{C} : \xi = \operatorname{Re} \lambda = -\frac{\psi(\operatorname{Im} \lambda)}{M}\}$ includes the spectrum $\sigma(\mathcal{A})$. Therefore, $\sigma(\mathcal{A}) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \leq -a_0\}$.

Thus, the curve $\Gamma_{\psi,2M} = \{\lambda \in \mathbb{C} : \xi = \operatorname{Re} \lambda = -\frac{\psi(\operatorname{Im} \lambda)}{2M}\}$ lies in $\rho(\mathcal{A})$; moreover, for all $\lambda \in \Gamma_{\psi,2M}$ we have

$$R(\lambda, \mathcal{A}) = R(\xi + i\tau, \mathcal{A}) = \sum_{n=0}^{\infty} \xi^n R(i\tau, \mathcal{A})^{n+1}.$$

Since

$$\|\xi R(i\tau, \mathcal{A})\| \leq |\xi| \frac{M}{\psi(\tau)} \leq \frac{\psi(\tau)}{2M} \frac{M}{\psi(\tau)} = \frac{1}{2}$$

for every $\lambda \in \Gamma_{\psi,2M}$, $\lambda = \xi + i\tau$, it follows that

$$\|R(\lambda, \mathcal{A})\| \leq \sum_{n=0}^{\infty} |\xi|^n \Big(\frac{M}{\psi(\tau)} \Big)^{n+1} = \frac{2M}{\psi(\tau)}.$$

Starting with this point, we employ the approach to constructing operator semigroups that was described in [1, Theorem 12.7.1].

Consider the integral

$$\frac{1}{2\pi i}\int_{C_\alpha(\omega)} e^{\lambda t}R(\lambda,\mathcal{A})\,d\lambda, \tag{3.8}$$

where $C_\alpha(\omega)$ is the closed contour passing through the points $A=a+i\omega$, $B=-\alpha\psi(\omega)+i\omega$, $C=-\alpha\psi(-\omega)+i\omega$, and $D=a-i\omega$, where $0<\alpha<\frac{1}{M}$. The part BC of $C_\alpha(\omega)$ is an arc of the curve $\Gamma_{\psi,\alpha}$, and the other parts are rectilinear segments. The operator $T(t,a)$ is precisely the integral along the piece DA. We must show that the integrals over the horizontal segments of $C_\alpha(\omega)$ tend to zero as $\omega\to\infty$. Clearly, the norms of these integrals are dominated by

$$\frac{2M}{\psi(\pm\omega)}\int_{-\infty}^{a} e^{\sigma t}\,d\sigma=\frac{2M}{t\psi(\pm\omega)}e^{at},$$

and these quantities tend to 0 as $\omega\to\infty$, the convergence being uniform on any interval of the form $(\varepsilon,\frac{1}{\varepsilon})$, $\varepsilon>0$. The integral along BC tends to

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{(-\alpha\psi(\tau)+i\tau)t}R(-\alpha\psi(\tau)+i\tau,\mathcal{A})(-\alpha\psi'(\tau)+i)\,d\tau, \tag{3.9}$$

which is estimated by a constant multiple of

$$\int_{-\infty}^{\infty} e^{-\alpha\psi(\tau)t}\frac{1}{\psi(\tau)}\,d\tau. \tag{3.10}$$

By property (iv) of a function $\psi\in\Psi$, the integral (3.9) converges uniformly in t on any interval of the form $(\varepsilon,\frac{1}{\varepsilon})$, where $\varepsilon>0$. As a result, we obtain

$$\lim_{\omega\to\infty}\sup_{t\in(\varepsilon,\frac{1}{\varepsilon})}\|T(t,\omega)-T(t)\|=0,$$

and the function $T:\mathbb{R}_+\to\mathcal{B}(X)$ is representable by the absolutely convergent integral

$$T(t)=-\frac{1}{2\pi i}\int_{\Gamma_{\psi,\alpha}} e^{\lambda t}R(\lambda,\mathcal{A})\,d\lambda,\quad 0<\alpha<\frac{1}{M},\ \ t>0. \tag{3.11}$$

Formula (3.11) implies (3.6). Differentiating under the integral sign in (3.11), we obtain

$$T'(t)=-\frac{1}{2\pi i}\int_{\Gamma_{\psi,\alpha}} e^{\lambda t}\lambda R(\lambda,\mathcal{A})\,d\lambda,\quad t>0. \tag{3.12}$$

We show that the integral in (3.12) converges absolutely. By (3.9), for $\lambda=-\alpha\psi(\tau)+i\tau\in\Gamma_{\psi,\alpha}$, the norm of $R(\lambda,\mathcal{A})$ can be estimated as follows: $\|R(-\alpha\psi(\tau)+i\tau,\mathcal{A})\|\le\frac{2M}{\psi(\tau)}$. Thus, the norm of the right-hand side in (3.12) does not exceed a constant multiple of $\int_{-\infty}^{\infty} e^{-\alpha\psi(\tau)t}|\tau|\,d\tau$. Next,

$$\int_{-\infty}^{\infty} e^{-\alpha\psi(\tau)t}|\tau|\,d\tau\le\varphi\Big(\frac{1}{2}at\Big)\int_{-\infty}^{\infty} e^{-\frac{1}{2}\alpha\psi(\tau)t}|\tau|\,d\tau,$$

where $\varphi(s)=\max_{\tau\in\mathbb{R}}|\tau|e^{-s\psi(\tau)}$, $s>0$. The function $\varphi(s)$ is bounded for $s>0$ because

$$\frac{1}{2}\tau e^{-s\psi(\tau)}<\int_{\frac{\tau}{2}}^{\tau} e^{-s\psi(\mu)}\,d\mu,$$

and the right-hand side of this inequality tends to zero as $\tau\to\infty$ by property (iv) of the functions belonging to Ψ. Hence, the integral in (3.12) converges absolutely, and the function $T:\mathbb{R}_+\to\mathcal{B}(X)$ is differentiable.

The semigroup property $T(t+s) = T(t)T(s)$ is deduced from the usual properties of the holomorphic functional calculus (see, e.g., [1, Theorem 5.11.2] and [7, Chapter I, §5]). The semigroup T is infinitely differentiable by [1, Theorem 10.3.5] (that result claims that a differentiable operator semigroup is infinitely differentiable).

It remains to show that $\mathcal{A}$ is the basic generator of the semigroup T (in the sense of Definition 2.2). Let $(x,y) \in \mathcal{A}$. Then $(y,x) \in \mathcal{A}^{-1}$, and $\mathcal{A}^{-1} \in \mathcal{B}(X)$ because $0 \in \rho(\mathcal{A})$ by assumption. Consequently, $x = \mathcal{A}^{-1}y$. Since the integral in (3.11) converges absolutely, from (3.5) and the Hilbert identity we deduce that

$$\begin{aligned} T(t)x - T(s)x &= -\frac{1}{2\pi i}\int_{\Gamma_{\psi,\alpha}} (e^{\lambda t} - e^{\lambda s}) R(\lambda,\mathcal{A})\mathcal{A}^{-1} y\, d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_{\psi,\alpha}} \frac{e^{\lambda t} - e^{\lambda s}}{\lambda}(R(\lambda,\mathcal{A}) - \mathcal{A}^{-1}) y\, d\lambda \\ &= \int_s^t T(\tau) y\, d\tau, \quad 0 < s \le t < \infty. \end{aligned}$$

Thus, condition 2 of Definition 1.1 is verified.

For $x_0 \in \mathcal{D}(A_0)$, the function $\tau \mapsto T(\tau)x_0 : \overline{\mathbb{R}}_+ \to X$ is continuous and $T'(\tau)x_0 = T(\tau)A_0x_0$, $\tau > 0$. Integrating over the interval $[s,t]$, where $0 < s < t < \infty$, we arrive at

$$T(t)x_0 - T(s)x_0 = \int_s^t T(\tau)A_0x_0\, d\tau.$$

This means that $(x_0, \mathcal{A}x_0) \in \mathbb{A}$.

Now, we prove the inclusion $D(\mathcal{A}) \subset \overline{\operatorname{Im} T}$. Lemmas 3.1 and 3.2 show that $\overline{\mathcal{D}(\mathcal{A}^m)} = \overline{\mathcal{D}(A)}$ for sufficiently large $m \in \mathbb{N}$, and formulas (3.2), (3.3), and (3.6) imply that $D(\mathcal{A}) \subset \overline{\mathcal{D}(\mathcal{A}^m)} \subset \overline{\operatorname{Im} T}$. By (3.5), $T(t)$ commutes with $\mathcal{A}$ for $t > 0$. Indeed, if $(x,y) \in \mathcal{A}$, then $(y,x) \in \mathcal{A}^{-1} \in \mathcal{B}(X)$; therefore, $\mathcal{A}^{-1}y = x$. Since, $T(t)\mathcal{A}^{-1} = \mathcal{A}^{-1}T(t)$, $t > 0$, we see that $(T(t)y,\ T(t)x) \in \mathcal{A}^{-1}$, whence $(T(t)x,\ T(t)y) \in \mathcal{A}$ for $t > 0$, proving that $T(t)$ commutes with $\mathcal{A}$. Thus, $\mathcal{A}$ satisfies all conditions of Definition 1.2. □

Remark 3.1. The claim of Theorem 3.1 remains true for any relation $\mathcal{A} \in LR(X)$ whose resolvent set includes the half-plane $\mathbb{C}_\beta = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \ge \beta\}$ with some $\beta \in \mathbb{R}$ and whose resolvent admits the estimates $\|R(i\tau + \beta, \mathcal{A})\| \le \frac{M}{\psi(\tau)}$, $\tau \in \mathbb{R}$, for some $\psi \in \Psi$ and $\|R(\lambda,\mathcal{A})\| \le M_1(1+|\lambda|)^\alpha$, $\lambda \in \mathbb{C}_\beta$, with some $M_1 > 0$ and $\alpha \ge -1$. In this case (3.5) and (3.6) are valid for $a > -a_0$, where $a_0 = \inf_{\tau\in\mathbb{R}} \frac{M}{\psi(\tau)} - \beta$.

Theorem 3.2. *In addition to conditions* 1) *and* 2) *of Theorem* 3.1*, suppose the following*:

4) $\sup_{\lambda>0} \|\lambda R(\lambda,\mathcal{A})\| < \infty$; $\sup_{\operatorname{Re}\lambda\ge 0} \|R(\lambda,\mathcal{A})\| < \infty$;

5) *vectors in* $\mathcal{A}0$ *separate vectors in* $\mathcal{A}^*0$.

Then $X = \overline{D(\mathcal{A})} \oplus \operatorname{Ker} T$*, the subspace* $X_0 = \overline{D(\mathcal{A})}$ *is invariant under* $T(t)$, $t > 0$*, and the restriction* $T_0 : \mathbb{R}_+ \to \mathcal{B}(X)$ *of* T *to* X_0 (*i.e.,* $T_0(t) = T(s)|X_0$, $t > 0$) *is a semigroup of class* $(A)_\infty$ *in the terminology of* [1, §10.6]. *Here* T *is defined by* (3.5).

Proof. By [6, Theorem 4.3] (or [5, Theorem 5.4.14]), using condition 5 and the first part of condition 4 we obtain the decomposition $X = \overline{D(\mathcal{A})} \oplus \operatorname{Ker} T = \overline{D(\mathcal{A})} \oplus \mathcal{A}_0$; moreover, the restriction of T to X_0 is not a degenerate semigroup, and the restriction of T to $\operatorname{Ker} T$ is a zero semigroup. Furthermore, putting $\mathcal{A}_0 = \mathcal{A}|X_0$, we have $\sigma(\mathcal{A}_0) = \sigma(\mathcal{A})$ by the result mentioned above. But $\mathcal{A}$ is a linear operator and its resolvent satisfies the assumptions of Theorem 3.2 (because so does the resolvent of $\mathcal{A}_0$). Consequently, Theorem 12.7.1 in [1] is applicable to the resolvent of $\mathcal{A}$, whence we see that T_0 is a semigroup of class $(A)_\infty$. □

Corollary 3.1. *Under the assumptions of Theorem* 3.2*, if*

$$\int_{|\tau|\geq 1} |\tau\phi(\tau)|^{-1}\, d\tau < \infty,$$

then the limit $\lim_{t\to 0+} T(t)x$ *exists for* $x \in D(\mathcal{A})$ *and* $\lim_{t\to 0+} T(t)x_0 = x_0$ *for every* $x_0 \in D(\mathcal{A}_0)$.

Proof. All statements follow from the final part of the proof of Theorem 3.2 combined with [1, Theorem 12.7.1]. □

Remark 3.2. In [8], it was shown that the Cauchy problem for the differential inclusion

$$\dot{x}(t) \in \mathcal{A}x(t), \quad x(0) = x_0 \in X, \tag{3.13}$$

has a unique weak solution provided the linear relation $\mathcal{A}$ satisfies the following conditions:

1) $\mathbb{C}_\alpha \subset \rho(\mathcal{A})$ for some $\alpha \in \mathbb{R}$;

2) $\lim_{\mathrm{Re}\,\lambda\to\infty} \frac{\ln\|R(\lambda,\mathcal{A})}{\mathrm{Re}\,\lambda} = 0$;

3) there exists a monotone increasing function $\varphi : [\alpha,\infty) \to \mathbb{R}_+$ such that $\lim_{t\to\infty}\varphi(t) = \infty$ and $\lim_{|\lambda|\to\infty} \frac{\ln\|R(\lambda,\mathcal{A})\|}{|\lambda|} = 0$ for all $\lambda \in U(\varphi) = \{\lambda \in \mathbb{C}_\alpha\ :\ \mathrm{Re}\,\lambda \leq \varphi(\mathrm{Im}\,\lambda)\}$. By a weak solution we mean a function $x(t)$ continuous on the interval $[0,\infty)$, strongly continuously differentiable, satisfying (3.13) for all $t > 0$, and also obeying the initial condition $x(0) = x_0$ (x_0 may fail to belong to the domain of $\mathcal{A}$).

We observe that, under the assumptions of any statement of the present paper, the above conditions 1)–3) are fulfilled. This ensures the uniqueness of an operator semigroup describing a weak solution of a differential inclusion.

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