Toda Lattice as an Integrable System and the Uniqueness of Toda's Potential

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Treating the integral of Hénon's type, it is shown that Toda's potential is the only possible one to make the system integrable provided the assumption of the nearest neighbour interaction. This method can be applied to the one-dimensional pairing potential system, and the condition to make the system integrable can be written down explicitly.

§1. Introduction

For the Toda lattice, many works have been done and many results have been known so far.[1] Yet, it will have some meaning to show that the same results are derived by the different methods. Further it may cast light to the mathematical structure of the Toda lattice. Here we use the method which we have applied to the one-dimensional N-particle system with inversely quadratic pair potential.[2] And under the periodic boundary condition we treat the Hénon's integrals,[3] which have the advantage that the mutual functional independence is obvious. From this we can derive the integrability condition and deduce the fact that if we assume that the integrals are Hénon's type and potential interaction acts only between nearest neighbours, then only the Toda lattice is possible. This procedure can be applied also to the one-dimensional pairing potential case, where we assume $V(x_i-x_j)=V(|x_i-x_j|)$, and a part of Calogero's results are confirmed.

§2. Hénon's integrals[3] and the uniqueness of Toda's potential

We assume the nearest-neighbour potential interaction with periodic boundary condition. Thus we take as the Hamiltonian of the system,

$$H=\sum_i \frac{p_i^2}{2}+\frac{1}{2}\sum_i (V(x_i-x_{i+1})+V(x_i-x_{i-1})). \tag{1}$$

Naturally $V(x_i-x_{i+1})=V(x_{i+1}-x_i)$, where the ordering of suffices is to be taken into consideration. To include the Toda's potential, we do not assume that $V(x_i-x_{i+1})=V(|x_i-x_{i+1}|)$. This means that V is a real valued function on the interval $(-\infty, \infty)$. Further $V(x_i-x_{i+1})$ or $V(x_{i+1}-x_i)$

means simply the potential energy between the i-th particle and $(i+1)$-th which is a function of difference of coordinates of each particle. We use this notation intentionally to extend to the pair potential case, where potential terms contain $V(x_i-x_j)$ and $V(x_j-x_i)$. Defining the operator $\mathcal{H}$ acting on A, the function of set $(\{p_i\}, \{x_i\})$, by

$$\begin{aligned}\{H, A\} &\equiv \mathcal{H}A \\ &= \Big[\sum_i p_i \frac{\partial}{\partial x_i} - \frac{1}{2}\sum_i \Big\{V'(x_i-x_{i+1})\Big(\frac{\partial}{\partial p_i}-\frac{\partial}{\partial p_{i+1}}\Big) \\ &\qquad + V'(x_i-x_{i-1})\Big(\frac{\partial}{\partial p_i}-\frac{\partial}{\partial p_{i-1}}\Big)\Big\}\Big]A, \end{aligned} \tag{2}$$

where { , } means the usual Poisson bracket and $V'(x_i-x_j)=\partial/\partial x_i V(x_i-x_j)$, we write the general equation of motion as

$$\frac{d}{dt}A=\mathcal{H}A.$$

Because of $V'(x_i-x_{i-1})=-V'(x_{i-1}-x_i)$, we have

$$\mathcal{H}=\sum_i p_i\frac{\partial}{\partial x_i}-\sum_i\Big\{V'(x_i-x_{i+1})\frac{\partial}{\partial p_i}+V'(x_i-x_{i-1})\frac{\partial}{\partial p_i}\Big\}. \tag{4}$$

As easily seen the only possible form of the integral whose leading term is the product Πp of all p's, is given by $I_N=\exp(-1/2\rho)\Pi p$, cf. Eq. (5) below. Now we should seek for the condition for the potential to make I_N really a integral.

For this purpose we introduce the operator ρ defined by

$$\rho=\sum_i(V(x_i-x_{i+1})+V(x_{i+1}-x_i))\frac{\partial^2}{\partial p_i\partial p_{i+1}} \tag{5}$$

and have the following identities:

$$\begin{aligned}[\mathcal{H},\rho] &= \sum_i\Big\{V'(x_i-x_{i+1})(p_i-p_{i+1})\frac{\partial^2}{\partial p_i\partial p_{i+1}} \\ &\qquad + V'(x_i-x_{i-1})(p_i-p_{i-1})\frac{\partial^2}{\partial p_i\partial p_{i-1}}\Big\} \\ &\quad -\sum_i\Big\{(V(x_i-x_{i+1})+V(x_{i+1}-x_i))\frac{\partial^2}{\partial p_i\partial x_{i+1}} \\ &\qquad +(V(x_i-x_{i-1})+V(x_{i-1}-x_i))\frac{\partial^2}{\partial p_i\partial x_{i-1}}\Big\},\end{aligned} \tag{6}$$

$$[[\mathcal{H}, \rho], \rho] = -2\sum_i \Big\{ (V(x_i - x_{i+1}) + V(x_{i+1} - x_i)) \times V'(x_i - x_{i+1}) \left(\frac{\partial^3}{\partial p_i \partial p_{i+1}^2} - \frac{\partial^3}{\partial p_i^2 \partial p_{i+1}} \right)$$

$$+ (V(x_i - x_{i-1}) + V(x_{i-1} - x_i)) V'(x_i - x_{i-1}) \left(\frac{\partial^3}{\partial p_i \partial p_{i-1}^2} - \frac{\partial^3}{\partial p_i^2 \partial p_{i-1}} \right) \Big\}$$

$$-4\sum_i \Big\{ (V(x_{i-1} - x_i) + V(x_i - x_{i-1})) V'(x_i - x_{i+1}) \frac{\partial^3}{\partial p_i \partial p_{i+1} \partial p_{i-1}}$$

$$+ (V(x_{i+1} - x_i) + V(x_i - x_{i+1})) V'(x_i - x_{i-1}) \frac{\partial^3}{\partial p_i \partial p_{i-1} \partial p_{i+1}} \Big\} \tag{7}$$

and

$$[[[\mathcal{H}, \rho], \rho], \rho] = 0, \tag{8}$$

where [,] means the commutator.

From these identities we have

$$e^{\rho/2}\mathcal{H}e^{-\rho/2} = \mathcal{H} - \frac{1}{2}[\mathcal{H}, \rho] + \frac{1}{8}[[\mathcal{H}, \rho], \rho]$$

$$= \sum_i p_i \frac{\partial}{\partial x_i} - \sum_i \Big\{ V'(x_i - x_{i+1}) \frac{\partial}{\partial p_i} + V'(x_i - x_{i-1}) \frac{\partial}{\partial p_i} \Big\}$$

$$-\frac{1}{2}\sum_i \Big\{ V'(x_i - x_{i+1})(p_i - p_{i+1}) \frac{\partial^2}{\partial p_i \partial p_{i+1}} + V'(x_i - x_{i-1})(p_i - p_{i-1}) \frac{\partial^2}{\partial p_i \partial p_{i-1}} \Big\}$$

$$+\frac{1}{2}\sum_i \Big\{ (V(x_i - x_{i+1}) + V(x_{i+1} - x_i)) \frac{\partial^2}{\partial p_i \partial x_{i+1}} + (V(x_i - x_{i-1}) + V(x_{i-1} - x_i)) \frac{\partial^2}{\partial p_i \partial x_{i-1}} \Big\}$$

$$-\frac{1}{4}\sum_i \Big\{ (V(x_i - x_{i+1}) + V(x_{i+1} - x_i)) V'(x_i - x_{i+1}) \left(\frac{\partial^3}{\partial p_i \partial p_{i+1}^2} - \frac{\partial^3}{\partial p_i^2 \partial p_{i+1}} \right)$$

$$+ (V(x_i - x_{i-1}) + V(x_{i-1} - x_i)) V'(x_i - x_{i-1}) \left(\frac{\partial^3}{\partial p_i \partial p_{i-1}^2} - \frac{\partial^3}{\partial p_i^2 \partial p_{i-1}} \right) \Big\}$$

$$-\frac{1}{2}\sum_i \Big\{ (V(x_{i-1} - x_i) + V(x_i - x_{i-1})) V'(x_i - x_{i+1}) \frac{\partial^3}{\partial p_i \partial p_{i+1} \partial p_{i-1}}$$

$$+ (V(x_{i+1} - x_i) + V(x_i - x_{i+1})) V'(x_i - x_{i-1}) \frac{\partial^3}{\partial p_i \partial p_{i-1} \partial p_{i+1}} \Big\}. \tag{9}$$

From this we get

$$e^{1/2\rho}\mathcal{H}e^{-1/2\rho}\Pi p = -\sum_i (V'(x_i-x_{i+1})+V'(x_i-x_{i-1}))\Pi^{(i)}p$$
$$-\frac{1}{2}\sum_i (V'(x_i-x_{i+1})(p_i-p_{i+1})\Pi^{(i,i+1)}p$$
$$+V'(x_i-x_{i-1})(p_i-p_{i-1})\Pi^{(i,i-1)}p)$$
$$-\frac{1}{2}\sum_i \left(2\frac{\partial}{\partial x_i}\{V(x_{i-1}-x_i)V(x_i-x_{i+1})\}\Pi^{(i-1,i,i+1)}p\right), \quad (10)$$

where $\Pi^{(i)}p$, $\Pi^{(i,j)}p$ and $\Pi^{(i,j,k)}p$ are the product of all p's except p_i, (p_i, p_j) and (p_i, p_j, p_k) respectively. Because the operator $e^{\rho/2}$ is non-singular, the necessary and sufficient condition for I_N to be an integral is that (10) is vanishing identically. As easily seen the first sum cancels the second sum. So the identical vanishing of the last sum is the desired condition, and for this the only possible way is the vanishing of each term of the sum:

$$\frac{\partial}{\partial x_i}(V(x_{i-1}-x_i)V(x_i-x_{i+1}))=0. \quad (11)$$

The solution of this functional equation is given by following:

$$V(x_i-x_{i+1})=Ae^{B(x_i-x_{i+1})}, \quad (12)$$

where A and B are some constants. For the physical system we prefer that A is positive. Thus Toda's potential is the only possible one which has the integral $e^{-\rho/2}\,\Pi p$ of Hénon's type. Other integrals of Hénon's type can be derived by the following way. Noting

$$\left[\rho, \sum_i \frac{\partial}{\partial x_i}\right]=0, \quad (13)$$

$$\left[\sum_i \frac{\partial}{\partial p_i}, \mathcal{H}\right]=\sum_i \frac{\partial}{\partial x_i} \quad (14)$$

and defining

$$I_{N-n}=\left(\sum_i \frac{\partial}{\partial p_i}\right)^n I_N=\left(\sum_i \frac{\partial}{\partial p_i}\right)^n e^{-\rho/2}\Pi p, \quad (15)$$

we have

$$\mathcal{H}I_{N-n}=\mathcal{H}\left(\sum_i \frac{\partial}{\partial p_i}\right)^n I_N$$
$$=\left(\sum_i \frac{\partial}{\partial p_i}\right)\mathcal{H}\left(\sum_i \frac{\partial}{\partial p_i}\right)^{n-1} I_N+\left(\sum_i \frac{\partial}{\partial x_i}\right)\left(\sum_i \frac{\partial}{\partial p_i}\right)^{n-1} I_N$$
$$=\left(\sum_i \frac{\partial}{\partial p_i}\right)\mathcal{H}I_{N-n+1}+\left(\sum_i \frac{\partial}{\partial p_i}\right)^{n-1} e^{-\rho/2}\left(\sum_i \frac{\partial}{\partial x_i}\right)\Pi p$$

$$=\left(\sum_i \frac{\partial}{\partial p_i}\right)\mathcal{H}I_{N-n+1}. \tag{16}$$

Because of (16), using the induction argument we have

$$\mathcal{H}I_{N-n}=0, \qquad n=0,1,2,\cdots,N-1. \tag{17}$$

Thus these I's are the integrals. Further functional independence of these I's is guaranteed by the form of the leading terms which are the fundamental symmetric functions of p's.

§3. Generalization to 1-dimensional pairing potential

We take as the Hamiltonian of the system the following form:

$$\tilde{H}=\sum_i \frac{p_i^2}{2}+\frac{1}{2}(\sum_i \sum_j)' V(x_i-x_j), \tag{18}$$

where prime means $i \neq j$. Then corresponding to the equation (4), we have

$$\tilde{\mathcal{H}}=\sum_i p_i \frac{\partial}{\partial x_i}-(\sum_i \sum_j)' V'(x_i-x_j)\frac{\partial}{\partial p_i}. \tag{19}$$

Corresponding to the definition (5), we define $\tilde{\rho}$ by the following:

$$\tilde{\rho}=(\sum_i \sum_j)' V(x_i-x_j)\frac{\partial^2}{\partial p_i \partial p_j}. \tag{20}$$

Then as before we have the following identities:

$$\begin{aligned}[\tilde{\mathcal{H}},\tilde{\rho}]=&(\sum_i \sum_j)'\left\{V'(x_i-x_j)(p_i-p_j)\frac{\partial^2}{\partial p_i \partial p_j}\right\}\\ &-(\sum_i \sum_j)'\left\{(V(x_i-x_j)+V(x_j-x_i))\frac{\partial^2}{\partial p_i \partial x_j}\right\},\end{aligned} \tag{21}$$

$$\begin{aligned}[[\tilde{\mathcal{H}},\tilde{\rho}],\tilde{\rho}]=&-2(\sum_i \sum_j)'\Big\{(V(x_i-x_j)+V(x_j-x_i))\\ &\times V'(x_i-x_j)\left(\frac{\partial^3}{\partial p_i \partial p_j^2}-\frac{\partial^3}{\partial p_i^2 \partial p_j}\right)\Big\}\\ &-4(\sum_i \sum_j \sum_k)'\Big\{(V(x_i-x_k)+V(x_k-x_i))\\ &\times V'(x_i-x_j)\frac{\partial^3}{\partial p_i \partial p_j \partial p_k}\Big\},\end{aligned} \tag{22}$$

where the prime of the last sum means $i \neq j \neq k \neq i$, and

$$[[[\tilde{\mathcal{H}},\tilde{\rho}],\tilde{\rho}],\tilde{\rho}]=0. \tag{23}$$

From these identities we get

$$e^{\beta/2}\mathscr{H}e^{-\beta/2}\Pi p = -(\sum_i \sum_j)' V'(x_i - x_j)\Pi^{(i)}p$$
$$-\frac{1}{2}(\sum_i \sum_j)' V'(x_i - x_j)(p_i - p_j)\Pi^{(i,j)}p$$
$$-\frac{1}{2}(\sum_i \sum_j \sum_k)'\{(V(x_i - x_k) + V(x_k - x_i))V'(x_i - x_j)\Pi^{(i,j,k)}p\}$$
$$= -\frac{1}{2}(\sum_i \sum_j \sum_k)'\{(V(x_i - x_k) + V(x_k - x_i))V'(x_i - x_j)\Pi^{(i,j,k)}p\}. \tag{24}$$

So the following condition

$$\sum^c \{(V(x_i - x_k) + V(x_k - x_i))V'(x_i - x_j)\}$$
$$= \sum^c \{(V(x_i - x_k) - V(x_k - x_j))V'(x_i - x_j)\} = 0 \tag{25}$$

is the condition for $e^{-\beta/2}\Pi p$ to be an integral, where $\sum^c$ denotes the cyclic sum of i, j, k. The solution of this functional equation should satisfy the following equation:

$$\frac{\partial}{\partial x_i} V(x_i - x_j) = A_{ijk} + B_{ijk} V(x_i - x_j), \tag{26}$$

where A_{ijk} and B_{ijk} are invariant with respect to cyclic permutations of i, j, k. Substituting (26) into (25) we have a kind of algebraic addition formula. From the Weierstrass-Phragmén's theorem[5] the solution must be a rational function of elliptic functions, whose explicit form was derived by Calogero[4] using the inverse scattering method.

As before, other $N-1$ integrals are obtained by the following:

$$\left(\sum \frac{\partial}{\partial p_i}\right)^n e^{-\beta/2}\Pi p, \qquad n = 0, 1, 2, \cdots, N-1.$$

References

1) For example see the review work of M. Toda himself:
M. Toda, Phys. Rep. **18C** (1975), 1.
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