Commun. Math. Phys. 165, 281–296 (1994)

Communications in Mathematical Physics

Toeplitz Quantization of Kähler Manifolds and $gl(N)$, $N \to \infty$ Limits

Martin Bordemann[1], **Eckhard Meinrenken**[2], **Martin Schlichenmaier**[3]

[1] Department of Physics, University of Freiburg, Hermann-Herder-Strasse 3, D-79104 Freiburg, Germany. E-mail address: mbor@ibm.ruf.uni-freiburg.de
[2] Department of Mathematics, M.I.T., Cambridge, Ma 02139, USA
E-mail address: mein@math.mit.edu
[3] Department of Mathematics and Computer Science, University of Mannheim, D-68131 Mannheim, Germany. E-mail address: schlichenmaier@math.uni-mannheim.de

Received: 6 October 1993/in revised form: 10 February 1994

Abstract: For general compact Kähler manifolds it is shown that both Toeplitz quantization and geometric quantization lead to a well-defined (by operator norm estimates) classical limit. This generalizes earlier results of the authors and Klimek and Lesniewski obtained for the torus and higher genus Riemann surfaces, respectively. We thereby arrive at an approximation of the Poisson algebra by a sequence of finite-dimensional matrix algebras $gl(N)$, $N \to \infty$.

1. Introduction

In a couple of papers titled "Quantum Riemann Surfaces" [24] Klimek and Lesniweski have recently proved a classical limit theorem for the Poisson algebra of smooth functions on a compact Riemann surface Σ of genus $g \geq 2$ (with Petersson Kähler structure) using the Toeplitz quantization procedure:

$$\lim_{\hbar \to 0} \|T_f^{(1/\hbar)}\| = \|f\|_\infty , \tag{1.1}$$

$$\lim_{\hbar \to 0} \left\| \frac{1}{\hbar} [T_f^{(1/\hbar)}, T_g^{(1/\hbar)}] - \mathrm{i} T_{\{f,g\}}^{(1/\hbar)} \right\| = 0 . \tag{1.2}$$

Here, $\frac{1}{\hbar} = 1, 2, \ldots$ are tensor powers of the quantizing Hermitian line bundle (L, h) over M, and the Toeplitz operators act on the Hilbert space of holomorphic sections of $L^{1/\hbar}$ as the holomorphic part of the operator that multiplies section with f.

As usual (1.2) gives the connection between the Poisson bracket of functions and the commutator of the associated operators and (1.1) prevents the theory from being empty. Compared to Berezin's covariant symbols [3] and to the concept of star products [2, 6, 9, 11], where the basic idea is the deformation of the algebraic structure on $C^\infty(M)$ using $\hbar$ as a formal deformation parameter, the emphasis lies here more on the approximation of $C^\infty(M)$ by operator algebras in norm sense. More generally, the estimates (1.1) and (1.2) above can be seen in the setting of approximating an (infinite-dimensional) Lie algebra $\mathfrak{L}$ by a family $(\mathfrak{L}_\alpha)$ of metrized Lie algebras indexed by some parameter α.

This concept does not only apply to the classical limit in quantization procedures, but also to other physical contexts. An important example is the Lie algebra $\mathrm{diff}_A\,\Sigma$ of all convergence-free or volume-preserving vector fields which plays a distinguished rôle both in two-dimensional hydrodynamics [1, 13] and in the theory of relativistic membrances [4, 23]. Its relation to the Poisson algebra of Σ is that the Poisson algebra is isomorphic (modulo the constant functions) to the Lie algebra of Hamiltonian vector fields on Σ, which in turn is (up to first de Rham cohomology) equal to $\mathrm{diff}_A\,\Sigma$. Originally starting from membrane theory (where this limit occurred in a phenomenological way as approximation of structure constants, see [23]), an axiomatic treatment of such an approximation scheme which was called $\mathfrak{L}_\alpha$-quasilimit was given in [5]. Roughly speaking, quasilimits can be seen as generalized projective limits with the homomorphisms $\mathfrak{L}_\alpha \to \mathfrak{L}_\beta$ replaced by certain asymptotic conditions. Apart from several examples the paper [5] also contains the relation to classical limits via geometric quantization on compact Kähler manifolds and the proof of (1.1) and (1.2) for the Poisson algebra on the $2n$-torus using theta functions (with characteristics). The above Toeplitz operators $T_f^{(1/\hbar)}$ were replaced by the operators of geometric quantization $Q_f^{(1/\hbar)}$, but the asymptotic results are equivalent according to Tuynman's relation $Q_f^{(1/\hbar)} = \mathrm{i}\,T^{(1/\hbar)}_{f-(\hbar/2)\Delta f}$.

The aim of this paper is to generalize the classical limit for Toeplitz quantization of the above Riemann surfaces to the general compact Kähler case (the "quantum Kähler manifolds"), i.e. to prove (1.1) and (1.2) in this context and to use them to show the following theorem (conjectured in [5]):

Theorem. *Let (M,ω) be a quantizable compact Kähler manifold, ω the Kähler form, $\mathscr{P}(M)$ the Poisson algebra of real valued C^∞-functions with respect to ω, L the quantum line bundle, and L^m its m^{th} tensor power. Let ω be rescaled (by multiplying it with a positive integer) in such a way that L is very ample. Then, with respect to the maps $f \to \mathrm{i}mT_f^{(m)}$ and $f \to mQ_f^{(m)}$ the Poisson algebra $\mathscr{P}(M)$ is a $u(\dim \Gamma_{\mathrm{hol}}(M, L^m))$-quasilimit $(m \to \infty)$ in both cases.*

The technical details entering the hypotheses of this theorem will be explained below. We believe that one can probably dispense with the condition that the bundle is very ample (i.e. avoid the rescaling).

The proof is largely based on the theory of generalized Toeplitz structures developed in the mid-seventies by Boutet de Monvel, Guillemin, and Sjöstrand in the framework of microlocal analysis [7, 8, 18]. In fact, the estimate (1.2) is an easy consequence of the symbol calculus for generalized Toeplitz operators, whereas the innocent looking (1.1) requires more efforts.

Let us give a rough outline of the arguments. Denote by U the dual line bundle to L, along with its Hermitian fibre metric, and by Q the unit disc bundle. Sections of L^m can be identified with functions on Q satisfying appropriate equivariance conditions. In this way, the direct sum of the spaces of holomorphic sections of L^m gets identified with a Hilbert subspace of $L^2(Q)$, called generalized Hardy space. As shown in [7, 8, 18], the orthogonal projector onto the Hardy space has good microlocal properties, and renders a ring of generalized Toeplitz operators on $L^2(Q)$ having properties similar to pseudo-differential operators. On the other hand, the spaces of holomorphic sections of L^m can be recovered using Fourier decomposition with respect to the natural circle action on the Hardy space, and the symbol calculus for the generalized Toeplitz operators gives the desired approximation results for the original problem.

The paper is organized as follows. In Sect. 2 we recall the notion of $\mathfrak{L}_\alpha$-quasilimit and describe its relation to geometric quantization for the convenience of the reader and to fix notation. In Sect. 3 we discuss the above theorem for projective Kähler manifolds and Riemann surfaces. In Sect. 4 we formulate the basic asymptotic results for partial Toeplitz operators (Eqs. (1.1) and (1.2) above) and explain why this implies the main theorem. Their proof is given in Sect. 5.

2. $\mathfrak{L}_\alpha$-Quasilimits and Geometric Quantization

We recall from [5] the definition of an $\mathfrak{L}_\alpha$-quasilimit. Let $(\mathfrak{L}, [\,,])$ be a real or a complex Lie algebra and $(\mathfrak{L}_\alpha, [\,,]_\alpha))_{\alpha\in I}$ a family of real resp. complex Lie algebras with index set I either $\mathbb{N}$ or other suitable subsets of $\mathbb{R}$. Let the Lie algebras $\mathfrak{L}_\alpha$ be equipped with metrics d_α (in our cases they are all coming from a norm) and let $(p_\alpha : \mathfrak{L} \to \mathfrak{L}_\alpha)_{\alpha\in I}$ be a family of linear maps.

Definition 2.1. $(\mathfrak{L}_\alpha, [\,,]_\alpha))_{\alpha\in I}$ *is called an approximating sequence for* $(\mathfrak{L}, [\,,])$ *and* $(\mathfrak{L}, [\,,])$ *is called an* $\mathfrak{L}_\alpha$*-quasilimit induced by* $(p_\alpha : \mathfrak{L} \to \mathfrak{L}_\alpha)_{\alpha\in I}$ *if*

(1) *all* p_α *for* $\alpha \gg 0$ *are surjective,*

(2) *if for all* $x, y \in \mathfrak{L}$ *we have* $d_\alpha(p_\alpha(x), p_\alpha(y)) \to 0$, *for* $\alpha \to \infty$ *then* $x = y$,

(3) *for all* $x, y \in \mathfrak{L}$ *we have* $d_\alpha(p_\alpha([x, y]), [p_\alpha(x), p_\alpha(y)]_\alpha) \to 0$, *for* $\alpha \to \infty$.

From (2) it follows that an element which is asymptotically zero is already zero and from (3) it follows that there is only one Lie product on L which is compatible with a given approximating sequence and a given system of maps (p_α). For examples we refer to [5, Sect. 3].

As was pointed out to us by Bost this definition is related to the notion of continuous fields of C^*-algebras as introduced in [12].

Let M be a compact Kähler manifold of complex dimension n with Kähler form ω. In particular, (M, ω) is a symplectic manifold. For every smooth function f on M the Hamiltonian vector field X_f is defined by $i_{X_f}(\omega) = df$. Let $\mathscr{P}(M)$ be the Lie algebra of smooth functions on M with the Lie bracket

$$\{f, g\} := df(X_g) = \omega(X_f, X_g)\,. \tag{2.1}$$

Now let (M, ω) be a quantizable manifold and L be a holomorphic quantum line bundle with fiber metric h and compatible covariant derivative ∇. For the explanation of the above terms we refer to [5, Sect. 4] for a quick review, resp. to [31, 32, 34] for detailed information.

The condition for L to be a quantum line bundle for (M, ω) says that the curvature of L is essentially equal to the symplectic form. More precisely for every pair of vector fields X, Y we have the prequantum condition

$$F(X, Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} = -\mathrm{i}\omega(X, Y)\,. \tag{2.2}$$

By this definition L is a positive line bundle. According to Kodaira's embedding theorem some tensor power L^m is "very ample", i.e. one gets a holomorphic embedding of M into a projective space using the holomorphic sections of L^m. After the choice of a basis $\varphi_0, \ldots, \varphi_N$ of $\Gamma_{\mathrm{hol}}(M, L^m)$ this embedding is given as

$$M \to \mathbb{P}^N\,, \qquad x \mapsto (\varphi_0(x) : \varphi_1(x) : \ldots : \varphi_N(x))\,.$$

Chow's theorem says that M is in fact a projective algebraic manifold [29, p. 60].

For every smooth function f on M the following prequantum operator P_f acting on the complex vector space $\Gamma(M, L)$ of all smooth global sections of L is formed $P_f := -\nabla_{X_f} + \mathrm{i} f \cdot 1$. This defines a map

$$P : \mathscr{P}(M) \to gl(\Gamma(M, L)), \qquad f \mapsto P_f .$$

By the prequantum condition (2.2) the map P is an injective Lie algebra homomorphism. Let $\Omega = \frac{1}{n!}\omega^n$ denote the symplectic volume form on M, and define the prequantum Hilbert space $L^2(M, L)$ as the completion of $\Gamma(M, L)$ with respect to the scalar product

$$\langle \varphi \mid \psi \rangle := \int_M h(\varphi, \psi)\,\Omega . \tag{2.3}$$

With respect to this scalar product P_f becomes an antihermitian operator of $\Gamma(M, L)$ for real valued f.

A second step in the geometric quantization scheme is the choice of a polarization. The canonical concept for Kähler manifolds is the separation into holomorphic and anti-holomorphic directions, called Kähler polarization. The quantum Hilbert space is the subspace $\Gamma_{\mathrm{hol}}(M, L)$ of holomorphic sections in $L^2(M, L)$. Due to compactness of M the space $\Gamma_{\mathrm{hol}}(M, L)$ is always finite dimensional. The quantum operator Q_f is defined as $Q_f := \Pi^{(1)} \circ P_f \circ \Pi^{(1)}$, where $\Pi^{(1)} : L^2(M, L) \to \Gamma_{\mathrm{hol}}(M, L)$ denotes orthogonal projection. The map $Q : f \mapsto Q_f$ is a linear map from $\mathscr{P}(M)$ to the finite dimensinal Lie algebra $u(\Gamma_{\mathrm{hol}}(M, L))$ of antihermitian operators in $\Gamma_{\mathrm{hol}}(M, L)$.

In this paper, however, we will be more concerned with Toeplitz quantization, defined as follows. For $f \in \mathscr{P}(M)$ the corresponding Toeplitz operator on $\Gamma_{\mathrm{hol}}(M, L)$ is the operator of multiplication M_f by f followed by orthogonal projection back to $\Gamma_{\mathrm{hol}}(M, L)$,

$$T_f := T(f) := \Pi^{(1)} \circ M_f \circ \Pi^{(1)} . \tag{2.4}$$

According to a result of Tuynman [32] (see also [5, Proposition 4.1]) one has

$$Q_f = \mathrm{i}T\big(f - \tfrac{1}{2}\Delta f\big) . \tag{2.5}$$

Here Δ is the Laplacian on functions calculated with respect to the Riemannian metric g coming from ω.

To obtain a family of finite dimensional Lie algebras associated to $\mathscr{P}(M)$ we consider everything for the m^{th} tensor power $L^m := L^{\otimes m}$ of the quantum line bundle for $m \in \mathbb{N}$. The quantum Hilbert space is thus $\Gamma_{\mathrm{hol}}(M, L^m)$, with scalar product

$$\langle \varphi \mid \psi \rangle := \int_M h^m(\varphi, \psi)\,\Omega , \qquad h^m := h \otimes \ldots \otimes h \quad (m \text{ factors}) , \tag{2.6}$$

and the prequantum operators $P_f^{(m)}$ define a representation of $(\mathscr{P}(M), m \cdot \omega)$. In order to render a representation of $(\mathscr{P}(M), \omega)$ they have to be rescaled to $\hat{P}_f^{(m)} := mP_f^{(m)} = -\nabla^{(m)}_{X_f} + \mathrm{i} m f$. The rescaled quantum operators are given as

$$\hat{Q}_f^{(m)} := \Pi^{(m)} \circ \hat{P}_f^{(m)} \circ \Pi^{(m)} , \tag{2.7}$$

with $\Pi^{(m)}$ the corresponding projection map. By Eq. (2.5) one has

$$\hat{Q}_f^{(m)} = \mathrm{i}T^{(m)}\left(f - \frac{1}{2m}\Delta f\right). \tag{2.8}$$

Note that neither $T^{(m)}$ nor the Laplacian are rescaled.

For the elements in $gl(\Gamma_{\text{hol}}(M, L^m))$ we take the rescaled norm

$$\|A\|_m := \frac{1}{m}\sup_{\varphi\neq 0}\frac{\|A\varphi\|}{\|\varphi\|}, \tag{2.9}$$

and $\|\ldots\|$ the operator norm.

For $\varphi, \psi \in \Gamma_{\text{hol}}(M, L^m)$ we obtain

$$\langle\varphi \mid T_f^{(m)}\psi\rangle = \langle\Pi^{(m)}\varphi \mid f\cdot\Pi^{(m)}\psi\rangle = \langle\varphi \mid f\cdot\psi\rangle = \int_M f h^m(\varphi,\psi)\,\Omega. \tag{2.10}$$

The settings for $m \in \mathbb{N}$ with $m \to \infty$,

$$(\mathscr{P}(M), \{\,,\}) \to (u(\Gamma_{\text{hol}}(M, L^m)), [\,,], \|\ldots\|_m), \qquad p_m : f \mapsto \hat{Q}_f^{(m)}, \tag{2.11}$$

$$(\mathscr{P}(M), \{\,,\}) \to (u(\Gamma_{\text{hol}}(M, L^m)), [\,,], \|\ldots\|_m), \qquad p_m : f \mapsto \mathrm{i}m\cdot T_f^{(m)}, \tag{2.12}$$

are exactly the settings examined in the scheme of $\mathfrak{L}_\alpha$-quasilimits. That m^{-1} is likely to play the role of $\hbar$ is already indicated by the formula for the dimension of $\Gamma_{\text{hol}}(M, L^m)$. Indeed, the Hirzebruch-Riemann-Roch theorem says that for m large, this dimension is a polynomial in m with leading term

$$\dim \Gamma_{\text{hol}}(M, L^m) = \frac{m^n}{(2\pi)^n}\,\mathrm{vol}(M) + O(m^{n-1}), \tag{2.13}$$

where vol(M) is the symplectic volume. But this is just what is to be expected from the uncertainty relation.

3. The Approximation Theorem

The following theorem will be proved in the remaining Sects. 4 and 5.

Theorem 3.1. *Let (M,ω) be a quantizable compact Kähler manifold, $\mathscr{P}(M)$ the Poisson algebra of real valued C^∞-functions with respect to ω, L the quantum line bundle, and L^m its m^{th} tensor power. Let ω be rescaled (by multiplying it with a positive integer) in such a way that L is very ample. Then, with respect to both settings* (2.11) *and* (2.12) *$\mathscr{P}(M)$ is a $u(\dim \Gamma_{\text{hol}}(M, L^m))$-quasilimit ($m \to \infty$).*

Let us illustrate the theorem by two important special classes of examples: The first class consists of the projective Kähler submanifolds. For the N-dimensional projective space $\mathbb{P}^N$ the Fubini-Study fundamental form ω_{FS} is defined as

$$\omega_{\text{FS}} := \mathrm{i}\,\frac{(1+|w|^2)\sum_{i=1}^{N} dw_i \wedge d\bar{w}_i - \sum_{i,j=1}^{N} \bar{w}_i w_j dw_i \wedge d\bar{w}_j}{(1+|w|^2)^2} \tag{3.1}$$

with respect to the local coordinates $w_i = z_i/z_0$, $i = 1, \ldots, N$ on the coordinate chart where the homogeneous coordinate $z_0 \neq 0$ (see for example [33]). It defines the standard Kähler form on $\mathbb{P}^N$ and it is up to the scalar factor $-\mathrm{i}$ the curvature form of the hyperplane bundle H. Hence, H is an associated quantum line bundle.

Now let $i: M \hookrightarrow \mathbb{P}^N$ be a projective Kähler submanifold of dimension n. The pullback $L = i^*(H)$ (resp. the restriction) of the hyperplane bundle H is a quantum line bundle associated to the pullback $i^*(\omega_{\mathrm{FS}})$ which is the Kähler form of M. The space of global holomorphic sections of L^m is generated by the restrictions of the homogeneous polynomials of degree m in $N+1$ variables. Note that they are in general not linearly independent when restricted to M. Formula (2.13) is the Hilbert polynomial of M, i.e. $n! \frac{\mathrm{vol}(M)}{(2\pi)^n}$ (which is a positive integer) is equal to the degree of M considered as a projective submanifold.

The second class of examples are Riemann surfaces with their "standard" Kähler forms. For the rest of this section let M be a compact Riemann surface with fixed complex structure. Depending on the type of the simply connected universal covering $\widetilde{M}$ of M the classes of Riemann surfaces can be divided into three subclasses (see [15, 29]).

Case 1. Here $\widetilde{M} = \mathbb{P}^1$, the projective line over $\mathbb{C}$, resp. the sphere S^2. In this case $M \cong \widetilde{M} = \mathbb{P}^1$. This isomorphism like all other isomorphisms appearing in the following is an analytic isomorphism. We use the standard covering of $\mathbb{P}^1$ by the open sets U_0 and U_1, $U_0 \cong U_1 \cong \mathbb{C}$,

$$U_0 := \{(z_0 : z_1) \mid z_0 \neq 0\}, \qquad U_1 := \{(z_0 : z_1) \mid z_1 \neq 0\}.$$

We take $z = z_1/z_0$ as coordinate for U_0, and $w = z_0/z_1$ as coordinate for U_1. The transition function is given as $w(z) = 1/z$. In the following we will describe every object by local functions in U_0. The Kähler form (3.1) specializes to

$$\omega_0(z) = \frac{\mathrm{i}}{(1+z\bar{z})^2}\, dz \wedge d\bar{z}. \tag{3.2}$$

The corresponding quantum line bundle is the hyperplane bundle L_0 with transition function $1/z$. Its global holomorphic sections are the elements of the vector space $\langle 1, z\rangle_{\mathbb{C}}$. For the tensor powers $L_0^m := L_0^{\otimes m}$ we obtain (for example by using the theorem of Riemann Roch [29]) $\dim \Gamma_{\mathrm{hol}}(\mathbb{P}^1, L_0^m) = m+1$. A basis is given by $1, z^1, z^2, \ldots, z^m$.

Case 2. $\widetilde{M} = \mathbb{C}$. In this case M is a one dimensional complex torus, e.g. $M \cong \mathbb{C}/\Gamma$, where $\Gamma = \langle 1, \tau\rangle_{\mathbb{Z}}$ $(\operatorname{Im}\tau > 0)$ is a two dimensional lattice in $\mathbb{C}$. The genus of M is equal to 1 and the Kähler form is given by

$$\omega_1(z) = \frac{\mathrm{i}\pi}{\operatorname{Im}\tau}\, dz \wedge d\bar{z}. \tag{3.3}$$

Here z is the coordinate on the covering. A corresponding quantum line bundle is the theta line bundle L_1 of degree 1. It depends on the complex structure of M, e.g. on τ. Its space of global sections is one dimensional and a basis element is given by the Riemann theta function (see [5, Sect. 5]). By the Riemann Roch theorem we get for the tensor powers L_1^m $\dim \Gamma_{\mathrm{hol}}(M, L_1^m) = m$. These spaces are generated by the theta functions with characteristics. Of course, L_1 is only ample. But $L^{\otimes 3}$ will be very ample [17].

Case 3. $\widetilde{M} = E$ with $E := \{z \in \mathbb{C} \mid |z| < 1\}$ the open unit disc. There exists a Fuchsian group D, i.e. a discrete subgroup satisfying some additional conditions (see [15]) of

$$SU(1,1) := \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in GL(2,\mathbb{C}) \mid |a|^2 - |b|^2 = 1 \right\},$$

such that $M \cong E/D$ (analytically). Here the elements $R \in SU(1,1)$ operate by fractional linear transformations

$$z \mapsto R(z) := \frac{az+b}{\bar{a}z+\bar{b}}$$

on E. This situation could equivalently be described by the upper half plane and the group $SL(2,\mathbb{R})$. As Kähler form on E we take

$$\omega = \frac{2\mathrm{i}}{(1-z\bar{z})^2}\, dz \wedge d\bar{z}\,. \tag{3.4}$$

Because $R'(z) = (\bar{b}z+\bar{a})^{-2}$ we obtain $\omega(R(z)) = \omega(z)$. Hence (3.4) is invariant under $SU(1,1)$ and defines a Kähler form ω_g on M.

An associated quantum line bundle L_g is the canonical line bundle K (i.e. the line bundle whose local sections are the local holomorphic differentials). Again, K resp. L_g depends on the complex structure, i.e. on the group D. For generic Riemann surfaces of genus $g > 2$ the bundle L_g is already very ample. In any case $L_g^{\otimes 3}$ will be very ample [27].

The bundles L_g^m are the m-canonical bundles. By the theorem of Riemann Roch we obtain

$$\dim \Gamma_{\text{hol}}(M, L_g^m) = \begin{cases} g\,, & m = 1\,, \\ (2m-1)(g-1)\,, & m \geq 2\,. \end{cases}$$

As in the $g = 1$ case the sections can be identified with functions on the covering space E which behave suitably under the operation of the group D. A holomorphic function f on E is called an automorphic form of weight[1] $2k$ for the group D if for every $R = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in D$,

$$f(R(z)) = (\bar{b}z+\bar{a})^{2k} \cdot f(z) = (R'(z))^{-k} \cdot f(z)\,.$$

From the definition it is clear that $f(z)\,(dz)^k = f(R(z))\,(d(R(z)))^k$. Hence, such an automorphic form of weight $2k$ defines a section of L_g^k. Conversely, every such section defines by pullback an automorphic form on E.

Note that in all the above cases the theorem also holds without the "very ample" condition, see [5, 24].

4. Approximation and Toeplitz Operators

Let (M,ω) be a quantizable compact Kähler manifold and L some quantum line bundle with metric h over M. We assume L to be very ample. Let $\mathscr{H}^{(m)} = \Gamma_{\text{hol}}(M, L^m)$ be the Hilbert space of holomorphic sections in L^m, with scalar product

[1] The definition of weight varies in literature. Our weight $2k$ is sometimes called weight k or dimension $-2k, \ldots$

(2.6). Recall the relation (2.8) between the quantum operators and the multiplication (Toeplitz) operators. We will show that Theorem 3.1 will follow from Theorem 4.1 and Theorem 4.2 below. In Sect. 5 we will prove these theorems.

First we will show that the surjectivity (property (1) in Definition 2.1) is always true, due to the following propositions.

Proposition 4.1. *The canonical linear mapping*

$$s^{(m)}\colon \mathrm{End}(\mathcal{H}^{(m)}) \to C^\infty(M) \quad \textit{defined by} \quad s^{(m)}(|\varphi\rangle\langle\psi|) := h^{(m)}(\varphi,\psi)\,, \qquad (4.1)$$

is an injection.

Proof. Let $e_1, e_2, \ldots, e_d$ be a basis for $\mathcal{H}^{(m)}$. In a local complex chart (V, z) these sections are represented by holomorphic functions $e_i(z)$. In this chart the d^2 sections $s^{(m)}(|e_i\rangle\langle e_j|)$ are given by the d^2 functions $h(z)\overline{e_i(z)}e_j(z)$ where h is some fixed positive function. Suppose that

$$\sum a_{ij}h(z)\overline{e_i(z)}e_j(z) = 0\,,$$

for some $a_{ij} \in \mathbb{C}$. After dividing by h, this can be analytically extended to $V \times V$:

$$\sum a_{ij}\overline{e_i(z)}e_j(w) = 0 \quad \forall z, w \in V\,.$$

It follows that $a_{ij} = 0$. □

Proposition 4.2. *The linear mappings $T^{(m)}$ and $\hat{Q}^{(m)}\colon C^\infty(M) \to \mathrm{End}(\mathcal{H}^{(m)})$ are surjections.*

Proof. For all $f \in C^\infty(M)$ and $A \in \mathrm{End}(\mathcal{H}^{(m)})$, one has for the Hilbert-Schmidt scalar product

$$\langle A \mid T_f^{(m)}\rangle = \mathrm{tr}(A^* \cdot T_f^{(m)}) = \int_M f(x)\, s^{(m)}(A^*)(x)\,\Omega(x) = \langle s^{(m)}(A), f\rangle_{L^2}\,. \qquad (4.2)$$

Suppose that A is orthogonal to the range of $T^{(m)}$. Then both sides of (4.2) vanish for all f, i.e. $s^{(m)}(A) = 0$. According to Proposition 4.1, this implies $A = 0$, hence $T^{(m)}$ is surjective. The analogous result for $\hat{Q}^{(m)}$ follows from $\hat{Q}^{(m)} = mT^{(m)} \circ \left(1 - \frac{1}{2m}\Delta\right)$, since $\left(1 - \frac{1}{2m}\Delta\right)$ is positive and elliptic. Hence, for every $g \in C^\infty$ there is a $f \in C^\infty$ with $\left(1 - \frac{1}{2m}\Delta\right) f = g$. □

Theorem 4.1. *For every $f \in \mathcal{P}(M)$ there is some $c > 0$ such that*

$$\|f\|_\infty - \frac{C}{m} \le \|T_f^{(m)}\| \le \|f\|_\infty \quad \textit{as} \quad m \to \infty\,. \qquad (4.3)$$

Here $\|f\|_\infty$ is the sup-norm of f on M and $\|T_f^{(m)}\|$ is the operator norm on $\mathcal{H}^{(m)}$. In particular,

$$\lim_{m\to\infty} \|T_f^{(m)}\| = \|f\|_\infty\,. \qquad (4.4)$$

Theorem 4.2. *For all $f, g \in \mathscr{P}(M)$,*

$$\|m[T_f^{(m)}, T_g^{(m)}] - \mathrm{i}T_{\{f,g\}}^{(m)}\| = O(m^{-1}) \quad as \quad m \to \infty\,. \tag{4.5}$$

From both theorems it follows immediately

$$\lim_{m\to\infty} \|[T_f^{(m)}, T_g^{(m)}]\| = 0\,. \tag{4.6}$$

Proof of Theorem 3.1. The required surjectivity is just Proposition 4.2. Obviously for the assignment $f \to \mathrm{i}m \cdot T_f^{(m)}$, by (4.4) and (4.5) the remaining two conditions are fulfilled. Hence for the setting (2.12) Theorem 3.1 is true. (Note, we use the rescaled operator norm $\|\dots\|_m$.) Using the relations (2.8) which connects the quantum operator with the Toeplitz operator it is easy to check (using (4.6)) that

$$\lim_{m\to\infty} \|\hat{Q}_f^{(m)}\|_m = \|f\|_\infty\,, \tag{4.7}$$

$$\lim_{m\to\infty} \|[\hat{Q}_f^{(m)}, \hat{Q}_g^{(m)}] - \hat{Q}_{\{f,g\}}^{(m)}\|_m = 0\,. \tag{4.8}$$

Hence, we obtain Theorem 3.1 also for the setting (2.11). □

Remark. In the case of Riemann surfaces Theorem 4.1 and 4.2 have been already proved by tedious calculations. Klimek and Lesniewski [24] did the case of genus $g \geq 2$. Our Theorem 4.1 corresponds to [24, II.], Theorem A and Theorem 4.2 corresponds to [24, II.], Corollary to Theorem B. Note that we defined our Poisson bracket (2.1) with the opposite sign of the bracket used in [24]. The case $g = 1$ has been done by the authors in [5] as a special case of n-dimensional complex algebraic tori. The authors (unpublished) also did the case $g = 0$ using asymptotics of binomials (Stirling formula, etc.).

Before we prove these theorems in Sect. 5 for the general setting we will give a more elementary proof of Theorem 4.1 for the first class of examples, the projective Kähler manifolds M. Let $i: M \hookrightarrow \mathbb{P}^N$ be a nonsingular projective variety, and $\pi: U \to M$ be the restriction of the tautological line bundle of $\mathbb{P}^N$ to M with its induced Hermitian structure k. The bundle U is the dual of L, the pullback of the hpyerplane bundle H, i.e. $U = L^* = i^*(H^*)$. Then L is a quantum bundle of (M, ω), where ω is the pullback of the Fubini-Study form of $\mathbb{P}^N$. Using the scalar product on $\mathbb{C}^{(N+1)}$ the metric k extends to a function on $U \times U$, holomorphic in the second argument and anti-holomorphic in the first. In particular, the Calabi (diastatic) function [9, 10].

$$D: M \times M \to \mathbb{R}_{\geq 0} \cup \{\infty\}\,, \qquad D(\pi(\lambda), \pi(\mu)) = -\log|k(\lambda, \mu)|^2 \tag{4.9}$$

(where we have to choose λ and μ with $k(\lambda, \lambda) = k(\mu, \mu) = 1$ representing the points of M) is well-defined and vanishes only along the diagonal.

Proof of Theorem 4.1 for these cases. The second inequality follows directly from the definition (2.4) of T_f. To proof the first, let $x_0 \in M$ be a point where $|f|$ assumes its supremum, and fix a $\lambda_0 \in \pi^{-1}(x_0)$ with $k(\lambda_0, \lambda_0) = 1$. Identifying holomorphic sections $\Phi^{(m)}$ of $U^{-m} = L^m$ with holomorphic functions $\tilde{\Phi}^{(m)}: U \to \mathbb{C}$ which are equivariant (i.e. which obey $\tilde{\Phi}^{(m)}(\alpha v) = \alpha^m \tilde{\Phi}^{(m)}(v)$), we define a sequence $\Phi^{(m)} \in \mathscr{H}^{(m)}$ by setting

$$\tilde{\Phi}^{(m)}(\lambda) = k(\lambda_0, \lambda)^m\,.$$

Note that $h^m(\Phi^{(m)}, \Phi^{(m)})(x) = \exp(-mD(x_0, x))$. (Recall, we chose λ such that $k(\lambda, \lambda) = 1$.) Hence, using Cauchy-Schwartz's inequality,

$$\|T_f^{(m)}\| \geq \frac{\|T_f^{(m)}\Phi^{(m)}\|}{\|\Phi^{(m)}\|} \geq \frac{|\langle \Phi^{(m)} | \, T_f^{(m)} \, | \Phi^{(m)} \rangle|}{\langle \Phi^{(m)} \mid \Phi^{(m)} \rangle}$$
$$= \frac{\left| \int_M f(x) h^m(\Phi^{(m)}, \Phi^{(m)})(x) \Omega(x) \right|}{\int_M h^m(\Phi^{(m)}, \Phi^{(m)})(x) \Omega(x)} = \frac{\left| \int_M f(x) e^{-mD(x_0,x)} \Omega(x) \right|}{\int_M e^{-mD(x_0,x)} \Omega(x)} .$$

Both integrands vanish exponentially (with respect to $m \to \infty$) outside $x = x_0$. Moreover, as a function of x the Calabi function has a nondegenerate critical point at $x = x_0$, i.e. one can apply the stationary phase theorem [22] to both integrals to conclude that

$$\|T_f^{(m)}\| \geq |f(x_0)| + O(m^{-1}) = \|f\|_\infty + O(m^{-1}) . \qquad \square$$

5. Proofs of Theorems 4.1 and 4.2

The proofs will follow from the theory of "global" Toeplitz operators as developed by Boutet de Monvel and Guillemin [7]. Let us review the necessary prerequisites from their book. Let (M, ω) be an n-dimensional Kähler manifold, $(U, k) := (L^*, h^{-1})$ be the dual of the quantum line bundle as above, and

$$\hat{k}: U \to \mathbb{R}_{\geq 0}, \qquad \hat{k}(\lambda) = k(\lambda, \lambda) .$$

Let $Q = \hat{k}^{-1}(1)$ be the unit circle bundle.

It is known (see e.g. [6]) that the 2-form $\mathrm{i}\partial\bar{\partial}\hat{k}$ on U is Kähler off the zero section. In particular, the unit disc bundle is strictly pseudoconvex.

The natural circle action makes Q into a principal S^1 bundle $\tau : Q \to M$, and the tensor powers of U may be viewed as associated bundles. Let $\mathrm{i}\alpha \in \mathrm{i}\Omega^1(Q)$ be the $\mathfrak{u}(1)$-valued connection 1-form corresponding to the Hermitian linear connection ∇ on U. $\left(\alpha \text{ is the restriction of the 1-form } \frac{1}{2\mathrm{i}}(\partial\hat{k} - \bar{\partial}\hat{k}) \text{ to the circle bundle.}\right)$ According to the prequantum condition, $d\alpha = \tau^*\omega$, and $\nu = \frac{1}{2\pi}\tau^*\Omega \wedge \alpha$ is a volume form on Q. The generalized Hardy space $\mathscr{H}$ is defined as the closure in $L^2(Q, \nu)$ of the subspace of all $f \in C^\infty(Q)$ that extend to holomorphic functions on the disc bundle. $\mathscr{H}$ is preserved under the circle action and thus splits into a completed direct sum $\mathscr{H} = \sum_{m=0}^{\infty} \mathscr{H}^{(m)}$, where $c \in S^1$ acts on $\mathscr{H}^{(m)}$ by multiplication with c^m. Under the identification of sections of L^m with functions on Q satisfying the equivariance condition $\phi(c\lambda) = c^m\phi(\lambda), (c \in S^1)$, the Fourier sectors $\mathscr{H}^{(m)}$ coincide with the Hilbert spaces defined in Sect. 4. The orthogonal projector $\Pi : L^2(Q) \to \mathscr{H}$ is called the generalized Szegö projector.

We shall assume that L is very ample, i.e. that M can be embedded into some projective space $\mathbb{P}^N$ via the global holomorphic sections of L. In particular, L is the restriction (pullback) of the hyperplane bundle and U is the restriction of the tautological bundle. Away from the zero section the latter and hence U can be embedded into $\mathbb{C}^{N+1}$. The image of U is an affine cone, hence a Stein variety (with singularity at 0 coming from the collapse of the zero section). Under this condition

Π defines a Toeplitz structure in the sense of [7, p. 18] (see the remark at the end of Ref. [8]), with underlying symplectic submanifold of $T^*Q\backslash 0$ the positive cone over the graph of α:

$$\Sigma = \{t\alpha(\lambda) \mid \lambda \in Q, t > 0\} \subset T^*Q\backslash 0 . \tag{5.1}$$

(Here and in the following $T^*Q\backslash 0$ denotes the total space T^*Q with the zero section removed.) Let $\tau_\Sigma : \Sigma \to M$ denote the natural projection. A (global) Toeplitz operator of order k associated to (Σ, Π) is by definition an operator $A : \mathscr{H} \to \mathscr{H}$ of the form $A = \Pi R \Pi$, where R is a pseudo-differential operator of order k. The principal symbol of A is the restriction of the principal symbol of R (which is a function on T^*Q) to Σ. It was shown in [7] that Toeplitz operators form a ring, and that the principal symbol of Toeplitz operators is well defined and obeys the same rules as for pseudo-differential operators:

$$\sigma(A_1 A_2) = \sigma(A_1)\sigma(A_2), \qquad \sigma([A_1, A_2]) = \mathrm{i}\{\sigma(A_1), \sigma(A_2)\},$$

where the Poisson brackets are computed with respect to the symplectic structure on Σ.

The generator of the circle action $\frac{1}{\mathrm{i}}\frac{\partial}{\partial\varphi}$ gives a first order Toeplitz operator D_φ with symbol $\sigma(D_\varphi)(t\alpha(\lambda)) = t$. D_φ operates on $\mathscr{H}^{(m)}$ as multiplication by m. For $f \in \mathscr{P}(M)$ let M_f be the multiplication operator on $L^2(Q)$ and $T_f = \Pi M_f \Pi$. The symbol of T_f is the pullback of f to Σ. Being invariant under the circle action, T_f splits into a direct sum $T_f = \bigoplus_{m=0}^{\infty} T_f^{(m)}$. Identifying $\mathscr{H}^{(m)}$ with the space of holomorphic sections, the operator $T_f^{(m)}$ on $\mathscr{H}^{(m)}$ is just the Toeplitz quantization (multiplication) corresponding to f considered in the previous section.

Proof of Theorem 4.2. The commutator $[T_f, T_g]$ is a Toeplitz operator of order -1 with principal symbol $\mathrm{i}\{\tau_\Sigma^* f, \tau_\Sigma^* g\}_\Sigma (t\alpha(\lambda)) = \mathrm{i}t^{-1}\{f, g\}_M(\tau(\lambda))$. It follows that the S^1-invariant, first order Toeplitz operator

$$A := D_\varphi^2 [T_f, T_g] - \mathrm{i} D_\varphi T_{\{f,g\}}$$

has vanishing principal symbol, i.e. is in fact zeroth order. But zeroth order pseudo-differential operators on compact manifolds are bounded (see e.g. [16, p. 29], or [22]), and since Π is bounded, as an operator on $L^2(Q, \nu)$, it follows that A is bounded. Since $\|A^{(m)}\| \le \|A\|$ and

$$A^{(m)} = A \mid \mathscr{H}^{(m)} = m^2 [T_f^{(m)}, T_g^{(m)}] - \mathrm{i} m T_{\{f,g\}}^{(m)},$$

we are done. □

Remark. In a similar fashion, the theory in [7] leads to

(1) Let $f \in \mathscr{P}(M)$, $U^{(m)}(t) = \exp(-\mathrm{i} m t T_f^{(m)})$ the corresponding time evolution operator, and $g \in C^\infty(M)$. If F^t denotes the Hamiltonian flow for f, one has

$$\|U^{(m)}(t) T_g^{(m)} U^{(m)}(-t) - T_{(F^t)_* g}^{(m)}\| = O(m^{-1}) \quad (\text{for } m \to \infty).$$

This follows from the Egorov theorem for Toeplitz operators, see [7, p. 100].

(2) For all $f_1, f_2, \ldots, f_r \in C^\infty(M)$,

$$\|T_{f_1 \cdots f_r}^{(m)} - T_{f_1}^{(m)} \cdots T_{f_r}^{(m)}\| = O(m^{-1}) \quad (\text{for } m \to \infty).$$

(3) For all $f_1 \cdot f_2, \ldots, f_r \in C^\infty(M)$,

$$\frac{1}{\dim \mathscr{H}^{(m)}} \operatorname{tr}(T_{f_1}^{(m)} \ldots T_{f_r}^{(m)}) = \frac{1}{\operatorname{vol}(M)} \int f_1 \ldots f_r \Omega + O(m^{-1}).$$

For the proof, see Guillemin [18].

Proof of Theorem 4.1. The second inequality is obvious. To prove the first, we have to construct a sequence $\Phi^{(m)} \in \mathscr{H}^{(m)}$ such that

$$\frac{\|T_f^{(m)} \Phi^{(m)}\|}{\|\Phi^{(m)}\|} = \|f\|_\infty + O(m^{-1}). \tag{5.2}$$

The idea is to regard the $\Phi^{(m)}$ as Fourier modes (with respect to the S^1 action) of a single distribution $\Phi \in \mathscr{D}'(Q)$. Let $x_0 \in M$ be a point where $|f(x_0)| = \|f\|_\infty$, and let $\lambda_0 \in \tau^{-1}(x_0)$ be fixed. For $\lambda \in Q$, let

$$\Xi_\lambda := \{t\alpha(\lambda) \in T^*Q \mid t > 0\} \tag{5.3}$$

be the ray through $\alpha(\lambda)$.

We will look for a suitable Φ among those distributions which have a singularity at λ_0 in the direction of $\alpha(\lambda_0)$, i.e. whose wave front set [21] is contained in Ξ_λ for $\lambda = \lambda_0$. A class of distributions having this property is the space $I^r(Q, \Xi)$ of "Hermite distributions" studied in [7, 19]: Choose local coordinates $y = (y_1, \ldots, y_q)$, $q = \dim Q$ around λ such that, in the corresponding cotangent coordinates (y, η), the ray Ξ_λ is given by the equations $y_1 = \ldots = y_q = 0$, $\eta_2 = \ldots = \eta_q = 0$, $\eta_1 > 0$. Let us write $y' = (y_2, \ldots, y_q)$, $\eta' = (\eta_2, \ldots, \eta_q)$. Then the space $I^r(Q, \Xi_\lambda)$ consists of distributions Φ that can be written, $\operatorname{mod} C^\infty(Q)$, as oscillatory integrals

$$\Phi(y) = (2\pi)^{-q} \int e^{iy\eta} a\left(\eta_1, \frac{\eta'}{\sqrt{|\eta_1|}}\right) d^q\eta. \tag{5.4}$$

Here the amplitude $a(\eta_1, \eta')$ is smooth, vanishes for $\eta_1 < \varepsilon$ for some $\varepsilon > 0$, and admits an asymptotic expansion

$$a(\eta_1, \eta') \sim \sum_{j=0}^{\infty} a_j(\eta_1, \eta'), \tag{5.5}$$

where a_j is positively homogeneous of degree $r - \dfrac{j+q}{2}$ in η_1 for $\eta_1 \gg 0$ and a Schwartz function in η'. It can be shown that this definition does not depend on the particular choice of coordinates. In particular, we can assume that $\dfrac{\partial}{\partial y_1} = \dfrac{\partial}{\partial \varphi}$.

From [7], Theorem 11.1 and 9.4, $I^r(Q, \Xi_\lambda)$ is invariant under the Szegö projector Π and under zeroth order pseudo-differential operators. In particular, it is invariant under M_f, hence also under the Toeplitz operator T_f. Using that f has a critical point at x_0, the transport equation ([7], Theorem 10.2) shows that

$$(f - f(x_0))\Phi \in I^{r-1}(Q, \Xi) \quad \text{for} \quad \Phi \in I^r(Q, \Xi). \tag{5.6}$$

We will need the following lemma:

Lemma 1. *For all $\Phi \in I^r(Q, \Xi_\lambda)$, the Fourier modes $\Phi^{(m)}$ have finite norm admitting an asymptotic expansion*

$$\|\Phi^{(m)}\|^2 \sim \sum_{j=0}^{\infty} b_j m^{2r - \frac{q+j+1}{2}} \tag{5.7}$$

for $m \to \infty$ and vanish faster than any power for $m \to -\infty$. Moreover, the leading term b_0 depends only on the equivalence class in $I^r(Q, \Xi_\lambda)/I^{r-\frac{1}{2}}(Q, \Xi_\lambda)$, i.e. on its "principal symbol."

Let us postpone the proof of Lemma 1 for a moment, and explain how to make a particularly nice choice for $\Phi^{(m)}$.

Let $T^*_\lambda Q$ be the cotangent fiber. Since $T^*_\lambda Q \cap \Sigma = \Xi_\lambda$, Theorem 9.4 from [7] shows that Π maps the space $I^r(Q, T^*_\lambda Q - \{0\})$ of Fourier integrals into the space $I^r(Q, \Xi_\lambda)$. Applying this to the delta function $\delta_\lambda \in I^{q/2}(Q, T^*_\lambda Q - \{0\})$, we get some $e_\lambda = \Pi \delta_\lambda \in I^{q/2}(Q, \Xi_\lambda)$. The Fourier modes $e^{(m)}_\lambda$ of e_λ have finite norm according to Lemma 1, so they are in $\mathscr{H}^{(m)}$, and they satisfy for all $\Psi^{(m)} \in \mathscr{H}^{(m)}$,

$$\langle e^{(m)}_\lambda \mid \Psi^{(m)} \rangle = \langle \delta_\lambda | \, \Pi^{(m)} \, | \Psi^{(m)} \rangle = \Psi^{(m)}(\lambda) \,, \tag{5.8}$$

where again we have identified sections of L^m with equivariant functions. On the other hand, (5.8) characterizes the $e^{(m)}_\lambda$ by Riesz' Lemma, and in fact (5.8) is used as by Rawnsley [28] as the defining property of his "coherent states."

Lemma 2. *For all $\lambda \in Q$,*

$$\|e^{(m)}_\lambda\|^2 = (2\pi)^{-n} m^n + O(m^{n-\frac{1}{2}}) \,.$$

Proof. According to Lemma 1, the leading term depends only on the principal symbol of e_λ. As for any statement concerning principal symbol, it is therefore admissable to check the claim in a "model situation." Model Q as the unit circle bundle in the tautological line bundle over $\mathbb{P}^n$. In this model, the coherent states are explicitly known, and their squared norm is $\|e^{(m)}\|^2 = (2\pi)^{-n}(m+n)!/n!$ (see e.g. [28]), in accordance with the statement of the lemma. □

Let us now choose $\Phi^n = e^n_{\lambda_0}$. The two lemmas (together with (5.6)) show that

$$\frac{\|T^{(m)}_f \Phi^{(m)} - f(x_0)\Phi^{(m)}\|}{\|\Phi^{(m)}\|} = O(m^{-1}) \,.$$

But this clearly gives (5.2) by the triangle inquality. □

Remark. The fact that the coherent states $e^{(m)}_\lambda$ are Fourier modes of a Hermite distribution, together with Lemma 1, may be used to derive a number of their asymptotic properties by microanalytic means. For example:

(1) If $\tau(\lambda) \neq \tau(\mu)$, then

$$\langle e^{(m)}_\lambda, e^{(m)}_\lambda \rangle = O(m^{-\infty}) \,,$$

i.e. the coherent states are "peaked" at their base point.

(2) Let $f \in \mathscr{P}(M)$ and $U^{(m)}(t) = \exp(-\mathrm{i} mtT_f^{(m)})$ the corresponding time evolution operator. If F^t denotes the Hamiltonian flow for f, one has

$$\frac{\|U^{(m)}(t)e_\lambda^{(m)} - e_{F^t(\lambda)}^{(m)}\|}{\|e_\lambda^{(m)}\|} = O(m^{-\frac{1}{2}}),$$

i.e. the coherent states move according to the laws of classical mechanics.

Proof of Lemma 1. Consider the following distribution on S^1:

$$w(\varphi) = \sum_{m=-\infty}^{\infty} e^{\mathrm{i} m\varphi}\|\Phi^{(m)}\|^2 = \langle \Phi|\, e^{\mathrm{i}\varphi D_\varphi}\, |\Phi\rangle\,. \tag{5.9}$$

Since the singular support of Φ is λ_0 and the singular support of $e^{\mathrm{i}\varphi D_\varphi}\Phi$ is $e^{\mathrm{i}\varphi}\lambda_0$, the distribution w is well-defined and smooth away from $\varphi \in 2\pi\mathbb{Z}$. Let us study the singularity at $\varphi = 0$. (We may disregard the smooth part because the Fourier components of a smooth function on S^1 go to zero faster than any power.) Using the above local coordinates, one computes (mod smooth terms), using Parseval's identiy

$$\begin{aligned} w(\varphi) &= \int_Q \overline{\Phi(\lambda)}\Phi(e^{\mathrm{i}\varphi}\lambda)\,d\nu(\lambda) = (2\pi)^{-q}\int e^{\mathrm{i}\varphi\eta_1}\left|a\left(\eta_1, \frac{\eta'}{\sqrt{|\eta_1|}}\right)\right|^2 d\eta \\ &= (2\pi)^{-q}\int e^{\mathrm{i}\varphi\eta_1}|\eta_1|^{\frac{q-1}{2}}\left(\int |a(\eta_1,\eta')|^2\,d\eta'\right)d\eta_1\,. \end{aligned}$$

Since

$$g(\eta_1) = (2\pi)^{-(q-1)}|\eta_1|^{\frac{q-1}{2}}\int |a(\eta_1,\eta')|^2\,d\eta'$$

is a classical symbol of order $\dfrac{q-1}{2} + 2\left(r - \dfrac{1}{4}\right) = 2r + \dfrac{q}{2} - 1$ in the sense of Hörmander, this is a classical Fourier integral of order $2r+\dfrac{q}{2}-\dfrac{3}{4}$. The full distribution is mod C^∞

$$w(\varphi) = (2\pi)^{-1}\sum_{k\in\mathbb{Z}}\int e^{\mathrm{i}\tau(\varphi+2\pi k)}g(\tau)\,d\tau\,.$$

With Poisson's summation formula, this can be rewritten as a sum over the Fourier transforms:

$$w(\varphi) = \sum_{m\in\mathbb{Z}} g(m)e^{\mathrm{i} m\varphi}\,.$$

This shows that $\|\Phi^{(m)}\|^2 = g(m) \bmod m^{-\infty}$. The lemma now follows using the asymptotic expansion of the symbol g. □

Acknowledgement. We are very much indebted to J.B. Bost for useful hints and discussion. In particular, he suggested the use of the global Toeplitz operators.

References

1. Arnol'd, V.I.: Mathematical methods of classical mechanics. Berlin, Heidelberg, New York: Springer 1980
2. Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A., Sternheimer, D.: Deformation theory and quantization. Ann. Phys. **111**, 61–110 (part I), 111–151 (part II) (1978)
3. Berezin, F.A.: Quantization. Math. USSR Izv. **8**, no. 5, 1109–1165 (1974); Quantization in complex symmetric spaces. Math. USSR Izv. **9**, no. 2, 341–379 (1975); General concept of quantization. Commun. Math. Phys. **40**, 153–174 (1975)
4. Bordemann, M., Hoppe, J.: The dynamics of relativistic membranes. I: Reduction to 2-dimensional fluid dynamics. To appear in Phys. Lett. B
5. Bordemann, M., Hoppe, J., Schaller, P., Schlichenmaier, M.: $gl(\infty)$ and geometric quantization. Commun. Math. Phys. **138**, 209–244 (1991)
6. Bordemann, M., Meinrenken, E., Römer, H.: Total space quantization of Kähler manifolds. Preprint April 1993, Freiburg THEP 93/5
7. Boutet de Monvel, L., Guillemin, V.: The spectral theory of Toeplitz operators. Ann. Math. Studies, Nr. 99, Princeton, NJ: Princeton University Press 1981
8. Boutet de Monvel, Sjöstrand, J.: Sur la singularité des noyaux de Bergman et de Szegö. Astérisque **34–34**, 123–164 (1976)
9. Cahen, M., Gutt, S., Rawnsley, J.H.: Quantization of Kähler manifolds I: Geometric interpretation of Berezin's quantization. JGP **7**, no. 1, 45–65 (1990); Quantization of Kähler manifolds II. Preprint (1991)
10. Calabi, E.: Isometric imbedding of complex manifolds. Ann. Math. **58**, 1–23 (1953)
11. DeWilde, M., Lecomte, P.B.A.: Existence of star products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds. Lett. Math. Phys. **7**, 487–496 (1983)
12. Dixmier, J.C.: C^*-algebras. Amsterdam, New York, Oxford: North-Holland 1977
13. Ebin, D.G., Marsden, J.: Group of diffeomorphisms and the motion of an incompressible fluid. Ann. Math. **92**, 102–163 (1970)
14. Fairlie, D., Fletscher, P., Zachos, C.N.: Trigonometric structure constants for new infinite algebras. Phys. Lett. B **218**, 203 (1989)
15. Farkas, H.M., Kra, I.: Riemann surfaces. Berlin, Heidelberg, New York: Springer 1980
16. Gilkey, P.: Invariance theory, the heat equation, and the Atiyah-Singer index theorem. Wilmington: Publish or Perish 1984
17. Griffiths, Ph., Harris, J.: Principles of algebraic geometry. New York: John Wiley 1978
18. Guillemin, V.: Some classical theorems in spectral theory revisited. Seminars on singularities of solutions of linear partial differential equations. Ann. Math. Studies, Nr. 91, Hörmander, L., (ed.) Princeton, NJ: Princeton University Press 1979, pp. 219–259
19. Guillemin, V.: Symplectic spinors and partial differential equations. Géométrie symplectic et physique mathématique (Aix en Provence 1974), Coll. Int. CNRS 237, pp. 217–252
20. Guillemin, V., Uribe, A.: Circular symmetry and the trace formula. Invent. Math. **96**, 385–423 (1989)
21. Hörmander, L.: Fourier integral operators I. Acta Math. **127**, 79–183 (1971)
22. Hörmander, L.: The analysis of linear partial differential operators. Vol. I–IV, Berlin, Heidelberg, New York: Springer 1985
23. Hoppe, J.: Quantum theory of a relativistic surface .... MIT PhD Thesis 1982, Elem. Part. Res. J. (Kyoto) **83** (1989/1990), no. 3
24. Klimek, S., Lesniewski, A.: Quantum Riemann surfaces: I. The unit disc. Commun. Math. Phys. **146**, 103–122 (1992); Quantum Riemann Surfaces: II. The discrete series. Lett. Math. Phys. **24**, 125–139 (1992)
25. Meinrenken, E.: Semiklassische Näherungen und mikrolokale Analysis. Dissertation (in preparation), Fak. für Physik, University of Freiburg
26. Mumford, D.: Algebraic geometry I. Complex projective varieties. Berlin, Heidelberg, New York: Springer 1976
27. Mumford, D.: Curves and their Jacobians. Ann. Arbor: The University of Michigan Press 1976
28. Rawnsley, J.H.: Coherent states and Kähler manifolds. Quart. J. Math. Oxford **28**, no. 2, 403–415 (1977)
29. Schlichenmaier, M.: An introduction to Riemann surfaces, algebraic curves and moduli spaces. Lecture Notes in Physics **322**. Berlin, Heidelberg, New York: Springer 1989

30. Simon, B.: The classical limit of quantum partition functions. Commun. Math. Phys. **71**, 247–276 (1980)
31. Tuynman, G.M.: Generalized Bergman kernels and geometric quantization. J. Math. Phys. **28**, 573–583 (1987)
32. Tuynman, G.M.: Quantization: Towards a comparison between methods. J. Math. Phys. **28**, 2829–2840 (1987)
33. Wells, R.O.: Differential analysis on complex manifolds. Berlin, Heidelberg, New York: Springer 1980
34. Woodhouse, N.: Geometric quantization. Oxford: Clarendon Press 1980

Communicated by A. Jaffe