

# Topics in D-Geometry

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**Abstract.** We discuss the general theory of D-branes on Calabi-Yaus, recent results from the theory of boundary states, and new results on the spectrum of branes on the quintic CY. (Contribution to the proceedings of Strings '99 in Potsdam, Germany.)

## 1. Introduction

The present contribution consists of three parts. The first is a general summary of the theory of D-branes on Calabi-Yau; the second summarizes the works [5, 9] which connect the boundary state approach with large volume results; the third summarizes new results on lines of marginal stability on the quintic found in June 1999. The transparencies for this talk (which emphasize different parts of the material) are also available at [1].

For background material on “D-geometry,” see [11]. This term refers to the study of how the conventional geometry which describes branes in supergravity is generalized in the context of D-branes. As a point of departure we could consider any of the geometrical pictures which branes give us for the various terms in an effective action. Perhaps the simplest example is the following: the moduli space of a 0-brane at a point in a  $CY_3$  is the  $CY_3$  itself; the moduli space metric is just the Ricci-flat metric on the  $CY_3$ .

Examples of the “unconventional” geometry we have in mind include the following:

- (i) Stringy and quantum corrections will generally modify conventional geometric predictions. In particular, we can ask how a D-brane world volume action is affected by “stringy” ( $l_s$ ) corrections. An example is to find the moduli space metric for the D0-brane at a point; this provides a canonical non-Ricci flat metric for each point in CY moduli space. Qualitative effects visible at finite  $l_s$  include T-duality and mirror symmetry; we will discuss the latter below.
- (ii) Perturbative string compactification can be defined non-geometrically, by specifying an appropriate internal CFT. Some examples (such as Gepner models) turn out to have geometric interpretations, and this definition provides a concrete way to work in the “highly stringy” regime. Others such as asymmetric orbifolds do not have

known geometric interpretations; studying D-branes on these spaces will probably lead either to finding such interpretations or showing why they do not exist.

- (iii) D-brane world-volume theories include open strings stretching between pairs of branes, which in many cases provide alternate gauge theory origins for what are gravitational effects in the large distance limit. Orbifold resolution by quiver theories are an example in which non-trivial topology is reproduced as a classical gauge theory moduli space. The short distance gravitational interactions between D-branes are replaced by quantum gauge theory dynamics. In special cases (in the large  $N$  limit or for quantities protected by supersymmetry) this is believed to reproduce supergravity, but more generally provides another way of defining its stringy generalization.
- (iv) Noncommutative gauge theory arises on D-brane world-volumes in appropriate limits of string theory, such as compactification on a small torus with fixed background  $B$  field, or in Minkowski space with large  $B$  field. It seems quite likely that similar theories are relevant in curved backgrounds; finding concrete examples is an important problem for future work.

This is by no means a complete list but perhaps includes the most interesting points discovered so far. As each of them would form a topic in its own right, for the rest of the review we will focus on the following meta-question: to what extent do these effects lead to *qualitative* changes in the brane physics – and thus cannot be ignored? The way to study this question is to frame the alternative (null) hypothesis: the qualitative properties of brane theories (especially, the low energy effective action, dimension of the moduli space, types of singularities and so on) are the same as predicted by naive geometric considerations – and test it in examples. We will refer to this as the “geometric hypothesis” and make it more precise below.

## 2. D-branes on Calabi-Yaus

Quite a lot is known about D-branes in flat space (Minkowski or toroidal compactifications) and in K3 compactifications, where type II-heterotic duality and the large supersymmetry already suffice to give a good picture. The geometric hypothesis appears to be essentially true in these cases – the brane spectrum and moduli spaces can be described as the spectrum and moduli spaces of semistable coherent sheaves (a generalization of vector bundle which allows singularities corresponding to pointlike instantons) [25].

D-branes on Calabi-Yau threefolds are not so well understood and look quite interesting for a number of reasons. Physically, supersymmetry preserving branes will have  $\mathcal{N} = 1$ ,  $d = 4$  gauge theories on the world-volume which may be directly relevant for phenomenology. They generalize the strong coupling limit of heterotic string compactification but in some ways appear simpler than the  $(0, 2)$  sigma models which appear there. Many questions can be addressed using the highly developed theory of  $\mathcal{N} = 2$  supersymmetry and mirror symmetry.

An important difference with the cases of higher supersymmetry is that the spectrum of branes can depend on the particular vacuum (point in moduli space) under discussion. For example, in pure  $SU(2)$  gauge theory, we know that the strong coupling spectrum is quite different from the semiclassical spectrum; the purely electric “W bosons” are not present. Given  $\mathcal{N} = 2$  supersymmetry this dependence of spectrum on moduli is highly constrained: as is well known, the BPS spectrum can change only on lines of marginal stability defined by the condition  $\text{Im } Z(Q_1)/Z(Q_2) = 0$ . Thus the problem of finding the spectrum of wrapped branes on CY and deciding whether it too changes at string scales is non-trivial but accessible, as we will discuss in the next sections.

Supersymmetric (1/2 BPS) branes on a  $CY_3$  are divided into A and B branes depending on the boundary condition on the  $U(1)$  currents in the  $(2, 2)$  superconformal algebra (which determines which part of the world-sheet supersymmetry they preserve) [32]: either  $Q_L = +Q_R$  or  $Q_L = -Q_R$  is a consistent choice. The notation comes from topological field theory – an A brane is one whose open strings naturally couple to A-twisted topological theory and the Kähler moduli, while a B brane couples to complex structure moduli. Mirror symmetry will exchange the two – the spectrum and world-sheet theories of A branes on a CY  $\mathcal{M}$  is isomorphic to that of the B branes on its mirror  $\mathcal{W}$ .

If we consider branes defined by Dirichlet and Neumann boundary conditions in the non-linear sigma model with  $CY_3$  target, the B branes are  $2p$ -branes wrapped on holomorphic cycles and carrying holomorphic vector bundles (this is the case with direct analogy to the heterotic string), while the A branes are 3-branes wrapped on what are called special Lagrangian submanifolds (or sL-submanifolds; more below) [2]. At first this notation may seem backwards given the discussion in the previous paragraph, since the  $2p$ -cycles and the masses of B branes are controlled by Kähler moduli (and thus are calculable in the A-twisted topological closed string theory), while the 3-cycles and masses of A branes are controlled by complex structure moduli. Nevertheless it is correct – in going from the open to closed string channel the boundary conditions on the  $U(1)$  current change sign, interchanging A and B twistings.

This switch has important consequences, especially if we combine it with the known properties of CY sigma models. Specifically, the B twisted models receive no quantum corrections, while A twisted models receive world-sheet instanton corrections. Physically, this means that the  $\mathcal{N} = 2$  prepotential in compactified IIB theory, which depends only on complex structure moduli, is classically exact. This means that whereas B brane masses receive world-sheet instanton corrections, the large volume results for central charges and masses of A branes are already exact (this fact and mirror symmetry can then be used to determine B masses).

This means that lines of marginal stability for A branes are the same as in the large volume limit, and this fact strongly suggests that the spectrum of A branes is determined entirely by classical geometric considerations. Since we have not argued that the world-volume theory itself does not receive stringy corrections (indeed we expect it to), this

might seem to be an unjustified leap of faith at this point. Nevertheless there is a good argument for it, which we now summarize.

The classical geometric prediction is that each A brane is a 3-brane wrapped on a sL-submanifold. Now a sL-submanifold  $\Sigma$  of a CY  $n$ -fold is a Lagrangian submanifold with respect to the Kähler form:  $\omega|_{\Sigma} = 0$ , satisfying an additional constraint involving the holomorphic  $n$ -form: there exists a constant  $\theta$  such that

$$\text{Im } e^{i\theta}\Omega|_{\Sigma} = 0. \tag{2.1}$$

The constant  $\theta$  determines which of the original  $\mathcal{N} = 2$  supersymmetries remains unbroken; two branes of different  $\theta$  together break all supersymmetry.

While Lagrangian submanifolds are “floppy,” specified locally by an arbitrary function (in canonical coordinates,  $p_i = \partial f/\partial x^i$ ), the special Lagrangian condition determines this function up to a finite dimensional moduli space, which for a smooth CY has been shown to be smooth and of real dimension  $b^1 = \dim H^1(\Sigma, \mathbf{R})$  [30]. A D-brane configuration is specified by  $\Sigma$  and a flat  $U(1)$  gauge connection, leading to a moduli space of complex dimension  $b^1$ , which before taking stringy corrections into account is a torus fibration.

Interesting examples of sL-submanifolds of  $\mathbf{R}^6$  are known, but not too many are known for CY's. The only general construction known is as the fixed point of an involution, i.e.  $\text{Im } z^i = 0$  in a CICY. Even necessary or sufficient conditions for candidate cycles to support sL-submanifolds are not known. The subject is still rather new however and interest has picked up dramatically as a consequence of the proposal of Strominger, Yau and Zaslow that the mirror  $\mathcal{W}$  to a CY  $\mathcal{M}$  is just the moduli space of the D3-brane on  $\mathcal{M}$  mirror to the D0 on  $\mathcal{W}$ , which will be some (appropriately chosen)  $T^3$ . [39] A number of papers have shown the existence of  $T^3$  fibrations on particular CY's which can in principle be deformed to special Lagrangian fibrations. [22]

The question of how deformations of the CY itself affect the spectrum of sL-submanifolds has recently been studied by Joyce. [26] The part of this story relevant for complex structure deformations (also summarized in [27]) is as follows.

The natural geometric description of transitions between 3-brane configurations in six dimensions is for two intersecting 3-branes to intercommute, producing a single 3-brane, or the reverse. In the large volume limit, this process can be studied in the neighborhood of the intersection point, and the relevant question is: out of all configurations  $\Sigma_{\Theta}$  in  $\mathbf{R}^6$  which asymptote to two planes  $\Sigma_1$  and  $\Sigma_2$  at fixed angles  $\Theta$ , is the minimal volume surface the union of the two planes, or something else, and if so what?

This question was answered some years ago by use of calibrated geometry [24] and the result is known as the “angle theorem”: let  $\Sigma_1$  be the first plane and  $\bar{\Sigma}_2$  the orientation reversal of the second plane; out of  $SO(2n)$  rotations turning  $\Sigma_1$  into  $\bar{\Sigma}_2$  take the eigenvalues  $e^{i\theta_i}$  and let  $\theta = \sum \theta_i$ . If the minimal such  $\theta$  is greater than or equal to  $\pi$ , the volume cannot be reduced; while if  $\theta < \pi$  it can.

The surface of lower volume can be approximately described by use of an exact

sL-submanifold solution in  $\mathbf{R}^6$  with the prescribed asymptotics, which exists in the case  $\theta = \pi$ . One can try use this solution to lower the volume by orienting it to cross both of  $\Sigma_1$  and  $\Sigma_2$  near the intersection point; if it does so, the finite region between the intersections is guaranteed to have lower volume than the original planes. This will be possible exactly when  $\theta < \pi$ .

The angle theorem tells us which of two configurations is stable in terms of a local geometric condition (the same as the string theory condition for the intersection point to have an associated tachyon [3]), but the geometric picture furthermore implies that this can be tested just knowing the central charges for the two branes. This is because the relative angle is known given the phase of pullback of  $\Omega$  (locally  $dz^1 \wedge dz^2 \wedge dz^3$ ) to each brane, and  $\Omega$  must have constant phase on each brane. Thus decays take place just when  $Z(Q_1)$  and  $Z(Q_2)$  are colinear – this is exactly the standard marginal stability condition. These considerations tell us a little more – namely, which state (the single brane, or two branes) is stable on which side of the marginal stability line.

This geometrical picture of A brane decay and stability fits with the constraints following from the exact stringy prepotential and thus, despite the fact that other consequences of this geometrical picture may well be false for substringy branes, it is consistent to imagine that the spectrum is the geometric one. This is in contrast to the B description of the spectrum which must be modified by the stringy corrections to the prepotential. This is the first example of what we will call below the “modified geometric hypothesis.”

All of this tells us quite a bit about the dependence of the spectrum of 3-branes on the CY moduli, but does not substitute for the need to have some results on the spectrum in at least some part of moduli space. Since so little is known about 3-branes at present we instead take this from the large volume limit of the B brane spectrum as many mathematical results towards classifying holomorphic cycles and vector bundles are known.

The most basic of these is the following. Given a holomorphic vector bundle, the Donaldson-Uhlenbeck-Yau theorem gives necessary and sufficient conditions for the existence of a Yang-Mills connection preserving supersymmetry: it must be semistable. This is a somewhat complicated condition involving all holomorphic subbundles, but a simpler necessary condition is known which depends on the Chern character of the bundle (which corresponds to D-brane charge as  $Q_{6-2k} \equiv ch_k(F)$ , the  $2k$  form in  $\text{Tr } e^{F/2\pi}$ ) and the Kähler class:

$$\int (Q_6 Q_2 + \frac{1}{2} Q_4^2) \wedge \omega \geq 0. \quad (2.2)$$

On manifolds with  $b^{1,1} > 1$  this describes an explicit dependence of the spectrum on the Kähler class, as has been discussed by Sharpe. [37]

Since the prepotential determining the central charges of B branes receives world-sheet instanton corrections, it is fairly certain that this mathematical stability condition is modified in the stringy regime. This is quite interesting as it would mean that the condition for a bundle to be usable in superstring compactification is *not* always the

geometrical condition which has been implicitly assumed in the past.

Given a specific supersymmetric brane, we can try to derive its world-volume effective action and general considerations suggest that the simplest quantities to start with are the holomorphic ones: the superpotential and gauge kinetic term. The latter corresponds to the dilaton and in CY compactifications with zero NS field strength this only becomes non-trivial at string loop level (this is one of the invariants defined in [4]). However a superpotential can appear at tree level and indeed for multiple parallel branes we expect a generalization of the  $\text{tr } Z^1[Z^2, Z^3]$  superpotential of 3-branes in flat space. There are also known examples of superpotentials for single branes (see [5] for a discussion).

A plausible counterpart of the nonrenormalization theorem for the  $\mathcal{N} = 2$  prepotential is the following: the superpotential, being essentially a topological quantity in open string theory, depends only on the moduli of the appropriate twisted theory. [5] Specifically, an A brane superpotential depends only on Kähler moduli, while a B brane superpotential depends only on complex structure moduli, and furthermore is equal to the large volume result.

This comes close to showing that a B brane moduli space is the same as in the large volume limit, but not quite – the potential can also contain D terms. These would naturally depend on the Kähler moduli, as in the example of quiver theories. A natural generalization of the preceding conjecture is that these could be determined in the large volume limit from the A brane point of view.

As explained in [27], the D terms are related to the stability question. A world-volume description of the decay process of Joyce starts with the two intersecting 3-branes and  $U(1) \times U(1)$  gauge theory; the intersection comes with a chiral multiplet charged under both  $U(1)$ 's, and the dependence on complex structure moduli comes through an FI term for the relative  $U(1)$ . As one goes through the transition one goes from a supersymmetry breaking ground state with unbroken  $U(1) \times U(1)$  to a supersymmetry preserving ground state with broken relative  $U(1)$ . ‡

An analogous statement was already known on the B side. [37] Equality in (2.2) defines a boundary within the Kähler cone on which stability degenerates to semistability. This means that the connection on the brane becomes reducible, and an enhanced gauge symmetry appears, a phenomenon which in  $\mathcal{N} = 1$  theory can only arise from D terms as above. We see that this qualitative picture survives the stringy corrections, but the precise location of the boundary is different, in a way determined by the A picture geometry.

The upshot of the discussion is that mirror symmetry leads to a natural conjecture for a modified or “mirror geometric hypothesis” – some brane questions are geometric

‡ As in [6] this configuration can be shown by a probe analysis to be equivalent to a single 3-brane produced by intercommutation. [17] A difference with the case of 2-branes considered there is that the resulting 3-brane is not actually special Lagrangian. This is possible because near the transition it has a large extrinsic curvature and this is evidence that the detailed form of the special Lagrangian condition gets  $\alpha'$  corrections.

in the A picture, and others are geometric in the B picture. As is well known the prepotential in the complex structure sector is determined geometrically; this determines A brane central charges and stability and strongly motivates the claim that the spectrum of branes can be understood geometrically in the A picture. We can add to this the claim that the superpotential in the B twisted model is classical; this means that brane moduli spaces are largely determined by the geometry of the B picture. Finally, it may be possible to determine the D terms in the A picture and complete the story.

So far as I know, these conjectures are consistent with the evidence, but require much more testing. The most interesting tests are in the stringy regime, as we discuss next.

### 3. Boundary states and branes

Exactly solvable CFT's were a fruitful source of insight into compactification of closed string theory and are now beginning to teach us about branes in these compactifications. The fundamental notion is that of "boundary state," a CFT description of a boundary condition as a linear functional on the closed string Hilbert space. Reparameterization invariance and supersymmetry can be easily implemented by imposing operator constraints. One must then impose the condition that all annulus partition functions (associated with pairs of boundary states) have an open string Hilbert space interpretation (the multiplicities are integers); this condition was proposed by Cardy and can be solved for rational CFT's. D-brane ground states correspond to such boundary states (not much is known about the non-rational case; possibly additional unknown constraints must be satisfied).

The simplest and most studied models are orbifolds and orientifolds. In this case the general boundary state approach can be shown to reduce to the world-sheet prescription proposed in [12] – one introduces image D-branes on the cover and quotients by a simultaneous space-time and gauge action. The case of strings and branes near a single orbifold or orientifold singularity is particularly easy and one obtains quiver gauge theories as world-volume theories. For  $\mathbf{C}^3/\Gamma$  these have been much studied and among the noteworthy results are the following:

- (i) The resolution of these singularities is described in quiver gauge theory by FI terms coupling to Kähler moduli. [13, 35] If multiple resolutions with different topology are mathematically possible, they all appear to be accessible physically. [20]
- (ii) The resulting metrics are not Ricci flat. [14, 35] Although some caveats were made in that work, it can be shown that this statement is true at string tree level. [17]
- (iii) The quiver theory depends on the choice of representation of  $\Gamma$ ; the basic case is the regular representation, while non-regular representations correspond to branes wrapped around exceptional cycles (or "fractional branes"). [8]
- (iv) If we take D3-branes to get a 3+1 theory, the regular representation is distinguished by having zero beta function in the large  $N$  limit. [29]

- (v) These theories have supergravity duals corresponding to the quotients  $AdS_5 \times S^5/\Gamma$ . [28]

Recently Diaconescu and Gomis have studied the case of  $\mathbf{C}^3/\mathbf{Z}_3$  in detail. [9] Besides checking the equivalence between the boundary state approach and the proposal of [12], they determined the mapping between fractional branes and wrapped branes in the large volume limit, using techniques we will describe below. Additional summary of this example can be found in [1].

We now turn to Gepner models and the work [5]. Gepner models provide CFT models which are equivalent to CY compactification at special points in moduli space of enhanced discrete symmetry. The study of boundary states in these models was initiated by Recknagel and Schomerus [34]; they classified the subset of boundary states which can be obtained by separate boundary conditions in the individual  $\mathcal{N} = 2$  minimal model factors, for which Cardy's techniques apply. (See also [23], as well as [19] which uses the Landau-Ginsburg approach.)

Let us briefly summarize the spectrum of branes one obtains and the main result used in the analysis of [5] – the intersection form between two branes. Cardy's analysis (for diagonal modular invariant) produces boundary conditions in one-to-one correspondence with closed string primary fields; the spectrum of open strings with two such boundary conditions  $a$  and  $b$  is generated by primary fields in one-to-one correspondence with those on the right hand side of the (Verlinde) fusion rules  $\phi_a \phi_b \rightarrow N_{ab}^c \phi_c$ .

The  $A_k$   $\mathcal{N} = 2$  minimal model can be obtained as a deformation of the  $SU(2)_k$  WZW model, and its primary fields  $\phi_m^l$  are labelled similarly, by two integers  $0 \leq l \leq k$  (the  $SU(2)$  representation label) and  $0 \leq m < 2k + 4$  (the charge under the  $U(1)$  of  $\mathcal{N} = 2$ ) up to a  $\mathbf{Z}_2$  identification  $(l, m) \cong (k - l, m + k + 2)$ . The fusion rules are the product of  $U(1)$  fusion rules (i.e.  $\mathbf{Z}_{k+2}$  charge conservation) with  $SU(2)_k$  fusion rules.

Before implementing the GSO projection, the Gepner model boundary conditions are labelled by a set of such integers, and are all A boundary states (since they correspond to left-right symmetric fields). The GSO projection then restricts the closed string spectrum to (odd) integer total  $U(1)$  charge  $\sum m$ , while twisted states with  $m_L \neq m_R$  are added. The restriction has the effect of reducing the number of distinct A boundary states, while the twisted sectors provide new candidate B boundary states.

The final result for the  $(3)^5$  model is that all boundary states are labelled by a set of five  $L_i \in \{0, 1\}$ ; the A boundary states are also labelled by five  $M_i$  satisfying one relation and form representations of  $\mathbf{Z}_5^4 \times S_5$  discrete symmetry, while the B boundary states have a single  $M$  label and represent a  $\mathbf{Z}_5$  discrete symmetry. These are the known discrete symmetries of the CFT at the Gepner point; it is known to be equivalent to the Fermat quintic  $\sum_{i=1}^5 Z_i^5 = 0$  in  $\mathbf{P}^4$  with manifest  $\mathbf{Z}_5^4$  symmetry, at a special point in Kähler moduli space with quantum  $\mathbf{Z}_5$  symmetry.

The modified geometric hypothesis of section 2 would imply that these A branes are exactly the sL-submanifolds of the Fermat quintic and we can test this idea for the



known sL-submanifolds. These are obtained by taking a real section  $\text{Im } e^{2\pi i m_i/5} Z_i = 0$ : topologically these are  $\mathbf{RP}^3$ 's, which fall into the same representation of  $\mathbf{Z}_5^4 \times S_5$  as two sets of boundary states: those with all  $L_i = 0$  and those with all  $L_i = 1$ . How can we tell which (if either) is their counterpart?

A strong check of any proposed identification is that the geometric intersection number of a pair of 3-branes must agree with the quantity  $\text{Tr}_{ab}(-1)^F$  in this sector of the open string theory. [15] This can be seen by considering electric-magnetic charge quantization in the resulting  $d = 4$  theory. This computation is a special case of those in [34] and it turns out the  $L_i = 1$  states match this intersection form, while the  $L_i = 0$  states do not (they presumably correspond to some other sL-submanifolds). So far this is in agreement with both the original and the modified geometric hypothesis.

However, one also finds that the  $L_i = 1$  brane world-volumes have a massless chiral multiplet, and this disagrees with the geometric prediction of [30]. As discussed in [5] it is likely that this is lifted by a superpotential, but even so this contradicts the strongest form of the geometric hypothesis, in which both this massless field and the superpotential would have matched. It does not contradict the modified geometric hypothesis, which allows the A brane superpotential to depend on the Kähler form, and furthermore shows that massive fields in the large volume limit can come down to become (linearized) moduli. Such effects and even jumping of the dimension of moduli space are known to be possible in the B picture; perhaps this superpotential would be manifest in a mirror description.

Turning to the B branes, we have more intuition for which of these exist in the large volume limit: namely the condition (2.2) must be satisfied (if  $Q_6 \neq 0$ ; there is an analogous statement if  $Q_6 = 0$  but  $Q_4 \neq 0$ ). Although bundles on the quintic are by no means classified, various considerations suggest that generic charge vectors for which the discriminant (the left hand side of (2.2)) is sufficiently large will be associated to stable bundles.

Thus it is interesting to express the charges of the B boundary states in large volume terms, and compare. A precise form of this comparison is to choose a path in Kähler moduli space from the Gepner point to the large volume limit, and use the flat  $Sp(2r, \mathbf{Z})$  connection provided by special geometry to transport the charge lattices between the two regimes.

The Kähler moduli space and prepotential for the quintic is of course well known from the famous work of Candelas et. al. [7] which computed the periods of the three-form on the mirror. To review the structure of this moduli space: it is a Riemann sphere with three singularities, a large volume limit at  $z \rightarrow \infty$ , the Gepner point with a  $\mathbf{Z}_5$  quotient singularity at  $z = 0$ , and finally a ‘‘conifold’’ singularity at  $z = 1$  at which a three-cycle of the mirror degenerates (has zero period). It turns out [7, 21] that on the original quintic this is precisely the central charge of the ‘‘pure’’ (trivial gauge field) six-brane. The periods  $\Pi_i(z)$  can be obtained as solutions of Picard-Fuchs ODE’s or more explicitly as series expansions around each singularity, with radius of convergence determined by the locations of the others.

Two concrete results are now needed from this analysis. First, the mirror map gives us an appropriate basis for the large volume limit – central charges of the individual  $2p$ -branes. Second, given that the central charge of a brane with charge vector  $Q^i$  is  $Z = Q^i \Pi_i(z)$ , the transition functions of the flat connection on the charge lattice are simply the linear transformations between different bases  $\Pi_i(z)$  adapted to different regions of moduli space (these are connection formulas for generalized hypergeometric functions). This tells us what the  $2p$ -brane central charges will be at the Gepner point.

In principle these could already be compared with a precise computation of the central charges of our boundary states, but such a comparison will run into tricky problems of normalization. The best way to study the charges of D-branes – as was done in the very first example [33] – is to instead compute the interaction between two D-branes in the open string channel, as this is canonically normalized (it is a partition function). Indeed the simplest quantity of this type is the intersection form  $\text{Tr}_{ab}(-1)^F$  discussed above and thus the simplest way to proceed is to express the known large volume intersection form in terms of a natural basis at the Gepner point (one which represents the quantum  $\mathbf{Z}_5$  symmetry in a simple way) and compare this intersection form with the intersection forms of the boundary states.

It turns out that the resulting boundary state charges are simple when expressed using the basis first postulated by Candelas – the zero-brane period and its  $\mathbf{Z}_5$  images. The states of minimal charge are the five  $L_i = 0$  states; one of these turns out to be the pure six-brane  $\langle 6B | \equiv (1 \ 0 \ 0 \ 0)$ , and to get the others we just need to know the  $\mathbf{Z}_5$  monodromy in the large volume basis, which is given in [7]. In the conventions of [5] it is§

$$g \equiv (Q_6 \ Q_4 \ Q_2 \ Q_0) \rightarrow (Q_6 \ Q_4 \ Q_2 \ Q_0) \begin{pmatrix} -4 & -1 & -8 & 5 \\ -3 & 1 & 5 & 3 \\ 1 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix} \quad (3.1)$$

and thus the others are  $\langle 6B | g^M$ . The charges for states with  $L = \sum L_i > 0$  can be derived from these by using the fusion rules: essentially, they are  $\langle Q_6 | (1 + g)^L g^M$ .

One surprise of the result is that the D0-brane is not present (as a rational boundary state; this is not to say that it does not exist at the Gepner point). It appears that this is also consistent with the geometric hypothesis in the following sense: any location we might pick for the D0 would break some of  $\mathbf{Z}_5^4$ , but all of the rational B boundary states are singlets under  $\mathbf{Z}_5^4$ , so we should not find the D0 in this analysis.||

Looking at the charges of all of the boundary states, they appear to be consistent with the original geometric hypothesis, at least in the weak sense that they are all consistent with (2.2). Not too much more is known about vector bundles on the quintic so it is hard to be more precise.

§ Note that these are conventions in which the charge vectors include the factor  $\sqrt{\hat{A}}$ , which are not the conventions of (2.2). The latter are also given in [5]; they are the ones in which the large volume monodromy is simple but charges are not necessarily integral.

|| It appears that other Gepner models can contain the D0 as a boundary state.[10]

On the other hand, the monodromy (3.1) in general can take solutions of (2.2) into non-solutions, making it highly implausible that it is a symmetry of the entire brane spectrum. This is reminiscent of related phenomena in the study of  $\mathcal{N} = 2$  gauge theory, and we turn to this analogy.

#### 4. Marginal stability on the quintic

As we saw in the previous section, the D0-brane is not a rational boundary state for the Gepner quintic. This leads one to wonder whether it exists in the stringy regime at all, and more generally how much the spectrum of branes varies as we move around.

In generic  $\mathcal{N} = 2$ ,  $d = 4$  theories, the spectrum of BPS states depends on the moduli, but it varies in a highly constrained way. A state of charge  $Q$  will generically be stable under variations of the moduli, but there exist can lines of marginal stability (or “jumping lines”), on which the state can decay to BPS states of charge  $Q_1$  and  $Q_2$ , if the condition

$$|Z(Q)| = |Z(Q_1)| + |Z(Q_2)| \quad (4.1)$$

is satisfied. Here  $Z(Q) = Q \cdot \Pi(z)$  is the central charge in terms of a vector of periods  $\Pi(z)$  at a point  $z$  in moduli space; for the A branes these are the periods of the three-form  $\Pi = \int \Omega$  (normalized to  $\int \Omega \wedge \bar{\Omega} = 1$ ).

The most familiar examples are supersymmetric gauge theories, which have been studied in great detail. For example, pure  $SU(2)$   $\mathcal{N} = 2$  gauge theory (the original Seiberg-Witten solution) has a line of marginal stability which goes through the massless monopole and dyon points and separates the strong and weak coupling limits. The strong coupling BPS spectrum consists only of the monopole and dyon, the two states responsible for the singularities. This phenomenon was necessary as otherwise monodromies around the massless monopole point would produce states with arbitrarily large electric charge, which are not present in the known semiclassical spectrum.

Besides the known semiclassical spectrum, a number of constraints follow from the solution for the prepotential and justify this result. The primary constraint is the physical correspondance between singularities and massless states [38]: if  $Z(Q)$  vanishes at some  $z$ , either there is a corresponding singularity which we can think of as coming from integrating out this state at nearby points, or else the state must not exist at  $z$ . If it exists at some  $z'$ , there must be a line of marginal stability separating  $z$  and  $z'$ . This is quite strong as it turns out that the ratio of the two periods  $a_D/a$  assumes all possible real values (in all the asymptotically free  $SU(2)$  theories in fact) and thus every charge can be constrained. One sees this most easily by combining the result (easily verified numerically) that  $\text{Im } a_D/a$  changes sign between weak and strong coupling regimes with the  $SL(2, \mathbf{Z})$  transformation properties of  $a_D/a$  (which force the line  $\text{Im } a_D/a = 0$  to connect the massless monopole and dyon points).

Our earlier observation that the  $\mathbf{Z}_5$  monodromy obtained by encircling the Gepner point in the quintic does not fit well with the known constraints on the large volume

spectrum is our first suggestion that similar phenomena will obtain here. There is also a qualitative similarity to the change of sign of  $\text{Im } a_D/a$ . Let the conifold point be  $z_c = 1$ : here the six-brane becomes massless,  $\Pi_6 \sim z - z_c$ . Although the other periods are not analytic here, they are still continuous:  $\Pi \sim (z - z_c) \log(z - z_c) + \text{regular}$ . Thus just as in gauge theory,  $\text{Im } \Pi_6/\Pi_0$  changes sign as we go through this point.

This starts to suggest that the gauge theory picture with its drastic change in the spectrum might also be possible here. Unfortunately few of the other elements of the story there have been developed for the quintic (or indeed any CY) moduli space. In particular, the appropriate analogs of  $SL(2, \mathbf{Z})$  and the fundamental region are not known, making it difficult to get a good global picture of the moduli space.

The boundary state results show us that the answer will not be as simple as that for gauge theory – the spectrum will not collapse simply to the states which can become massless. We should also not assume that all of the boundary states exist at large volume.

To study this one can simply follow all of the central charges for boundary states out from the Gepner point to the large volume limit, to see what happens. One expects more marginal stability lines in the neighborhood of the conifold point, so to minimize the possibilities for decay we choose the trajectory  $z$  real and negative opposite to it in moduli space. We then numerically integrated the Picard-Fuchs equations (and checked the results against the series expansions of [7]) to get the periods and thus the BPS masses.

Using these to compute the masses of BPS branes with the charges of all rational boundary states produces a surprise: one of them has its period go through zero! In other words, there exists a BPS state at the Gepner point whose mass appears to go to zero at a non-singular point  $X$  in moduli space. (Readers who want proof that this is not an error of numerics or conventions will find a semi-qualitative proof in the appendix.)

This in itself is not inconsistent as long as there exists a line of marginal stability separating the point  $X$  from the Gepner point. At this point we run into one of the main difficulties in studying these questions for CY: there are an infinite number of candidate marginal stability lines, and we need more knowledge about the BPS spectrum to decide which are real (i.e. the decay takes place, which requires the states of charge  $Q_1$  and  $Q_2$  to actually exist on the line). This is closely related to the fact that at a generic point in moduli space, there exist charges  $Q$  such that  $|Z(Q)| < \epsilon$  for any positive  $\epsilon$ , no matter how small. Consider the Gepner point: there the periods are the fifth roots of unity, so the set of  $Z(Q)$  is a  $Z_5$  symmetric lattice embedded in the complex plane.

Although we have not as yet found the true marginal stability lines, we can at least try to postulate a pair of charges  $Q_1$  and  $Q_2$  into which the problematic state can decay and whose masses do not cross zero on the way to large volume. This is not hard to do, and thus the existence of such a marginal stability line seems perfectly plausible – there seems no reason to doubt the consistency of the theory.

Thus we have proof of the existence of at least one marginal stability line; given that we have two points at which  $Z(Q)$  vanishes for “simple” charges  $Q$  it is quite likely

that many other true marginal stability lines pass through these points.

An even stronger consideration of this type is to follow large volume branes to the Gepner point: it is easy to find charge vectors satisfying (2.2) whose period goes through zero on this axis. If it is true that these generally correspond to stable bundles, we have many more examples.

All this starts to be significant evidence for the claim that the BPS spectrum is rather different in the stringy regime.

#### 4.1. A note on attractor points

A question related to marginal stability but somewhat simpler has arisen in the study of BPS black holes in CY compactification. It has been shown [18] that the entropy of such black holes is governed by the “attractor mechanism.” Given a black hole of large charge  $Q$ , the consistency condition for a covariantly constant spinor is a first order equation which is just gradient flow on the moduli space to a minimum of the quantity  $S(z) = |Q \cdot \Pi(z)|$ ; the entropy is the minimal value  $S_{min}(Q)$ .

For some  $Q$ , it is possible that  $S_{min}(Q) = 0$ . In the previously known examples (such as the state which goes massless at the conifold point), the state existed at the minimizing point in moduli space and produced a singularity in the moduli space metric, modifying the discussion. What we have found here is a  $Q$  for which the discussion above leads to a contradiction (as noted in [31]) – the attractor equation breaks down (has no sensible solution) before reaching the horizon, so this is not a failure of supergravity. Indeed, this could be interpreted as an argument that such black holes cannot exist, and an observation consistent with this idea is that (at least in some cases) the condition  $S_{min}(Q) = 0$  reduces to the negation of (2.2) in the large volume limit on the quintic,

However, we have found a particle with (small) charge  $Q$  and  $S_{min}(Q) = 0$  at the Gepner point, so we have a paradox. We can take  $N$  of these particles and put them into a small region of space, using only total energy  $Nm + \epsilon$ . For  $N$  sufficiently large, one would certainly expect that they form a black hole  $NQ$ , for which the previous argument applies.

What is going on ? The resolution will almost certainly use the fact that – as a single brane – the object in question was unstable at the minimizing point. One scenario is that the final stable object is a bound state of two black holes of charges  $NQ_1$  and  $NQ_2$  with a hard core repulsive potential. [16] This would evade the previously cited argument, which assumed a spherically symmetric configuration.

It seems likely that more surprises along this line await us.

## 5. Conclusions and further directions

D-branes have played a central role in the study of superstring and M theory duality. Quite a lot has been understood about compactifications with enhanced supersymmetry, but eventually we will need to deal with the physical cases of  $\mathcal{N} = 0$  (and hopefully

$\mathcal{N} = 1$ !) supersymmetry in four dimensions.

A large class of  $\mathcal{N} = 1$  supersymmetric string compactifications can be obtained by using D-branes on Calabi-Yaus. Many of these are related to known constructions (F theory or the strong coupling limit of heterotic strings) but what I have tried to show here is that we can make further progress by using special properties of the weak type II string coupling limit, namely the close relation between D-brane theories which fill different parts of Minkowski space (e.g. D3 and D0-branes), and the powerful tools of mirror symmetry and exactly solvable CFT.

A reasonable goal for the current work is to settle the geometric hypothesis and modified geometric hypothesis as described here – namely, to show that the superpotential and D terms depend only on complex and Kähler data for B branes (the reverse for A branes) and answer the following questions:

- (i) Are all A branes 3-branes wrapped on sL-submanifold's, even for a stringy CY ?  
Are all marginal stability lines and decays described by the local intercommutation of 3-branes ?
- (ii) If so, do the mirror symmetry predictions for the spectrum of B branes agree with geometric predictions at large volume? What does the semistability condition translate into in the A picture ?
- (iii) Is the spectrum of B branes on stringy CY's very different from the large volume spectrum (as the results here suggest) ? If so, is it finite or perhaps characterized by simple inequalities analogous to (2.2) ?
- (iv) Is knowledge of the large volume spectrum and the exact prepotential enough to determine the spectrum throughout moduli space (using consistency arguments of the sort which worked for supersymmetric gauge theory) ?
- (v) Can we make a complete statement about the potential and moduli space on these branes (presumably combining B picture information to get the superpotential and A picture to get the D terms) ?
- (vi) Can we extend this picture to finite string coupling, perhaps by making contact with the heterotic string limits of the same models ?

A longer term goal will be to understand terms in the effective action which are not so strongly constrained by supersymmetry, such as the D0-metric on a CY.

Perhaps interesting non-supersymmetric models can be obtained by considering non-BPS space-filling brane configurations, along the lines of [36, 27].

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## Appendix

We give a semi-qualitative argument for the vanishing period, using the results for the periods of the mirror to the quintic in [7]. They are functions of a complex modulus  $\psi$  which covers the Riemann sphere with three punctures.  $\psi \rightarrow \infty$  is the large volume limit, with  $t = B + iV = -\frac{5}{2\pi i} \log(5\psi)$ .  $\psi = 1$  is the conifold point, and  $\psi = 0$  is the Gepner point.  $\psi \rightarrow \alpha\psi$  with  $\alpha = e^{2\pi i/5}$  is the  $\mathbf{Z}_5$  quantum symmetry of the Gepner point – it leads to the same bulk theory but acts as an  $Sp(4, \mathbf{Z})$  monodromy on the brane spectrum.

Candelas et al. use a basis  $\omega_k(\psi)$  where  $\omega_0$  is the 0-brane period in the large volume limit and the others are its images under the  $\mathbf{Z}_5$  of the Gepner point. These are multi-valued on the  $\psi$  plane and thus it is necessary to take care with the domains of definition.

There are three lines along which the periods have simple reality properties. We define the line  $A$  to be  $\psi = x$  real with  $x > 1$ , the line  $B$  as  $\psi = x$  real satisfying  $0 \leq x < 1$ , and the line  $C$  as  $\psi = e^{2\pi i/10}x$  with  $x$  real and positive.

From the explicit series expansions for the periods it is easy to check the following qualitative properties:

- (i) Near the Gepner point,  $\omega_j(\psi) \rightarrow -\alpha^{2+j}C\psi$  with  $C = \Gamma(1/5)/\Gamma(4/5)^4$  a positive real constant.
- (ii) In the large volume limit,  $\omega_j(\psi) \sim \frac{S_{j3}}{6}t^3$  where  $S_{j3} = 0, 5, -15, 15, -5$  for  $j = 0, 1, 2, 3, 4$ .
- (iii) Along  $B$  we have  $(\omega_j(x))^* = \omega_{1-j}(x)$  and along  $C$  we have  $(\omega_j(\alpha^{1/2}x))^* = \omega_{-j}(\alpha^{1/2}x)$  (this must be checked using both large and small volume expansions).

From [5], one can check that the period

$$\Pi_X = \omega_1 - \omega_4 \tag{5.1}$$

is the central charge of a B boundary state  $L_1 = 1$ ,  $L_i = 0$  for  $i > 1$ .

We now argue that  $\Pi_X$  will have a zero along the axis C. From (iii) we see that  $\Pi_X$  is purely imaginary along this axis, so if the imaginary part changes sign between the Gepner and large volume limits it must have a zero. This can be checked explicitly given the limiting behaviors we quoted.

A way to see that this was inevitable is to consider the behavior of the six-brane period  $\Pi_6 = \omega_1 - \omega_0$  on the loop  $ABC$  in moduli space. At large volume,  $\Pi_6 \sim_{\psi \sim \alpha^{1/2}\infty} \frac{5}{6}t^3 \sim \frac{5}{6} - iV^3$  so it comes in from negative imaginary infinity towards zero. Along A and B  $\Pi_6$  is purely imaginary and as we know it crosses zero at  $\psi = 1$  (the conifold point) and comes out the other side, to reach its value at the Gepner point  $\Pi_6(\psi) \sim_{\psi \sim 0} C(\alpha^2 - \alpha^3)\psi = C2i \sin \frac{\pi}{5}\psi$ . As we come back along the axis C, we know that the six-brane does not become massless anywhere, so  $\Pi_6$  must move off into the complex plane to avoid the origin, finally joining the same asymptotics  $\Pi_6 \sim -5/6iV^3$  we had at large positive  $\psi$ .

This behavior implies that  $\Pi_6$  must cross the real axis at some point, and since  $\omega_0$  is real all along  $C$ ,  $\omega_1$  must become real at this point.

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