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Tomographic reconstruction of 2-D vector fields: application to flow imaging

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SUMMARY

We examine the problem of reconstructing a 2-D vector field $\mathbf{v}(x, y)$ throughout a bounded region D from the line integrals of $\mathbf{v}(x, y)$ through D. This problem arises in the 2-D mapping of fluid-flow in a region D from acoustic travel-time measurements through D. For an arbitrary vector field, the reconstruction problem is in general underdetermined since $\mathbf{v}(x, y)$ has two independent components, $v_x(x, y)$ and $v_y(x, y)$. However, under the constraint that $\mathbf{v}$ is divergenceless ($\nabla \cdot \mathbf{v} = 0$) in D, we show that the vector reconstruction problem can be solved uniquely. For incompressible fluid flow, a divergenceless velocity field follows under the assumption of no sources or sinks in D.

A vector central-slice theorem is derived, which is a generalization of the well-known 'scalar' central-slice theorem that plays a fundamental role in conventional tomography. The key to the solution to the vector tomography problem is the decomposition of the field $\mathbf{v}(x, y)$ into its irrotational and solenoidal components: $\mathbf{v} = \nabla\Phi + \nabla \times \boldsymbol{\Psi}$, where $\Phi(x, y)$ and $\boldsymbol{\Psi}(x, y)$ are scalar and vector potentials. We show that the solenoidal component $\nabla \times \boldsymbol{\Psi}$ can be uniquely reconstructed from the line integrals of $\mathbf{v}$ through D, whereas the irrotational component $\nabla\Phi$ cannot. However, when the field is divergenceless in D, the scalar potential Φ solves Laplace's equation in D and can be determined by the values of $\mathbf{v}$ on the boundary of D. An explicit formula for Φ from the boundary values of $\mathbf{v}$ is derived. Consequently, $\mathbf{v}(x, y)$ can be uniquely recovered throughout the region of reconstruction from the following information: line-integral measurements of $\mathbf{v}$ through this region and $\mathbf{v}$ measured on the boundary of this region.

Key words: acoustic tomography, central-slice theorem, flow imaging, ocean acoustic tomography, tomography, vector-field tomography.

1 INTRODUCTION

Consider the problem of reconstructing a 2-D vector field $\mathbf{v}(x, y)$ from line-integral measurements of the form

$$T(\mathbf{a}, \mathbf{b}) = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v}(x, y) \cdot d\mathbf{l}, \tag{1}$$

in which $d\mathbf{l} = \boldsymbol{\tau} dl$, where $\boldsymbol{\tau}$ is a unit vector along the line joining the points $\mathbf{a}$ and $\mathbf{b}$ and dl is an element of path length. Equation (1) is thus the line integral of the *component* of $\mathbf{v}(x, y)$ along the line joining $\mathbf{a}$ and $\mathbf{b}$. We also assume that the vector field $\mathbf{v}(x, y)$ is defined on a bounded, convex domain D in the x–y plane with boundary ∂D, and that $\mathbf{a}$ and $\mathbf{b}$ lie on ∂D, as illustrated in Fig. 1. (The restriction to a convex domain guarantees that the integration path joining $\mathbf{a}$ and $\mathbf{b}$ lies entirely in D for any $\mathbf{a}$ and $\mathbf{b}$ on the boundary.)

We show below that the problem of reconstructing a 2-D fluid velocity field, $\mathbf{v}(x, y)$, in a bounded region D from acoustic travel-time measurements through D reduces to the above reconstruction problem under the assumption of straight-line propagation of the acoustic signals (i.e. neglecting refraction). Applications of fluid-flow imaging exist in a variety of scientific and engineering disciplines (see e.g. Johnson *et al.* 1977a, 1977b). One potentially important application of flow imaging is the reconstruction of ocean current distributions from reciprocal acoustic travel-time measurements. This idea was first proposed by Munk & Wunsch (1979) and later elaborated on by these and other authors (see e.g. Munk & Wunsch 1982; Eisler *et al.* 1984; Munk 1986; Howe *et al.* 1987). The present two-dimensional theory is applicable to the latter problem under conditions when the vertical component of current flow is negligible compared to the horizontal flow component and when the domain of reconstruction is sufficiently limited so that the propagation paths may be regarded as straight. In the next section, we show that acoustic travel-time measurements through a region of fluid flow can be cast in the form of equation (1). Before this, however, we summarize the steps employed in solving the vector tomography problem defined by equation (1).

A central-slice theorem is first derived for the vector

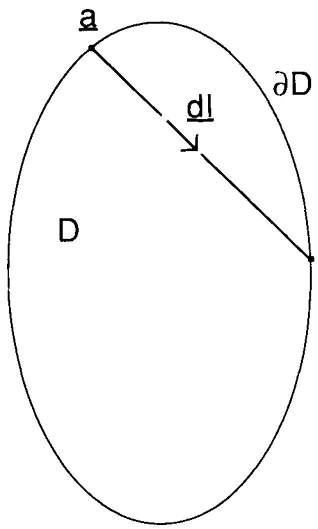

Figure 1. Line-integral measurements of a two-dimensional vector field [equation (1)] are made between points on the boundary ∂D of the domain D.

tomographic-reconstruction problem analogous to the scalar central-slice theorem well known in conventional tomography. The vector central-slice theorem relates the 1-D Fourier transform of the path-integral measurements given by equation (1) to the 2-D Fourier transforms of the x and y components, $v_x(x, y)$ and $v_y(x, y)$, of $\mathbf{v}(x, y)$, where

$$\mathbf{v}(x, y) = v_x(x, y)\hat{x} + v_y(x, y)\hat{y}, \tag{2}$$

$\hat{x}$ and $\hat{y}$ being unit vectors. We then show that the vector central-slice theorem simplifies when the vector field $\mathbf{v}(x, y)$ is separated into its irrotational and solenoidal vector components. Any 'well-behaved' vector field can be uniquely decomposed in this way by defining scalar and vector potential functions $\Phi(x, y)$ and $\boldsymbol{\Psi}(x, y)$ such that

$$\mathbf{v}(x, y) = \nabla\Phi(x, y) + \nabla \times \boldsymbol{\Psi}(x, y), \tag{3}$$

where $\nabla\Phi$ and $\nabla \times \boldsymbol{\Psi}$ are, respectively, the irrotational and solenoidal parts of $\mathbf{v}$. Equation (3) is called Helmholtz's theorem; conditions for the uniqueness of the Helmholtz decomposition are that $\mathbf{v}$ is finite, continuous and vanishes at infinity (Morse & Feshbach 1953, pp. 52–53). Since $\mathbf{v}$ is confined to the x–y plane, a single component of the vector potential $\boldsymbol{\Psi}$ in the z-direction is sufficient to define $\nabla \times \boldsymbol{\Psi}$ uniquely; i.e. we can write $\boldsymbol{\Psi}(x, y) = \Psi(x, y)\hat{z}$. Thus, in two dimensions, $\mathbf{v}(x, y)$ is entirely determined by the two scalar functions $\Phi(x, y)$ and $\Psi(x, y)$ which, as we shall see, is a more convenient representation than the component representation (2).

With the decomposition (3), we then show that the solenoidal component $\nabla \times \boldsymbol{\Psi}$ can be uniquely reconstructed tomographically, i.e. from line integrals of the type (1); in fact, the problem can be formulated so that the irrotational component $\nabla\Phi$ gives no contribution to the path-integral (1), leaving only the contribution from the solenoidal part $\nabla \times \boldsymbol{\Psi}$. However, we also show that, if $\nabla \cdot \mathbf{v} = 0$ in D, then the scalar potential Φ can be uniquely reconstructed throughout D from measurements of the field $\mathbf{v}$ only on the boundary ∂D. The latter result is a consequence of the fact that, if $\mathbf{v}$ is divergenceless in D, then Φ solves Laplace's equation $[\nabla^2\Phi = 0]$ in D, and hence Φ is determined by its boundary values on ∂D. This implies that the vector field $\mathbf{v}(x, y)$ can be reconstructed from the following information: line-integral measurements of $\mathbf{v}$, as defined by equation (1), between all pairs of points on the boundary ∂D and measurements of the field $\mathbf{v}$ on the boundary ∂D. An analytical example which illustrates the procedure for reconstructing both $\nabla\Phi$ and $\nabla \times \boldsymbol{\Psi}$ from line-integral and boundary measurements is given in the Appendix.

A formulation of the vector-field reconstruction problem was also given by Johnson *et al.* (1977, 1977b), but they proceed directly to a numerical solution using iterative image-reconstruction algorithms. Johnson *et al.* also point out the possibility of invisible flow, i.e. for which the right-hand side of equation (1) gives zero. Invisible flow can arise from flow distributions generated by sources or sinks in the field $\mathbf{v}$ (i.e. when $\nabla \cdot \mathbf{v} \neq 0$). Johnson *et al.* suggest that the fact that source- or sink-generated flow can result in no contribution to the line integral (1) is a consequence of the symmetry of a diverging field, which results in equal and opposite contributions to the line integral that add to zero. However, this statement is generally correct only when the integration paths extend from $-\infty$ to ∞; for finite integration paths, such fields will not always give a zero line integral because the opposite contributions may not be equal. In this paper, we show that the constraint of a divergenceless field ($\nabla \cdot \mathbf{v} = 0$) in D (i.e. assuming incompressible flow with no sources or sinks in D) is sufficient to eliminate invisible flow; under these conditions, the 2-D reconstruction problem may be uniquely solved.

2 TOMOGRAPHIC IMAGING OF FLUID FLOW

Let D denote a bounded and convex region of 2-D fluid flow with boundary ∂D,* in which the flow is defined by the vector velocity field $\mathbf{u}(x, y)$. We examine the problem of recovering the vector field $\mathbf{u}(x, y)$ from acoustic travel-time measurements through D. Let $c(x, y)$ denote the local acoustic speed in D. As shown below, the travel-time measurements also provide sufficient information to recover independently the scalar function $c(x, y)$. We assume that the magnitude of fluid flow $|\mathbf{u}|$ is everywhere much less than the acoustic speed c. We also assume that the variations in c are sufficiently small and/or the path lengths sufficiently short so that the ray paths may be regarded as straight.† Then, letting $\boldsymbol{\tau}$ denote the unit tangent vector along the acoustic ray path, the travel-time, $t(\mathbf{a}, \mathbf{b})$, of an acoustic signal between $\mathbf{a}$ and $\mathbf{b}$ is

$$t(\mathbf{a}, \mathbf{b}) = \int_{\mathbf{a}}^{\mathbf{b}} \frac{dl}{c(x, y) + \mathbf{u}(x, y) \cdot \boldsymbol{\tau}} \approx \int_{\mathbf{a}}^{\mathbf{b}} \frac{dl}{c(x, y)} - \int_{\mathbf{a}}^{\mathbf{b}} \frac{\mathbf{u}(x, y) \cdot \boldsymbol{\tau}}{c(x, y)^2}\, dl, \tag{4}$$

to first order in $|\mathbf{u}|/c$, where dl is an element of path length along the ray.

* ∂D is not a physical boundary; it is defined by the locus of points over which the sources and receivers are distributed.

† If this is not the case, see, for example, Norton & Linzer (1982) for a first-order perturbation correction to the travel-time measurements due to refraction.

Now transmitting the acoustic signal in the opposite direction from $\mathbf{b}$ to $\mathbf{a}$ gives $t(\mathbf{b}, \mathbf{a})$ in which $\boldsymbol{\tau}$ is replaced by $-\boldsymbol{\tau}$. Adding this to and subtracting from $t(\mathbf{a}, \mathbf{b})$ results in

$$t(\mathbf{a}, \mathbf{b}) + t(\mathbf{b}, \mathbf{a}) = 2\int_{\mathbf{a}}^{\mathbf{b}} \frac{dl}{c(x, y)}, \tag{5}$$

$$t(\mathbf{a}, \mathbf{b}) - t(\mathbf{b}, \mathbf{a}) = -2\int_{\mathbf{a}}^{\mathbf{b}} \frac{\mathbf{u}(x, y) \cdot \boldsymbol{\tau}}{c(x, y)^2} dl. \tag{6}$$

Johnson *et al.* (1977a, 1977b) obtained equations identical to equations (5) and (6), but proceed from here directly to a numerical solution; in more recent years the above equations have also been independently pointed out by other authors particularly in oceanography (see e.g. Munk & Wunsch 1982; Munk 1986; Howe *et al.* 1987, and references therein). Note the equation (5) can be used to recover the acoustic speed distribution $c(x, y)$ independently of flow using standard time-of-flight tomography (Greenleaf *et al.* 1975). The function $c(x, y)$ could then, in turn, be substituted into equation (6). For our purposes, the sound speed $c(x, y)$ is presumed known. Finally, equation (6) reduces to equation (1) by letting

$$T(\mathbf{a}, \mathbf{b}) \equiv t(\mathbf{a}, \mathbf{b}) - t(\mathbf{b}, \mathbf{a}),$$

$$\mathbf{v}(x, y) \equiv -\frac{2\mathbf{u}(x, y)}{c(x, y)^2} \quad \text{and} \quad d\mathbf{l} \equiv \boldsymbol{\tau} dl.$$

3 SOLUTION TO THE VECTOR RECONSTRUCTION PROBLEM

Assuming straight-ray propagation, we parametrize the line joining $\mathbf{a}$ and $\mathbf{b}$ by the polar coordinates (ρ, ϕ) where the line between $\mathbf{a}$ and $\mathbf{b}$ is now denoted by $L(\rho, \phi)$, as shown in Fig. 2. Equation (1) then becomes

$$T_\phi(\rho) = \int_{L(\rho, \phi)} \mathbf{v}(x, y) \cdot d\mathbf{l}. \tag{7}$$

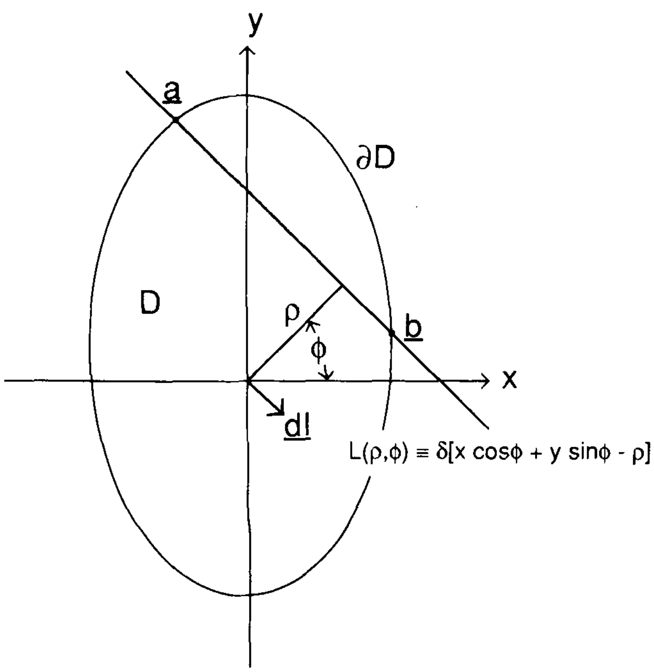

Figure 2. The line L is parameterized by its distance from the origin ρ and angle ϕ from the x-axis.

Since the line-integral measurements are made along paths only through the region of interest D, it is convenient to set

$$\mathbf{v}(x, y) \equiv 0 \text{ for } (x, y) \text{ outside of } D, \tag{8}$$

where $\mathbf{v}(x, y)$ is equal to the physical field inside D. Henceforth in this paper $\mathbf{v}$ shall now refer to the 'truncated field' defined by equation (8) unless otherwise indicated. Note, of course, that defining $\mathbf{v}$ equal to zero outside of D is justified since we can only reconstruct $\mathbf{v}$ within the measurements domain D in any case; presumably nothing is known about the physical field outside of D. Equation (8) has several consequences. First, it is now permissible and convenient to extend the integration path along the line $L(\rho, \phi)$ in equation (7) from $-\infty$ to ∞. Second, we apply the Helmholtz decomposition $\mathbf{v} = \nabla\Phi + \nabla \times \boldsymbol{\Psi}$ to the truncated field ($\mathbf{v}$ zero outside of D), *not* the physical field ($\mathbf{v}$ non-zero outside of D). This satisfies among other things the requirement that $\mathbf{v}$ vanish at infinity which is needed for the uniqueness of the Helmholtz decomposition (Morse & Feshbach 1953)*. Finally, all quantities, such as $\mathbf{v}$ and the potentials Φ and Ψ, are square integrable and thus Fourier transformable.

Now, from Fig. 2,

$$d\mathbf{l} = dl \sin\phi\hat{x} - dl \cos\phi\hat{y}. \tag{9}$$

Using equations (2) and (9) in equation (7) then gives

$$\begin{aligned} T_\phi(\rho) &= \sin\phi \int_{L(\rho,\phi)} v_x(x, y)\, dl - \cos\phi \int_{L(\rho,\phi)} v_y(x, y)\, dl \\ &= \sin\phi \iint_D dx\, dy v_x(x, y)\delta(x\cos\phi + y\sin\phi - \rho) \\ &\quad -\cos\phi \iint_D dx\, dy v_y(x, y)\delta(x\cos\phi + y\sin\phi - \rho), \end{aligned} \tag{10}$$

where $\delta(\cdot)$ is the (1-D) Dirac delta function. In equation (10), the argument of the delta function, $x\cos\phi + y\sin\phi - \rho$, is zero for points on the line $L(\rho, \phi)$, as required. Next, the Fourier transform of $T_\phi(\rho)$ with respect to ρ is

$$\tilde{T}_\phi(k) = \int_{-\infty}^{\infty} T_\phi(\rho)e^{-ik\rho}\, d\rho. \tag{11}$$

Substituting equation (10) into (11) and interchanging orders of integration gives

$$\begin{aligned} \tilde{T}_\phi(k) &= \sin\phi \iint_D dx\, dy v_x(x, y)e^{-ik(x\cos\phi + y\sin\phi)} \\ &\quad - \cos\phi \iint_D dx\, dy v_y(x, y)e^{-ik(x\cos\phi + y\sin\phi)}. \end{aligned} \tag{12}$$

Defining the 2-D Fourier transforms of the components

* Strictly speaking, the uniqueness of the Helmholtz decomposition also depends on continuity of $\mathbf{v}$, whereas the truncated field is discontinuous on the boundary of D by virtue of equation (8). If, however, $\mathbf{v}$ is continuous in D, then the nonuniqueness applies only to the values of $\mathbf{v}$ on the boundary. This is not a problem, since we are only concerned with reconstructing $\mathbf{v}$ in the interior of D, and presumably the physical (non-truncated) field is accessible to measurement on the boundary.

$v_x(x, y)$ and $v_y(x, y)$ as

$$\tilde{v}_x(u, v) = \iint_D dx\, dy v_x(x, y) e^{-i(ux+vy)} \tag{13}$$

$$\tilde{v}_y(u, v) = \iint_D dx\, dy v_y(x, y) e^{-i(ux+vy)} \tag{14}$$

and comparing equations (13) and (14) to equation (12) shows that

$$\tilde{T}_\phi(k) = \sin\phi \tilde{v}_x(k\cos\phi, k\sin\phi) - \cos\phi \tilde{v}_y(k\cos\phi, k\sin\phi). \tag{15}$$

This is the central-slice theorem for 2-D vector fields. The function $\tilde{T}_\phi(k)$ on the left of equation (15) is known (i.e. $\tilde{T}_\phi$ is the Fourier transform of the line-integral measurements (1)) and one desires to determine $\tilde{v}_x(u, v)$ and $\tilde{v}_y(u, v)$ on the right, from which the components $v_x(x, y)$ and $v_y(x, y)$ can then be obtained by inverse Fourier transforming equations (13) and (14). As it stands, the formula for $\tilde{T}_\phi(k)$ given by equation (15) is underdetermined since v_x and v_y are in general arbitrary and independent functions. This suggests that an additional constraint, such as $\nabla \cdot \mathbf{v} = 0$, might eliminate the ambiguity in equation (15), resulting in a unique solution. We later show this to be the case.

We proceed by decomposing the (truncated) field $\mathbf{v}(x, y)$ into irrotational and solenoidal components, as in equation (3): $\mathbf{v} = \nabla\Phi + \nabla \times \boldsymbol{\Psi}$. Next, Fourier transform $\mathbf{v}$ and substitute the result into the right-hand side of equation (15). In doing so, we shall see that the contribution from the irrotational component, $\nabla\Phi$, disappears.

As noted earlier, for a 2-D vector field confined to the x–y plane, the vector potential $\boldsymbol{\Psi}$ may be defined as $\boldsymbol{\Psi} = \Psi\hat{z}$. Then,

$$\nabla \times \boldsymbol{\Psi} = \hat{x}\frac{\partial\Psi}{\partial y} - \hat{y}\frac{\partial\Psi}{\partial x}. \tag{16}$$

Also, in two dimensions the irrotational component reads

$$\nabla\Phi = \hat{x}\frac{\partial\Phi}{\partial x} + \hat{y}\frac{\partial\Phi}{\partial y}. \tag{17}$$

On adding equations (16) and (17), the x and y components of $\mathbf{v}$ are

$$v_x(x, y) = \frac{\partial\Psi}{\partial y} + \frac{\partial\Phi}{\partial x} \tag{18a}$$

$$v_y(x, y) = -\frac{\partial\Psi}{\partial x} + \frac{\partial\Phi}{\partial y}. \tag{18b}$$

Now, let $\tilde{\Phi}(u, v)$ and $\tilde{\Psi}(u, v)$ denote, respectively, the 2-D Fourier transforms of $\Phi(x, y)$ and $\Psi(x, y)$. Upon Fourier transforming equations (18), we obtain

$$\tilde{v}_x(u, v) = iv\tilde{\Psi}(u, v) + iu\tilde{\Phi}(u, v) \tag{19a}$$

$$\tilde{v}_y(u, v) = -iu\tilde{\Psi}(u, v) + iv\tilde{\Phi}(u, v). \tag{19b}$$

Finally, substituting equations (19) into the central-slice theorem (15) results in

$$\begin{aligned}\tilde{T}_\phi(k) = {} & ik\sin^2\phi\tilde{\Psi}(k\cos\phi, k\sin\phi) \\ & + ik\sin\phi\cos\phi\tilde{\Phi}(k\cos\phi, k\sin\phi) \\ & + ik\cos^2\phi\tilde{\Psi}(k\cos\phi, k\sin\phi) \\ & - ik\cos\phi\sin\phi\tilde{\Phi}(k\cos\phi, k\sin\phi),\end{aligned}$$

which reduces to

$$\tilde{T}_\phi(k) = ik\tilde{\Psi}(k\cos\phi, k\sin\phi). \tag{20}$$

Equation (20) shows that the Fourier transform of the solenoidal component, $\tilde{\Psi}(u, v)$, and hence, $\nabla \times \boldsymbol{\Psi}(x, y)$, is determined uniquely from the line-integral data $\tilde{T}_\phi(k)$. Note, however, that the contribution from the irrotational component, $\nabla\Phi$, has disappeared in equation (20). This is significant, since it allows us to reconstruct the solenoidal component separately from the irrotational component. We shall later show how the component $\nabla\Phi$ can be derived from the boundary values of $\mathbf{v}$ on ∂D.

The disappearance of the irrotational component in equation (20) could also have been anticipated from a well-known theorem from vector calculus that states that the line integral of the gradient of a scalar field depends only on its end points (i.e. is path independent); specifically,

$$\int_{\mathbf{a}}^{\mathbf{b}} \nabla\Phi \cdot d\mathbf{l} = \Phi(\mathbf{b}) - \Phi(\mathbf{a}). \tag{21}$$

Thus, substituting $\mathbf{v} = \nabla\Phi + \nabla \times \boldsymbol{\Psi}$ into equation (1) gives, in view of equation (21),

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v} \cdot d\mathbf{l} = \Phi(\mathbf{b}) - \Phi(\mathbf{a}) + \int_{\mathbf{a}}^{\mathbf{b}} \nabla \times \boldsymbol{\Psi} \cdot d\mathbf{l}. \tag{22}$$

Now letting $|\mathbf{a}|$ and $|\mathbf{b}| \to \infty$ and using the property* that $\Phi(\mathbf{r}) \to 0$ as $|\mathbf{r}| \to \infty$, equation (22) becomes

$$\int_{-\infty}^{\infty} \mathbf{v} \cdot d\mathbf{l} = \int_{-\infty}^{\infty} \nabla \times \boldsymbol{\Psi} \cdot d\mathbf{l}, \tag{23}$$

where the infinite limits are meant to denote integration along the *infinite* line passing through **a** and **b**. Also, note that

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{v} \cdot d\mathbf{l} = \int_{-\infty}^{\infty} \mathbf{v} \cdot d\mathbf{l} = \int_{-\infty}^{\infty} \nabla \times \boldsymbol{\Psi} \cdot d\mathbf{l}, \tag{24}$$

where the first equality follows from equation (8) and the second equality is equation (23). This shows again that the central-slice theorem (20) holds only for the solenoidal part of $\mathbf{v}$, since any contribution from the scalar potential $\boldsymbol{\Phi}$ disappears in the integration to infinity. We emphasize that it is the *finite* path integral on the far left in equation (24) that is measured; equation (24) shows, however, that the measurement is equal to the infinite path integral of $\nabla \times \boldsymbol{\Psi}$ on the far right of equation (24), and the latter is required for the central-slice theorem (20) [i.e. the quantity $\tilde{T}_\phi(k)$ in equation (20) is the Fourier transform of the right-hand integral in equation (24)]. Finally, is is interesting to note

* This can be shown to follow when $\mathbf{v}$ is bounded and has finite support; see equation (8).

that in general

$$\int_{\mathbf{a}}^{\mathbf{b}} \nabla \times \boldsymbol{\Psi} \cdot d\mathbf{l} \neq \int_{-\infty}^{\infty} \nabla \times \boldsymbol{\Psi} \cdot d\mathbf{l},$$

since equation (8) does not hold for $\nabla \times \boldsymbol{\Psi}$, i.e. unlike $\mathbf{v}$, the field $\nabla \times \boldsymbol{\Psi}$ alone is not in general zero outside the domain D; one must add $\nabla\Phi$ to $\nabla \times \boldsymbol{\Psi}$ to obtain zero in the region outside of D, as required by equation (8). This is illustrated in the analytical example given in the Appendix. The above inequality is also clear by comparing equations (22) and (24).

We can view the above argument in another way. It is evident from this equation that the irrotational component $\nabla\Phi$ does contribute to the path-integral measurement [the left-hand integral in equation (22)] for *finite* **a** and **b**, as does the solenoidal part $\nabla \times \boldsymbol{\Psi}$. Note that, however, one has the freedom to select any integration end points, **a** and **b**, as long as they lie outside of D, since $\mathbf{v}$ vanishes outside of D [equation (8)]. The crucial point is that the choice $|\mathbf{a}| \to \infty$ and $|\mathbf{b}| \to \infty$ conveniently eliminates any contribution to the measurement from the irrotational part, $\nabla\Phi$, leaving only the contribution from $\nabla \times \boldsymbol{\Psi}$, as is shown by equation (23). Thus, as noted, the central-slice theorem (20) allows us to reconstruct separately (and uniquely) the solenoidal part of $\mathbf{v}$.

In computing $\nabla \times \boldsymbol{\Psi}$ from equation (20), one can first perform the 2-D inverse Fourier transform of $\tilde{\Psi}(k\cos\phi, k\sin\phi) = \tilde{T}_\phi(k)/ik$ and then differentiate Ψ according to equation (16). Alternatively, one can perform the differentiation in Fourier space before inverse Fourier transforming, as follows. Letting $\mathscr{F}\{\cdot\}_{u,v}$ and $\mathscr{F}^{-1}\{\cdot\}_{x,y}$ denote respectively the 2-D Fourier and inverse Fourier transforms, then Fourier transforming equation (16) gives

$$\begin{aligned}\mathscr{F}\{\nabla \times \boldsymbol{\Psi}\}_{u,v} &= \hat{x}iv\tilde{\Psi}(u,v) - \hat{y}iu\tilde{\Psi}(u,v)\\ &= (\hat{x}\sin\phi - \hat{y}\cos\phi)ik\tilde{\Psi}(k\cos\phi, k\sin\phi)\\ &= (\hat{x}\sin\phi - \hat{y}\cos\phi)\tilde{T}_\phi(k)\end{aligned}$$

using $u = k\cos\phi$ and $v = k\sin\phi$ and substituting equation (20). Thus

$$\nabla \times \boldsymbol{\Psi} = \mathscr{F}^{-1}\{(\hat{x}\sin\phi - \hat{y}\cos\phi)\tilde{T}_\phi(k)\}_{x,y}.$$

Writing the 2-D inverse Fourier transform in polar form (in which $x = r\cos\theta$ and $y = r\sin\theta$), this may be written

$$\nabla \times \boldsymbol{\Psi}(r,\theta) = \frac{1}{(2\pi)^2}\int_0^\infty k\,dk \times \int_0^{2\pi} d\phi(\hat{x}\sin\phi - \hat{y}\cos\phi)\tilde{T}_\phi(k)\exp[ikr\cos(\theta-\phi)]. \tag{25}$$

The above considerations show that no central-slice theorem exists for the irrotational component $\nabla\Phi$. This component can, however, be recovered from the values of $\mathbf{v}$ on the boundary of the measurement domain D if $\mathbf{v}$ is divergenceless in D. (The actual physical field can, of course, have nonzero divergence outside of D.) First, note that if $\nabla \cdot \mathbf{v} = 0$ in D, then setting the divergence of equation (3) to zero [and using $\nabla \cdot \nabla \times \boldsymbol{\Psi} \equiv 0$] gives Laplace's equation

$$\nabla^2\Phi = 0, \tag{26}$$

which holds in the interior of D. Thus, since Φ solves Laplace's equation in D, Φ is uniquely determined by its values on the boundary ∂D. This implies that tomography is in fact unnecessary for the special case of a purely irrotational field $\nabla\Phi$ when $\nabla \cdot \mathbf{v} = 0$ in D (i.e. for potential flow in the absence of sources and sinks in D), since boundary information alone is evidently sufficient to determine Φ, and hence $\nabla\Phi$, everywhere in D. The latter is a special case, however, and our present concern is in reconstructing an arbitrary flow field $\mathbf{v} = \nabla\Phi + \nabla \times \boldsymbol{\Psi}$. We have already seen that the solenoidal component $\nabla \times \boldsymbol{\Psi}$ is determined by the path-integral measurements (1) through the central-slice theorem (20).

Our remaining task is to derive the scalar potential Φ when $\mathbf{v} = \nabla\Phi + \nabla \times \boldsymbol{\Psi}$. We now show that Φ, and hence the irrotational component, $\nabla\Phi$, can be computed in the interior of the domain D from measurements of $\mathbf{v}$ on the boundary ∂D. In this computation, $\nabla \times \boldsymbol{\Psi}$ gives no contribution. To show this, we first examine the general procedure for deriving the scalar and vector potential Φ and $\boldsymbol{\Psi}$ from $\mathbf{v}$ defined on the domain D. In particular, it is possible to show (Morse & Feshbach 1953, pp. 52–53)

$$\Phi = -\nabla \cdot \mathbf{U} \quad \text{and} \quad \boldsymbol{\Psi} = \nabla \times \mathbf{U}, \tag{27}$$

where

$$\mathbf{U}(x,y) \equiv \iint_D dx'\,dy'\mathbf{v}(x',y')G(x,y\,|\,x',y'), \tag{28}$$

and $G(x,y\,|\,x',y')$ is the Green's function for the Laplacian; for example, in two dimensions, G solves

$$\nabla^2 G(x,y\,|\,x',y') = -\delta(x-x', y-y'),$$

and is given by

$$G(x,y\,|\,x',y') = -\frac{1}{4\pi}\ln[(x-x')^2 + (y-y')^2]. \tag{29}$$

One can verify the relations (27) and (28) by noting that $\mathbf{U}$ is a solution to the vector Poisson equation,

$$\nabla^2\mathbf{U} = \begin{cases} -\mathbf{v} & (x,y) \text{ inside } D\\ 0 & (x,y) \text{ outside } D,\end{cases}$$

and using the vector identity (Morse & Feshbach 1953)

$$\nabla^2\mathbf{U} = \nabla[\nabla \cdot \mathbf{U}] - \nabla \times [\nabla \times \mathbf{U}].$$

From equations (27) and (28),

$$\begin{aligned}\Phi(x,y) &= -\nabla \cdot \mathbf{U}(x,y)\\ &= -\iint_D dx'\,dy'\mathbf{v}(x',y') \cdot \nabla G(x,y\,|\,x',y').\end{aligned} \tag{30}$$

But $\nabla G = -\nabla' G$, where ∇ and ∇' denote the gradient with respect to (x,y) and (x',y'). Thus, equation (30) can be written

$$\Phi(x,y) = \iint_D dx'\,dy'\mathbf{v}(x',y') \cdot \nabla' G(x,y\,|\,x',y'). \tag{31}$$

Now substituting the identity

$$\mathbf{v} \cdot \nabla' G = \nabla' \cdot (G\mathbf{v}) - G(\nabla' \cdot \mathbf{v})$$

into equation (31) and employing the divergence theorem to write the integral of the first term on the right as a boundary integral around ∂D, equation (31) becomes

$$\Phi = \oint_{\partial D} G\mathbf{v}\cdot\mathbf{n}\,ds - \iint_D dx'\,dy' G(\nabla'\cdot\mathbf{v}),$$

where $\mathbf{n}$ is the unit outward normal on the boundary curve ∂D and ds is an element of arc length along ∂D. Now assuming that $\nabla'\cdot\mathbf{v}=0$ in D, the last integral vanishes and, on using equation (29) for G, we finally obtain

$$\Phi(x, y) = -\frac{1}{4\pi}\oint_{\partial D} \ln\left[(x-x')^2 + (y-y')^2\right]\mathbf{v}(x', y')\cdot\mathbf{n}\,ds. \tag{32}$$

This integral gives the scalar potential Φ at every point in D and on ∂D in terms of the value of the vector field $\mathbf{v}(x, y)$ on the boundary ∂D. Hence, the irrotational component $\nabla\Phi$ can be computed throughout D by differentiating equation (32). Taking the gradient of equation (32) and differentiating under the integral sign gives

$$\nabla\Phi(x, y) = -\frac{1}{4\pi}\oint_{\partial D} \frac{(x-x')\hat{x} + (y-y')\hat{y}}{(x-x')^2 + (y-y')^2}\mathbf{v}(x', y')\cdot\mathbf{n}\,ds. \tag{33}$$

Equations (25) and (33) complete the reconstruction problem. For illustration, the Appendix contains a simple reconstruction problem for which all steps can be carried out analytically.

The assumption that $\mathbf{v}$ is divergenceless deserves some discussion. The condition follows from the continuity equation $\nabla\cdot(\rho\mathbf{v})=0$ for steady flow under the assumptions of an incompressible fluid (constant density ρ) and no sources or sinks in the region under reconstruction. In 2-D flow, sources or sinks can be ruled out when there is no creation or destruction of fluid in the region of interest. In a 'pseudo' 2-D problem, where one attempts to reconstruct a 2-D slice from a region of 3-D flow, sources or sinks in the plane of interest can arise from upwelling or downwelling of fluid from below or above the plane. The preceding theory applies, however, if one is willing to assume a negligible vertical component of flow compared to horizontal flow. More generally, the above theory is still applicable in the presence of vertical flow provided this flow leads to no horizontal divergence of $\mathbf{v}$. To see this, assume that $\nabla^{(3)}\cdot\mathbf{v}$ represents the 3-D divergence of a 3-D field $\mathbf{v}$ and let $\nabla^{(2)}\cdot\mathbf{v}$ denote the transverse divergence in, say, the x–y plane. Then assuming no 3-D sources or sinks and incompressible flow, we have $\nabla^{(3)}\cdot\mathbf{v} = \nabla^{(2)}\cdot\mathbf{v} + \partial v_z/\partial z = 0$, which implies that $\partial v_z/\partial z$ must vanish to guarantee the condition $\nabla^{(2)}\cdot\mathbf{v}=0$ demanded by the 2-D tomographic problem.

4 EXTENSION TO THREE DIMENSIONS

The generalization of the basic two-dimensional central-slice theorem given by equation (15) to three dimensions can be easily derived. Moreover, under the assumption of no sources and sinks in a bounded 3-D domain D, the 3-D analogue of equation (32) for computing the 3-D scalar potential Φ can also be derived. However, the three-dimensional analogue of the central-slice theorem (20) for the solenoidal component $\nabla\times\Psi$ does not exist since the 3-D vector potential Ψ is uniquely specified by a minimum of two independent components [Ψ actually has three components, but one of these can be effectively eliminated by the choice of gauge $\nabla\cdot\Psi=0$ (Jackson 1962)]. Because of the latter fact, the two-dimensional solution developed in this paper does not appear to generalize directly to three dimensions. That is, a complete set of path integral measurements (i.e. from all directions in 3-D space) through a 3-D domain D together with the boundary measurements of $\mathbf{v}$ on D and the 3-D constraint $\nabla\cdot\mathbf{v}=0$ in D are evidently insufficient to determine the three-dimensional problem uniquely.

To state this another way, the reason that the above approach succeeds in two dimensions is that, in 2-D, there are two unknown functions to reconstruct, i.e. the components $v_x(x, y)$ and $v_y(x, y)$, but we have one central-slice theorem [equation (15)] and one constraint ($\nabla\cdot\mathbf{v}=0$). This constraint equation together with the central slice theorem are sufficient to determine the two functions v_x and v_y uniquely. In three dimensions there are three unknown component functions, but one (3-D) central-slice theorem and one (3-D) constraint equation, which together are insufficient to determine the three functions v_x, v_y and v_z uniquely. One additional constraint is needed. For example, the added constraint v_z = constant would suffice. Other physically-motivated constraints might also be useful in uniquely determining the solution to the three-dimensional problem.

5 CONCLUSION

In this paper, we examine the problem of reconstructing a 2-D vector field from its line integrals over a bounded domain D. A vector central-slice theorem is derived, which is a generalization of the well-known central-slice theorem from conventional 'scalar' tomography. Of key importance in the analysis, however, is the decomposition of the vector field into its irrotational and solenoidal components: $\mathbf{v} = \nabla\Phi + \nabla\times\Psi$, where Φ and Ψ are scalar and vector potentials. In particular, we show that the solenoidal component $\nabla\times\Psi$ can be uniquely reconstructed from the line integrals of $\mathbf{v}$, whereas the irrotational component $\nabla\Phi$ cannot be recovered in this way. The latter component can, however, be reconstructed if the field $\mathbf{v}$ is divergenceless within the measurement domain D. A divergenceless velocity field is implied if the fluid is incompressible and no sources or sinks are assumed to exist in D. In the latter case, the scalar potential Φ solves Laplace's equation in the interior of D and is determined by its values on the boundary of D. We derive an explicit formula that gives Φ, and hence $\nabla\Phi$, throughout D in terms of $\mathbf{v}$ on the boundary of D. As a result, both components of $\mathbf{v}$ can be recovered uniquely from the line integrals of $\mathbf{v}$ through the measurement region and the values of $\mathbf{v}$ on the boundary of this region.

An important application of the above theory is the reconstruction of 2-D fluid flow fields from reciprocal acoustic travel-time measurements.

REFERENCES

Eisler, T. J., Porter, D. L., New, R. & Calderone, D., 1984. Resolution, bias, and variance in tomographic estimates of sound speed and currents, *J. geophys. Res.*, **89,** 10469–10478.

Gradshteyn, I. S. & Ryzhik, I. M., 1965. *Tables of Integrals, Series and Products,* 4th edn, Academic Press, New York.

Greenleaf, J. F., Johnson, S. A., Samayoa, W. F. & Duck, F. A. 1975. Algebraic reconstruction of spatial distributions of acoustic velocities in tissue from their time-of-flight profiles, *Acoustical Holography,* vol. 6, ed. Booth, N., pp. 71–90, Plenum Press, New York.

Howe, B. M., Worchester, P. F. & Munk, W., 1987. Ocean acoustic tomography: mesoscale velocity. *J. geophys. Res.*, **92,** 3785–3805.

Jackson, J. D., 1962. *Classsical Electrodynamics,* Wiley, New York.

Johnson, S. A., Greenleaf, J. F., Hansen, C. R., Samayoa, W. F., Tanaka, M., Lent, A., Christensen, D. A. & Woolley, R. L., 1977a. Reconstructing three-dimensional fluid velocity vector fields from acoustic transmission measurements, *Acoustical Holography,* vol. 7, ed. Kessler, L. W. pp. 307–326, Plenum Press, New York.

Johnson, S. A., Greenleaf, J. F., Tanaka, M. & Flandro, G., 1977b. Reconstructing three-dimensional temperature and fluid velocity vector fields from acoustic transmission measurements, *Proc. 1977 Ann. Mtg of the Instrument Society of America,* National Bureau of Standards Special Publication 484, August 1977.

Morse, P. M. & Feshbach, H., 1953. *Methods of Theoretical Physics,* McGraw-Hill, New York.

Munk, W., 1986. Acoustic monitoring of ocean gyres, *J. Fluid Mech.*, **173,** 43–53.

Munk, W. & Wunsch, C., 1979. Ocean acoustic tomography: a scheme for large-scale monitoring, *Deep-Sea Res.*, **26A,** 123–161.

Munk, W. & Wunsch, C., 1982. Observing the ocean in the 1990s, *Phil. Trans. R. Soc. Lond.* **A, 307,** 439–464.

Norton, S. J. & Linzer, M., 1982. Correcting for ray refraction in ultrasonic tomography: a perturbation approach, *Ultrasonic Imaging,* **4,** 201–233.

APPENDIX

For illustration, we consider below a vector reconstruction problem that can be solved analytically. Consider a constant flow field $\mathbf{v}$ in the x-direction given by

$$\mathbf{v}(x, y) = v_0\hat{x}, \qquad v_0 = \text{constant}, \tag{A1}$$

where the domain of reconstruction D is the interior of a circle of diameter R centered at the origin. The above field satisfies $\nabla \cdot \mathbf{v} = 0$ in D, as we require. Inserting this distribution into equation (7) [or equation (10)] gives the line-integral measurements $T_\phi(\rho)$. Using $\mathbf{v} \cdot d\mathbf{l} = v_0 \sin\phi\, dl$, equation (7) becomes, in view of equation (8),

$$T_\phi(\rho) = v_0 \sin\phi \int_{L(\rho,\phi)} \operatorname{circ}(r/R)\, dl,$$

where

$$\operatorname{circ}(r) \equiv \begin{cases} 1 & \text{for} \quad r \le 1 \\ 0 & \text{otherwise,} \end{cases}$$

and $r = \sqrt{x^2 + y^2}$. Performing the above integration gives

$$T_\phi(\rho) = \begin{cases} 2v_0 \sin\phi \sqrt{R^2 - \rho^2} & \text{for} \quad |\rho| < R \\ 0 & \text{for} \quad |\rho| > R. \end{cases} \tag{A2}$$

Equation (A2) constitutes our line-integral data. Note that in the scalar version of this problem, the directionally-dependent factor $\sin\phi$ would not appear, since the scalar problem is axially symmetric (i.e. $T_\phi(\rho)$ would not depend on ϕ).

From equation (A2) we can obtain the solenoidal component of $\mathbf{v}$ through equations (11) and (20). As additional data, we shall also assume that $\mathbf{v}$ is measured on the boundary $r = R$. From the boundary measurements of $\mathbf{v}$ the irrotational component of $\mathbf{v}$ may be derived by means of equation (32).

To obtain the vector-potential function Ψ, Fourier transform equation (A2) with respect to ρ as in equation (11), giving

$$\begin{aligned} \tilde{T}_\phi(k) &= 2v_0 \sin\phi \int_{-R}^{R} \sqrt{R^2 - \rho^2}\, e^{-ik\rho}\, d\rho \\ &= 2v_0 \sin\phi \left[\frac{\pi R J_1(kR)}{k}\right], \end{aligned} \tag{A3}$$

where $J_1(\cdot)$ is the Bessel function of order one. Now using equation (20) to solve for $\tilde{\Psi}$, we obtain

$$\tilde{\Psi}(k\cos\phi, k\sin\phi) = \frac{\tilde{T}_\phi(k)}{ik} = -2\pi i v_0 R \sin\phi \frac{J_1(kR)}{k^2}.$$

Next take the 2-D inverse Fourier transform of $\tilde{\Psi}$. Writing the 2-D inverse transform in polar form [in which $(x, y) = (r\cos\theta, r\sin\theta)$] for convenience, we have

$$\begin{aligned} &\Psi(r\cos\theta, r\sin\theta) \\ &= \frac{1}{(2\pi)^2} \int_0^\infty k\, dk \\ &\quad \times \int_0^{2\pi} d\phi\, \tilde{\Psi}(k\cos\phi, k\sin\phi) \exp[ikr\cos(\theta - \phi)] \\ &= -\frac{iv_0 R}{2\pi} \int_0^\infty dk \frac{J_1(kR)}{k} \int_0^{2\pi} d\phi \sin\phi \exp[ikr\cos(\theta - \phi)]. \end{aligned} \tag{A4}$$

With the aid of integral tables (Gradshteyn & Ryzhik 1965) and after some manipulation, we find

$$\int_0^{2\pi} d\phi \sin\phi \exp[ikr\cos(\theta - \phi)] = 2\pi i \sin\theta J_1(kr).$$

Equation (A4) then becomes

$$\Psi(r\cos\theta, r\sin\theta) = v_0 R \sin\theta \int_0^\infty \frac{dk}{k} J_1(kR) J_1(kr). \tag{A5}$$

Using the Bessel function identity $J_1(kr) = kr[J_0(kr) + J_2(kr)]/2$ in equation (A5) and the relation [Gradshteyn & Ryzhik 1965, p. 667]

$$\int_0^\infty J_n(xa) J_{n-1}(xb)\, dx = \begin{cases} b^{n-1}/a^n & \text{for} \quad b \le a \\ 0 & \text{for} \quad b > a, \end{cases}$$

equation (A5) reduces to

$$\Psi(r\cos\theta, r\sin\theta) = \begin{cases} \dfrac{v_0 r \sin\theta}{2} & \text{for} \quad r \le R \\ \dfrac{v_0 R^2 \sin\theta}{2r} & \text{for} \quad r > R, \end{cases}$$

or, in rectangular coordinates,

$$\Psi(x, y) = \begin{cases} \dfrac{v_0 y}{2} & \text{for } r \le R \\ \dfrac{v_0 R^2 y}{2r^2} & \text{for } r > R, \end{cases} \tag{A6}$$

where $r = \sqrt{x^2 + y^2}$.

We next compute the scalar potential Φ from the values of $\mathbf{v}$ on the boundary of D using formula (32). In evaluating equation (32) for this problem, polar coordinates are again convenient. The Green's function in equation (32) may be expressed in polar form as follows (Morse & Feshbach 1953, p. 1188):

$$\begin{aligned} G(r, \theta \mid r', \theta') &= -\frac{1}{4\pi} \ln [R^2 + r^2 - 2rR \cos(\theta - \theta')] \\ &= -\frac{1}{4\pi} \ln r^2 + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{R}{r}\right)^n \cos[n(\theta - \theta')], \quad r > R \\ &= -\frac{1}{4\pi} \ln R^2 + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{R}\right)^n \cos[n(\theta - \theta')], \quad R > r. \end{aligned} \tag{A7}$$

Then, noting that $\mathbf{v}(x', y') \cdot \mathbf{n} = v_0 \cos \theta'$, equation (32) becomes

$$\Phi(r \cos \theta, r \sin \theta) = \int_0^{2\pi} G(r, \theta \mid R, \theta') v_0 \cos \theta' R \, d\theta'. \tag{A8}$$

Finally, on substituting equation (A7) into (A8) and performing the θ' integration from 0 to 2π, only the term in the sums in equation (A7) for which $n = 1$ survives; we then have after a little algebra

$$\Phi(r \cos \theta, r \sin \theta) = \begin{cases} \dfrac{v_0 r \cos \theta}{2} & \text{for } r \le R \\ \dfrac{v_0 R^2 \cos \theta}{2r} & \text{for } r > R, \end{cases}$$

or, in rectangular coordinates,

$$\Phi(x, y) = \begin{cases} \dfrac{v_0 x}{2} & \text{for } r \le R \\ \dfrac{v_0 R^2 x}{2r^2} & \text{for } r > R. \end{cases} \tag{A9}$$

Now compute the total field $\mathbf{v} = \nabla\Phi + \nabla \times \boldsymbol{\Psi}$ by differentiating the potential functions Φ and Ψ. From equation (A6),

$$\nabla \times \boldsymbol{\Psi} = \hat{x} \frac{\partial \Psi}{\partial y} - \hat{y} \frac{\partial \Psi}{\partial x} =$$

$$\begin{cases} \hat{x} \dfrac{v_0}{2} & \text{for } r < R \\ \dfrac{v_0 R^2}{2r^4} [\hat{x}(x^2 - y^2) + 2\hat{y}(xy)] & \text{for } r > R, \end{cases}$$

and from equation (A9),

$$\nabla\Phi = \hat{x} \frac{\partial \Phi}{\partial x} + \hat{y} \frac{\partial \Phi}{\partial y}$$

$$= \begin{cases} \hat{x} \dfrac{v_0}{2} & \text{for } r < R \\ \dfrac{v_0 R^2}{2r^4} [\hat{x}(y^2 - x^2) - 2\hat{y}(xy)] & \text{for } r > R. \end{cases}$$

Adding then gives

$$\mathbf{v} = \nabla\Phi + \nabla \times \boldsymbol{\Psi} = \begin{cases} \hat{x} v_0 & \text{for } r < R \\ 0 & \text{for } r > R, \end{cases}$$

which agrees with equation (A1) inside D and is zero outside D. Note that, although $\mathbf{v}$ is identically zero outside D, the individual components $\nabla\Phi$ and $\nabla \times \boldsymbol{\Psi}$ are nonzero there, since the potentials Φ and Ψ are nonzero outside D. Also note that Φ solves Laplace's equation (26) in D, as required, but not outside of D. This behavior is a consequence of truncating the original vector field $\mathbf{v}$ outside D, as indicated by equation (8). In general, the potentials Φ and $\boldsymbol{\Psi}$ have no physical meaning outside D.

To check the above results, one can substitute equation (A1) for $\mathbf{v}$ directly into equation (28) and compute $\mathbf{U}(x, y)$. Each component of $\mathbf{U}(x, y)$ can then be evaluated analytically using Gauss's theorem (Jackson 1962). The potential functions Φ and Ψ then follow on differentiating $\mathbf{U}$ as defined in equations (27). The results are found to agree with equations (A6) and (A9), as expected.