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TOEPLITZ OPERATORS ON THE SEGAL-BARGMANN SPACE

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ABSTRACT. In this paper, we give a complete characterization of those functions on $2n$-dimensional Euclidean space for which the Berezin-Toeplitz quantizations admit a symbol calculus modulo the compact operators. The functions in question are characterized by a condition of "small oscillation at infinity".

1. Introduction. We consider the Toeplitz operators on the Segal-Bargmann space $H^2(\mathbf{C}^n, d\mu)$ of Gaussian square-integrable entire functions on $\mathbf{C}^n$. Such operators have been studied by Berezin and others [**4**, **5**, **10**, **11**] and arise naturally as "anti-Wick quantization operators". Via the Schrödinger representation [**7**, **10**], there is a natural equivalence between Topelitz operators on $H^2(\mathbf{C}^n, d\mu)$ and a generalization of pseudodifferential operators on $L^2(\mathbf{R}^n, dv)$, the so-called Weyl quantization [**9**, **10**].

Let P be the orthogonal projection operator $L^2(\mathbf{C}^n, d\mu)$ onto $H^2(\mathbf{C}^n, d\mu)$ with $d\mu(z) = (2\pi)^{-n} e^{-|z|^2/2} dv(z)$ and $dv(z)$ ordinary Lebesgue measure on $\mathbf{C}^n$. For f in $L^\infty(\mathbf{C}^n)$, the multiplication operator M_f on $L^2(\mathbf{C}^n, d\mu)$ is defined by $M_f h = fh$. The *Toeplitz operator* T_f is defined, for h in $H^2(\mathbf{C}^n, d\mu)$, by

$$T_f h = PM_f h = P(fh).$$

In this paper, we complete the program, begun in [**6**], of determining the largest $*$-algebra Q in $L^\infty(\mathbf{C}^n)$ for which $T_f T_g - T_{fg}$ is a compact operator for all f, g in Q. Functions in Q are characterized by a condition of "small oscillation at infinity".

It should be noted that the Weyl unitary operators [**7**] which generate the Segal-Bargmann representation of the Heisenberg group on $\mathbf{C}^n$ [**1**, **14**, **15**] take the form $W_\lambda = T_{e_\lambda(z)\exp\{|\lambda|^2/4\}}$ for λ in $\mathbf{C}^n$, where [**6**] $e_\lambda(z) = \exp\{i \operatorname{Im} \overline{\lambda} \cdot z\}$ and $\overline{\lambda} \cdot z = \overline{\lambda}_1 z_1 + \overline{\lambda}_2 z_2 + \cdots + \overline{\lambda}_n z_n$ for $z = (z_1, z_2, \ldots, z_n)$ in $\mathbf{C}^n$ and $|\lambda|^2 = |\lambda_1|^2 + |\lambda_2|^2 + \cdots + |\lambda_n|^2$. Thus, the C^*-algebra $CCR(\mathbf{C}^n)$ generated by the $\{W_\lambda\}$ is just the closure, in the operator norm, of

$$\{T_f \colon f \text{ a trigonometric polynomial on } \mathbf{C}^n = \mathbf{R}^{2n}\}$$

[**6**]. Since $CCR(\mathbf{C}^n)$ is known to be simple [**7**], nonconstant trigonometric polynomials *cannot* be in Q. On the other hand, we shall see that Q contains all functions, such as $e^{i\sqrt{|z|}}$, which oscillate "less than linearly". We shall also show that Q is closely related to $CCR(\mathbf{C}^n)$ in a more direct way.

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For a precise statement of the main results, we require several definitions

$$\Gamma = \{f \in L^\infty(\mathbf{C}^n)\colon H_f \equiv (I-P)M_fP \text{ is compact}\},$$
$$B = \{f \in L^\infty(\mathbf{C}^n)\colon PM_fP \text{ is compact}\}.$$

We also have the function algebras:

$$ESV = \left\{ f \in L^\infty(\mathbf{C}^n)\colon \operatorname*{Lim}_{R\to\infty} \sup_{\substack{|z-w|\le 1\\ |z|\ge R}} |f(z)-f(w)| = 0 \right\},$$
$$V = \{f \in L^\infty(\mathbf{C}^n)\colon \operatorname*{Lim}_{|z|\to\infty} f(z) = 0\}$$
$$BC = \text{ Bounded continuous functions on } \mathbf{C}^n.$$

We ignore sets of measure zero in the above definitions of ESV and V. We write $BCESV = BC \cap ESV$ and $C_0 = BC \cap V$.

For f in $L^\infty(\mathbf{C}^n)$, we make use of the convolution transform

$$\tilde{f}(a) = (2\pi)^{-n} \int f(z) e^{-|z-a|^2/2}\, dv(z).$$

This transform is the Berezin symbol of the operator T_f [**3**] and is also the solution of the heat equation on $\mathbf{C}^n = \mathbf{R}^{2n}$ at time $t = \frac{1}{2}$ with initial values f [**4**, **8**, **10**].

Let $\mathcal{K}$ denote the ideal of all compact operators on the relevant Hilbert space, H. Let π be the usual quotient map from $B(H)$ onto $B(H)/\mathcal{K}$ where $B(H)$ is the algebra of all bounded operators on H. We denote by $\tau(Q)$ the C^*-algebra generated by all T_f with f in Q.

Our main results can now be summarized.

PROPOSITION A. *$Q = \{f \in L^\infty(\mathbf{C}^n)$: $(I-P)M_fP$ and $(I-P)M_{\bar{f}}P$ are compact$\}$. For f in Q, $T_gT_f - T_{gf}$ and $T_fT_g - T_{gf}$ are in $\mathcal{K}$ for all g in $L^\infty(\mathbf{C}^n)$. Q is the unique maximal $*$-subalgebra of $L^\infty(\mathbf{C}^n)$ with the property that $T_fT_g - T_{fg}$ is compact for all f, g in Q.*

THEOREM B. *$(I-P)M_fP$ is compact if and only if $(I-P)M_{\bar{f}}P$ is compact. Moreover, $Q = \Gamma = ESV + Q \cap B$.*

THEOREM C. *The ideal $Q \cap B$ is given by $Q \cap B = \{f \in L^\infty(C^n)\colon \widetilde{|f|^2} \in C_0\}$.*

THEOREM D. *The commutant of $\pi\{CCR(\mathbf{C}^n)\}$ in $B[H^2(\mathbf{C}^n, d\mu)]/\mathcal{K}$ is $\pi\{\tau(Q)\}$. Equivalently, $[A, W_\lambda]$ is in $\mathcal{K}$ for all λ in $\mathbf{C}^n$ if and only if $A - T_f$ is in $\mathcal{K}$ for some f in ESV.*

THEOREM E. $\pi\{\tau(Q)\} \simeq Q/Q \cap B \simeq ESV/V \simeq BCESV/C_0$.

It should be pointed out that the algebra Q is the homolog of the algebra QC of quasi-continuous functions in the case of Toeplitz operators on the unit circle. Moreover, Γ is the homolog of the algebra $H^\infty + C$. Of course, on the circle, $QC \neq H^\infty + C$. The absence of nonconstant bounded entire functions on $\mathbf{C}^n$ seems to be reflected in the fact that $Q = \Gamma$.

A critical ingredient in our analysis is an averaging operation over the Segal-Bargmann representation of the Heisenberg group given, for A in $B\{H^2(d\mu)\}$, by

$$\hat{A} = \int W_\lambda^* A W_\lambda\, d\mu(\lambda).$$

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In Theorem 6 of §3, we discuss some useful properties of $\hat{A}$ and relate $\hat{A}$ to the Berezin symbol $\tilde{A}$ [**3**].

We recall that $H^2(d\mu)$ has the reproducing kernels $e^{\bar{a}\cdot z/2}$ so, for g in $H^2(d\mu)$,

$$g(a) = \langle g, e^{\bar{a}\cdot z/2}\rangle = (2\pi)^{-n} \int g(z) e^{a\cdot \bar{z}/2} e^{-|z|^2/2}\, dv(z).$$

Normalizing, we have $k_a(z) = e^{\bar{a}\cdot z/2 - |a|^2/4}$ with $\|k_a\| = 1$ in $H^2(d\mu)$. In terms of the k_a, the Berezin symbol of any operator A on $H^2(d\mu)$ is defined by [**3**]

$$\tilde{A}(a) = \langle Ak_a, k_a\rangle.$$

It is known that $\tilde{A}(a)$ is a smooth function which is uniquely determined by A. Moreover, it is not hard to check that $\tilde{A}$ is in C_0 for A compact and, for all bounded A, $\hat{A} = T_{\tilde{A}}$ where $\hat{A}$ is the average over the Heisenberg group defined above.

Let $\tilde{f}^{(m)}$ denote the mth iterate of $\tilde{f}$. The main idea in the proofs of Theorems B, C, D, E is to note that ESV is characterized by $f - \tilde{f}^{(m)} \in V$ for all $m > 0$ and use the fact that $\tilde{f}^{(m)}$ is Lipschitz with modulus of continuity converging to 0 as $m \to \infty$. We also use the fact that $\int K(a) d\nu(a)$ is compact whenever $K(a)$ is a uniformly bounded weakly measurable compact operator valued function and ν is a positive measure of finite total mass.

We remark that T_f is bounded for f in a larger class than L^∞. In particular, $M_f P$ is bounded if f is measurable and $\widetilde{|f|^2}$ is bounded.

In §2 of this paper, we discuss some analytic preliminaries. The function-theoretic properties of $\tilde{f}$ are discussed and the class ESV is described in terms of $\tilde{f}$. In §3, Theorems B, C, D are proved. In §4, the algebra $\tau(Q)$ is analyzed using earlier results. The index theory of $\tau(Q)$ is described. Finally, in §5, we discuss extensions and generalizations.

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2. Preliminary results. We now discuss some analytic preliminaries. Beyond the definitions in §1, we will use the space

$$\Lambda(\varepsilon) = \{f \in BC\colon |f(a) - f(b)| \le \varepsilon |a - b|, \text{ all } a, b\}.$$

We note that Γ, B, Q, ESV, V, BC, C_0 are all closed. V is an ideal in L^∞ and C_0 is an ideal in BC. ESV and BC are conjugate-closed algebras. It is easy to check, as in [**6**], that Γ is an algebra and that B is a Γ module so that $\Gamma \cap B$ is an ideal in Γ.

We begin by sketching the proof of

PROPOSITION A. *$Q = \{f \in L^\infty\colon (I-P)M_f P$ and $(I-P)M_{\bar{f}}P$ are compact$\}$. For f in Q, $T_g T_f - T_{gf}$ and $T_f T_g - T_{gf}$ are in $\mathcal{K}$ for all g in L^∞. Q is the unique maximal $*$-subalgebra of $L^\infty(\mathbf{C}^n)$ with the property that $T_f T_g - T_{fg}$ is in $\mathcal{K}$ for all f, g in Q.*

PROOF. See [**6**]. For completeness, note that

$$\begin{aligned} PM_{gf}P &= PM_g\{P + (I-P)\}M_f P \\ &= (PM_gP)(PM_fP) + PM_g\{(I-P)M_fP\}. \end{aligned}$$

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It follows at once that $T_gT_f - T_{gf}$ is in $\mathcal{K}$ for f in Γ and g in L^∞. On the other hand

$$\{(I-P)M_gP\}^*\{(I-P)M_gP\} = PM_{|g|^2}P - (PM_{\bar{g}}P)(PM_gP)$$

so $T_{\bar{g}}T_g - T_{|g|^2}$ is in $\mathcal{K}$ if and only if g is in Γ. The desired result follows at once.

For $d\mu(z) = (2\pi)^{-n}e^{-|z|^2/2}\,dv(z)$ on $\mathbf{C}^n$, we recall that the subspace $H^2(d\mu)$ consists of all entire functions in $L^2(d\mu)$. For g in $H^2(d\mu)$, we have the reproducing kernels $e^{\bar{a}\cdot z/2}$ with

$$g(a) = \langle g(z), e^{\bar{a}\cdot z/2}\rangle = (2\pi)^{-n}\int g(z)e^{a\cdot\bar{z}/2}e^{-|z|^2/2}\,dv(z).$$

If P is the orthogonal projection operator from $L^2(d\mu)$ onto $H^2(d\mu)$ it follows that, for b in $L^2(d\mu)$

$$(Pb)(z) = (2\pi)^{-n}\int b(w)e^{z\cdot\bar{w}/2}e^{-|w|^2/2}\,dv(w).$$

Denoting by M_f the operator of "multiplication by f" on $L^2(d\mu)$, we will need to estimate the norms of $[M_f, P] = M_fP - PM_f$ and $PM_{|f|^2}P$. Such estimates can be obtained by using the unitary map from $L^2(d\mu)$ to $L^2((2\pi)^{-n}\,dv)$ given by

$$(Ug)(z) = e^{-|z|^2/4}g(z).$$

PROPOSITION 1. *For b in $L^2((2\pi)^{-n}\,dv)$,*

$$U[M_f, P]U^*b(z) = (2\pi)^{-n}\int k(f,z,w)b(w)\,dv(w),$$
$$UPM_{|f|^2}PU^*b(z) = (2\pi)^{-n}\int h(|f|^2,z,w)b(w)\,dv(w)$$

where

$$k(f,z,w) = [f(z)-f(w)]\exp\{-|z-w|^2/4 + i\operatorname{Im}\bar{w}\cdot z/2\},$$
$$h(|f|^2,z,w) = (2\pi)^{-n}e^{-|z-w|^2/8} \times \int |f(u)|^2\exp\left\{-\left|u-\left(\frac{z+w}{2}\right)\right|^2\Big/2 + i\operatorname{Im}\left(\frac{z-w}{2}\cdot\bar{u}\right)\right\}dv(u).$$

PROOF. Direct calculation.

For f in $L^\infty(\mathbf{C}^n)$, we consider some properties of the convolution transform

$$\tilde{f}(a) = (2\pi)^{-n}\int f(z)e^{-|z-a|^2/2}\,dv(z).$$

We denote by $\tilde{f}^{(m)}$ the mth iterate of this transform. The map $f\to\tilde{f}$ is a smoothing operator which is clearly related to the heat equation on $\mathbf{C}^n = \mathbf{R}^{2n}$. In fact,

$$\tilde{f}(t,a) = (4\pi t)^{-n}\int f(z)e^{-|z-a|^2/4t}\,dv(z)$$

is the unique solution of the heat equation with initial values (at $t=0$) $f(z)$ [**8**]. Thus, $\tilde{f}(a) = \tilde{f}(\frac{1}{2}, a)$ is the solution of the initial value problem for $f(z)$ at $t = \frac{1}{2}$.

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We will need one estimate

LEMMA 2. *For f in L^∞, we have*

$$|\tilde{f}^{(m)}(a) - \tilde{f}^{(m)}(b)| \le 2(2\pi)^{-1/2}\|f\|_\infty m^{-1/2}|a-b|.$$

PROOF. The first step is to note that

$$\tilde{f}(a) - \tilde{f}(b) = (2\pi)^{-n}\int f\left(z + \frac{a+b}{2}\right)\left[e^{-|z-(a-b)/2|^2/2} - e^{-|z+(a-b)/2|^2/2}\right] dv(z).$$

It follows that

$$|\tilde{f}(a) - \tilde{f}(b)| \le (2\pi)^{-n}\|f\|_\infty \int |e^{-|z-(a-b)/2|^2/2} - e^{-|z+(a-b)/2|^2/2}|\, dv(z).$$

Careful but routine analysis shows that the right-hand side of the last inequality is exactly equal to

$$2(2\pi)^{-1/2}\|f\|_\infty \int_{-|a-b|/2}^{+|a-b|/2} e^{-x^2/2}\, dx.$$

It is immediate that

$$|\tilde{f}(a) - \tilde{f}(b)| \le 2(2\pi)^{-1/2}\|f\|_\infty |a-b|.$$

Using the semigroup property of the heat kernel (or direct calculation) we see that

$$\tilde{f}^{(m)}(a) = \tilde{f}(m/2, a) = (2\pi m)^{-n}\int f(z) e^{-|z-a|^2/2m}\, dv(z).$$

It follows that $\tilde{f}^{(m)}(a) = \tilde{g}(a/\sqrt{m})$ with $g(z) = f(z\sqrt{m})$. The desired result follows by applying the Lipschitz estimate above to $\tilde{g}$.

In view of the central role played by the algebra ESV in our analysis, we next provide some useful examples.

THEOREM 3. *The algebra ESV includes* (i) $\hat{g}(z) = g(z/|z|)$ *for g continuous complex-valued on* $S^{2n-1} = \{z : |z| = 1\}$, (ii) $f(|z|)$ *for f in BC_rESV (see* [**6**]), (iii) $\{V + \lambda 1 \colon \lambda \in \mathbf{C}\}$.

PROOF. (i) can be checked directly, using the uniform continuity of g on S^{2n-1}. We note that, for $|z-w| \le 1$,

$$\left|\frac{z}{|z|} - \frac{w}{|z|}\right| \le \frac{1}{|z|}, \qquad \left|\frac{w}{|z|} - \frac{w}{|w|}\right| \le \frac{|w|\,||w|-|z||}{|w|\,|z|} \le \frac{1}{|z|},$$

so that

$$\left|\frac{z}{|z|} - \frac{w}{|w|}\right| \le \frac{2}{|z|}.$$

(ii) follows directly from the corresponding definition of BC_rESV [**6**] as the radial version of $BCESV$ defined above.

(iii) is immediate from the definition of ESV.

REMARK. It follows from Theorem 3 and discussion in [**6**] that $\exp(i\sqrt{|z|})$ is in ESV. On the other hand, $\exp(i\,\mathrm{Im}(\overline{\lambda}\cdot z))$ is not in ESV unless $\lambda = 0$.

The following lemma exhibits the strong interaction between ESV and the transform $\tilde{f}$.

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LEMMA 4. *For f in ESV, $f - \tilde{f}$ is in V.*

PROOF. We write

$$f(a) - \tilde{f}(a) = (2\pi)^{-n} \int [f(a) - f(z)] e^{-|z-a|^2/2}\, dv(z)$$
$$= (2\pi)^{-n} \int [f(a) - f(a+z)] e^{-|z|^2/2}\, dv(z).$$

Thus, for $\varepsilon > 0$ and $N = N(\varepsilon)$ large enough

$$(2\pi)^{-n} \int_{|z| \geq N} |f(a) - f(a+z)| e^{-|z|^2/2}\, dv(z)$$
$$\leq (2\pi)^{-n} 2\|f\|_\infty \int_{|z|\geq N} e^{-|z|^2/2}\, dv(z) < \frac{\varepsilon}{2}$$

and

$$|f(a) - \tilde{f}(a)| < \frac{\varepsilon}{2} + (2\pi)^{-n} \int_{|z|<N} |f(a) - f(a+z)| e^{-|z|^2/2}\, dv(z).$$

Now, using the definition of ESV, there is an $R(\varepsilon) > 0$ so that $|f(a) - f(a+z)| < \varepsilon/2$ for $|z| < N$ whenever $|a| > R(\varepsilon)$. The desired result follows at once.

We can now establish

THEOREM 5. *The following conditions are equivalent*

(i) $f \in ESV$,

(ii) $f - \tilde{f} \in V$,

(iii) $f - \tilde{f}^{(m)} \in V$ *for all* $m \geq 1$,

(iv) $f \in \bigcap_{\varepsilon>0}(\Lambda(\varepsilon) + V)$.

PROOF. ((i)→(ii)) If f is in ESV then $f - \tilde{f}$ is in V by Lemma 4.

((ii)→(iii)) If $f - \tilde{f} \in V$ then, by Theorem 3 and Lemma 4, $f - \tilde{f}^{(2)} \in V$. Iteration and addition show that $f - \tilde{f}^{(m)} \in V$ for all $m \geq 1$.

((iii)→(iv)) Suppose $f - \tilde{f}^{(m)} \in V$ for all $m \geq 1$. By Lemma 2, $f \in \bigcap_{\varepsilon>0}(\Lambda(\varepsilon) + V)$.

((iv)→(i)) Suppose $f \in \bigcap_{\varepsilon>0}(\Lambda(\varepsilon)+V)$. Then for each $\varepsilon > 0$ we have $f = g_\varepsilon + h_\varepsilon$ for g_ε in $\Lambda(\varepsilon)$ and h_ε in V. Suppose that $|h_\varepsilon(z)| < \varepsilon$ whenever $|z| > R(\varepsilon)$. Then

$$|f(a) - f(b)| \leq |g_{\varepsilon/3}(a) - g_{\varepsilon/3}(b)| + |h_{\varepsilon/3}(a)| + |h_{\varepsilon/3}(b)|$$
$$\leq (\varepsilon/3)|a - b| + |h_{\varepsilon/3}(a)| + |h_{\varepsilon/3}(b)|.$$

Thus, for $|a-b| \leq 1$ and $|a| > R(\varepsilon/3)+1$, we have $|b| > R(\varepsilon/3)$ and $|h_{\varepsilon/3}(a)| < \varepsilon/3$, $|h_{\varepsilon/3}(b)| < \varepsilon/3$ so $|f(a) - f(b)| < \varepsilon$. Thus, f is in ESV.

COROLLARY. *The following conditions are equivalent:*

(i) f *is in* $BCESV$,

(ii) $f - \tilde{f} \in C_0$,

(iii) $f - \tilde{f}^{(m)} \in C_0$ *for all* $m \geq 1$,

(iv) $f \in \bigcap_{\varepsilon>0}(\Lambda(\varepsilon) + C_0)$.

PROOF. Clear.

REMARK. Theorem 5 implies that the class ESV has some significance in the classical analysis of the initial value problem for the heat equation.

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3. The symbol calculus for Toeplitz operators. We begin with a discussion of the Berezin symbol [**3**] and a related averaging operation over a representation of the Heisenberg group. This averaging operation appears to be of some independent interest and is extremely useful in our subsequent analysis.

On $H^2(d\mu)$, we have the unitary operator-valued map

$$a \to W_a = e^{iT_{\mathrm{Im}(\bar{a}\cdot z)}} = T_{\exp\{|a|^2/4+i\,\mathrm{Im}(\bar{a}\cdot z)\}}$$

for a in $\mathbf{C}^n$ (see [**6**]). The map $a \to W_a$ extends to a map from $\mathbf{C}^n$ to unitary operators acting on $L^2(d\mu)$ by the formula [**1, 6**] $W_a = k_a(z)t_a$ where $k_a(z) = \exp\{\bar{a}\cdot z/2 - |a|^2/4\}$ and $(t_a f)(z) = f(z-a)$ for f in $L^2(d\mu)$. For $e_a(z) = \exp\{i\,\mathrm{Im}(\bar{a}\cdot z)\}$, we also consider the unitary operator-valued map $a \to M_{e_a}$ on $L^2(d\mu)$.

It is not hard to check that both $a \to W_a$ and $a \to M_{e_a}$ are weakly continuous on $L^2(d\mu)$. Using the identities

$$W_a W_b = e_a(b/2)W_{a+b}, \quad M_{e_a}M_{e_b} = M_{e_{a+b}},$$

it follows that $a \to W_a$, $a \to W_a^*$, $a \to M_{e_a}$, $a \to M_{e_a}^*$ are all strongly continuous maps.

For α, β complex numbers of modulus one, we now have representations of the Heisenberg group on $L^2(d\mu)$ and $H^2(d\mu)$ via $(\alpha, a) \to \alpha W_a$ and the foregoing identities. Note that the multiplication law for the Heisenberg group is just

$$(\alpha, a)\cdot(\beta, b) = (\alpha\beta e_a(b/2), a+b).$$

Of course, as is well known [**7**], the representation on $L^2(d\mu)$ is reducible while the representation on $H^2(d\mu)$ is irreducible.

For A a bounded operator on $L^2(d\mu)$ or $H^2(d\mu)$, we can now define an averaging operation by

$$\hat{A} = \int W_a^* A W_a \, d\mu(a).$$

Note that the integrand is strongly continuous in a and uniformly bounded for each fixed A. For a discussion of such integrals, see [**7**]. We note that $\hat{A}$ is determined by

$$\langle \hat{A}f, g\rangle = \int \langle W_a^* A W_a f, g\rangle \, d\mu(a).$$

Recall that we defined

$$\tilde{f}(\lambda) = (2\pi)^{-n}\int f(z)e^{-|z-\lambda|^2/2}\, dv(z).$$

On $H^2(d\mu)$ we have the Berezin symbol [**3**]

$$\tilde{A}(\lambda) = \langle Ak_\lambda, k_\lambda\rangle$$

for any bounded operator A. It was shown in [**3**] that $\tilde{A}(\lambda)$ is always a bounded smooth function which A determines *uniquely* (for any polynomials p, q, $\langle Ap, q\rangle$ is obtained by evaluating appropriate derivatives of $\tilde{A}(\lambda)$ at $\lambda = 0$). Since $\{k_\lambda\}$ converges weakly to 0 as $|\lambda| \to \infty$, it is easy to see that for A in $\mathcal{K}$ (a compact operator) $\tilde{A}(\lambda)$ is in C_0. It is easy to check [**3**] that $\tilde{T}_f = \tilde{f}$.

The relation between $\hat{A}$ and $\tilde{A}$ can now be determined.

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THEOREM 6. *We have $\hat{A} = T_{\tilde{A}}$ for all bounded operators A on $H^2(d\mu)$. The map $A \to \hat{A}$ is a 1-1 norm-decreasing positive linear map from all bounded operators to Toeplitz operators with symbols in BC. We have $\hat{T}_f = T_{\tilde{f}}$ on $H^2(d\mu)$ while, on $L^2(d\mu)$, $\hat{M}_f = M_{\tilde{f}}$ and $\widehat{AP} = \hat{A}P$, $\widehat{PA} = P\hat{A}$.*

PROOF. By direct calculation $W_a k_\lambda = e_a(\lambda/2)k_{\lambda+a}$ so $\tilde{\hat{A}} = \tilde{T}_{\tilde{A}}$ and $\hat{A} = T_{\tilde{A}}$. That $A \to \hat{A}$ is 1-1 follows from the unicity of $\tilde{A}$ and the fact that the symbol of a Toeplitz operator uniquely determines the operator ($T_f = 0$ if and only if $f = 0$ [**6**]). The remaining observations are checked easily.

REMARK. Theorem 6 shows that $A \to \hat{A}$ is almost a conditional expectation from all bounded operators to Toeplitz operators with BC symbols. Using $\hat{T}_f = T_{\tilde{f}}$, it is clear that repeated application of ˆ increasingly smooths the symbol by Lemma 2. This property of ˆ will be used in what follows and should have other applications. The well-known irreducibility of the $\{W_a\colon a \in \mathbf{C}^n\}$ on $H^2(d\mu)$ follows easily from Theorem 6 since $[A, W_a] = 0$ for all a implies $A = \hat{A} = T_{\tilde{A}}$ so $\tilde{A} = \tilde{A}^{(2)}$ and, by iteration, $\tilde{A} = \tilde{A}^{(m)}$. Lemma 2 then implies that $\tilde{A}(\lambda)$ is a constant function so that A is a scalar multiple of I.

Using the Berezin symbol, it is easy to show

THEOREM 7. *For f in B, $\tilde{f}$ is in C_0.*

PROOF. Recall that $\langle T_f k_\lambda, k_\lambda\rangle = \tilde{T}_f(\lambda) =' \tilde{f}(\lambda)$. Now $k_\lambda \rightharpoonup 0$ (weakly) as $|\lambda| \to \infty$ so compactness of T_f implies that $T_f k_\lambda \to 0$ (strongly) and so $\tilde{f} \in C_0$.

We also have

THEOREM 8. *$ESV \cap B = V$.*

PROOF. It is a direct calculation in [**6**] that for f in L^∞ with compact support, $f \in B$. It follows from the fact that B is closed that $V \subset B$ and, hence, $V \subset ESV \cap B$.

For the converse, suppose $f \in ESV \cap B$. By Lemma 4, $f - \tilde{f} \in V$ while Theorem 7 implies $\tilde{f} \in C_0$. It follows immediately that $f \in V$.

We also have

LEMMA 9. *$V \subset Q \cap B$.*

PROOF. By a direct operator-theoretic argument

$$\Gamma \cap B = \{f\colon |f|^2 \in B\} = Q \cap B.$$

Moreover, $f \in V$ if and only if $|f|^2 \in V$. By Theorem 8, $V \subset B$ so, for f in V, $|f|^2$ is in B and f is in $Q \cap B$.

We will need

LEMMA 10. *$f \in Q$ if and only if $[M_f, P]$ is compact.*

PROOF. If $[M_f, P]$ is compact then $(I - P)M_f P$ and $(I - P)M_{\bar{f}} P$ are also compact so f is in $\Gamma \cap \overline{\Gamma} = Q$.

For the converse, note that for f in Q we have $(I - P)M_f P$ and $(I - P)M_{\bar{f}} P$ compact. Hence, $PM_f(I - P)$ is compact so

$$(I - P)M_f P - PM_f(I - P) = [M_f, P]$$

is compact.

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Recall that U is the unitary transformation from $L^2(d\mu)$ onto $L^2((2\pi)^{-n}\,dv)$ given by

$$(Ug)(z) = e^{-|z|^2/4} g(z).$$

As customary, $\mathcal{K}$ denotes the ideal of compact operators.

THEOREM 11. $ESV \subset Q$.

PROOF. If f is in ESV then, by Theorem 5, $f = g_\varepsilon + h_\varepsilon$ with g_ε in $\Lambda(\varepsilon)$ and h_ε in V. By Proposition 1,

$$U[M_{g_\varepsilon}, P]U^* b(z) = (2\pi)^{-n} \int k(g_\varepsilon, z, w) b(w)\, dv(w)$$

and

$$|k(g_\varepsilon, z, w)| \le \varepsilon |z - w| \exp\{-|z-w|^2/4\}$$

so

$$|U[M_{g_\varepsilon}, P]U^* b(z)| \le \varepsilon (2\pi)^{-n} \int |z-w| e^{-|z-w|^2/4} |b(w)|\, dv(w).$$

Let

$$(Bb)(z) = (2\pi)^{-n} \int e^{-|z-w|^2/4} |z-w| b(w)\, dv(w).$$

Then B is a bounded convolution operator. In fact,

$$\|B\| = (2\pi)^{-n} \int e^{-|w|^2/4} |w|\, dv(w).$$

It follows that

$$\|[M_{g_\varepsilon}, P]\| \le \varepsilon \|B\|.$$

Recall that, by Lemma 9, $M_{h_\varepsilon} P$ and $P M_{h_\varepsilon}$ are compact operators. It follows that

$$\|[M_f, P] + \mathcal{K}\| \le \varepsilon \|B\|$$

and, since $\varepsilon > 0$ is arbitrary, that $[M_f, P]$ is compact. An application of Lemma 10 completes the proof.

REMARK. It should be pointed out that Theorem 11 can also be obtained as an application of results in [**11**].

Suppose that X is a Borel space with ν a positive measure on X and $\nu(X)$ finite. Suppose further that $A(x)$ is a weakly measurable function on X with range contained in the bounded operators on a separable Hilbert space H. Recall that $\int A(x)\, dv(x) = A$ is a bounded operator on H defined, for f, g in H, by

$$\langle Af, g\rangle \equiv \int \langle A(x) f, g\rangle\, d\nu(x).$$

The next lemma is essential for our analysis. We thank William Zame for this simplified variant of our original version.

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LEMMA 12. *If* $\|A(x)\| \leq M$ *and* $A(x)$ *is a compact operator for all* x *in* X *then* $\int A(x)\, d\nu(x)$ *is also compact.*

PROOF. For $\langle e_k \colon k = 1, 2, 3, \dots \rangle$ an orthonormal basis for H, write P_k for the orthogonal projection operator with range spanned by $\langle e_1, e_2, \dots, e_k \rangle$. Clearly, $P_k A(x) P_k - A(x)$ is weakly measurable. Given $\varepsilon > 0$,

$$E_k = \{x \in X \colon \|P_k A(x) P_k - A(x)\| < \varepsilon\}$$

is a measurable set since

$$\|P_k A(x) P_k - A(x)\| = \sup_{f, g \in D} \langle [P_k A(x) P_k - A(x)] f, g \rangle$$

for D a dense countable subset of the unit ball of H.

Note that $\bigcup_{k \geq 1} E_k = X$ since $A(x)$ is compact for all x in X. We define

$$E'_k = E_k \Big\backslash \bigcup_{j=1}^{k-1} E'_j, \qquad E'_1 = E_1,$$

so that the E'_k are measurable and disjoint with

$$\bigcup_{k \geq 1} E_k = \bigcup_{k \geq 1} E'_k = X.$$

Since $\nu(X)$ is finite, there is an m so that $\sum_{k > m} \nu(E'_k) < \varepsilon / M$. We now have

$$\begin{aligned} \int_X A(x)\, d\nu(x) = & \sum_{k=1}^{m} \int_{E'_k} P_k A(x) P_k \, d\nu(x) \\ & + \sum_{k=1}^{m} \int_{E'_k} [A(x) - P_k A(x) P_k]\, d\nu(x) \\ & + \int_{\bigcup_{k>m} E'_k} A(x)\, d\nu(x). \end{aligned}$$

The last two terms on the right have norms less than $\varepsilon \nu(X)$ and ε respectively while the first term has range contained in the range of P_m. Since $\varepsilon > 0$ was arbitrary, the desired result follows immediately.

THEOREM 13. *We have* $\Gamma \subset ESV + Q \cap B$.

PROOF. Note that

$$Q \cap B = \{f \colon |f|^2 \in B\} = \Gamma \cap B.$$

We will show, for f in Γ, that $f - \tilde{f}$ is in $Q \cap B$ and that $\tilde{f}$ is in ESV.

Since $T_{\tilde{f}} = \hat{T}_f$ by Theorem 6, it follows that

$$T_{f - \tilde{f}} = \int [T_f, W_a^*] W_a P \, d\mu(a)$$

on $H^2(d\mu)$. Writing $H_f = (I - P) M_f P$ and letting

$$d\tilde{\mu}(a) = e^{|a|^2/4}\, d\mu(a),$$

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direct calculation shows that

$$T_{f-\tilde{f}} = \int PM_{e_{-a}}H_fW_aP\,d\tilde{\mu}(a) - H^*_{\tilde{f}}\int M_{e_{-a}}PW_aP\,d\tilde{\mu}(a).$$

The fact that $d\tilde{\mu}$ has finite total mass and Lemma 12 together with the fact that H_f is compact for f in Γ allow us to conclude that the first integral is a compact operator.

Next, we compute

$$\begin{aligned} J &= \int M_{e_{-a}}PW_a\,d\tilde{\mu}(a) \\ &= \int M_{e_{-a}}Pk_a(z)t_aP\,d\tilde{\mu}(a) \\ &= \int e_{-a}e^{\bar{a}\cdot z/2}t_aP\,d\mu(a) \\ &= \int e^{a\cdot\bar{z}/2}t_aP\,d\mu(a) = P. \end{aligned}$$

The last step uses Fubini's theorem and the fact that the $d\mu(a)$ integral of an analytic function of a is just the constant term in the McLaurin expansion. It follows that

$$H^*_{\tilde{f}}\int M_{e_{-a}}PW_aP\,d\tilde{\mu}(a) = H^*_{\tilde{f}}J = PM_f(I-P)P = 0$$

so the second integral in the expression for $T_{f-\tilde{f}}$ is zero and $T_{f-\tilde{f}}$ is compact. Hence $f-\tilde{f}$ is in B.

Next, using the fact that $[W_a, P] = 0$ on $L^2(d\mu)$ and Theorem 6, it is not hard to see for $H_f = (I-P)M_fP$ that $\hat{H}_f = H_{\tilde{f}}$. It follows from Lemma 12, that for f in Γ, since H_f is compact, $\hat{H}_f$ must be compact and so $\tilde{f}$ is in Γ. Thus, $f-\tilde{f}$ is in $\Gamma\cap B = Q\cap B$.

Since $f-\tilde{f}$ is in B, $\tilde{f}-\tilde{f}^{(2)}$ is in C_0 by Theorem 7. It follows immediately from Theorem 5 that $\tilde{f}$ is in ESV.

Finally, we have the characterization

THEOREM B. $\Gamma = ESV + Q\cap B = Q$.

PROOF. Combining Theorems 11 and 13 we see that $\Gamma\subset ESV + Q\cap B\subset Q$. But $Q = \Gamma\cap\overline{\Gamma}\subset\Gamma$ so the inclusions above must be equalities.

Next, we establish a useful relation between $|f(z)|^2$ and $\widetilde{|f|^2}(a)$ where, as earlier

$$\tilde{f}(a) = (2\pi)^{-n}\int f(z)e^{-|z-a|^2/2}\,dv(z).$$

We have

LEMMA 14. $\left\|\widetilde{|f|^2}\right\|_\infty \le \left\|PM_{|f|^2}P\right\| \le 4^n\left\|\widetilde{|f|^2}\right\|_\infty.$

PROOF. Note first that for $k_\lambda(z) = e^{\overline{\lambda}\cdot z/2-|\lambda|^2/4}$ we have

$$\langle PM_{|f|^2}Pk_\lambda, k_\lambda\rangle = \widetilde{|f|^2}(\lambda).$$

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Since $\|k_\lambda\|_2 = 1$, it follows that

$$\|PM_{|f|^2}P\| \geq \left\|\widetilde{|f|^2}\right\|_\infty .$$

The remaining estimate is more subtle. Using Proposition 1, we find that for b in $L^2((2\pi)^{-n}\,dv)$

$$UPM_{|f|^2}PU^*b(z) = (2\pi)^{-n}\int h(|f|^2, z, w)b(w)\,dv(w)$$

where

$$\begin{aligned}|h(|f|^2, z, w)| &\leq (2\pi)^{-n}e^{-|z-w|^2/8}\int |f(u)|^2 e^{-|u-(z+w)/2|^2/2}\,dv(u)\\ &\leq e^{-|z-w|^2/8}\left\|\widetilde{|f|^2}\right\|_\infty .\end{aligned}$$

It follows that

$$\begin{aligned}\left|UPM_{|f|^2}PU^*b(z)\right| &\leq \left\|\widetilde{|f|^2}\right\|_\infty (2\pi)^{-n}\int e^{-|z-w|^2/8}|b(w)|\,dv(w)\\ &\leq \left\|\widetilde{|f|^2}\right\|_\infty (A|b|)(z)\end{aligned}$$

where

$$(Ab)(z) = (2\pi)^{-n}\int e^{-|z-w|^2/8}b(w)\,dv(w).$$

Thus we have

$$\left\|UPM_{|f|^2}PU^*b\right\|_2 \leq \|A\| \left\|\widetilde{|f|^2}\right\|_\infty \|b\|_2$$

and so

$$\|PM_{|f|^2}P\| \leq \|A\| \left\|\widetilde{|f|^2}\right\|_\infty .$$

An easy computation shows that the convolution operator A has $\|A\| = 4^n$ and the desired estimate follows.

REMARK. Using the lemma above, it is not hard to check that $\|M_fP\|$ is finite if and only if $\widetilde{|f|^2}$ is bounded *even if f is not in L^∞*. We will return to this point in the last section of this paper.

We can now give a complete characterization of $Q \cap B$.

THEOREM C. $Q\cap B = \{f \in L^\infty : \widetilde{|f|^2} \in C_0\}$.

PROOF. If f is in $Q\cap B$ then M_fP is compact so $(M_fP)^*(M_fP) = PM_{|f|^2}P$ is compact and $\langle PM_{|f|^2}Pk_\lambda, k_\lambda\rangle = \widetilde{|f|^2}(\lambda)$ is in C_0.

For the converse, suppose $\widetilde{|f|^2}$ is in C_0. By Lemma 9, $V \subset Q\cap B$. Let χ_ρ be the characteristic function of $\{z : |z| > \rho\}$. Then

$$M_fP = M_{f\chi_\rho}P + M_{f(1-\chi_\rho)}P$$

and $f(1-\chi_\rho)$ is in V so $M_{f(1-\chi_\rho)}P$ is compact. Hence, for M_fP to be compact (and f to be in $Q\cap B$) it suffices to show that $\mathrm{Lim}_{\rho\to\infty}\|M_{f\chi_\rho}P\| = 0$. Using Lemma 14, it is enough to check for $g_\rho(\lambda) = \widetilde{\chi_\rho|f|^2}(\lambda)$ that $\mathrm{Lim}_{\rho\to\infty}\|g_\rho\|_\infty = 0$. Note that the functions g_ρ are in C_0 since $\widetilde{|f|^2}$ is in C_0. Moreover, the g_ρ are nonnegative

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with $g_{\rho'}(a) \leq g_\rho(a)$ whenever $\rho' > \rho$ and $\mathrm{Lim}_{\rho\to\infty} g_\rho(a) = 0$ for each a in $\mathbf{C}^n$. An elementary variant of Dini's Theorem completes the proof.

We have seen that V is in $Q \cap B$ while, for f in $Q \cap B$, $\widetilde{|f|^2} \in C_0$. This raises the question of whether $Q \cap B$ is larger than V. The anser is "yes" by a construction which exhibits a somewhat surprising property of the heat equation.

EXAMPLE. On $\mathbf{C}$, let D_j be the open unit disc of radius j^{-1} centered at j for $j = 1, 2, 3, \ldots$ along the real axis. Let $N = \bigcup_j D_j$ and let f be a continuous nonnegative real-valued function on $\mathbf{C}$ with $0 \leq f(z) \leq 1$, $f(j) = 1$ ($j = 1, 2, 3, \ldots$) and support (f) contained in N. Then f is clearly not in V but $\tilde{f}$ and $\widetilde{f^2}$ *are* in C_0 by straightforward estimates.

Finally, we have

THEOREM D. *For A a bounded operator on $H^2(d\mu)$, $[A, W_a]$ is compact for all a in $\mathbf{C}^n$ if and only if $A - T_f$ is compact for some f in ESV.*

PROOF. If $A - T_f$ is compact for f in ESV then Proposition A, Theorem 11, and the fact that W_a is a Toeplitz operator (discussed earlier) imply that $[A, W_a]$ is compact for all a.

For the converse, note by Theorem 6, $\hat{A} = T_f$ with $f = \tilde{A}(\lambda)$. Thus, we have

$$A - T_f = A - \hat{A} = \int (A - W_a^* A W_a)\, d\mu(a) = \int [A, W_a^*] W_a \, d\mu(a).$$

By Lemma 12 and the fact that $W_a^* = W_{-a}$, we see that the last integral is a compact operator. It follows that $\tilde{A} - \tilde{T}_f = f - \tilde{f}$ is in V. It follows from Theorem 5 that f is in ESV.

4. The algebra $\tau(Q)$. We now use the analysis of §3 to determine the structure of $\tau(Q)$. We first identify some function algebra relations which are implicit in §3.

THEOREM 15. *There are C^*-algebra isomorphisms*

$$Q/Q \cap B \simeq ESV/V \simeq BCESV/C_0.$$

PROOF. Direct consequence of Theorem B ($Q = ESV + Q \cap B$) and Theorem 8 ($ESV \cap B = V$). We also use the fact that $f - \tilde{f}$ is in V for f in ESV so that $ESV = BCESV + V$.

We use the standard notation of $\tau(X)$ for the C^*-algebra generated by $\{T_f : f \in X\}$. We now have

THEOREM 16. *$\tau(Q)$ contains $\mathcal{K}$ and the map $\psi(f) = \pi(T_f)$ induces a $*$-isomorphism between $BCESV/C_0$ and $\tau(Q)/\mathcal{K}$.*

PROOF. Note that for χ_ρ the characteristic function of $\{z : |z| > \rho\}$ we have, as $\rho \to \infty$, $T_{f(1-\chi_\rho)} \rightharpoonup T_f$ weakly for all f in L^∞. Since $f(1 - \chi_\rho)$ is in V it follows that $\tau(L^\infty)$ is contained in the weak closure of $\tau(V)$. But $\tau(L^\infty)$ contains $\{W_a : a \text{ in } \mathbf{C}^n\}$ (see §1) and this set is irreducible by an earlier remark. It follows that $\tau(L^\infty)$, $\tau(Q)$, and $\tau(V)$ are also irreducible. Since $\tau(V) \subset \mathcal{K}$, it follows from standard C^*-algebra results that $\mathcal{K} = \tau(V)$.

Next, using Proposition A and Theorems 13 and 15, we see that $\tau(Q)$ is the closure of

$$S = \{T_f + K : f \in BCESV,\ K \in \mathcal{K}\}.$$

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For f in $BCESV$, let $\psi(f) = \pi(T_f)$. Then ψ is a $*$-homomorphism from $BCESV$ *onto* $\tau(Q)/\mathcal{K}$ since the image of any $*$-homomorphism is closed. It follows that $\tau(Q) = S$ and that

$$BCESV/\ker\psi \simeq \tau(Q)/\mathcal{K}.$$

Finally, we note that, by Theorem 8,

$$\begin{aligned}\ker\psi &= \{f \in BCESV : T_f \in \mathcal{K}\} = BCESV \cap B \\ &= BC \cap ESV \cap B = BC \cap V = C_0.\end{aligned}$$

Combining Theorems 15 and 16, we have

THEOREM E. $\pi\{\tau(Q)\} \simeq Q/Q \cap B \simeq ESV/V \simeq BCESV/C_0$.

In the rest of this section, we consider the Fredholm theory and index problem for $\tau(Q)$. The following lemma and Theorem 18 appear in [**12**].

LEMMA 17. *For f in $BCESV$, the following conditions are equivalent:* (i) *there is a g in $BCESV$ with $gf - 1$ in C_0, and* (ii) *for some $R > 0$ there is an $m > 0$ with $|f(z)| \geq m$ for all z with $|z| \geq R$.*

PROOF. (i)$\rightarrow$(ii). Suppose $gf - 1 = h \in C_0$. If there is a sequence $\{z_k\}$ with $|z_k| \to \infty$ and $|f(z_k)| < \varepsilon$ for each $\varepsilon > 0$, then

$$|1 + h(z_k)| = |g(z_k)|\,|f(z_k)| \leq \varepsilon\|g\|_\infty$$

and, for ε small and k large, we have a contradiction.

(ii)$\rightarrow$(i). There are two cases depending on the dimension of $\mathbf{C}^n$.

CASE 1. $n = 1$. If $f(z)$ has winding number r on $|z| = R$ then the function $f(z)(\bar{z}/|z|)^r$ on $|z| \geq R$ extends to $F(z)$ on $\mathbf{C}^1$ with $|F(z)| \geq m' > 0$ and F in $BCESV$ by a standard homotopy argument. It follows that $1/F$ is in $BCESV$ and $(1/F)f = (z/|z|)^r$ for $|z| \geq R$. Let

$$G(z) = \begin{cases} (\bar{z}/|z|)^r, & |z| \geq R, \\ (\bar{z}/R)^r, & |z| < R. \end{cases}$$

Then $(G/F)f - 1 = 0$ for $|z| \geq R$ and $g = G/F$ will do.

CASE 2. $n > 1$. Here, $f(z)$ on $|z| \geq R$ extends to a continuous $F(z)$ on $\mathbf{C}^n$ with $|F(z)| \geq m' > 0$ and $F(z)$ in $BCESV$ by a standard homotopy argument. It follows that $1/F$ is in $BCESV$ and, for $g = 1/F$, $gf - 1 = 0$ on $|z| \geq R$.

Now let $\sigma(x)$ denote the spectrum of x for x in any Banach algebra with identity. We will be concerned with the abelian C^*-algebra $BCESV/C_0$.

THEOREM 18. *For f in $BCESV$ and $[f]$ the class of f in $BCESV/C_0$, we have*

$$\sigma([f]) = \bigcap_{R>0} \operatorname{closure}[f(z : |z| \geq R)].$$

PROOF. Recall that λ is in $\sigma([f])$ if and only if there is no $[g]$ in $BCESV/C_0$ with $[g][f - \lambda 1] = [1]$, or, equivalently, if and only if there is no g in $BCESV$ with $g(f - \lambda 1) - 1 \in C_0$. The desired result follows immediately from Lemma 17.

COROLLARY 1. *$\sigma([f])$ is connected for all $[f]$ in $BCESV/C_0$.*

PROOF. By Theorem 18, $\sigma([f])$ is the intersection of a nested family of compact connected sets and is, therefore, connected.

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Let $\mathcal{M}$ be the maximal ideal space of $BCESV/C_0$. We now have

COROLLARY 2. *$\mathcal{M}$ is connected.*

PROOF. If not, by standard facts, $BCESV/C_0$ would have a nontrivial idempotent element with spectrum $\{0,1\}$. This is impossible by Corollary 1.

Recall that $\sigma_e(A) \equiv \sigma[\pi(A)]$ for π the quotient map from $B(H)$ onto $B(H)/\mathcal{K}$.

COROLLARY 3. *For f in Q, $\sigma_e(T_f)$ is connected.*

PROOF. Easy from Corollary 1 and Theorem 16.

REMARK. Since $CCR(\mathbf{C}^n)$ contains nontrivial projections [**13**], $\sigma_e(T_f)$ must be disconnected for some trigonometric polynomials f.

Finally, we can establish an index theorem for $\tau(Q)$ along familiar lines. Using the characterization of $S = \tau(Q)$ in the proof of Theorem 16, it suffices to consider T_f for f in $BCESV$.

THEOREM 19. *For f in $BCESV$, T_f is Fredholm if and only if $|f(z)| \geq m > 0$ for all z with $|z| \geq R$ for some R. For such f, index $(T_f) = -$ winding number $(f|_{|z|=R})$ when $n = 1$ and index $(T_f) = 0$ for $n > 1$.*

PROOF. It is easy to check that T_f is Fredholm if and only if $[T_f]$ is invertible in $\tau(Q)/\mathcal{K}$. By Lemma 17 and Theorem 16 this is true if and only if $|f(z)| \geq m > 0$ for all z with $|z| \geq R$ for some R.

Suppose f in $BCESV$ satisfies $|f(z)| \geq m > 0$ for all Z with $|z| \geq R$. For $n \geq 2$, as noted before $f - g$ is in C_0 for some g in $BCESV$ with $|g(z)| \geq m' > 0$ for *all* z. For $n = 1$, there is an integer r and there is a g in $BCESV$ with $|g(z)| \geq m' > 0$ for *all* z so that $(\bar{z}/|z|)^r f(z) - g(z)$ is in V. Moreover, g has winding number zero around any circle in $\mathbf{C}$. An easy calculation in [**6**] shows that index $T_{(\bar{z}/|z|)^r} = r$, which is the winding number of f around $|z| = R$. The previous discussion of $\tau(Q)$ and standard Fredholm theory show that $r +$ index $T_f =$ index T_g for $n = 1$ while index $T_f =$ index T_g for $n \geq 2$. Thus, it will suffice to check that index $T_g = 0$.

Since $|g|$ is bounded below and g is in $BCESV$, $1/|g|$ is also in $BCESV$ and, for $G = g/|g|$

$$\text{index } T_G = \text{index } T_g + \text{index } T_{1/|g|}.$$

Since $t(1/|g|) + (1-t)1$ is an arc of invertible elements in $BCESV$, index $T_{1/|g|} = 0$ and we need only check that index $T_G = 0$ for G in $BCESV$ with $|G| = 1$.

By monodromy, G has a continuous argument F on $\mathbf{C}^n$ (of course, F need not be bounded) so $G(z) = \exp\{iF(z)\}$. We can check that $F(z)$ is an ESV-like function in the sense that

$$\operatorname*{Lim}_{\rho\to\infty} \sup_{\substack{|z|\geq\rho \\ |w-z|\leq 1}} |F(z) - F(w)| = 0.$$

Choose $\delta(\varepsilon)$ so that $|e^{ia} - 1| < \delta(\varepsilon)$ and $|a| \leq 1$ implies $|a| < \varepsilon$. For $\varepsilon > 0$ given, consider a fixed z with $|z|$ large enough that $|G(z) - G(w)| < \delta(\varepsilon)$ for $|w - z| \leq 1$. It follows that for all such w, there is an integer-valued function $k(w)$ with $|F(z) - F(w) - 2\pi k(w)| < \varepsilon$. By continuity of F, $k(z) = 0$ and $k(\cdot)$ must be constant so $|F(z) - F(w)| < \varepsilon$. It follows that, for any integer $m > 0$, $H_m = \exp\{iF/m\}$ is in $BCESV$ with $(H_m)^m = G$. Hence,

$$\text{index } T_G = \text{index } T_{H_m}^m = m \text{ index } T_{H_m}$$

and so index $T_G = 0$.

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EXAMPLE. Consider the function on $\mathbf{C}$

$$f(z) = \begin{cases} z, & |z| < 1, \\ z/|z|, & |z| \geq 1. \end{cases}$$

It is easy to check that f is in $BCESV$ and T_f is Fredholm with index $(T_f) = -1$.

5. Extensions and generalizations. In this section, we discuss some extensions and possible generalizations of our results.

We note, first, that Theorems B and C hold when L^∞ is replaced by the function space

$$L = \{f \text{ measurable: } \widetilde{|f|^2} \in BC\}$$

in the definition of Γ, B, Q. For f in L, Lemma 14 shows that T_f is a bounded operator. As for f in L^∞, $f = \tilde{f} + (f - \tilde{f})$ gives a decomposition of Q as $ESV + Q \cap B$. Note that ESV can be defined as before: the unbounded part of f is absorbed in $Q \cap B$.

Analogs of our results are likely to hold for the classical domains. In particular, the Bergman space of the unit disc D, in $\mathbf{C}^1$, has been profitably studied. The group $SL(2, \mathbf{R})$ acts on $H^2(D, dA)$ by linear fractional transformations and this group plays a role like that of the Heisenberg group on $H^2(\mathbf{C}^n, d\mu)$. K. H. Zhu has recently obtained a characterization of the algebra Q on $H^2(D, dA)$ in terms of oscillation near the boundary [**16**]. The following result is useful in the analysis of Q on general classical domains.

THEOREM 20. *Let Ω be a bounded Cartan domain in $\mathbf{C}^n$ with dv the usual volume measure. Suppose P is the usual orthogonal projection from $L^2(\Omega, dv)$ onto the Bergman subspace of holomorphic functions, $H^2(\Omega, dv)$. Then $P|_{L^\infty(\Omega)}$ is a compact operator from the Banach space $L^\infty(\Omega)$ to $H^2(\Omega, dv)$.*

PROOF. Let E be the injection of $L^\infty(\Omega)$ into $L^2(\Omega, dv)$. Then $P|_{L^\infty(\Omega)} = PE$ and, for M_{χ_σ} the operator of multiplication by the characteristic function of the compact set σ, $\sigma \subset \Omega$, we have

$$PE = PM_{\chi_\sigma}E + PM_{\chi_{\Omega\setminus\sigma}}E.$$

Note that PM_{χ_σ} is a compact operator since P is an integral operator with smooth kernel away from the boundary $\partial\Omega$. Choose σ so that $v(\Omega \setminus \sigma) < \varepsilon$. Then, for $\|f\|_\infty \leq 1$, we have

$$\left\|PM_{\chi_{\Omega\setminus\sigma}}Ef\right\|_2 = \left\|PM_f\chi_{\Omega\setminus\sigma}\right\|_2 \leq \left\|\chi_{\Omega\setminus\sigma}\right\|_2 < \sqrt{\varepsilon}$$

so that $\left\|PM_{\chi_{\Omega\setminus\sigma}}E\right\| < \sqrt{\varepsilon}$. Hence, PE is a norm limit of compact operators.

From the viewpoint of quantum mechanics, it may be of interest to extend our results to "infinitely many complex variables" (see [**2, 15**]). This extension appears to work and the results remain approximately the same. We expect to treat this problem in a subsequent note.

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