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## Topoelectrical-circuit realization of topological corner modes

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Ronny

**Abstract:** Quantized electric quadrupole insulators have recently been proposed as novel quantum states of matter in two spatial dimensions. Gapped otherwise, they can feature zero-dimensional topological corner mid-gap states protected by the bulk spectral gap, reflection symmetries and a spectral symmetry. Here we introduce a topoelectrical circuit design for realizing such corner modes experimentally and report measurements in which the modes appear as topological boundary resonances in the corner impedance profile of the circuit. Whereas the quantized bulk quadrupole moment of an electronic crystal does not have a direct analogue in the classical topoelectrical-circuit framework, the corner modes inherit the identical form from the quantum case. Due to the flexibility and tunability of electrical circuits, they are an ideal platform for studying the reflection symmetry-protected character of corner modes in detail. Our work therefore establishes an instance where topoelectrical circuitry is employed to bridge the gap between quantum theoretical modelling and the experimental realization of topological band structures.

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# Topoelectrical circuit realization of topological corner modes

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Quantized electric quadrupole insulators have been recently proposed as unprecedented quantum states of matter in two spatial dimensions. Gapped otherwise, they may feature zero-dimensional topological corner midgap states protected by the bulk spectral gap, reflection symmetries, and a spectral symmetry. We develop and measure a topoelectrical circuit design for such corner modes which manifest themselves as topological boundary resonances in the corner impedance profile of the circuit. While the quantized bulk quadrupole moment of the electronic crystal does not have a direct analogue in the classical topoelectrical circuit framework, the corner modes inherit the identical form from the quantum case and, due to the accessibility and tunability of electrical circuits, lend themselves to a detailed study of their reflection symmetry-protected character. Our work therefore establishes an instance where topoelectrical circuitry is employed to bridge the gap between quantum theoretical modeling and the experimental realization of topological band structures.

**Introduction** — The Berry phase provides a powerful language to describe the topological character of band structures and single-particle systems<sup>1,2</sup>. Manifestly, it allows to treat fermionic and bosonic quantum systems on the same footing. Furthermore, the Berry phase concept is not tied to Hilbert space, but applies to the connectivity of any given coordinate space, and as such accounts for classical degrees of freedom as well<sup>3</sup>. It is thus intuitive that, with the discovery of various topological quantum states of matter such as quantum Hall<sup>4</sup> and quantum spin Hall effect<sup>5</sup>, classical systems with similar phenomenology could also be identified. This was initiated in the context of photonics<sup>6</sup>, and subsequently transferred to other fields such as mechanics<sup>7</sup>, acoustics<sup>8</sup>, and electronics<sup>9</sup>. Even though spectra and eigenstates of the single particle problem, including edge modes, might look similar or even identical, it is the fundamental degrees of freedom which pose the central distinction between quantum systems and their designed classical analogues. First, quantization phenomena deriving from topological invariants usually necessitate the non-commutativity of phase space and as such are often reserved to quantum systems. Second, internal symmetries pivotal to the protection of a topological phase might not carry over to classical systems as the degrees of freedom are changed. For instance, this applies to time-reversal symmetry  $T$  as the protecting symmetry of the quantum spin Hall effect, where the half integer spin of electrons implies Kramer's degeneracy due to  $T^2 = -1$  in the quantum case, while it does not in the classical case  $T^2 = 1$ . Whereas the classical counterpropagating edge modes might still be detectable, there is no particular topological protection left, rendering the classical system much more vulnerable to perturbations<sup>9,10</sup>.

From this perspective, at least two directions appear as most promising to develop classical topological band structure models that are universally stable beyond fine-tuning. The first is the realization of classical analogues to topological semimetals<sup>11-16</sup>, where the extensive edge mode degeneracy suggests unambiguous persistent spectral edge features also in the presence of small perturbations. The second is

to focus on topologically insulating quantum electronic states where either no protecting symmetries are needed such as for the quantum Hall effect<sup>6</sup>, or where the protecting symmetries obey the same algebraic relations in the classical and quantum mechanical case.

Electric quadrupole insulators<sup>17</sup> fall in the latter category. While the quantum case is most suitably constructed from the viewpoint of quantized multipole moments of an electronic crystal, the complementary protecting symmetry perspective is most intuitive for the classical system design. The symmetry group that protects the quantization of the quadrupole moment includes two non-commuting reflection symmetries  $M_x$  and  $M_y$  as well as a  $C_4$  rotation symmetry. In particular, they obey  $M_{x,y}^2 = 1$ , and as such directly carry over to the classical degrees of freedom. In analogy to the relation between the quantization of bulk dipole moment (which is quantized to half-integer values by inversion symmetry) and the appearance of protected end states in the topological Su-Schrieffer-Heeger model, an additional spectral symmetry, the chiral symmetry, is needed to pin the topological boundary modes in the middle of the bulk energy gap. All these symmetries are realized in the microscopic model given in Ref. 17. Hence, the only task is to implement the hopping model given by a four site unit cell and real, but sign-changing hybridization elements. Due to recent progress in implementing waveguide elements that invert the sign of hybridization<sup>18</sup>, the complexity of this model could recently be captured by a photonic cavity lattice structure<sup>19</sup>. We turn to topoelectrical circuits to realize the quadrupole insulators in a classical environment.

**Linear circuit theory and topology** — We consider non-dissipative linear electric circuits, i.e., circuits made of capacitors and inductors. Labeling the nodes of a circuit by  $a = 1, 2, \dots$ , the response of the circuit at frequency  $\omega$  is given by Kirchhoff's law

$$I_a(\omega) = \sum_{b=1,2,\dots} J_{ab}(\omega) V_b(\omega) \quad (1)$$

that relates the voltages  $V_a$  to the currents  $I_a$  via the grounded

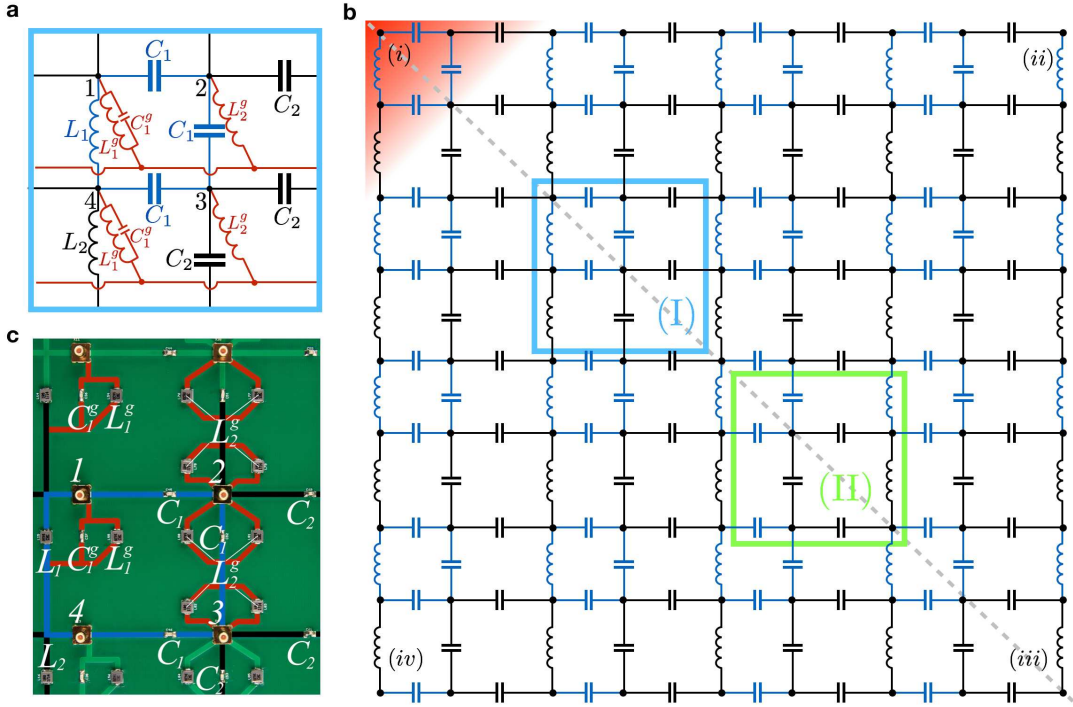


FIG. 1. Electrical circuit exhibiting a topological corner state with nodes of the circuit indicated by black dots. a) Unit cell of the circuit. Blue and black circuit elements correspond to weak and strong bonds in a tight-binding or mechanical analogue of the circuit. Red circuit elements connect to the ground. All capacitor-inductor pairs have the same resonance frequency  $\omega_0 = 1/\sqrt{L_1 C_1} = 1/\sqrt{L_2 C_2} = 1/\sqrt{L_1^g C_1^g}$ . b) Layout of the full circuit which has been realized experimentally. The corners (i) and (iii) are invariant under the mirror symmetry that leaves the dashed green line invariant. They are compatible with the bulk unit cell choices (I) and (II), respectively, which correspond to an interchange of strong and weak bonds. As a consequence we expect a topological bound state at corner (i) but not at corner (iii). c) Unit cell of the experimentally realized circuit.

circuit Laplacian

$$J_{ab}(\omega) = i\omega C_{ab} - \frac{i}{\omega} W_{ab}. \quad (2)$$

Here, the off-diagonal components of the matrix  $C$  contain the capacity  $C_{ab}$  between nodes  $a \neq b$ , while its diagonal component is given by the total node capacitance

$$C_{aa} = -C_{a0} - \sum_{b=1,2,\dots} C_{ab} \quad (3)$$

including the capacitance  $C_{a0}$  between node  $a$  and the ground. Similarly, the off-diagonal components of the matrix  $W$  contain the inverse inductivity  $W_{ab} = L_{ab}^{-1}$  between nodes  $a \neq b$ , while its diagonal components are given by the total node inductivity

$$W_{aa} = -L_{a0}^{-1} - \sum_{b=1,2,\dots} L_{ab}^{-1} \quad (4)$$

including the inductivity  $L_{a0}$  between node  $a$  and the ground.

At fixed frequency  $\omega$ ,  $J_{ab}(\omega)$  determines the linear response of the circuit in that the impedance  $Z_{ab}$  between two nodes  $a$  and  $b$  is given by

$$Z_{ab}(\omega) = G_{aa}(\omega) + G_{bb}(\omega) - G_{ab}(\omega) - G_{ba}(\omega), \quad (5)$$

where  $G(\omega) = J^{-1}(\omega)$  is the circuit Green's function. The impedance is thus dominated by the smallest eigenvalues  $j_n(\omega)$  of  $J(\omega)$  at this given frequency, provided that the sites  $a$  and  $b$  are in the support of the corresponding eigenfunctions.

In turn, frequencies  $\omega$  for which an exact zero eigenvalue  $j_n(\omega) = 0$  exists correspond to eigenmodes of the circuit. They are determined by the equations of motion satisfied by the electric potential  $\phi_a(t)$  at node  $a$

$$\sum_{b=1,2,\dots} C_{ab} \frac{d^2}{dt^2} \phi_b(t) + \sum_{b=1,2,\dots} W_{ab} \phi_b(t) = 0. \quad (6)$$

The spectrum  $\omega^2$  of eigenmodes of the circuit is thus given by the spectrum of the dynamical matrix

$$D = C^{-1/2} W C^{-1/2}, \quad (7)$$

with matrix multiplication implied.

We now explain why topological properties can be defined for the matrices  $J(\omega)$  and  $D$  that describe the physics of the circuit. In order to define topological properties of a physical system, the notions of *locality* and *adiabaticity* (enabled by spectral gaps) are of central importance. Locality naturally arises when we consider circuits in which the nodes  $a$  are arranged in a (in the case at hand two-dimensional) lattice. This

also allows to define spatial symmetry transformations. Adiabaticity in turn follows from the spectral continuity of  $J(\omega)$  as a function of  $\omega$ , that is, if a specific frequency  $\omega_0$  lies in a gap in the spectrum of  $D$ , the spectrum of  $J(\omega_0)$  also has a gap around zero eigenvalues. Furthermore, a spectrally isolated eigenvalue (which may be a topological bound state) of  $D$  at frequency  $\omega_0$  is in correspondence with a spectrally isolated zero mode of  $J(\omega_0)$ .

Due to these relations between  $J(\omega)$  and  $D$ , protected boundary modes of a circuit can arise from the topological properties of either matrix. In this work, we choose to build a two-dimensional circuit for which the topology of  $J(\omega_0)$  at a specific frequency  $\omega_0$  protects corner modes. The topological protection of spectrally isolated zero modes always requires a spectral (chiral or particle-hole) symmetry that relates eigenvalues of equal magnitude and opposite sign. Spectrally and locally isolated eigenstates of this symmetry, if present, are protected in that they are pinned to the eigenvalue zero. As an eigenstate of  $J(\omega)$ , such a state naturally dominates the linear response of the circuit.

**Circuit with corner states** —To realize a quadrupole insulator with topologically protected corner states, the system should have two *anticommuting* mirror symmetries, as well as a  $\hat{C}_4$  rotation symmetry in the bulk. The fundamental mirror symmetries in classical systems *commute*. To build a classical analogue of an electric quadrupole insulator, we thus devise a circuit that has an *emergent* pair of anticommuting mirror symmetries  $\hat{M}_x$  and  $\hat{M}_y$  for modes near a specific frequency  $\omega_0$ . This means that  $J(\omega_0)$  commutes exactly with  $\hat{M}_x$  and  $\hat{M}_y$  and the eigenspaces of  $D$  are approximately invariant under  $\hat{M}_x$  and  $\hat{M}_y$  for frequencies near  $\omega_0$ .

We first discuss the bulk properties of a periodically repeating circuit unit cell, depicted in Fig. 1, before considering boundary modes. The circuit unit cell contains four sites denoted by pairs  $(i, j) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ . We use two pairs of capacitors and inductors  $(C_1, L_1)$  and  $(C_2, L_2)$  which have the same resonance frequency  $\omega_0 = 1/\sqrt{L_1 C_1} = 1/\sqrt{L_2 C_2}$  to couple these sites. The latter equality is automatically satisfied if we set  $C_2 = \lambda C_1$ ,  $L_2 = L_1/\lambda$  for some real positive parameter  $\lambda$ . Sites 1 and 4 are connected to the ground via an LC circuit with  $C_1^g = 2C_1$  and  $L_1^g = L_1/2$  such that it has the same resonance frequency  $\omega_0$ . Sites 2 and 3 are connected to the ground via an inductivity  $L_2^g = L_1/[2(1 + \lambda)]$ . In this setup, the circuit is parametrized by the parameters  $\omega_0$  and  $\lambda$ .

We now describe the circuit with periodic boundary conditions in momentum space. The Fourier components of the matrix  $J_\lambda(\omega)$ , denoted by  $\tilde{J}_\lambda(\omega, \mathbf{k})$ , are  $4 \times 4$  matrices that satisfy

$$\begin{aligned} M_x \tilde{J}_\lambda(\omega_0, k_x, k_y) M_x^{-1} &= \tilde{J}_\lambda(\omega_0, -k_x, k_y), \\ M_y \tilde{J}_\lambda(\omega_0, k_x, k_y) M_y^{-1} &= \tilde{J}_\lambda(\omega_0, k_x, -k_y), \\ C_4 \tilde{J}_\lambda(\omega_0, k_x, k_y) C_4^{-1} &= \tilde{J}_\lambda(\omega_0, k_y, -k_x), \end{aligned} \quad (8)$$

where  $M_x = \sigma_1 \tau_3$ ,  $M_y = \sigma_1 \tau_1$ , and  $2C_4 = (\sigma_1 + i\sigma_2)\tau_0 + (\sigma_1 - i\sigma_2)(i\tau_2)$  are the representations of the symmetries satisfying  $M_x M_y = -M_y M_x$  and  $C_4 M_x C_4^{-1} = M_y$ . Here,  $\sigma_\mu$

and  $\tau_\mu$ ,  $\mu = 0, 1, 2, 3$  are the  $2 \times 2$  identity matrix and the three Pauli matrices acting on the  $i$  and  $j$  sublattice index, respectively. Note that the circuit is then also invariant under the combined symmetries  $\hat{M}_{x\bar{y}} = C_4 M_x$  and  $\hat{M}_{xy} = C_4 M_y$  that map  $(x, y) \rightarrow (-y, -x)$  and  $(x, y) \rightarrow (y, x)$ , respectively. In addition,  $\tilde{J}_\lambda(\omega_0, \mathbf{k})$  has a chiral symmetry  $\mathcal{C} = \sigma_3 \tau_0$ , which by  $\mathcal{C} \tilde{J}_\lambda(\omega_0, \mathbf{k}) \mathcal{C}^{-1} = -\tilde{J}_\lambda(\omega_0, \mathbf{k})$  implies a spectral symmetry. Up to an overall factor of  $i$ , the circuit Laplacian  $\tilde{J}_\lambda(\omega_0, \mathbf{k})$  takes exactly the same form as the Bloch Hamiltonian matrix of the quadrupole insulator introduced in Ref. 17 (see Methods section). For  $\lambda \neq 1$  the spectrum of  $\tilde{J}(\omega_0, \mathbf{k})$  is gapped, and the gapless point  $\lambda = 1$  corresponds to a topological phase transition between a quadrupole circuit for  $\lambda > 1$  and a trivial circuit for  $\lambda < 1$ .

We now turn to a circuit with open boundary conditions to realize topologically protected corner modes. In general, two criteria must be met to realize a topological bulk-boundary correspondence. First, the symmetries which protect the topological character may not be broken by the boundary. Second, the system termination must be compatible with the choice of bulk unit cell for which a topological invariant has been defined, i.e., the boundary should not cut through unit cells. We demonstrate all of these properties on a single circuit by choosing different boundary terminations as follows.

In order for the open system to obey the chiral symmetry  $\mathcal{C}$ , the diagonal elements of  $J(\omega)$  need to vanish at  $\omega_0$ . This holds for all bulk sites by the construction of the model. Imposing this symmetry also for edge and corner sites in an open geometry fixes the circuit elements (capacitor and or inductor) that connect each site to the ground. (See the supplemental material for the specific grounding that was used for the open circuit.)

With this condition imposed on the boundary sites, we terminate the upper left edge of the circuit in a way compatible with the choice of bulk unit cell denoted as (I) in Fig. 1 c). The lower left circuit termination is chosen to be compatible with the unit cell denoted as (II) in Fig. 1 c). This edge termination preserves the mirror symmetry  $\hat{M}_{x\bar{y}} = C_4 M_x$  and breaks all other spatial symmetries mentioned above. Topological corner modes could thus potentially be protected at the upper left and the lower right corner, which are invariant under  $\hat{M}_{x\bar{y}}$ , but not at the other two corners. However, the bulk circuit Laplacians which correspond to the two choices of unit cell (I) and (II) satisfy  $\tilde{J}_\lambda^{(\text{II})}(\omega_0, \mathbf{k}) = \lambda \tilde{J}_\lambda^{(\text{I})}(\omega_0, \mathbf{k})$  for an appropriate labeling of unit cell sites. Recalling that the topological phase transition occurs at  $\lambda = 1$ , this implies that when  $\tilde{J}^{(\text{I})}(\omega_0, \mathbf{k})$  is in a topological phase,  $\tilde{J}^{(\text{II})}(\omega_0, \mathbf{k})$  is trivial and vice versa. As a result, our choice of boundary termination renders one corner topological (the upper left one for  $\lambda > 1$ ) and the opposite corner trivial.

We thus expect that for  $\lambda > 1$  and at eigenfrequency  $\omega_0$  the circuit depicted in Fig. 1 c) supports a localized topological corner state at the upper left corner, and none at the lower right corner. We further note that the corner mode should be an exact eigenstate of the  $\hat{M}_{x\bar{y}}$  symmetry. We will now present impedance measurements that support this expectation.

**Experimental results** — For the experimental realization

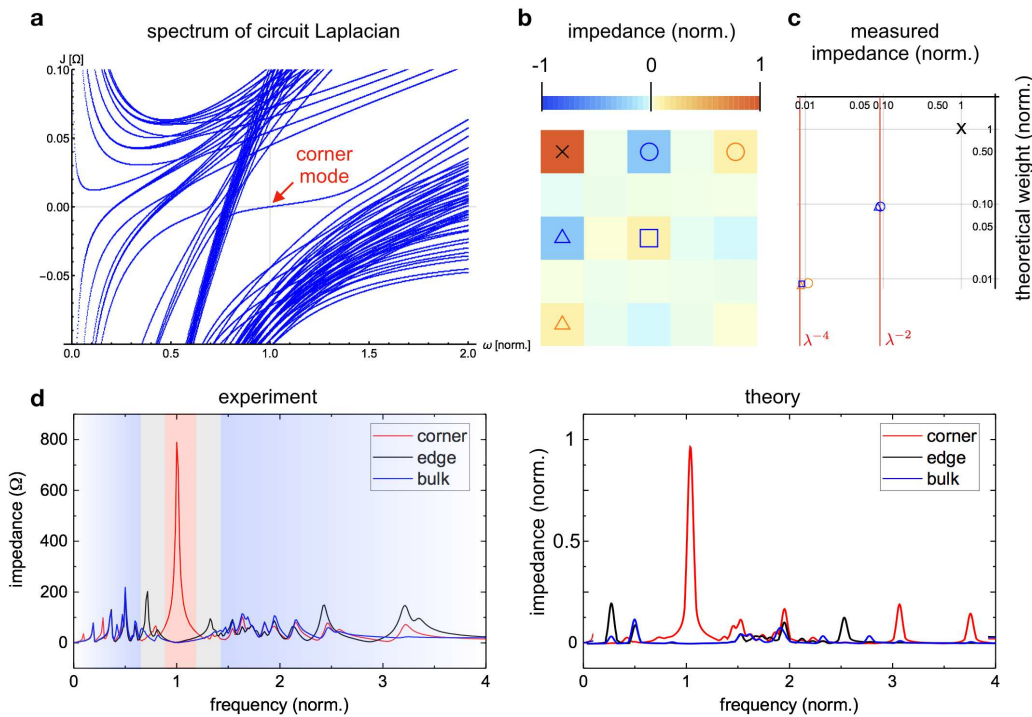


FIG. 2. Comparison of experimental and theoretical results for the circuit spectrum and corner mode. (a) Theoretical spectrum of the circuit Laplacian  $J(\omega)$  as a function of the driving frequency. All frequency scales are normalized to the resonance frequency  $\omega_0$ . An isolated mode crossing the gap, which corresponds to a zero energy eigenvalue of  $J(\omega)$  at  $\omega = \omega_0$  is clearly visible. It corresponds to the topological corner mode. The calculation includes a random disorder of 1% for all capacitors and 2% for all inductors. (b) Theoretical weight distribution of the eigenstate of  $J(\omega_0)$  that corresponds to the corner mode, where only the circuit nodes near the corner are shown. (c) Comparison between the experimental corner mode impedance at  $\omega = \omega_0$ , measured between nearest neighbor nodes along the horizontal and vertical edges, and along the diagonal, and the theoretically computed weight of the corner mode eigenstate. Both decay with the decay constant  $\lambda = 3.3$  set by the ratio of alternating capacitors/inductors. (d) Frequency scan (normalized with respect to  $\omega_0$ ) of the impedance between two nearest-neighbor sites at the corner, at the edge, and in the bulk. Both the experimental and theoretical curves show the corner state resonance isolated in the gap of bulk and edge states.

of topological corner modes a circuit board with  $4.5 \times 4.5$  unit cells was designed. The line spacing on the board was chosen large enough such that spurious inductive coupling between the circuit elements was below our measurement resolution. All impedance measurements were performed with a HP 4194A Impedance/Gain-Phase Analyzer in a full differential configuration. In order to achieve a clearly resolvable corner state resonance on the superimposed resistive background of the bulk states (i. e., the combined impedance contribution of our RLC circuit), which is of the order of a few hundreds of milli-ohm, the values of the circuit elements were chosen for the resonance frequency to be in the MHz-range. The ratio  $\lambda$  between the capacitors/inductors was set to 3.3.

Figure 2 compares the experimental data with the theoretical predictions, finding excellent agreement between the two. It demonstrates the existence of a spectrally and spatially localized topological corner state. In Fig. 2 a) the frequency-dependent spectrum of the circuit Laplacian shows the isolated corner mode and illustrates the connection between a (bulk and edge) spectral gap of  $J(\omega)$  at fixed frequency  $\omega$  and a gap in the spectrum of the dynamical matrix  $D$ , which corre-

sponds to a range of frequencies without zero modes of  $J(\omega)$ . In Fig. 2 b) and c) the corner mode at  $\omega = \omega_0$  is mapped out with single-site resolution. The exponential decay of the measured impedance is in excellent correspondence with the theoretical expectation. The experimental demonstration that the corner mode is indeed a spectrally isolated is contained in Fig. 2 d).

**Physical interpretation of corner modes** — Along the  $x$  and  $y$  directions, the circuit corresponds to a collection of connected pairs of linear circuits with alternating capacitors and inductors, respectively. With the appropriate boundary conditions discussed previously, electric charge on the capacitors forms “dimerized”, isolated oscillators as described in Ref. 16 and 20. Note that the capacitances alternate between  $C_1$  and  $C_2$  with  $C_1 < C_2$ . Therefore, by virtue of being an eigenmode of the circuit Laplacian in terms of potential and current profile where every second node exhibits no current and accordingly no potential difference<sup>16</sup>, a fixed amount of charge  $Q$  between each pair of capacitors give rise to a potential difference  $V_1 > V_2$ , since  $Q = V_1 C_1 = V_2 C_2$ . With appropriate boundary conditions, we can thus infer the exist-

tence of a boundary mode of anti-phase currents that is decaying exponentially by a factor of  $1/\lambda = C_1/C_2$  per unit cell. This boundary mode is equivalent to an edge state in the one-dimensional Su-Schrieffer-Heeger (SSH) model.

A novel feature of our measured corner mode is that this mode is *not* the result of edge polarization, i.e., even though the Laplacian eigenstate form of the corner mode suggest a similar in  $x$  and  $y$  direction, it cannot be arrived at by combining SSH models along the different edges. This hints at topological *quadrupole* polarization in the given circuit, as opposed to dipole polarization in the SSH case. It is instructive to decompose the given circuit in terms of pairs of vertical and horizontal SSH-type circuit chains, where we see both SSH chains built by capacitors as well as their dual form built by inductors in each unit cell string along the  $x$  or  $y$  axis. The alternating  $L$ -type and  $C$ -type SSH chains within the unit cell then are arranged such that their edge charge polarizations cancel. To see this concretely, we turn to frequency space, where a voltage difference equals  $Q/C$  across a capacitor  $C$ , but takes the form  $L\ddot{Q} \rightarrow -\omega^2 LQ$  across an inductor  $L$ . By identifying  $1/C \equiv -\omega^2 L$ , we notice that the  $L$ -type dual chain amounts to effectively "negative" capacitances. For  $\omega \rightarrow \omega_0$  this then gives the same absolute but sign-reversed capacitance, and the dipolar SSH-type polarization cancels out in each unit cell.

**Discussion** — A fundamental difference between classical topological systems (e.g., of mechanical degrees of freedom, electrical circuits, photonic metamaterials) and topological insulators made of fermions is that the topology is manifested in the excitations of classical systems, but not directly by their bulk properties as in fermionic systems (see the Methods section for a more detailed discussion.) For example, a fermionic electric quadrupole insulator has a quantized *bulk* quadrupole moment that is an – in principle measurable – characteristic of its (zero temperature) ground state. (A more canonical example is the bulk Hall conductivity of an integer quantum Hall effect.) To measure the analogous observable in the bulk of a classical system will require the excitation of all its eigenstates below the bulk gap.

In contrast, topological *boundary* modes are in principle as accessible for measurements in classical as in fermionic quantum systems, since they correspond to spectrally isolated excitations. For this reason, we have focused on the boundary characteristics of the topological circuit in this work.

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## METHODS AND APPENDICES

### A. Impedance response and circuit Green's function

The signature of a nontrivial topological phase often lies in its response to an external perturbation. In electronic topological systems for instance, a nontrivial Chern number corresponds to a nonvanishing quantized Hall response, as epitomized by the Kubo formula. In circuits, however, the Kubo formula does not apply as there is no quantum excitation from a Fermi sea. Below, we shall derive the appropriate analog of the Kubo formula for circuits, which shall characterize the so-called *topoelectrical* response.

Define  $V_a$  and  $I_a$  to be the voltage and external input current on node  $a$  of a circuit. By Kirchhoff's law,

$$\dot{I}_a = C_{ab}\ddot{V}_b + \sigma_{ab}\dot{V}_b + W_{ab}V_b \quad (9)$$

where  $C_{ab}$ ,  $\sigma_{ab}$  and  $W_{ab}$  are the Laplacian matrices of capacitances, conductances and inverse inductances, and the summation over repeated indices is implied. For a mode  $V(t) \sim V(0)e^{i\omega t}$  at frequency  $\omega$ , Eq. (9) takes the form

$$I_a = \left( i\omega C_{ab} + \sigma_{ab} - \frac{i}{\omega} W_{ab} \right) V_b = J_{ab}(\omega) V_b \quad (10)$$

where  $J_{ab}(\omega)$  is the (grounded) circuit Laplacian.

The most natural measurement on a circuit is the impedance response  $Z_{ab}(\omega)$ , which is the ratio of the voltage between two nodes  $a$  and  $b$  due to a current  $I_j = I_0(\delta_{j,a} - \delta_{j,b})$  that enters through  $a$  and exits at  $b$ . Mathematically,  $Z_{ab}(\omega)$  simply involves the inversion of Eq. (10):

$$\begin{aligned} Z_{ab}(\omega) &= \frac{V_a - V_b}{I_0} \\ &= \sum_i \frac{G_{ai}(\omega)I_i - G_{bi}(\omega)I_i}{I_0} \\ &= G_{aa}(\omega) + G_{bb}(\omega) - G_{ab}(\omega) - G_{ba}(\omega) \\ &= \sum_n \frac{|\phi_n(a) - \phi_n(b)|^2}{j_n(\omega)} \end{aligned} \quad (11)$$

where  $J_{ab}(\omega) = \sum_n j_n(\omega) |\phi_n(a)\rangle \langle \phi_n(b)|$  is the expansion of the Laplacian into its eigenmodes (the  $\omega$  dependence of the eigenmodes is left implicit), with the Green's function  $G_{ab}(\omega) = \sum_n \frac{1}{j_n(\omega)} |\phi_n(a)\rangle \langle \phi_n(b)|$  being its inverse. When the circuit is ungrounded, an overall shift of the potential cannot be felt, and the corresponding zero eigenspace should be excluded in the definition of the Green's function.

Equation (11) describes the impedance between any two nodes purely in terms of the eigenmodes and eigenvalues of the Laplacian. Most notably, it suggests that circuit resonances (divergences of the impedance) occur whenever there are nontrivial zero eigenvalues  $j_n$ . In a realistic circuit with unavoidable disorder, the strength of such resonances depend on the density of such zero eigenmodes, as well as whether there is any mechanism that pins them to zero.

A quintessential example of a strong protected resonance is a *topoelectrical* resonance, which occurs due to topologically

protected zero modes of the circuit Laplacian. Due to the localization of these modes at the boundary, such resonances can be easily identified through extremely large resonances at the boundary but not the interior of the circuit lattice. In this paper, the corner modes are such an example.

The circuit Laplacian in momentum space  $\tilde{J}_\lambda(\omega_0, \mathbf{k})$  is given by

$$\begin{aligned} \tilde{J}_\lambda(\omega_0, \mathbf{k}) = i\sqrt{\frac{c}{l}} & [(1 + \lambda \cos k_x)\sigma_1\tau_0 \\ & + (1 + \lambda \cos k_y)\sigma_2\tau_2 \\ & - \lambda \sin k_x \sigma_2\tau_3 \\ & + \lambda \sin k_y \sigma_2\tau_1], \end{aligned} \quad (12)$$

which has, up to an overall factor of  $i$ , the same form as the model for an electric quadrupole insulator defined in Ref. 17.

### B. Mapping to an effective Dirac problem and boundary modes

In the main text, we showed that the admittance matrix  $J(\omega_0)$  possesses the required symmetries to define the topological characteristics of a quadrupole insulator. In this section we demonstrate that in the corresponding dynamical matrix  $D$ , the same symmetry properties are emergent for frequencies near  $\omega_0$ , but globally realized. We derive the effective Dirac form of the matrix  $D$  and explicitly show that it implies the existence of corner modes.

We denote by  $\tilde{C}(k_x, k_y)$  and  $\tilde{W}(k_x, k_y)$  the Fourier components of the matrices  $C$  and  $W$  defined in the main text for a circuit with periodic boundary conditions. To show that  $\hat{M}_x$  and  $\hat{M}_y$  defined in Eq. (8) are emergent symmetries of the dynamical matrix  $\tilde{D}(k_x, k_y) = \tilde{C}^{-1/2}(k_x, k_y)\tilde{W}(k_x, k_y)\tilde{C}^{-1/2}(k_x, k_y)$  we note that the spectrum of  $\tilde{D}(k_x, k_y)$  is gapless for  $\lambda = 1$  with a linear band touching point near  $(k_x, k_y) = (\pi, \pi)$ , but is gapped for  $\lambda \neq 1$ . This motivates to expand  $\tilde{D}(k_x, k_y)$  to linear order in  $(1 - \lambda)$  and the deviations  $(p_x, p_y)$  of  $\mathbf{k}$  from  $(\pi, \pi)$ . The resulting effective dynamical matrix  $\mathcal{D}(p_x, p_y)$  takes Dirac form

$$\begin{aligned} \mathcal{D}(p_x, p_y) = \omega_0^2\sigma_0\tau_0 + \frac{\omega_0^2}{4} & (p_x\sigma_2\tau_3 - p_y\sigma_2\tau_1) \\ & + \frac{\omega_0^2}{4}(1 - \lambda)(\sigma_1\tau_0 + \sigma_2\tau_2), \end{aligned} \quad (13)$$

where the term proportional to  $(1 - \lambda)$  is a mass term. The spectrum of  $\mathcal{D}(p_x, p_y)$  is symmetric about  $\omega_0^2$ . This is a result of the chiral symmetry  $\mathcal{C} = \sigma_3\tau_0$  which anticommutes with  $\mathcal{D}(p_x, p_y)$ . If this symmetry is not broken by a boundary in the range of frequencies near  $\omega_0$ , topological boundary modes will be pinned to the frequency  $\omega_0$ .

We are searching for an explicit analytical solution to the localized corner state within the respective Dirac equation. For simplicity, we set  $\omega_0 = 2$  and remove the overall energy shift  $\omega_0^2$  from the Dirac operator. Further we substitute  $(1 - \lambda)\sigma_1\tau_0$

by  $\Delta \sin \phi \sigma_1\tau_0$  and  $(1 - \lambda)\sigma_2\tau_2$  by  $\Delta \cos \phi \sigma_2\tau_2$  so that the operator reads

$$D = p_x\sigma_2\tau_3 - p_y\sigma_2\tau_1 + \Delta(\sin \phi \sigma_1\tau_0 + \cos \phi \sigma_2\tau_2), \quad (14)$$

where  $\phi = \pi/4$  and  $\phi = -3\pi/4$  holds inside and outside of the material, respectively. We now equip  $\phi$  with a position dependence to model a corner. The bulk symmetries  $\hat{M}_x$ ,  $\hat{M}_y$  and  $\hat{C}_4$  are all broken locally by the corner. The only symmetry that leaves the corner invariant is the diagonal mirror symmetry  $\hat{M}_{x\bar{y}} = C_4 M_x$  that sends  $(x, y) \rightarrow (y, x)$  and is represented by

$$M_{x\bar{y}} = \frac{1}{2}(\sigma_0 + \sigma_3)\tau_3 + \frac{1}{2}(\sigma_0 - \sigma_3)\tau_1. \quad (15)$$

Also, the system respects chiral symmetry for any choice of  $\phi$ . We now endow  $\phi$  with a spatial dependence and note that  $M_{x\bar{y}}$  symmetry is preserved if

$$\phi(x, y) = -\phi(y, x) + \pi/2 \text{ mod } 2\pi. \quad (16)$$

If we parametrize  $x = r\cos\varphi$ ,  $y = r\sin\varphi$ , the condition translates into one on the  $\varphi$  dependence of  $\phi$ . Specifically

$$\phi(\varphi) = -\phi(-\varphi + \pi/2) + \pi/2 \text{ mod } 2\pi. \quad (17)$$

The choice  $\phi_1(\varphi) = \varphi$  is consistent with this symmetry, and so is

$$\phi_2(\varphi) = \arctan\left(\frac{\varphi}{\lambda}\right) + \arctan\left(\frac{\varphi - \pi/2}{\lambda}\right) + \frac{\pi}{4}. \quad (18)$$

In the limit  $\lambda \rightarrow 0$ ,  $\phi_2(\varphi)$  realizes a corner with the non-trivial part of the system located in the upper right quadrant. For  $\phi_1(\varphi)$ , in contrast, the operator (14) is equivalent to the Hamiltonian that describes a vortex in an  $s$ -wave superconducting surface state of a three-dimensional topological insulator<sup>21,22</sup>. The latter supports a spectrally isolated zero energy mode localized at the origin. It is protected to lie at zero energy by the chiral symmetry. We can now choose any interpolation between  $\phi_1(\varphi)$  and  $\phi_2(\varphi)$  to connect these two situations: since chiral symmetry cannot be broken by the interpolation, the zero mode has to remain also in the system with a corner.

### C. Topological index: Mirror-graded winding number

Here we define the bulk topological invariant for a topological quadrupole insulator as a mirror-symmetry graded winding number. This index is valid if the model has diagonal mirror symmetry (e.g.,  $M_{x\bar{y}}$ ) and chiral symmetry  $\mathcal{C}$ . The latter is in any case required to pin topological corner modes to eigenvalue zero. Our topological invariant is complementary to the characterization of multipole insulators in terms of Wilson loops that was given in Ref. 17.

Consider a  $\mathbf{k}$ -dependent matrix (being for example a Bloch Hamiltonian, or an admittance matrix)  $R(\mathbf{k})$  that

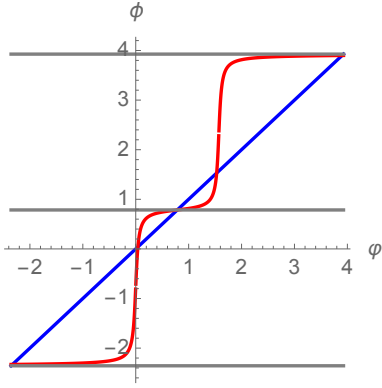


FIG. 3. Two dependencies of the operator  $\mathcal{D}$  from Eq. (14) on the angular variable  $\varphi$  that mimic a superconducting vortex (blue) and the corner of an electric quadrupole insulator (red). The existence of a zero mode in the former implies the existence of a corner mode in the latter.

both obeys  $\mathcal{C}$ , i.e.,  $\mathcal{C}R(\mathbf{k})\mathcal{C}^{-1} = -R(\mathbf{k})$ , and  $M_{x\bar{y}}$ , i.e.,  $M_{x\bar{y}}R(k_x, k_y)M_{x\bar{y}}^{-1} = -R(k_y, k_x)$  and let  $[\mathcal{C}, M_{x\bar{y}}] = 0$ . The occupied bands of  $R(k, k)$  can then be divided in a subspace with mirror eigenvalues  $\pm 1$  (or  $\pm i$  for spinful mirror symmetry). Using this grading, we can bring  $R(k, k)$  to the form

$$R(k, k) = \begin{pmatrix} 0 & q_+(k) & 0 & 0 \\ q_+(k)^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & q_-(k) \\ 0 & 0 & q_-(k)^\dagger & 0 \end{pmatrix}, \quad (19)$$

where the first half acts on the  $+1$  mirror subspace, while the second half acts on the  $-1$  mirror subspace. For  $R(k, k)$  to be gapped, all eigenvalues of  $q_\pm(k)$  need to be nonzero. We can thus define a ‘spectrally flattened’ pair of unitary matrices  $\tilde{q}_\pm(k)$  which share the eigenstates and phase of the eigenvalues with  $q_\pm(k)$ , but have eigenvalues of absolute value 1. We can now define the winding numbers

$$\nu_\pm := \frac{i}{2\pi} \int_0^{2\pi} dk \operatorname{tr} \tilde{q}_\pm^\dagger(k) \partial_k \tilde{q}_\pm(k), \quad (20)$$

which are quantized to be integers. For a system with vanishing dipole moment, the net winding number  $\nu_+ + \nu_-$  must vanish in any direction of momentum space. Hence, for the systems of interest to us  $\nu_+ = -\nu_-$ , and we can use

$$\nu := \frac{\nu_+ - \nu_-}{2} \in \mathbb{Z} \quad (21)$$

as a topological invariant. The number of topological corner modes is equal to the parity of  $\nu$ .

We now demonstrate this topological invariant for the admittance matrix realized in our electrical circuit. Up to prefactors, the matrix takes the form

$$\begin{aligned} R(\mathbf{k}) = & (1 + \lambda \cos k_x) \sigma_1 \tau_0 \\ & + (1 + \lambda \cos k_y) \sigma_2 \tau_2 \\ & - \lambda \sin k_x \sigma_2 \tau_0 \\ & + \lambda \sin k_y \sigma_2 \tau_1, \end{aligned} \quad (22)$$

and  $\mathcal{C} = \sigma_3 \tau_0$ , while  $M_{x\bar{y}} = \frac{1}{2}(\sigma_0 + \sigma_3) \tau_3 + \frac{1}{2}(\sigma_0 - \sigma_3) \tau_1$ . The mirror-eigenvalue graded off-diagonal components of  $R(k, k)$  are scalars in this case and can be computed as

$$q_\pm(k) = \sqrt{2} (1 + \lambda e^{\mp ik}). \quad (23)$$

Clearly, for  $\lambda > 1$ , they have winding number  $\nu_\pm = \pm 1$  and thus  $\nu = +1$ , corresponding to the topologically nontrivial phase with corner modes. In contrast, for  $\lambda < 1$  we find  $\nu_\pm = 0$  and thus  $\nu = 0$ , corresponding to the topologically trivial phase.

#### D. Experimental circuit implementation

The circuit board was fit with ceramic chip capacitors [WCAP-CSGP Ceramic Capacitors 0805 (Würth Elektronik)] with 1 nF and 3.3 nF capacity, respectively, and SMD power inductors with low serial resistance RDC [WE-LHMI SMD Power Inductor (Würth Elektronik)] with 3.3  $\mu\text{H}$  ( $R_{\text{DC}} < 76 \text{ m}\Omega$ ) and 1  $\mu\text{H}$  ( $R_{\text{DC}} < 27 \text{ m}\Omega$ ) inductivity. All components were pre-characterized with the HP 4194A to obtain tolerances of below 2% of the nominal component values. The HP 4194A was also used to measure differential impedance spectra between the nodes. Therefore a differential four terminal measurement between the trivial node in the lower right corner and the nodes of interest in the upper left (i. e., the topological corner) was performed. The analyzer’s compensation algorithm was used to cancel out the impedance contribution caused by the measurement feed lines.

#### E. Dipole and quadrupole polarization

In this subsection, we present how the dipole and quadrupole topological polarization can be expressed in terms of Bloch eigenfunctions and the Berry connection.

##### 1. Dipole polarization, Wannier functions and projected density operator

In the continuum, the dipole polarization  $p_i = \int x_i \rho(\mathbf{x}) d\mathbf{x}$  gives us the expectation value of the center of mass with respect to a density operator  $\rho$ . On a two-dimensional lattice, its definition should be modified in two ways. Firstly,  $\rho$  should be replaced by the band projector  $P = \sum_{n, \mathbf{k}} |u_{\mathbf{k}}^n\rangle \langle u_{\mathbf{k}}^n|$ , where  $|u_{\mathbf{k}}^n\rangle = u_{\mathbf{k}}^n | \mathbf{k} \rangle$  is the  $n^{\text{th}}$  occupied Bloch eigenstate with quasimomentum  $\mathbf{k} = (k_x, k_y)$ . Secondly<sup>23</sup>,  $x$  should be replaced by the periodic position operator  $\hat{X} = e^{2\pi i \hat{x}/L_x} = \sum_x e^{2\pi i x/L_x} |x\rangle \langle x|$ , where  $|x\rangle$  denotes a state at site  $x$ , and  $L_x$  is the total number of sites. We can thus rewrite the polarization operator as

$$\begin{aligned} \tilde{\rho} &= P \hat{X} P \\ &= P e^{iQ\hat{x}} P, \end{aligned} \quad (24)$$

which may also be interpreted as the projected density operator at momentum  $Q = \frac{2\pi}{L_x}$ . When  $P$  trivially projects onto



all bands,  $\tilde{\rho} = \hat{X}$  simply gives the periodic position. When  $P$  is nontrivial, the eigenvalues and eigenvectors of  $\tilde{\rho}$  respectively give the polarization spectrum and Wannier functions. It is well-known that the polarization spectral flow tells us the net number of edge modes leaving the band(s). Note that these edge modes exist even in classical lattice systems, where band projectors cannot be physically realized as filled Fermi seas.

Since the density operator satisfies

$$e^{iQ\hat{x}} = \sum_{\mathbf{k}} |\mathbf{k} + Q\hat{e}_x\rangle \langle \mathbf{k}|, \quad (25)$$

the projected density operator takes the form

$$\begin{aligned} \tilde{\rho} &= \sum_{n,m,\mathbf{k}} |u_{\mathbf{k}+Q\hat{e}_x}^n\rangle \langle u_{\mathbf{k}+Q\hat{e}_x}^m| \langle u_{\mathbf{k}}^m| \langle u_{\mathbf{k}}^n| \\ &\approx \sum_{n,m,\mathbf{k}} [e^{iQA_x(\mathbf{k})}]_{nm} |u_{\mathbf{k}+Q\hat{e}_x}^n\rangle \langle u_{\mathbf{k}}^m|, \end{aligned} \quad (26)$$

with equality in the  $L_x \rightarrow \infty$  limit. In this limit, the matrix  $U_{nm}(\mathbf{k}) = \langle u_{\mathbf{k}+Q\hat{e}_x}^n | u_{\mathbf{k}}^m \rangle$  is unitary and tends towards  $[e^{iQA_x(\mathbf{k})}]_{nm}$ , where  $A_x(\mathbf{k}) = -i\langle u_{\mathbf{k}}^n | \partial_{k_x} u_{\mathbf{k}}^m \rangle$  is the non-abelian Berry connection. In this form, it is easy to guess the form of eigenvectors  $|W_{k_y}^s\rangle$  of  $\tilde{\rho}$ , which are also known as the Wannier functions. Note that  $k_x$  does no longer enter as an index, since  $\tilde{\rho}$  is not diagonal in it. As  $\tilde{\rho}$  implements both the momentum translation  $\mathbf{k} \rightarrow \mathbf{k} + Q\hat{e}_x$  and the internal rotation  $U_{nm}(\mathbf{k})$ , an eigenvector must contain compensatory factors such that it transforms covariantly under simultaneous translation and rotation. For this, it should be proportional to the Wilson line  $\Phi(k_x, k_y) = U(0, k_y) \dots U(k_x - Q\hat{e}_x, k_y) U(k_x, k_y) = \mathcal{P}e^{i\int_0^{k_x} A_x(p_x, k_y) dp_x}$ , where  $\mathcal{P}$  is the path ordering operator, as well as a power of  $e^{-ik_x}$ :

$$\begin{aligned} |W^s(k_y)\rangle &= \sum_{k_x} e^{-ik_x\theta_s(k_y)/(2\pi)} \Phi(k_x, k_y) |W_0^s(k_y)\rangle \\ &= \sum_{m,n,k_x} e^{-ik_x\theta_s(k_y)/(2\pi)} [\Phi(k_x, k_y)]_{mn} \\ &\quad \times |u_{\mathbf{k}}^m\rangle \langle u_{\mathbf{k}}^n | W_0^s(k_y)\rangle. \end{aligned} \quad (27)$$

Since the righthand side of Eq. (28) should be invariant under  $k_x \rightarrow k_x + 2\pi$ , it follows that  $e^{i\theta_s(k_y)}$  and  $|W_0^s(k_y)\rangle$  are respectively the eigenvalues and eigenvectors of the Wilson loop operator

$$\Phi(2\pi, k_y) = \mathcal{P}e^{i\int_0^{2\pi} A_x(p_x, k_y) dp_x}. \quad (28)$$

Through direct substitution of Eq. (28) into Eq. (26) it may then be verified that the eigenvalues of  $|W^s(k_y)\rangle$  are given by  $e^{i\theta(k_y)/L_x}$ .

To summarize, the Wilson loop operator  $\Phi(2\pi, k_y)$  is closely related to the projected density operator  $\tilde{\rho}$ , which is also diagonal in  $k_y$ . Their eigenvalues are given by  $e^{i\theta(k_y)}$  and  $e^{i\theta(k_y)/L_x}$  respectively. Given an eigenvector  $|W_0^s(k_y)\rangle$  of  $\Phi(2\pi, k_y)$ , one can construct the eigenvector  $|W^s(k_y)\rangle$  of  $\tilde{\rho}$  via Eq. (28). However, to do so, knowledge of the Wilson line  $\Phi(k_x, k_y)$  at all  $k_x$  is required. In this sense, the physical polarization eigenvectors (Wannier functions) carry “more” information than what is obtainable from the Wilson loop alone.

## 2. Nested Wilson loop and quadrupolar polarization

If the Wannier polarization ( $\tilde{\rho}$ ) spectrum is gapped, one can perform a nested Wilson loop computation to reveal a possible quadrupole moment.

In general, the total polarization is given by  $-i \log \text{Tr} \Phi$ , where  $\Phi$  is the Wilson loop operator. In the nested Wilson loop computed over the eigenstates  $|W^s(k_y)\rangle$  of  $\tilde{\rho}$ , the gapped cases allow for evaluation of the polarization of one sector at a time, where the total polarization simplifies to

$$\begin{aligned} p^s &= -\frac{1}{(2\pi)^2} \text{Tr} \int_{BZ} A_y^s(\mathbf{k}) d^2\mathbf{k} \\ &= i \frac{1}{(2\pi)^2} \text{Tr} \int_{BZ} \langle W^s(k_y) | \partial_{k_y} W^s(k_y) \rangle d^2\mathbf{k}, \end{aligned} \quad (29)$$

where  $A_y^s(\mathbf{k})$  is the Berry connection of  $|W^s(k_y)\rangle$ . To express  $p^s$  explicitly in terms of the Berry connections  $A_x, A_y$  of the original Bloch eigenstates  $|u_{\mathbf{k}}^m\rangle$ , one notes that if  $|W^s(k_y)\rangle = \sum_m M_{\mathbf{k}}^{ms} |u_{\mathbf{k}}^m\rangle$ ,

$$\begin{aligned} p^s &= -\frac{1}{(2\pi)^2} \text{Tr} \int_{BZ} [(MM^\dagger)A_y - iM^\dagger \partial_{k_y} M] d^2\mathbf{k} \\ &= \frac{i}{(2\pi)^2} \int_{BZ} \left[ \sum_{mm'} (M_{\mathbf{k}}^{m's})^* \langle u_{\mathbf{k}}^{m'} | \partial_{k_y} u_{\mathbf{k}}^m \rangle M_{\mathbf{k}}^{ms} + \sum_m (M_{\mathbf{k}}^{m's})^* \partial_{k_y} M_{\mathbf{k}}^{ms} \right] d^2\mathbf{k} \end{aligned} \quad (30)$$

where, from Eq. 28,

$$M_{\mathbf{k}}^{ms} = \sum_{k_x} e^{-ik_x\theta_s(k_y)/(2\pi)} \langle u_{\mathbf{k}}^m | \Phi(k_x, k_y) | W_0^s(k_y) \rangle \quad (31)$$

with  $\Phi(k_x, k_y) = \mathcal{P}e^{i\int_0^{k_x} A_x(p_x, k_y) dp_x}$ , and  $e^{i\theta_s(k_y)}$ ,  $|W_0^s(k_y)\rangle$  being the  $s^{\text{th}}$  eigenvalue and eigenvector of  $\Phi(2\pi, k_y)$ .

## 3. Multipolar polarizations in a classical environment

It is instructive to understand how bulk topological polarization is *indirectly* but faithfully manifested in classical systems. We first connect topological boundary modes with band projectors by observing that they, by virtue of residing within the bulk gap, are necessarily properties of pro-

jectors that demarcate a set of negative eigenvalue bands of the impedance operator  $\hat{J}$  from its complement. Indeed, the electric polarization in  $x$  direction of a crystal is given by the spectral flow of the eigenspectrum of the density operator<sup>24,25</sup>  $\tilde{\rho} = \hat{P}e^{i2\pi\hat{x}/L_x}\hat{P}$ , with  $\hat{P}$  the projector onto the filled subspace of bulk bands. To identify this spectral flow with physical quantities, we consider the adiabatic deformation

$$e^{i2\pi\hat{x}/L_x} \rightarrow \hat{R} \quad (32)$$

where  $\hat{R}$  is the projector onto a *real-space* region  $R$ . Under this deformation to the operator  $\hat{P}\hat{R}\hat{P}$ , the initially equally spaced polarization bands adiabatically accumulate near 1 and 0, the eigenvalues of  $\hat{R}$ , with the exception of those that traverse this interval due to nontrivial spectral flow.

The next observation is that since  $\hat{P}$  and  $\hat{R}$  are projectors,  $\hat{P}\hat{R}\hat{P}$  and  $\hat{R}\hat{P}\hat{R}$  have identical nontrivial eigenvalues and eigenmodes<sup>25</sup>. Now,  $\hat{R}\hat{P}\hat{R}$  is the band projector  $\hat{P}$  pro-

jected onto region  $R$  (i. e., with open boundary conditions). A further adiabatic interpolation

$$\hat{R}\hat{P}\hat{R} \rightarrow \hat{R}\hat{J}\hat{R} \quad (33)$$

completes the deformation to the Laplacian with open boundary conditions  $\hat{R}\hat{J}\hat{R}$ . Importantly, midgap states in the polarization spectrum are adiabatically mapped to midgap states in the Laplacian spectrum. Since midgap states exist within a bulk gap they must necessarily be boundary states.

Via this deformation, we can re-interpret real-space polarization as polarization in “admittance-space”, i.e. along the axis where eigenvalues of the Laplacian  $J$  reside. Hence, to summarize, the “dipole moment” for dipole polarization is classically manifested as the existence of midgap states that, by definition, are necessarily “polarized” at the boundary. This holds analogously for quadrupole moments as detailed in Sec. E 2.

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