Tomographic entropy for spin systems

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Abstract. We study the properties of spin tomograms and tomographic entropy and information. The specific structure of joint probability distributions of standard probability theory are compared with the spin-tomogram properties for two-qubits.

1. Introduction

As it was shown in [1, 2, 3, 4, 5] for a particle with spin j, the spin states can be described by spin tomograms. The tomograms are probability distributions of random spin projection onto the direction in a space labeled by a point on a sphere. For two spins, the description of quantum spin states by means of joint tomographic probability distributions was introduced in [6] on the example of a top.

In this approach, one can use standard probabilities instead of vectors in the Hilbert space or density operators. Thus, all the problems of quantum states can be reformulated as corresponding problems of standard probability theory. The advantage of the probability representation of quantum mechanics (tomographic-probability approach) was used in [7, 8, 9] to introduce such characteristics of probability as Shannon entropy and Shannon information [10] to be new characteristics of quantum states called tomographic entropy and tomographic information (or probability representation entropy and information).

The probability representation for continuous variables and quantum Fokker–Planck-type evolution equation for quantum systems, which is equivalent to the Schrödinger equation [11] or von Neumann equation [12] was introduced in [13].

The application of the probability representation to study entanglement [14] was suggested in [15, 16, 17] (it was also employed in some problems in [18, 19, 20]. In [21] it was proposed to apply the probability description of quantum states to the problem of quantum nonlocality. The Bell's inequality [22] violation problem for two qubits was reduced to the problem of standard probability theory of describing the joint probability distribution for two random variables [23].

The aim of this paper is to review new aspects of the probability representation of [7, 23], to study the entropy and information of quantum spin states, and to visualize the probability-theory meaning of Bell's inequality for two qubits. The Bell inequalities in the case of mixed

quantum states were considered in [24]. The partial scaling transform as the entanglement criterion for symplectic tomograms of Gaussian states was suggested in [25].

2. Joint probability distributions

In this section, we remind some elementary properties of probability distributions in standard probability theory, which are relevant to the properties of spin tomograms.

Given four positive numbers $P_{\mathrm{I}}, P_{\mathrm{II}}, P_{\mathrm{III}}, P_{\mathrm{IV}}$ satisfying the equality

$$P_{\mathrm{I}} + P_{\mathrm{II}} + P_{\mathrm{III}} + P_{\mathrm{IV}} = 1. \tag{1}$$

The four numbers can be considered as the probability distribution of one random variable $P(X)$. The meaning of the probability distribution can be different in various physical (and others) problems where the four numbers appear.

Let us now to consider indices of the four numbers labeled by double indices, namely,

$$\mathrm{I} \to 11, \qquad \mathrm{II} \to 12, \qquad \mathrm{III} \to 21, \qquad \mathrm{IV} \to 22.$$

We obtain the same four numbers given in the form

$$P_{jk} \equiv P(j,k), \qquad j,k = 1,2$$

satisfying the condition

$$\sum_{j,k=1}^{2} P(j,k) = 1. \tag{2}$$

Now the four numbers could be interpreted as a given probability distribution for two random variables $P(X,Y)$ associated with indices j and k.

The given numbers have intrinsic characteristics, for example, the function

$$\xi_P(\lambda) = -\sum_{j,k=1}^{2} P(j,k) \exp\left(\lambda \ln P(j,k)\right) = \langle \exp \lambda \ln P(j,k) \rangle, \tag{3}$$

which can be called the generating function since

$$\xi_P(\lambda) = -1 - \lambda\left(\sum_{j,k=1}^{2} P(j,k) \ln P(j,k)\right) - \cdots - \frac{\lambda^n}{n!} \sum_{j,k=1}^{2} P(j,k)\left(\ln P(j,k)\right)^n - \cdots. \tag{4}$$

The coefficient in front of λ is Shannon entropy

$$H = -\sum_{j,k=1}^{2} P(j,k) \ln P(j,k) \tag{5}$$

associated with given probability distributions. Other terms in the series determine the higher momenta of the function $\ln P(j,k)$.

Obviously, the same entropy can be obtained using the initial labeling of the four numbers as $P_{\mathrm{I}}, P_{\mathrm{II}}, P_{\mathrm{III}}$, and P_{IV} associating the set of numbers as the probability distribution for one random variable.

One can address the question.

What is a special characteristics of the probability distributions corresponding to the absence of correlations of two random variables?

In this case, one has

$$P(j,k) = \mathcal{P}(j) \cdot Q(k), \tag{6}$$

where two positive numbers $\mathcal{P}(j)$ and two positive numbers $Q(k)$ satisfy the condition

$$\sum_{j=1}^{2} \mathcal{P}(j) = 1, \qquad \sum_{k=1}^{2} Q(k) = 1. \tag{7}$$

In the initial notation, we assume that the four positive numbers can be presented in factorized form

$$P_{\text{I}} = \mathcal{P}(1)Q(1), \qquad P_{\text{III}} = \mathcal{P}(2)Q(1),$$

$$\tag{8}$$

$$P_{\text{II}} = \mathcal{P}(1)Q(2), \qquad P_{\text{IV}} = \mathcal{P}(2)Q(2).$$

One sees that, in this notation, the structure of positive numbers can be interpreted as a characteristics of the probability distribution of one random variable (not joint probability distribution of two variables without their correlations).

Notation (8) implies a reformulation of the question we addressed.

What is a special characteristics of the probability distribution of one random variable for which relation (8) takes place?

At this stage, we do not point out the problem of the absence of correlations.

In fact, we address the problem — what is specific in given set of four positive numbers (quartets), if they can be represented in factorized form (8) where relations (7) are also taken into account?

To answer the question, let us take four different representatives of the set of all the "quartets" which can be given in factorized form (8), namely,

$$P_{\text{I}}^{(\lambda,\mu)} = \mathcal{P}^{(\lambda)}(1)Q^{(\mu)}(1), \qquad P_{\text{III}}^{(\lambda,\mu)} = \mathcal{P}^{(\lambda)}(2)Q^{(\mu)}(1),$$

$$\tag{9}$$

$$P_{\text{II}}^{(\lambda,\mu)} = \mathcal{P}^{(\lambda)}(1)Q^{(\mu)}(2), \qquad P_{\text{IV}}^{(\lambda,\mu)} = \mathcal{P}^{(\lambda)}(2)Q^{(\mu)}(2).$$

Here the parameters λ and μ take four values which we denote as $a, b, c,$ and d belonging to some domain. In fact, the physical motivation for considering the distributions labeled by two parameters μ and ν is related to the comparison of the distributions and their characteristics with probabilistic properties of Bell's inequalities where the parameters μ and ν will be identified with directions of two spins in the space (see below Section 5). But formally, at this stage of the discussion, the parameters μ and ν are used to label a family of probability distributions without any connection with physical problems.

Thus, one can say that we consider probability distributions labeled by λ and μ.

Let us interpret now the four numbers $P^{(\lambda,\mu)}(j,k)$ for given λ and μ as the joint probability distribution of dichotomic random variables M_1 and M_2 which take values ± 1. Namely, we interpret $\mathcal{P}^{(\lambda)}(1)$ as the probability to have the first random variable $M_1 = +1$, $\mathcal{P}^{(\lambda)}(2)$ as the probability of the first random variable to take the value -1. The same interpretation is valued for numbers $Q^{(\mu)}(1)$ and $Q^{(\mu)}(2)$ associated with the second random variable M_2.

Having in mind this interpretation, we calculate four expectation values for the product of two random variables M_1 and M_2 for the probability distributions $P^{(\lambda,\mu)}$.

We have four numbers:

$$I_{ab} = \langle M_1 M_2 \rangle_{\lambda=a,\mu=b}$$

$$= \mathcal{P}^{(a)}(1)Q^{(b)}(1) - \mathcal{P}^{(a)}(1)Q^{(b)}(2) - \mathcal{P}^{(a)}(2)Q^{(b)}(1) + \mathcal{P}^{(a)}(2)Q^{(b)}(2),$$

$$I_{ac} = \langle M_1 M_2 \rangle_{\lambda=a,\mu=c}$$

$$= \mathcal{P}_1^{(a)}(1)Q^{(c)}(1) - \mathcal{P}^{(a)}(1)Q^{(c)}(2) - \mathcal{P}^{(a)}(2)Q^{(c)}(1) + \mathcal{P}^{(a)}(2)Q^{(c)}(2),$$

$$(10)$$

$$I_{dc} = \langle M_1 M_2 \rangle_{\lambda=d,\mu=c}$$

$$= \mathcal{P}^{(d)}(1)Q^{(c)}(1) - \mathcal{P}^{(d)}(1)Q^{(c)}(2) - \mathcal{P}^{(d)}(2)Q^{(c)}(1) + \mathcal{P}^{(d)}(2)Q^{(c)}(2),$$

$$I_{db} = \langle M_1 M_2 \rangle_{\lambda=d,\mu=b}$$

$$= \mathcal{P}^{(d)}(1)Q^{(b)}(1) - \mathcal{P}^{(d)}(1)Q^{(b)}(2) - \mathcal{P}^{(d)}(2)Q^{(b)}(1) + \mathcal{P}^{(d)}(2)Q^{(b)}(2).$$

One can rewrite these numbers without interpreting them as mean values in the form where the initial notation was used:

$$I_{ab} = P_{\mathrm{I}}^{(a,b)} - P_{\mathrm{II}}^{(a,b)} - P_{\mathrm{III}}^{(a,b)} + P_{\mathrm{IV}}^{(a,b)},$$

$$I_{ac} = P_{\mathrm{I}}^{(a,c)} - P_{\mathrm{II}}^{(a,c)} - P_{\mathrm{III}}^{(a,c)} + P_{\mathrm{IV}}^{(a,c)},$$

$$(11)$$

$$I_{db} = P_{\mathrm{I}}^{(d,b)} - P_{\mathrm{II}}^{(d,b)} - P_{\mathrm{III}}^{(d,b)} + P_{\mathrm{IV}}^{(d,b)},$$

$$I_{dc} = P_{\mathrm{I}}^{(d,c)} - P_{\mathrm{II}}^{(d,c)} - P_{\mathrm{III}}^{(d,c)} + P_{\mathrm{IV}}^{(d,c)},$$

with the positive numbers $P_{\mathrm{I}}^{(\lambda,\mu)}$, $P_{\mathrm{II}}^{(\lambda,\mu)}$, $P_{\mathrm{III}}^{(\lambda,\mu)}$, and $P_{\mathrm{IV}}^{(\lambda,\mu)}$ being of the form (9).

Now we take the linear combination of the four numbers obtained as follows:

$$I = I_{ab} + I_{ac} + I_{db} - I_{dc}. \tag{12}$$

One can easily see that

$$|I| \leq 2. \tag{13}$$

This inequality follows from the existence of decomposition (9) for given four distribution functions labeled by the parameters (a,b), (a,c), (d,b), and (d,c).

Let us give a numerical example where four probability distributions have the form

$$P_{\mathrm{I}}^{(a,b)} = \frac{1}{200}, \quad P_{\mathrm{II}}^{(a,b)} = \frac{19}{200}, \quad P_{\mathrm{III}}^{(a,b)} = \frac{9}{200}, \quad P_{\mathrm{IV}}^{(a,b)} = \frac{171}{200},$$

$$P_{\mathrm{I}}^{(a,c)} = \frac{1}{300}, \quad P_{\mathrm{II}}^{(a,c)} = \frac{29}{300}, \quad P_{\mathrm{III}}^{(a,c)} = \frac{9}{300}, \quad P_{\mathrm{IV}}^{(a,c)} = \frac{261}{300},$$

$$(14)$$

$$P_{\mathrm{I}}^{(d,b)} = \frac{1}{160}, \quad P_{\mathrm{II}}^{(d,b)} = \frac{19}{160}, \quad P_{\mathrm{III}}^{(d,b)} = \frac{7}{160}, \quad P_{\mathrm{IV}}^{(d,b)} = \frac{133}{160},$$

$$P_{\mathrm{I}}^{(d,c)} = \frac{1}{240}, \quad P_{\mathrm{II}}^{(d,c)} = \frac{29}{240}, \quad P_{\mathrm{III}}^{(d,c)} = \frac{7}{240}, \quad P_{\mathrm{IV}}^{(d,c)} = \frac{203}{240}.$$

One can address the question: Whether a decomposition of the form (9) exists or not? To answer the question, one can use the criterion (13).

The sum I in Eq. (12) calculated for these four probability distributions yields

$$I = \frac{173}{120}.$$

Inequality $|I| \leq 2$ is necessary condition for the possibility to represent the family of probability distributions in the form (9). If one finds that this inequality is violated, the decomposition (9) is impossible. But this condition is not sufficient condition.

Since the number $|I| \leq 2$, one can hope that decomposition (9) really exists. In fact, one can check that the four probability distributions can be connected by Eqs. (8) and (9) with the other four probability distributions determined by

$$\mathcal{P}^{(a)}(1) = \frac{1}{10}, \quad \mathcal{P}^{(d)}(1) = \frac{1}{8}, \quad Q^{(b)}(1) = \frac{1}{20}, \quad Q^{(c)}(1) = \frac{1}{30}. \tag{15}$$

The numerical example was specially chosen (numbers (15) were chosen) to get the family of four distributions (14). The inverse problem of finding the decomposition (9) for given family of probability distributions can have solution if $|I| \leq 2$ but we have no explicit expression for such a solution even in the case where the solution exists.

Inequality (13) obviously takes place also for convex sum generalization of Eq. (8). Namely, for

$$P_{\text{I}} = \sum_q p_q \mathcal{P}^{(q)}(1) Q^{(q)}(1), \quad P_{\text{II}} = \sum_q p_q \mathcal{P}^{(q)}(1) Q^{(q)}(2),$$

$$\tag{16}$$

$$P_{\text{III}} = \sum_q p_q \mathcal{P}^{(q)}(2) Q^{(q)}(2), \quad P_{\text{IV}} = \sum_q p_q \mathcal{P}^{(q)}(2) Q^{(q)}(2),$$

where

$$p_q \geq 0 \qquad \text{and} \qquad \sum_q p_q = 1,$$

the proof of inequality (13) is straightforward.

One can call such sets of positive numbers as "separable" ones. In the following sections, we apply the clarified properties of joint probability distributions in classical probability theory to quantum mechanical problem of entangled states. To do this, we consider the quantum states within the framework of the probability representation. In this representation, the quantum state is described by the probability distribution instead of the wave function or density matrix. This representation is convenient to consider entanglement in composed systems.

3. Tomography for two-spin particle

Below we consider two subsystems of a composed system and remind the spin tomographic approach.

Let us consider two particles with spin j_1 and j_2, respectively. The basis in the Hilbert space of states can be given by the product vector

$$|m_1 m_2\rangle = |j_1 m_1\rangle |j_2 m_2\rangle. \tag{17}$$

The density matrix ρ of a system state can be mapped onto the spin tomogram [6, 17]

$$\omega(m_1, m_2, \vec{n}_1, \vec{n}_2) = \langle m_1 m_2 | D^{(j_1)\dagger}(u_1) \otimes D^{(j_2)\dagger}(u_2) \rho D^{(j_2)}(u_2) \otimes D^{(j_1)}(u_1) | m_1 m_2 \rangle. \tag{18}$$

Here $D^{(j)}(u)$ is the matrix of irreducible representation of the $SU(2)$ group.

The matrix u (also the matrices u_1 and u_2 in (18) is 2×2 unitary matrix depending on three Euler angles φ, θ and ψ, i.e.,

$$u = \begin{pmatrix} \cos \frac{\theta}{2} \, e^{i(\varphi+\psi)/2} & \sin \frac{\theta}{2} \, e^{i(\varphi-\psi)/2} \\ -\sin \frac{\theta}{2} \, e^{-i(\varphi-\psi)/2} & \cos \frac{\theta}{2} \, e^{-i(\varphi+\psi)/2} \end{pmatrix}.$$

The unit vectors $\vec{n}_1$ and $\vec{n}_2$ are determined by two of three Euler angles parametrizing the 2×2-matrix u of the $SU(2)$ group.

The tomogram is the joint probability distribution function for two discrete random variables m_1 and m_2, which are spin projections on the directions $\vec{n_1}$ and $\vec{n_2}$, respectively.

The function is normalized

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \omega(m_1, m_2, \vec{n_1}, \vec{n_2}) = 1. \tag{19}$$

The tomographic entropy $S(\vec{n_1}, \vec{n_2})$ can be associated with this probability distribution function

$$S(\vec{n_1}, \vec{n_2}) = -\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \omega(m_1, m_2, \vec{n_1}, \vec{n_2}) \ln \omega(m_1, m_2, \vec{n_1}, \vec{n_2}). \tag{20}$$

The tomographic entropy depends on the points on two spheres determined by unit vectors $\vec{n_1}$ and $\vec{n_2}$. The tomographic probability for two particles determines the tomographic probability for one particle, e.g.,

$$\omega(m_1, \vec{n_1}) = \sum_{m_2=-j_2}^{j_2} \omega(m_1, m_2, \vec{n_1}, \vec{n_2}). \tag{21}$$

The dependence on vector $\vec{n}_2$ on the left-hand side of (21) disappears in view of the sum over m_2. This property follows due to the structure of formula (18) and the orthogonality condition for matrix elements of irreducible representations of the group $SU(2)$.

The entropy for one particle reads

$$S(\vec{n}) = -\sum_{m=-j}^{j} \omega(m, \vec{n}) \ln \omega(m, \vec{n}). \tag{22}$$

In [25] the unitary spin tomogram was introduced, in view of the relationship

$$\omega\Big(m_1, m_2, U(n)\Big) = \langle m_1 m_2 | U^+(n) \rho U(n) | m_1 m_2 \rangle,$$

$$n = (2j_1 + 1)(2j_2 + 1). \tag{23}$$

This tomogram is the joint probability distribution function depending on the unitary group element $U(n)$.

It is normalized for each group element, i.e.,

$$\sum_{m_1=j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \omega\Big(m_1, m_2, U(n)\Big) = 1. \tag{24}$$

Also

$$\int \omega\Big(m_1, m_2, U(n)\Big) \, dU(n) = 1, \tag{25}$$

where $\int dU(n) = 1$.

The meaning of integration over the group element $U(n)$ in (25) is the standard integral over n^2 parameters determining the unitary matrix $U(n)$ and the integration is fulfilled using the Haar measure. We applied the normalization by including the unitary-group volume into the measure.

The joint tomographic probability (23) determines the tomographic probability for one particle depending on the unitary group element

$$\omega_1\Big(m_1, U(n)\Big) = \sum_{m_2=-j_2}^{j_2} \omega\Big(m_1, m_2, U(n)\Big). \tag{26}$$

An analogous probability can be obtained for the second spin

$$\omega_2\Big(m_2, U(n)\Big) = \sum_{m_1=-j_1}^{j_1} \omega\Big(m_1, m_2, U(n)\Big). \tag{27}$$

We can associate with the tomographic probability the entropy, which is the function on the unitary group

$$S_1\Big(U(n)\Big) = -\sum_{m_1=-j_1}^{j_1} \omega\Big(m_1, U(n)\Big) \ln \omega\Big(m_1, U(n)\Big). \tag{28}$$

Also the tomographic entropy related to the tomogram (23) depends on the unitary-group element

$$S\Big(U(n)\Big) = -\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \omega\Big(m_1, m_2, U(n)\Big) \ln \omega\Big(m_1, m_2, U(n)\Big). \tag{29}$$

This entropy depends on the unitary-group parameters.

For the matrix

$$U(n) = D^{(j_1)}(u_1) \otimes D^{(j_2)}(u_2),$$

the entropy (29) coincides with the entropy (20).

We can construct also the conditional probability distribution for the first spin

$$\omega_{C1}\Big(m_1, U(n)\Big) = \frac{\omega\Big(m_1, m_2, U(n)\Big)}{\sum_{m_1=-j_1}^{j_1} \omega\Big(m_1, m_2, U(n)\Big)}. \tag{30}$$

An analogous formula can be written for the second spin.

4. Information, entropy and their relation to von Neumann entropy

Let tomographic information $I_{Y \leftrightarrow X}$, where X is the j_1-spin system and Y is the j_2-spin system, be given by a function on the unitary group

$$I_{j_1 \leftrightarrow j_2}\Big(U(n)\Big) = \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \omega\Big(m_1, m_2, U(n)\Big) \ln \frac{\omega\Big(m_1, m_2, U(n)\Big)}{\omega_1\Big(m_1, U(n)\Big)\omega_2\Big(m_2, U(n)\Big)}. \tag{31}$$

Here $\omega_2\Big(m_2, U(n)\Big)$ is given by (26) with the replacement $1 \leftrightarrow 2$.

Information introduced is just Shannon information applied to system of two spins within the framework of the probability representation of quantum mechanics. The generalization of Shannon information to multiqudit system is straightforward.

Information for a system of two spins has the same meaning that any Shannon information associated with a joint probability distribution. For quantum spin state, this information is an additional characteristics of the spin correlations.

The unitary spin entropy

$$S\big(U(n)\big) = -\sum_{m=-j}^{j} \omega\big(m, U(n)\big) \ln \omega\big(m, U(n)\big) \qquad (32)$$

for the case of the spin state of a single particle defines the von Neumann entropy of this state

$$S_N = -\mathrm{Tr}\left[\rho \ln \rho\right]. \qquad (33)$$

In fact, there exist elements of the unitary group $U^{(0)}(n)$, $n = 2j + 1$, which diagonalize the density matrix ρ. For these elements $U^{(0)}(n)$, the tomogram is equal to the probability-distribution function, which exactly coincides with eigenvalues of the density matrix. This means that the tomographic entropy (32) for these values of unitary group elements is equal to the von Neumann entropy of the spin state, i.e.,

$$S\big(U^{(0)}(n)\big) = S_N. \qquad (34)$$

On the other hand, it is obvious that for the elements $U^{(0)}(n)$ the tomographic entropy takes the minimum possible value. It follows from the property that the probability distributions determined by the density-matrix diagonal elements for the matrix obtained by means of unitary rotation of the basis are smoother than the distributions provided by eigenvalues of the density matrix [17].

We will demonstrate this property of the probability distribution given by diagonal elements of the hermitian positive density matrix

$$\rho = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}. \qquad (35)$$

The von Neumann entropy for this spin state reads

$$S_N = -p \ln p - (1-p) \ln(1-p). \qquad (36)$$

The tomogram of the state ρ is given by diagonal elements of the matrix

$$\rho_u = \begin{pmatrix} U_{11}^* & U_{21}^* \\ U_{12}^* & U_{22}^* \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix} \begin{pmatrix} U_{11} & U_{21} \\ U_{12} & U_{22}^* \end{pmatrix}, \qquad (37)$$

where the unitary matrix U_{jk} is parametrized by Euler angles. Thus one has

$$w\left(m = \frac{1}{2}, \vec{n}\right) = |U_{11}|^2 p + |U_{21}|^2 (1-p) = \cos^2 \frac{\theta_2}{2} p + \sin^2 \frac{\theta_2}{2} (1-p),$$
$$\qquad (38)$$
$$w\left(m = -\frac{1}{2}, \vec{n}\right) = |U_{12}|^2 p + |U_{22}|^2 (1-p) = \sin^2 \frac{\theta_2}{2} p + \cos^2 \frac{\theta_2}{2} (1-p).$$

One can see that the probability distribution determined by the numbers $w\left(m = \frac{1}{2}, \vec{n}\right)$ and $w\left(m = -\frac{1}{2}, \vec{n}\right)$ is smoother than the probability distribution determined by numbers p and $(1-p)$. This means that the numbers $w\left(m = \frac{1}{2}, \vec{n}\right)$ and $w\left(m = -\frac{1}{2}, \vec{n}\right)$ are contained in the interval between two numbers p and $(1-p)$. Consequently, entropy related to tomographic-probability distribution is greater than the von Neumann entropy. The analogous consideration can be repeated for $n \times n$-unitary matrix $U(n)$.

Thus the von Neumann entropy is the minimum of tomographic entropy.

One can introduce complete mutual information for two spins using the relationship

$$I_{j_2 \leftrightarrow j_1}\big(U(n)\big) = S_1\big(U(n)\big) + S_2\big(U(n)\big) - S\big(U(n)\big), \qquad (39)$$

where the tomographic entropies are functions on the unitary group and the information introduced is also the function on the unitary group.

5. Bell's inequality

Separable states of two spin-1/2 particles are defined as the states with the density operator of the form

$$\hat{\rho}(1,2) = \sum_j p_j \hat{\rho}_1^{(j)}(1) \otimes \hat{\rho}_2^{(j)}(2), \quad p_j \geq 0, \quad \sum_j p_j = 1. \tag{40}$$

In the tomographic probability representation, this definition is equivalent to the property of the state tomogram [15, 16, 17]

$$w(m_1, m_2, \vec{n}_1, \vec{n}_2) = \sum_j p_j w_1^{(j)}(m_1, \vec{n}_1) w_2^{(j)}(m_2, \vec{n}_2). \tag{41}$$

For the case of $(\vec{n}_1 = \vec{a}, \vec{n}_2 = \vec{b})$, $(\vec{n}_1 = \vec{a}, \vec{n}_2 = \vec{c})$, $(\vec{n}_1 = \vec{d}, \vec{n}_2 = \vec{b})$, and $(\vec{n}_1 = \vec{d}, \vec{n}_2 = \vec{c})$, one has four distributions of the form discussed in Section 2. Thus, one has the inequality for numbers $M_1 = 2m_1$, $M_2 = 2m_2$ of the form given by Eq. (13) which is the property of four distributions of the form (41), i.e., (13). It is just the Bell's inequality related to quantum locality (or hidden variables) approach.

In the tomographic approach, this inequality is the standard property of distribution functions of the form (8) (and the convex sum generalizations of the form (8)).

Two qubit states with the density matrix

$$P = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{42}$$

in the standard basis is determined by the tomographic probability [23] of the form (see Appendix)

$$w\left(\frac{1}{2}, \frac{1}{2}, \vec{n}_1, \vec{n}_2\right) = w\left(-\frac{1}{2}, -\frac{1}{2}, \vec{n}_1, \vec{n}_2\right)$$
$$= \frac{1}{2}\left(\cos^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2} + \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}\right) + \frac{1}{4}\sin\theta_1\sin\theta_2\cos(\varphi_1 + \varphi_2),$$

$$\tag{43}$$

$$w\left(\frac{1}{2}, -\frac{1}{2}, \vec{n}_1, \vec{n}_2\right) = w\left(-\frac{1}{2}, \frac{1}{2}, \vec{n}_1, \vec{n}_2\right)$$
$$= \frac{1}{2}\left(\cos^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2} + \sin^2\frac{\theta_1}{2}\cos^2\frac{\theta_2}{2}\right) - \frac{1}{4}\sin\theta_1\sin\theta_2\cos(\varphi_1 + \varphi_2),$$

where the vectors

$$(\vec{n})_{1,2} = (\sin\theta_{1,2}\cos\varphi_{1,2}, \sin\theta_{1,2}\sin\varphi_{1,2}, \cos\theta_{1,2}). \tag{44}$$

Considering four probability distributions for the angles

$$\theta_a = \pi/6, \theta_b = 0, \theta_c = \pi/3, \theta_d = -\pi/6,,$$

$$\varphi_a = \pi/4, \varphi_b = -\pi/4, \varphi_c = -\pi/4, \varphi_d = \pi/4,$$

and calculating the number I we obtain

$$I = 2.6 > 2. \tag{45}$$

This inequality means that the state with tomogram (43) or density operator (42) is entangled. On the other hand, it is the simple property of classical probability distributions which do not

satisfy relation (8) (and their convex sum generalizations). It is known that the upper limit, which can take the number I for two qubits, is the number $2\sqrt{2}$. One can address the following question: Is there the possibility to find four distributions with larger than $2\sqrt{2}$ value of number I? In fact, there obviously exist such four distributions of the form, e.g.,

$$
\begin{aligned}
P_{\mathrm{I}}^{(a,b)} &= 1, \quad P_{\mathrm{II}}^{(a,b)} = P_{\mathrm{III}}^{(a,b)} = P_{\mathrm{IV}}^{(a,b)} = 0, \\
P_{\mathrm{I}}^{(a,c)} &= P_{\mathrm{II}}^{(a,c)} = P_{\mathrm{III}}^{(a,c)} = 0, \quad P_{\mathrm{IV}}^{(a,c)} = 1,
\end{aligned}
$$

$$
\begin{aligned}
P_{\mathrm{I}}^{(d,b)} &= P_{\mathrm{IV}}^{(d,b)} = \frac{1}{2}, \quad P_{\mathrm{III}}^{(d,b)} = P_{\mathrm{II}}^{(d,b)} = 0, \\
P_{\mathrm{I}}^{(d,c)} &= P_{\mathrm{II}}^{(d,c)} = P_{\mathrm{IV}}^{(d,c)} = 0, \quad P_{\mathrm{III}}^{(d,c)} = 1.
\end{aligned}
\tag{46}
$$

For these four distributions one has

$$
I = 4 > 2\sqrt{2}.
\tag{47}
$$

These distributions cannot be expressed in the form (8). There exists also the property of the distributions that they cannot be obtained from the two-qubit-state tomogram. Such distributions can correspond to "supercorrelated" states which have stronger correlations of variables M_1 and M_2 than purely quantum correlations corresponding to entangled states. One can try to observe the correlations of this kind. If experimentally one can obtain

$$
I > 2\sqrt{2},
$$

this value corresponds to classically possible correlations of the nature, which is not related to separable quantum states. For sure, such correlations contradict to the quantum-mechanical description of both separable and entangled states but violation of the Bell's inequality in this case is from "upper" side.

6. Conclusions

To resume, we point out the main aspects of our approach presented in the paper.

We applied to analysis of the properties of quantum states the tomographic method (or the probability representation of quantum mechanics). Within the framework of the probability representation, the quantum states are described by standard positive probability distributions. Using notions and procedures of the probability theory, we introduced tomographic entropy and information. Entropy and information discussed can be used to study the properties of multipartite quantum states from a new point of view. The von Neumann entropy of spin states was shown to be the minimum of the tomographic entropy introduced. Analyzing the simplest joint probability distribution given by four positive numbers, we got the criterion of their decomposition into products of the probability distributions given by two positive numbers.

Comparing such kind of "separable" joint distributions (and their convex sum generalizations) with tomographic probability distributions describing quantum states of two qubits, we found that this criterion is exactly the Bell's inequality which usually is associated with quantum properties of spin systems. We have shown that there exists another inequality for the joint probability distribution which can be compared with the quantum-mechanical calculations. This inequality provides larger value of spin correlations which is available in a system of two qubits with the maximum possible entanglement. The relation of properties of the tomographic entropy with such "supercorrelated" joint probability distributions needs further investigations.

To conclude, we reviewed the tomographic-probability approach to entropy of spin states and to Bell's inequalities. A new result on the possibility of the analysis of two-spin supercorrelations (violating the quantum-mechanical description of two-spin states) was obtained.

Appendix
Here we provide steps to calculate the joint probability distribution (43).

We take the 4×4-matrix of tensor product of two unitary 2×2-matrices parametrized by Euler angles

$$u(4) = u_1 \otimes u_2. \tag{48}$$

In view of the definition of two-spin-state tomogram, first we construct the 4×4-matrix

$$\rho_u = u^\dagger(4) P u(4), \tag{49}$$

where P is matrix (42).

By calculating the diagonal elements of the matrix ρ_u, using Euler angle parametrization of matrices u_1 and u_2, we obtain

$$
\begin{aligned}
(\rho_u)_{11} = (\rho_u)_{44} &= \frac{1}{2} \left(\cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \right. \\
&\left. + \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \right) + \frac{1}{4} \sin \theta_1 \sin \theta_2 \cos(\varphi_1 + \varphi_2), \\
\\
(\rho_u)_{22} = (\rho_u)_{33} &= \frac{1}{2} \left(\cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \right. \\
&\left. + \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \right) - \frac{1}{4} \sin \theta_1 \sin \theta_2 \cos(\varphi_1 + \varphi_2).
\end{aligned}
\tag{50}
$$

Since the diagonal matrix elements determine the tomographic probability distribution for two qubits, one has

$$
\begin{aligned}
(\rho_u)_{11} &= w\left(\frac{1}{2}, \frac{1}{2}, \vec{n}_1, \vec{n}_2 \right), \\
(\rho_u)_{44} &= w\left(-\frac{1}{2}, -\frac{1}{2}, \vec{n}_1, \vec{n}_2 \right), \\
(\rho_u)_{22} &= w\left(\frac{1}{2}, -\frac{1}{2}, \vec{n}_1, \vec{n}_2 \right), \\
(\rho_u)_{33} &= w\left(-\frac{1}{2}, \frac{1}{2}, \vec{n}_1, \vec{n}_2 \right).
\end{aligned}
$$

In view of (50), one has explicit expression for (43).

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