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# TOLERANCE OF NONLINEARITIES IN LINEAR OPTICAL REGULATORS WITH INTEGRAL FEEDBACK*

The letter generalises some of the known theory of optimal regulators with proportional plus integral feedback. It is also shown that the fact of optimality implies that the regulators have some very desirable properties from an engineering point of view.

In Reference 1, the theory of linear optimal-control systems with input-derivative constraints is solved; i.e. for a system (time-invariant for simplicity) having state equations

$$\dot{x} = Fx + Gu \qquad (1)$$

and performance index

$$V\{x(t_0), u(.), t_0\} = \int_{t_0}^{\infty} (x'Qx + u'Ru + \dot{u}'S\dot{u})dt \qquad (2)$$

an optimal-control law $u^*(.)$ is determined. In Reference 2, it is shown that this control law may be realised by proportional plus integral feedback (at least for single-input systems), and thus the closed-loop system has the property that its equilibrium point (or operating point) is not affected by constant-input disturbances of unknown magnitude. This is, of course, an interesting and useful result, but, from an engineering viewpoint, the fact of optimality is only significant if it means desirable properties, such as a prescribed degree of stability, good phase and gain margins, good sensitivity properties and the ability to tolerate nonlinearities without going unstable. This letter shows that, as for the standard regulator,[3] these desirable properties are, in fact, assured to within very reasonable limits by virtue of optimality (irrespective of the particular $Q$, $R$ and $S$ chosen within the constraints set by the theory). We proceed by first deriving results for optimal regulators with integral feedback in a more general and a simpler form than that given in Reference 2.

Augmenting the system of eqn. 1 with integrators at the inputs and defining new variables and constants

$$x_1 = \begin{bmatrix} x \\ u \end{bmatrix} \qquad u_1 = \dot{u} \qquad (3)$$

$$F_1 = \begin{bmatrix} F & G \\ 0 & 0 \end{bmatrix} \quad G_1 = \begin{bmatrix} 0 \\ I \end{bmatrix} \quad R_1 = S \quad Q_1 = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \qquad (4)$$

results in an augmented system

$$\dot{x}_1 + F_1 x_1 + G_1 u_1 \qquad (5)$$

and performance index

$$V\{x_1(t_0), u_1(.), t_0\} = \int_{t_0}^{\infty} (u_1' R_1 u_1 + x_1' Q_1 x_1)dt \qquad (6)$$

Applying standard regulator theory to eqns. 5 and 6 (for $[F_1G_1]$ completely controllable and $[F_1D_1]$ completely observable for any $D_1$ such that $D_1D_1' = Q_1$) gives that the optimal-control law $u_1^*$:

$$u_1^* = -R_1^{-1}G_1'\bar{P}x_1 \qquad (7)$$

where $\bar{P} = \lim_{t \to -\infty} P(t, T)$, with $P(., T)$ the solution of

$$-\dot{P} = PF_1 + F_1'P - PG_1R_1^{-1}G_1'P + Q_1 \quad P(T, T) = 0 \qquad (8)$$

The assumption of complete controllability ensures that $\bar{P}$ exists, and the assumption of complete observability ensures that the closed-loop system $\dot{x}_1 = (F_1 - G_1R_1^{-1}G_1'\bar{P})x$ is asymptotically stable.

Partitioning $\bar{P}$ as $\begin{bmatrix} \bar{P}_{11} & \bar{P}_{21}' \\ \bar{P}_{21} & \bar{P}_{22} \end{bmatrix}$, and applying the definitions of eqns. 3 and 4 to the expression for $u_1^*$ in eqn. 7, gives the optimal control $u^*$ for the system of eqn. 1 and the index of eqn. 2 as follows:

$$\dot{u}^* = u_1^* = -R_1^{-1}G_1'\bar{P}x_1 = -S^{-1}\bar{P}_{21}x - S^{-1}\bar{P}_{22}u^* \quad (9)$$

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Now when $(G'G)$ is positive definite, or, equivalently, when the rank of $G$ is equal to the number of system inputs (as will usually be the case), $u$ may be expressed using eqn. 1 as $u = (G'G)^{-1} \times G'(\dot{x} - Fx)$. Eqn. 9 may then be written as

$$\dot{u}^* = K_1'x + K_2'u^* \qquad (10)$$

$$\dot{u}^* = K_3'\dot{x} + K_4x \qquad (11)$$

$$K_1' = -S^{-1}\bar{P}_{21} \quad K_2' = -S^{-1}\bar{P}_{22} \quad K_3' = K_2'(G'G)^{-1}G' \quad K_4' = K_1' - K_3'F \qquad (12)$$

We observe that $u^*$ may be realised (see eqn. 11) by proportional plus integral state-variable feedback, and thus

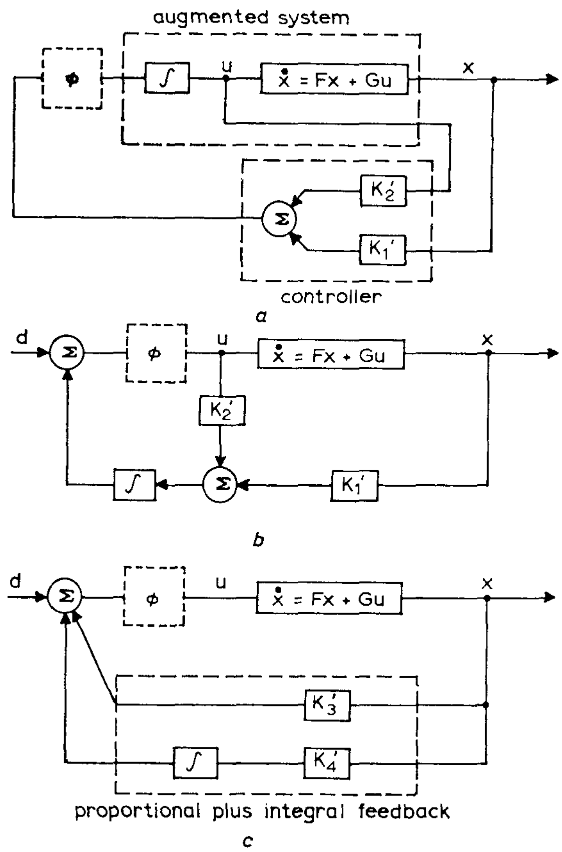


**Fig. 1** *Various forms for the one optimal regulator*

constant-input disturbances will not affect the equilibrium (or operating) point of the regulator. The expressions for $K_3$ and $K_4$ are a generalisation and a simplification on the corresponding ones given in References 2.

We now ask if systems with the control laws of eqns. 10 or 11 will be asymptotically stable when there are non-

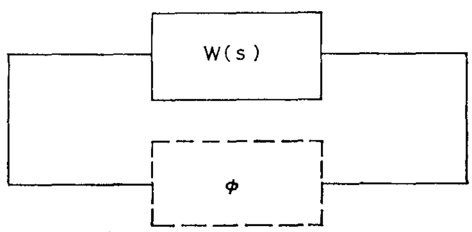


**Fig. 2** *Linear system with feedback nonlinearities*

linearities at the input transducers to the plant of eqn. 1. Such a question has been asked for standard regulators in Reference 4. Applying the theory of Reference 4 gives that the system of Fig. 1*a* is asymptotically stable, where $\phi$ is an arbitrary (possibly time-varying) nonlinearity in the sector $[\frac{1}{2}, \infty)$. Now all three systems in Fig. 1 can be rearranged as a linear subsystem $W(s)$ with feedback nonlinearities $\phi$, as in

Fig. 2. We note that the transfer functions for the three linear subsystems denoted by $W_a(s)$, $W_b(s)$ and $W_c(s)$, respectively, are

$$W_a(s) = s^{-1}K_1'(sI - F)^{-1}G + s^{-1}K_2' \quad . \quad . \quad . \quad (14)$$

$$\begin{aligned} W_b(s) &= K_1'(sI - F)^{-1}Gs^{-1} + K_2's^{-1} \\ &= sW(s)s^{-1} = W_a(a) \\ W_c(s) &= (s^{-1}K_4' + K_3')(sI - F)^{-1}G \\ &= \{s^{-1}K_1' + s^{-1}K_3'(sI - F)\}(sI - F)^{-1}G \\ &= s^{-1}K_3'G + s^{-1}K_1'(sI - F)^{-1}G \\ &= W_a(s) \end{aligned}$$

We conclude that, since all three systems of Fig. 1 have the form of a linear subsystem $W_a(s)$ given by eqn. 14 with feedback nonlinearities $\phi$ and since the system of Fig. 1*a* is asymptotically stable from known regulator theory,[4] so also are the regulators of Figs. 1*b* and *c*.

The above results also indicate that the gain-margin and phase-margin properties of single-input linear regulators and their ability to tolerate time delays within the closed loop and still remain asymptotically stable carry over to the optimal regulators with integral feedback. Moreover, the fact that the systems of Figs. 1*b* and *c* are standard regulators in disguise indicate that they have good sensitivity properties and that they can be designed quite simply to have a prescribed degree of stability (see Reference 3 for background details).

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