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Topological Approach for Rough Sets by Using J-Nearly Concepts via Ideals

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Abstract. Topological concepts and methods have been applied as useful tools to study computer science, information systems and rough set. Rough set was introduced by Pawlak. Its core concept is upper and lower approximation operations, which are the operations induced by an equivalent relation on a domain. They can also be seen as a closure operator and a interior operator of the topology induced by an equivalent relation on a domain. This paper explores rough set theory from the point of view of topology. I generalize the notions of rough sets based on the topological space. The set approximations are defined by using the new topological notions namely $\mathcal{I}$-J-nearly open sets. The topological properties of the present approximations are introduced and compared to the previous one and shown to be more general.

1. Introduction

A rough approximating a subset of the set of objects is a pair of dual approximation operator, called a lower and an upper approximation in term of equivalence classes. Rough sets are defined trough their dual set approximations in Pawlak approximation space [37, 38]. These approximations are the most important concepts in rough set theory. However, equivalence relations in Pawlak rough sets are too restrictive for theoretical and practical aspects. To enlarge the application scope of rough set theory, researchers have proposed many kinds of generalization of rough sets. Based on this reason, they replaced equivalence classes to some other models, for examples, [18, 20, 23, 34, 46, 47]. In recent times, lots of researchers are interested to generalize this theory in many fields of applications [30, 36, 52, 54]. The concept of topology shows up naturally in almost every branch of mathematics [19, 33, 40, 44]. This has made topology one of the great unifying ideas of mathematics. Ordinary topology now has been used in many subfields of artificial intelligence, such as knowledge representation, spatial reasoning etc. Pawlak's upper and lower approximations are the closure and the interior of a set. Several works in topological space and rough set theory have been proposed. In particular, the most important topological generalization of rough sets was presented by Skowron [45] and Wiweger [50]. Introducing the concept of topological space appears seldom in few papers published recently [4, 25, 27, 28, 39, 41, 43, 53]. In the past few years mathematicians

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turned their attention towards to near (or nearly) open concept as generalization of open sets to topological spaces (see: [1, 9, 10, 26, 31, 35, 42] for details). Amer et al. [6] used the J-nearly open concepts and introduced the notions of J-nearly approximations. More recently, Hosny [15] generalized J-nearly open sets by introducing new sort of open sets namely, $\delta\beta_J$-open sets and $\bigwedge_{\beta_J}$-sets and used these open sets to present $\delta\beta_J$-approximations and $\bigwedge_{\beta_J} - a$pproximations as a generalization of $J - n$early approximations. Ideal is a fundamental concept in studying the topological problems. The notion of ideal topological spaces was first studied by Kuratowski [24] and Vaidyanathaswamy [49] which is one of the important areas of research in the branch of mathematics. After them different mathematicians applied the concept of ideals in topological spaces (see: [7, 11, 14, 17, 32]). The interest in the idealized version of many general topological properties has grown drastically in the past 20 years. Few researchers [16, 21, 48] interesting in applying the concept of ideals in rough set theory.

In this paper, the notion of ideals is used to present the concepts of J-nearly open sets with respect ideals. These concepts generalize the usual notions of J-nearly open sets. Moreover, the notions of J-nearly approximations in terms of ideals are introduced as a generalization of J-nearly approximations. Pawlak rough sets are expressed in topological and ideals concepts. The current paper is structured as follows. Section 2 contains some fundamental definitions and properties of rough sets, J-neighborhood spaces, near open sets, J-nearly approximations and ideals, which are needed in this paper or are beneficial to the subsequent study. The aim of Section 3 is to present the concepts of j-near open sets with respect ideals namely, $\mathcal{I}$-α_J-open, $\mathcal{I}$-J-preopen, $\mathcal{I}$-J-semiopen and $\mathcal{I}$-β_J-open sets. These definitions are different from α_J-open, J-preopen, J-semiopen and β_J-open sets and more general. If $\mathcal{I} = \{\phi\}$, then the current definitions are coincided with Amer et al.'s [6] definitions. So, Amer et al.'s [6] definitions are special case of the current definitions. The main properties and the relationships among of these concepts are studied. Moreover, it is showed that the concepts of $\mathcal{I}$-α_J-open sets and $\mathcal{I}$-J-preopen sets are independent although every α_J-open set is J-preopen set in [6] which can considered to be one of the deviation between the present generalization and the previous one (see Examples 3.2 and 3.3). The object of Section 4 is to propose a new approximations. These approximations are based on J-nearly open sets with respect ideals. Several topological properties of the current approximation and the connections among them are also studied in this section. The relationships between these approximations, Abd El-Monsef et al.'s approximations 2.4 [2] and Amer et al.'s approximations 2.7 [6] are revealed through Theorem 4.1 and Corollary 4.1. The goal of Section 5 is to define membership functions via ideal. It is compared to the previous one such as Abd El-Monsef et al. [3] and Lin [29] (see Proposition 5.1 and Lemma 5.2) and shown to be more accurate. A concluding remark is given in the last section.

2. Preliminaries

Definition 2.1. *[17] Let X be a non-empty set.* $\mathcal{I} \neq \phi$, $\mathcal{I} \in P(X)$ *is an ideal on X, if*

1. $A \in \mathcal{I}$ *and* $B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,
2. $A \in \mathcal{I}$ *and* $B \subseteq A \Rightarrow B \in \mathcal{I}$.

i.e., $\mathcal{I}$ is closed under finite unions and subsets.

Definition 2.2. *[2] Let X be a non-empty finite set and* ρ *be an arbitrary binary relation on X. The J-neighborhood of* $x \in X$ *(J-nbd)* $(n_J(x))$, $J \in \{R, L, <R>, <L>, I, U, <I>, <U>\}$, *defined as:*

1. *R-nbd:* $n_R(x) = \{y \in X : x\rho y\}$.
2. *L-nbd:* $n_L(x) = \{y \in X : y\rho x\}$.

3. $< R >$-*nbd:* $n_{<R>}(x) = \cap_{x \in n_R(y)} n_R(y)$.
4. $< L >$-*nbd:* $n_{<L>}(x) = \cap_{x \in n_L(y)} n_L(y)$.
5. *I-nbd:* $n_I(x)$ *is the intersection of* $n_R(x)$ *and* $n_L(x)$.
6. *U-nbd:* $n_U(x)$ *is the union of* $n_R(x)$ *and* $n_L(x)$.
7. $< I >$-*nbd:* $n_{<I>}(x)$ *is the intersection of* $n_{<R>}(x)$ *and* $n_{<I>}(x)$.
8. $< U >$-*nbd:* $n_{<U>}(x)$ *is the union of* $n_{<R>}(x)$ *and* $n_{<I>}(x)$.

From the following concepts and throughout this paper $J \in \{R, L, < R >, < L >, I, U, < I >, < U >\}$.

Remark 2.1. *It should be noted that the concept of J-nbd of* $x \in X(n_J(x))$, *in [2] is the same as the notion of*

1. *the after set and fore sets in [8] if* $J = R, L$ *respectively.*
2. *the intersection of after set and fore sets and their union in [51] if* $J = I, U$ *respectively.*
3. *the minimal right set and the minimal left set in [5] if* $J =< R >, < L >$ *respectively.*
4. *the intersection of minimal right set and minimal left set in [22] if* $J =< I >$.

Definition 2.3. *[2] Let X be a non-empty finite set,* ρ *be an arbitrary binary relation on X and* $\xi_J : X \to P(X)$ *assigns each x in X its J-nbd in* $P(X)$. (X, ρ, ξ_J) *is a J-neighborhood space (J-nbdS).*

Theorem 2.1. *[2] Let* (X, ρ, ξ_J) *be a J-nbdS, and* $A \subseteq X$. *Then,* $\tau_J = \{A \subseteq X : \forall a \in A, n_J(a) \subseteq A\}$ *is a topology on X. The elements of* τ_J *are called J-open set and the complement of J-open set is J-closed set. The family* Γ_J *of all J-closed sets defined by* $\Gamma_J = \{F \subseteq X : F' \in \tau_J\}, F'$ *is the complement of F.*

Definition 2.4. *[2] Let* (X, ρ, ξ_J) *be a J-nbdS and* $A \subseteq X$. *The J-lower, J-upper approximations, J-boundary regions and J-accuracy of A are:*
$\underline{\rho}_J(A)$ *is the union of all J-open sets which are subset of* $A = int_J(A)$, *where* $int_J(A)$ *represents J-interior of A.*
$\overline{\rho}_J(A)$ *is the intersection of all J-closed sets which are superset of* $A = cl_J(A)$, *where* $cl_J(A)$ *represents J-closure of A.*
$B_J(A) = \overline{\rho}_J(A) - \underline{\rho}_J(A)$.
$\sigma_J(A) = \frac{|\underline{\rho}_J(A)|}{|\overline{\rho}_J(A)|}$, *where* $|\overline{\rho}_J(A)| \neq 0$.

Definition 2.5. *[2] Let* (X, ρ, ξ_J) *be a J-nbdS.* $A \subseteq X$ *is J-exact if* $\overline{\rho}_J(A) = \underline{\rho}_J(A)$. *Otherwise, A is J-rough.*

Definition 2.6. *[6] Let* (X, ρ, ξ_J) *be a J-nbdS.* $A \subseteq X$ *is*

1. *J-preopen (*P_J*-open), if* $int_J(cl_J(A)) \supseteq A$.
2. *J-semiopen (*S_J*-open), if* $cl_J(int_J(A)) \supseteq A$.
3. α_J*-open, if* $A \subseteq int_J[cl_J(int_J(A))]$.
4. β_J*-open (semi preopen), if* $A \subseteq cl_J[int_J(cl_J(A))]$.

These sets are called J-nearly open sets, the families of J-nearly open sets of X denoted by $\zeta_J O(X)$, *the complements of the J-nearly open sets are called J-nearly closed sets and the families of J-nearly closed sets of X denoted by* $\zeta_J C(X)$, $\forall \zeta \in \{P, S, \alpha, \beta\}$.

Remark 2.2. *[6] According to the results in [12, 13], the implications between* $\tau_J, \Gamma_J, \zeta_J O(X)$ *and* $\zeta_J C(X)$ *are in the following figure.*

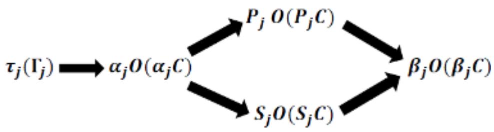

Figure 1: The relationships between $\tau_J, \Gamma_J, \zeta_J O(X)$ and $\zeta_J C(X)$.

From the following concepts and throughout this paper $\zeta \in \{p, s, \alpha, \beta\}$.

Definition 2.7. *[6] Let (X, ρ, ξ_J) be a J-nbdS, $A \subseteq X$. The J-nearly lower, J-nearly upper approximations, J-nearly boundary regions and J-nearly accuracy of A are:*
$\underline{\rho}_J^{\zeta}(A)$ is the union of all J-nearly open sets which are subset of A = J-nearly interior of A.
$\overline{\rho}_J^{\zeta}(A)$ is the intersection of all J-nearly closed sets which are superset of A = J-nearly closure of A.
$B_J^{\zeta}(A) = \overline{\rho}_J^{\zeta}(A) - \underline{\rho}_J^{\zeta}(A)$.
$\sigma_J^{\zeta}(A) = \frac{|\underline{\rho}_J^{\zeta}(A)|}{|\overline{\rho}_J^{\zeta}(A)|}$, *where* $|\overline{\rho}_J^{\zeta}(A)| \neq 0$.

Definition 2.8. *[6] Let (X, ρ, ξ_J) be a J-nbdS. $A \subseteq X$ is J-nearly exact if $\overline{\rho}_J^{\zeta}(A) = \underline{\rho}_J^{\zeta}(A)$. Otherwise, A is J-nearly rough.*

Theorem 2.2. *[6] Let (X, ρ, ξ_J) be a J-nbdS and $A \subseteq X$. Then, $\underline{\rho}_J(A) \subseteq \underline{\rho}_J^{\zeta}(A) \subseteq A \subseteq \overline{\rho}_J^{\zeta}(A) \subseteq \overline{\rho}_J(A)$.*

Proposition 2.1. *[6] Let (X, ρ, ξ_J) be a J-nbdS and $A \subseteq X$. Then,*

1. $\underline{\rho}_J^{\alpha}(A) \subseteq \underline{\rho}_J^{p}(A) \subseteq \underline{\rho}_J^{\beta}(A)$.
2. $\underline{\rho}_J^{\alpha}(A) \subseteq \underline{\rho}_J^{s}(A) \subseteq \underline{\rho}_J^{\beta}(A)$.
3. $\overline{\rho}_J^{\beta}(A) \subseteq \overline{\rho}_J^{p}(A) \subseteq \overline{\rho}_J^{\alpha}(A)$.
4. $\overline{\rho}_J^{\beta}(A) \subseteq \overline{\rho}_J^{s}(A) \subseteq \overline{\rho}_J^{\alpha}(A)$.

Corollary 2.1. *[6] Let (X, ρ, ξ_J) be a J-nbdS, and $A \subseteq X$. Then,*

1. $B_J^{\beta}(A) \subseteq B_J^{p}(A) \subseteq B_J^{\alpha}(A)$.
2. $B_J^{\beta}(A) \subseteq B_J^{s}(A) \subseteq B_J^{\alpha}(A)$.
3. $\sigma_J^{\alpha}(A) \leqslant \sigma_J^{p}(A) \leqslant \sigma_J^{\beta}(A)$.
4. $\sigma_J^{\alpha}(A) \leqslant \sigma_J^{s}(A) \leqslant \sigma_J^{\beta}(A)$.

Definition 2.9. *[3] Let (X, ρ, ξ_j) be a J-nbdS, $x \in X$ and $A \subseteq X$*

1. *if $x \in \underline{\rho}_J(A)$, then x is J-surely belongs to A, denoted by $x \underline{\in}_J A$*
2. *if $x \in \overline{\rho}_J(A)$, then x is J-possibly belongs to A, denoted by $x \overline{\in}_J A$*

Definition 2.10. *[3] Let (X, ρ, ξ_J) be a J-nbdS, $x \in X$ and $A \subseteq X$:*

1. *if $x \in \underline{\rho}_J^{\zeta}(A)$, then x is J-nearly surely (ζ_J-surely) belongs to A, denoted by $x \underline{\in}_J^{\zeta} A$*
2. *if $x \in \overline{\rho}_J^{\zeta}(A)$, then x is J-nearly possibly (ζ_J-possibly) belongs to A, denoted by $x \overline{\in}_J^{\zeta} A$*

Proposition 2.2. *[3] Let (X, ρ, ξ_J) be a J-nbdS and $A \subseteq X$. Then,*

1. *If $x \underline{\in}_J A$, then $x \underline{\in}_J^{\zeta} A$.*

2. *If* $x\,\overline{\in}^{\zeta}_{J}A$, *then* $x\,\overline{\in}_{J}A$.

Definition 2.11. *[3] Let* (X,ρ,ξ_J) *be a J-nbdS and* $A \subseteq X$ *and* $x \in X$. *The J-rough membership functions of A are defined by* $\mu^J_A \to [0,1]$, *where*

$$\mu^J_A(x) = \frac{|\cap n_J(x) \cap A|}{|\cap n_J(x)|}$$

and $|A|$ *is cardinality of A.*

Definition 2.12. *[3] Let* (X,ρ,ξ_J) *be a J-nbdS,* $A \subseteq X$ *and* $x \in X$. *The J-rough nearly membership functions of A are defined by* $\mu^{\zeta_J}_A \to [0,1]$, *where*

$$\mu^{\mathcal{I}-\zeta_J}_A(x) = \{\begin{matrix} 1 & if\ 1 \in \psi^{\zeta_J}_A(x). \\ min(\psi^{\zeta_J}_A(x)) & otherwise. \end{matrix}\}.$$

$$and\ \psi^{\zeta_J}_A(x) = \frac{|\zeta_J(x) \cap A|}{|\zeta_J(x)|}, x \in \zeta_J(x), \zeta_J(x) \in \zeta_J O(X).$$

Lemma 2.1. *[3] Let* (X,ρ,ξ_J) *be a J-nbdS and* $A \subseteq X$. *Then,*

1. $\mu^J_A(x) = 1 \Rightarrow \mu^{\zeta_J}_A(x) = 1, \forall\, x \in X.$
2. $\mu^J_A(x) = 0 \Rightarrow \mu^{\zeta_J}_A(x) = 0, \forall\, x \in X.$

3. *J*-nearly open sets via ideals

In this section, the concepts of $\mathcal{I}$-*J*-nearly open sets are presented as a generalization of *J*-nearly open sets. These concepts are based on the notions of ideals. The main topological properties of these concepts and the connections among them are also studied.

Definition 3.1. *Let* (X,ρ,ξ_J) *be a J-nbdS and* $\mathcal{I}$ *be an ideal on X.* $A \subseteq X$ *is called*

1. $\mathcal{I}$-α_J*-open, if* $\exists G \in \tau_J$ *such that* $(A - int_J(cl_J((G)) \in \mathcal{I}$ *and* $(G - A) \in \mathcal{I}$.
2. $\mathcal{I}$*-J-Preopen (briefly* $\mathcal{I}$-P_J*-open), if* $\exists G \in \tau_J$ *such that* $(A - G) \in \mathcal{I}$ *and* $(G - cl_J(A)) \in \mathcal{I}$.
3. $\mathcal{I}$*-J-Semi open (briefly* $\mathcal{I}$-S_J*-open), if* $\exists G \in \tau_J$ *such that* $(A - cl_J(G)) \in \mathcal{I}$ *and* $(G - A) \in \mathcal{I}$.
4. $\mathcal{I}$-β_J*-open, if* $\exists G \in \tau_J$ *such that* $(A - cl_J(G)) \in \mathcal{I}$ *and* $(G - cl_J(A)) \in \mathcal{I}$.

These sets are called $\mathcal{I}$*-J-nearly open sets, the complement of the* $\mathcal{I}$*-J-nearly open sets are called* $\mathcal{I}$*-J-nearly closed sets, the families of* $\mathcal{I}$*-J-nearly open sets of X denoted by* $\mathcal{I}$-$\zeta_J O(X)$ *and the families of* $\mathcal{I}$*-J-nearly closed sets of X denoted by* $\mathcal{I}$-$\zeta_J C(X)$, $\forall \zeta \in \{P,S,\alpha,\beta\}$.

The following proposition shows that the concepts of $\mathcal{I}$-*J*-nearly open sets are stronger than the concepts of *J*-nearly open sets. Consequently, the current Definition 3.1 is a generalization of Definition 2.6 [6].

Proposition 3.1. *Let* (X,ρ,ξ_J) *be a J-nbdS and* $\mathcal{I}$ *be an ideal on X. Then, the following implications hold:*

1. *Every* α_J*-open is* $\mathcal{I}$-α_J*-open.*
2. *Every* P_J*-open is* $\mathcal{I}$-P_J*-open.*
3. *Every* S_J*-open is* $\mathcal{I}$-S_J*-open.*
4. *Every* β_J*-open is* $\mathcal{I}$-β_J*-open.*

Proof. By using Definitions 2.6 and 3.1.

Remark 3.1. *The following example shows that the converse of Proposition 3.1 is not necessarily true.*

Example 3.1. *Let* $X = \{a,b,c\}$, $\rho = \{(a,a),(a,b),(b,b),(c,a),(c,b)\}$ *and* $\mathcal{I} = \{\phi,\{b\},\{c\},\{b,c\}\}$. *Then,* $\tau_R = \{X,\phi,\{b\},\{a,b\}\}$. *It is clear that* $A = \{c\}$ *is an* $\mathcal{I}$-β_R*-open (respectively,* $\mathcal{I}$-S_R*-open,* $\mathcal{I}$-P_R*-open,* $\mathcal{I}$-α_R*-open) set, but it is not a* β_R*-open (respectively,* S_R*-open,* P_R*-open,* α_R*-open) set.*

The following theorem shows that Amer et al.'s [6] definitions are special case of the current definitions.

Theorem 3.1. *Let (X, ρ, ξ_J) be a J-nbdS and $\mathcal{I}$ be an ideal on X. If $\mathcal{I} = \phi$ in the current Definition 3.1, then we get Amer et al. 's Definition 2.6 [6].*

Proof. Straightforward.

The following proposition shows the relationships among the $\mathcal{I}$-J-nearly open sets.

Proposition 3.2. *Let (X, ρ, ξ_J) be a J-nbdS and $\mathcal{I}$ be an ideal on X. Then, the following implications hold:*

$$\begin{array}{ccc} \mathcal{I}\text{-}\alpha_J\text{-open} & & \mathcal{I}\text{-}P_J\text{-open} \\ \Downarrow & & \Downarrow \\ \mathcal{I}\text{-}S_J\text{-open} & \Rightarrow & \mathcal{I}\text{-}\beta_J\text{-open.} \end{array}$$

Proof. Straightforward by Definition 3.1.

Remark 3.2. *The reverse implications of Proposition 3.2 are not necessarily true as shown in the following examples.*

Example 3.2. *Let the topological space (X, τ_R) be in Example 3.1 with $\mathcal{I} = \{\phi, \{b\}\}$. Then, $\{c\}$ is $\mathcal{I}$-α_R-open, but it is not $\mathcal{I}$-P_R-open set.*

Example 3.3. *Let $X = \{a, b, c, d\}$, $\rho = \{(a, a), (a, c), (b, a), (b, b), (b, c), (c, a), (c, c), (d, a), (d, d)\}$ and $\mathcal{I} = \{\phi, \{d\}\}$. Then, $\tau_R = \{X, \phi, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. It is clear that $\{a\}$ is $\mathcal{I}$-P_R-open ($\mathcal{I}$-β_R-open) set, but it is neither $\mathcal{I}$-α_R-open set nor $\mathcal{I}$-S_R-open set.*

Example 3.4. *Let $X = \{a, b, c\}$, $\rho = \{(a, a), (a, b), (a, c), (b, b), (c, c)\}$ and $\mathcal{I} = \{\phi, \{c\}\}$. Thus, $\tau_R = \{X, \phi, \{b\}, \{c\}, \{b, c\}\}$. Then, $\{a\}$ is $\mathcal{I}$-S_R-open ($\mathcal{I}$-β_R-open) set, but it is not $\mathcal{I}$-α_R-open set.*

Remark 3.3. *According to Examples 3.2 and 3.3 the concepts of $\mathcal{I}$-α_J-open sets and $\mathcal{I}$-P_J-open sets are independent although every α_J-open set is P_J-open set in [6]. Additionally, the concepts of $\mathcal{I}$-S_J-open sets and $\mathcal{I}$-P_J-open sets are not comparable as shown in Examples 3.3 and 3.4.*

Proposition 3.3. *Let (X, ρ, ξ_J) be a J-nbdS and let $\mathcal{I}$ be an ideal on X. Then, the following implications hold:*

$$\begin{array}{ccccc} \tau_J(\Gamma_J) & \Rightarrow & \mathcal{I}\text{-}\alpha_J O(\mathcal{I}\text{-}\alpha_J C) & & \mathcal{I}\text{-}P_J O(\mathcal{I}\text{-}P_J C) \\ & & \Downarrow & & \Downarrow \\ & & \mathcal{I}\text{-}S_J O(\mathcal{I}\text{-}S_J C) & \Rightarrow & \mathcal{I}\text{-}\beta_J O(\mathcal{I}\text{-}\beta_J C). \end{array}$$

Proof. By Remark 2.2 [6], and Propositions 3.1 and 3.2, the proof is obvious.

Remark 3.4. *Example 3.1 shows that the reverse implication " $\tau_J(\Gamma_J) \Rightarrow \mathcal{I}$-$\alpha_J O(\mathcal{I}$-$\alpha_J C)$" is not necessarily true. It is clear that $A = \{c\} \in \mathcal{I}$-$\alpha_R O(X), \notin \tau_R$.*

Theorem 3.2. *Let (X, ρ, ξ_J) be a J-nbdS and let $\mathcal{I}$ be an ideal on X. Then, the union of two $\mathcal{I}$-α_J-open (respectively, $\mathcal{I}$-S_J-open, $\mathcal{I}$-P_J-open, $\mathcal{I}$-β_J-open) sets is also $\mathcal{I}$-α_J-open (respectively, $\mathcal{I}$-S_J-open, $\mathcal{I}$-P_J-open, $\mathcal{I}$-β_J-open) set.*

Proof. Let A and B be $\mathcal{I}$-α_J-open sets. Then, $\exists G, H$ such that $(A - int_J cl_J(G)) \in \mathcal{I}, (G - A) \in \mathcal{I}, (B - int_J cl_J(H)) \in \mathcal{I}$ and $(H - B) \in \mathcal{I}$. Hence, $(G - (A \cup B)) \subseteq (G - A) \in \mathcal{I}$, $(H - (A \cup B)) \subseteq (H - B) \in \mathcal{I}$ and so, $(G - (A \cup B)) \cup (H - (A \cup B)) \in \mathcal{I}$. Let $W = G \cup H$, then $(W - (A \cup B)) \in \mathcal{I}$. Also, $(A - int_J cl_J(W)) \subseteq (A - int_J cl_J(G)) \in \mathcal{I}$ and $(B - int_J cl_J(W)) \subseteq (B - int_J cl_J(H)) \in \mathcal{I}$. Then, $(A - int_J cl_J(W)) \cup (B - int_J cl_J(W)) \in \mathcal{I}$ and so $((A \cup B) - int_J cl_J(W)) \subseteq (A - int_J cl_J(G)) \cup (B - int_J cl_J(H)) \in \mathcal{I}$. Thus, $A \cup B$ is $\mathcal{I}$-α_J-open set. The rest of the proof is similar.

Remark 3.5. *The family $\mathcal{I}$-$\zeta_J O(X)$, in a space X do not form a topology as the intersection of two $\mathcal{I}$-α_J-open (respectively, $\mathcal{I}$-S_J-open, $\mathcal{I}$-P_J-open, $\mathcal{I}$-β_J-open) sets is not $\mathcal{I}$-α_J-open (respectively, $\mathcal{I}$-S_J-open, $\mathcal{I}$-P_J-open, $\mathcal{I}$-β_J-open) set as*

1. *in Example 3.3, $A = \{c, d\}, B = \{a, d\} \in \mathcal{I}$-$P_R O(X)$, but $A \cap B = \{d\} \notin \mathcal{I}$-$P_R O(X)$.*
2. *in Example 3.4, let $\mathcal{I} = \{\phi, \{a\}\}$. Then,*

   i. $A = \{a, b\}, B = \{a, c\} \in \mathcal{I}$-$\alpha_R O(X)$, *but* $A \cap B = \{a\} \notin \mathcal{I}$-$\alpha_R O(X)$.

   ii. $A = \{a, b\}, B = \{a, c\} \in \mathcal{I}$-$S_R O(X)$, *but* $A \cap B = \{a\} \notin \mathcal{I}$-$S_R O(X)$.

   iii. $A = \{a, b\}, B = \{a, c\} \in \mathcal{I}$-$\beta_R O(X)$, *but* $A \cap B = \{a\} \notin \mathcal{I}$-$\beta_R O(X)$.

Remark 3.6. *Let (X, ρ, ξ_J) be a J-nbdS, $\mathcal{I}$ be an ideal on X and $A \subseteq X$. Then the following statements are not true in general:*

1. $\mathcal{I}$-$\zeta_U O(X) \subseteq \mathcal{I}$-$\zeta_R O(X) \subseteq \mathcal{I}$-$\zeta_I O(X)$.
2. $\mathcal{I}$-$\zeta_U O(X) \subseteq \mathcal{I}$-$\zeta_L O(X) \subseteq \mathcal{I}$-$\zeta_I O(X)$.
3. $\mathcal{I}$-$\zeta_{<U>} O(X) \subseteq \mathcal{I}$-$\zeta_{<R>} O(X) \subseteq \mathcal{I}$-$\zeta_{<I>} O(X)$.
4. $\mathcal{I}$-$\zeta_{<U>} O(X) \subseteq \mathcal{I}$-$\zeta_{<L>} O(X) \subseteq \mathcal{I}$-$\zeta_{<I>} O(X)$.
5. $\mathcal{I}$-$\zeta_R O(X)$ *is the dual of* $\mathcal{I}$-$\zeta_L O(X)$.
6. $\mathcal{I}$-$\zeta_{<R>} O(X)$ *is the dual of* $\mathcal{I}$-$\zeta_{<L>} O(X)$.

So, the relationships among $\mathcal{I}$-ζ_J- open sets are not comparable as in Example 3.3:

1. $\mathcal{I}$-$P_R O(X) = P(X) - \{\{b\}, \{d\}, \{b, d\}\}$.
2. $\mathcal{I}$-$P_L O(X) = \{X, \phi, \{b\}, \{d\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$.
3. $\mathcal{I}$-$P_I O(X) = P(X)$.
4. $\mathcal{I}$-$P_U O(X) = P(X)$.
5. $\mathcal{I}$-$P_{<R>} O(X) = \{X, \phi, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$.
6. $\mathcal{I}$-$P_{<L>} O(X) = \{X, \phi, \{b\}, \{d\}, \{a, b\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$.
7. $\mathcal{I}$-$P_{<I>} O(X) = \{X, \phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$.
8. $\mathcal{I}$-$P_{<U>} O(X) = P(X) - \{\{d\}\}$.

It is clear that

- $\mathcal{I}$-$P_U O(X) \nsubseteq \mathcal{I}$-$P_L O(X)$.
- $\mathcal{I}$-$P_U O(X) \nsubseteq \mathcal{I}$-$P_R O(X)$.
- $\mathcal{I}$-$P_{<U>} O(X) \nsubseteq \mathcal{I}$-$P_{<R>} O(X)$.
- $\mathcal{I}$-$P_{<L>} O(X) \nsubseteq \mathcal{I}$-$P_{<I>} O(X)$.
- $\mathcal{I}$-$P_R O(X)$ is not the dual of $\mathcal{I}$-$P_L O(X)$ and $\mathcal{I}$-$P_{<R>} O(X)$ is not the dual of $\mathcal{I}$-$P_{<L>} O(X)$.
- Similarly, $\mathcal{I}$-$P_L O(X) \nsubseteq \mathcal{I}$-$P_I O(X)$, $\mathcal{I}$-$P_R O(X) \nsubseteq \mathcal{I}$-$P_I O(X)$, $\mathcal{I}$-$P_{<R>} O(X) \nsubseteq \mathcal{I}$-$P_{<I>} O(X)$, and $\mathcal{I}$-$P_{<U>} O(X) \nsubseteq \mathcal{I}$-$P_{<I>} O(X)$.

4. *J*-nearly approximations via ideals

The purpose of this section is to generalize J-nearly approximations to $\mathcal{I}$-J-nearly approximations. The current approximations are based on $\mathcal{I}$-J-nearly open sets. The properties of the new approximations are studied.

Definition 4.1. *Let (X, ρ, ξ_J) be a J-nbdS, $\mathcal{I}$ be an ideal on X and $A \subseteq X$. The $\mathcal{I}$-J-nearly lower, $\mathcal{I}$-J-nearly upper approximations, $\mathcal{I}$-J-nearly boundary regions and $\mathcal{I}$-J-nearly accuracy of A are:*
$\underline{\rho}_J^{\mathcal{I}-\zeta}(A) = \cup\{G \in \mathcal{I}\text{-}\zeta_J O(X) : G \subseteq A\} = \mathcal{I}$-*J-nearly interior of A.*
$\overline{\rho}_J^{\mathcal{I}-\zeta}(A) = \cap\{H \in \mathcal{I}\text{-}\zeta_J C(X) : A \subseteq H\} = \mathcal{I}$-*J-nearly closure of A.*
$B_J^{\mathcal{I}-\zeta}(A) = \overline{\rho}_J^{\mathcal{I}-\zeta}(A) - \underline{\rho}_J^{\mathcal{I}-\zeta}(A)$.
$\sigma_J^{\mathcal{I}-\zeta}(A) = \frac{|\underline{\rho}_J^{\mathcal{I}-\zeta}(A)|}{|\overline{\rho}_J^{\mathcal{I}-\zeta}(A)|}$, *where* $|\overline{\rho}_J^{\mathcal{I}-\zeta}(A)| \neq 0$.

The following proposition studies the main properties of the current $\mathcal{I}$-J-nearly lower and $\mathcal{I}$-J-nearly upper approximations.

Proposition 4.1. *Let (X, ρ, ξ_J) be a J-nbdS, $\mathcal{I}$ be an ideal on X and $A, B \subseteq X$. Then,*

1. $\underline{\rho}_J^{\mathcal{I}-\zeta}(A) \subseteq A \subseteq \overline{\rho}_J^{\mathcal{I}-\zeta}(A)$ *equality hold if* $A = \phi$ *or* X.
2. $A \subseteq B \Rightarrow \overline{\rho}_J^{\mathcal{I}-\zeta}(A) \subseteq \overline{\rho}_J^{\mathcal{I}-\zeta}(B)$.
3. $A \subseteq B \Rightarrow \underline{\rho}_J^{\mathcal{I}-\zeta}(A) \subseteq \underline{\rho}_J^{\mathcal{I}-\zeta}(B)$.
4. $\overline{\rho}_J^{\mathcal{I}-\zeta}(A \cap B) \subseteq \overline{\rho}_J^{\mathcal{I}-\zeta}(A) \cap \overline{\rho}_J^{\mathcal{I}-\zeta}(B)$.
5. $\underline{\rho}_J^{\mathcal{I}-\zeta}(A \cup B) \supseteq \underline{\rho}_J^{\mathcal{I}-\zeta}(A) \cup \underline{\rho}_J^{\mathcal{I}-\zeta}(B)$.
6. $\overline{\rho}_J^{\mathcal{I}-\zeta}(A \cup B) \supseteq \overline{\rho}_J^{\mathcal{I}-\zeta}(A) \cup \overline{\rho}_J^{\mathcal{I}-\zeta}(B)$.
7. $\underline{\rho}_J^{\mathcal{I}-\zeta}(A \cap B) \subseteq \underline{\rho}_J^{\mathcal{I}-\zeta}(A) \cap \underline{\rho}_J^{\mathcal{I}-\zeta}(B)$.
8. $\underline{\rho}_J^{\mathcal{I}-\zeta}(A) = (\overline{\rho}_J^{\mathcal{I}-\zeta}(A'))'$, $\overline{\rho}_J^{\mathcal{I}-\zeta}(A) = (\underline{\rho}_J^{\mathcal{I}-\zeta}(A'))'$.
9. $\overline{\rho}_J^{\mathcal{I}-\zeta}(\overline{\rho}_J^{\mathcal{I}-\zeta}(A)) = \overline{\rho}_J^{\mathcal{I}-\zeta}(A)$.
10. $\underline{\rho}_J^{\mathcal{I}-\zeta}(\underline{\rho}_J^{\mathcal{I}-\zeta}(A)) = \underline{\rho}_J^{\mathcal{I}-\zeta}(A)$.
11. $\underline{\rho}_J^{\mathcal{I}-\zeta}(\underline{\rho}_J^{\mathcal{I}-\zeta}(A)) \subseteq \overline{\rho}_J^{\mathcal{I}-\zeta}(\underline{\rho}_J^{\mathcal{I}-\zeta}(A))$.
12. $\underline{\rho}_J^{\mathcal{I}-\zeta}(\overline{\rho}_J^{\mathcal{I}-\zeta}(A)) \subseteq \overline{\rho}_J^{\mathcal{I}-\zeta}(\overline{\rho}_J^{\mathcal{I}-\zeta}(A))$.
13. $x \in \overline{\rho}_J^{\mathcal{I}-\zeta}(A) \Leftrightarrow G \cap A \neq \phi, \forall G \in \mathcal{I}\text{-}\zeta_J O(X), x \in G$.
14. $x \in \underline{\rho}_J^{\mathcal{I}-\zeta}(A) \Leftrightarrow \exists G \in \mathcal{I}\text{-}\zeta_J O(X), x \in G, G \subseteq A$.

The proof of this proposition is simple using $\mathcal{I}$-J-nearly interior and $\mathcal{I}$-J-nearly closure, so I omit it.

Remark 4.1. *Example 3.3 shows that*

1. *the inclusion in Proposition 4.1 parts 1, 4, 5, 6, 7, 11 and 12 can not be replaced by equality relation:*
   (i) *for part 1, if* $A = \{a\}, \underline{\rho}_L^{\mathcal{I}-P}(A) = \phi, \overline{\rho}_L^{\mathcal{I}-P}(A) = \{a, c\}$, *then* $\overline{\rho}_L^{\mathcal{I}-P}(A) \nsubseteq A \nsubseteq \underline{\rho}_L^{\mathcal{I}-P}(A)$.
   (ii) *for part 4, if* $A = \{a\}, B = \{b\}, A \cap B = \phi, \overline{\rho}_L^{\mathcal{I}-P}(A) = \{a, c\}, \overline{\rho}_L^{\mathcal{I}-P}(B)$ $= \{a, b, c\}, \overline{\rho}_L^{\mathcal{I}-P}(A \cap B) = \phi$, *then* $\overline{\rho}_L^{\mathcal{I}-P}(A) \cap \overline{\rho}_L^{\mathcal{I}-P}(B) = \{a, c\} \nsubseteq \phi = \overline{\rho}_L^{\mathcal{I}-P}(A \cap B)$.
   (iii) *for part 5, if* $A = \{c, d\}, B = \{a, b, c\}, A \cup B = X, \underline{\rho}_L^{\mathcal{I}-P}(A) = \{d\}, \underline{\rho}_L^{\mathcal{I}-P}(B) = \{b\}, \underline{\rho}_L^{\mathcal{I}-P}(A \cup B) = X$, *then* $\underline{\rho}_L^{\mathcal{I}-P}(A \cup B) = X \nsubseteq \{b, d\} = \underline{\rho}_L^{\mathcal{I}-P}(A) \cup \underline{\rho}_L^{\mathcal{I}-P}(B)$.

(iv) *for part 11, if* $A = \{b\}$, $\underline{\rho}_L^{\mathcal{I}-P}(\underline{\rho}_L^{\mathcal{I}-P}(A)) = A$, $\overline{\rho}_L^{\mathcal{I}-P}(\underline{\rho}_L^{\mathcal{I}-P}(A)) = \{a,b,c\}$, *then* $\overline{\rho}_L^{\mathcal{I}-P}(\underline{\rho}_L^{\mathcal{I}-P}(A)) \nsubseteq \underline{\rho}_L^{\mathcal{I}-P}(\underline{\rho}_L^{\mathcal{I}-P}(A))$.

(v) *for part 12, if* $A = \{a,b,c\}$, $\overline{\rho}_L^{\mathcal{I}-P}(\overline{\rho}_L^{\mathcal{I}-P}(A)) = A$, $\underline{\rho}_L^{\mathcal{I}-P}(\overline{\rho}_L^{\mathcal{I}-P}(A)) = \{b\}$, *then* $\overline{\rho}_L^{\mathcal{I}-P}(\overline{\rho}_L^{\mathcal{I}-P}(A)) \nsubseteq \underline{\rho}_L^{\mathcal{I}-P}(\overline{\rho}_L^{\mathcal{I}-P}(A))$.

2. *the converse of parts 2 and 3 is not necessarily true:*

(i) *for part 2, if* $A = \{a\}, B = \{b\}$, *then* $\overline{\rho}_L^{\mathcal{I}-P}(A) = \{a,c\}$, $\overline{\rho}_L^{\mathcal{I}-P}(B) = \{a,b,c\}$. *Therefore,* $\overline{\rho}_L^{\mathcal{I}-P}(A) \subseteq \overline{\rho}_L^{\mathcal{I}-P}(B)$, *but* $A \nsubseteq B$.

(ii) *for part 3, if* $A = \{a,c,d\}, B = \{b,c,d\}$, *then* $\underline{\rho}_L^{\mathcal{I}-P}(A) = \{d\}$, $\underline{\rho}_L^{\mathcal{I}-P}(B) = \{b,d\}$. *Therefore,* $\underline{\rho}_L^{\mathcal{I}-P}(A) \subseteq \underline{\rho}_L^{\mathcal{I}-P}(B)$, *but* $A \nsubseteq B$.

Definition 4.2. *Let* (X, ρ, ξ_J) *be a J-nbdS,* $\mathcal{I}$ *be an ideal on X, and* $A \subseteq X$. *A is* $\mathcal{I}$-ζ_J*-nearly definable (*$\mathcal{I}$-ζ_J*-nearly exact) set if* $\overline{\rho}_J^{\mathcal{I}-\zeta}(A) = \underline{\rho}_J^{\mathcal{I}-\zeta}(A)$. *Otherwise, A is* $\mathcal{I}$-ζ_J*-nearly rough set.*

In Example 3.3 $A = \{d\}$ is $\mathcal{I}$-P_L-exact, while $B = \{b\}$ is $\mathcal{I}$-P_L-rough.

Remark 4.2. *Let* (X, ρ, ξ_J) *be a J-nbdS,* $\mathcal{I}$ *be an ideal on X and* $A \subseteq X$. *Then the intersection of two* $\mathcal{I}$-ζ_J*-rough sets need not to be* $\mathcal{I}$-ζ_J*-rough set as in Example 3.3* $\{b,d\}$ *and* $\{c,d\}$*, are* $\mathcal{I}$-α_L*-rough sets,* $\{b,d\} \cap \{c,d\} = \{d\}$ *is not* $\mathcal{I}$-α_L*-rough set.*

The following theorem and corollary present the relationships between the current approximations in Definition 4.1 and the previous one in Definitions 2.4 [2] and 2.7 [6].

Theorem 4.1. *Let* (X, ρ, ξ_J) *be a J-nbdS,* $\mathcal{I}$ *be an ideal on X and* $A \subseteq X$. *Then*

1. $\underline{\rho}_J^{\zeta}(A) \subseteq \underline{\rho}_J^{\mathcal{I}-\zeta}(A)$.
2. $\underline{\rho}_J(A) \subseteq \underline{\rho}_J^{\mathcal{I}-\zeta}(A)$.
3. $\overline{\rho}_J^{\mathcal{I}-\zeta}(A) \subseteq \overline{\rho}_J^{\zeta}(A)$.
4. $\overline{\rho}_J^{\mathcal{I}-\zeta}(A) \subseteq \overline{\rho}_J(A)$.

Proof.

(1) $\underline{\rho}_J^{\zeta}(A) = \cup\{G \in \zeta_J O(X) : G \subseteq A\} \subseteq \cup\{G \in \mathcal{I}\text{-}\zeta_J O(X) : G \subseteq A\} = \underline{\rho}_J^{\mathcal{I}-\zeta}(A)$ (by Proposition 3.2).

(2) By Theorem 2.2, $\underline{\rho}_J(A) \subseteq \underline{\rho}_J^{\zeta}(A)$, and by (1) $\underline{\rho}_J^{\zeta}(A) \subseteq \underline{\rho}_J^{\mathcal{I}-\zeta}(A)$. Hence, $\underline{\rho}_J(A) \subseteq \underline{\rho}_J^{\mathcal{I}-\zeta}(A)$.

(3) and (4) Similar to (1) and (2).

Corollary 4.1. *Let* (X, ρ, ξ_J) *be a J-nbdS,* $\mathcal{I}$ *be an ideal on U and* $A \subseteq X$. *Then*

1. $B_J^{\mathcal{I}-\zeta}(A) \subseteq B_J^{\zeta}(A)$.
2. $B_J^{\zeta}(A) \subseteq B_J(A)$.
3. $\sigma_J^{\zeta}(A) \leqslant \sigma_J^{\mathcal{I}-\zeta}(A)$.
4. $\sigma_J(A) \leqslant \sigma_J^{\mathcal{I}-\zeta}(A)$.

Corollary 4.2. *Let* (X, ρ, ξ_J) *be a J-nbdS,* $\mathcal{I}$ *be an ideal on X and* $A \subseteq X$. *Then*

1. *Every* ζ_J*-nearly exact subset in X is* $\mathcal{I}$-ζ_J*-nearly exact.*
2. *Every J-exact subset in X is* $\mathcal{I}$-ζ_J*-nearly exact.*
3. *Every* $\mathcal{I}$-ζ_J*-nearly rough subset in U is* ζ_J*-nearly rough.*
4. *Every* $\mathcal{I}$-ζ_J*-nearly rough subset in U is J-rough.*

The converse of parts of Corollary 4.2 is not necessarily true as in Example 3.3:

1. If $A = \{a\}$, then it is $\mathcal{I}$-β_L-exact, but it is neither β_L-exact nor L-exact.
2. If $A = \{d\}$, then it is l-rough, but it is neither $\mathcal{I}$-β_L-rough nor β_L-rough.

The following proposition and corollary are introduced the relationships between the $\mathcal{I}$-J-nearly lower (upper) approximations, $\mathcal{I}$-J-nearly boundary regions and $\mathcal{I}$-J-nearly accuracy.

Proposition 4.2. *Let (X, ρ, ξ_J) be a J-nbdS, $\mathcal{I}$ be an ideal on X and $A \subseteq X$. Then*

1. $\underline{\rho}_J^{\mathcal{I}-P}(A) \subseteq \underline{\rho}_J^{\mathcal{I}-\beta}(A)$.
2. $\underline{\rho}_J^{\mathcal{I}-\alpha}(A) \subseteq \underline{\rho}_J^{\mathcal{I}-S}(A) \subseteq \underline{\rho}_J^{\mathcal{I}-\beta}(A)$.
3. $\overline{\rho}_J^{\mathcal{I}-\beta}(A) \subseteq \overline{\rho}_J^{\mathcal{I}-P}(A)$.
4. $\overline{\rho}_J^{\mathcal{I}-\beta}(A) \subseteq \overline{\rho}_J^{\mathcal{I}-S}(A) \subseteq \overline{\rho}_J^{\mathcal{I}-\alpha}(A)$.

Proof. By Proposition 3.2, the proof is obvious.

Corollary 4.3. *Let (X, ρ, ξ_J) be a J-nbdS, $\mathcal{I}$ be an ideal on X and $A \subseteq X$. Then*

1. $B_J^{\mathcal{I}-\beta}(A) \subseteq B_J^{\mathcal{I}-P}(A)$.
2. $B_J^{\mathcal{I}-\beta}(A) \subseteq B_J^{\mathcal{I}-S}(A) \subseteq B_J^{\mathcal{I}-\alpha}(A)$.
3. $\sigma_J^{\mathcal{I}-P}(A) \leqslant \sigma_J^{\mathcal{I}-\beta}(A)$.
4. $\sigma_J^{\mathcal{I}-\alpha}(A) \leqslant \sigma_J^{\mathcal{I}-S}(A) \leqslant \sigma_J^{\mathcal{I}-\beta}(A)$.

Corollary 4.4. *Let (X, ρ, ξ_J) be a J-nbdS, $\mathcal{I}$ be an ideal on U and $A \subseteq X$. Then*

1. *A is J-exact $\Rightarrow$ A is $\mathcal{I}$-α_J-exact $\Rightarrow$ $\mathcal{I}$-S_J-exact $\Rightarrow$ $\mathcal{I}$-β_J-exact.*
2. *A is $\mathcal{I}$-P_J-exact $\Rightarrow$ $\mathcal{I}$-β_J-exact.*

Remark 4.3. *Example 3.3 shows that the converse of the implications in Corollaries 4.3, 4.4 and Proposition 4.2 is not true in general.*

Remark 4.4. *From the above results, it is noted that there are different methods to approximate the sets. The best of these methods is there are given by using $\mathcal{I}$-β_J in constructing the approximations of sets, since the boundary regions in this case are decreased (or canceled) by increasing the lower approximation and decreasing the upper approximation. Moreover, the $\mathcal{I}$-β_J-accuracy is more accurate than the other types.*

In Table 1, the lower, upper approximations, boundary region and accuracy are calculated by using Abd El-Monsef et al.'s approximations 2.4 [2], Amer et al.'s approximations 2.7 [6], and the current approximations in Definition 4.1 by using Example 3.3.

Table 1: Comparison between the boundary and accuracy by Abd El-Monsef et al.'s method 2.4 [2], Amer et al.'s method 2.7 [6] and the current method in Definition 4.1 at $\zeta = \beta, J = L$.

2*A	Abd El-Monsef et al.'s method 2.4 [2]				Amer et al.'s method 2.7 [6]				The current method in Definition 4.1			
	$\underline{\rho}_L(A)$	$\overline{\rho}_L(A)$	$B_L(A)$	$\sigma_L(A)$	$\underline{\rho}_L^{\beta}(A)$	$\overline{\rho}_L^{\beta}(A)$	$B_L^{\beta}(A)$	$\sigma_L^{\beta}(A)$	$\underline{\rho}_{\mathcal{I}}^{\beta_L}(A)$	$\overline{\rho}_{\mathcal{I}}^{\beta_L}(A)$	$B_{\mathcal{I}}^{\beta_L}(A)$	$\sigma_{\mathcal{I}}^{\beta_L}(A)$
$\{a\}$	ϕ	$\{a,c\}$	$\{a,c\}$	0	ϕ	$\{a\}$	$\{a\}$	0	$\{a\}$	$\{a\}$	ϕ	1
$\{b\}$	$\{b\}$	$\{a,b,c\}$	$\{a,c\}$	$\frac{1}{3}$	$\{b\}$	$\{b\}$	ϕ	1	$\{b\}$	$\{b\}$	ϕ	1
$\{c\}$	ϕ	$\{a,c\}$	$\{a,c\}$	0	ϕ	$\{c\}$	$\{c\}$	0	$\{c\}$	$\{c\}$	ϕ	1
$\{d\}$	$\{d\}$	$\{a,c,d\}$	$\{a,c\}$	$\frac{1}{3}$	$\{d\}$	$\{d\}$	ϕ	1	$\{d\}$	$\{d\}$	ϕ	1
$\{a,b\}$	$\{b\}$	$\{a,b,c\}$	$\{a,c\}$	$\frac{1}{3}$	$\{a,b\}$	$\{a,b\}$	ϕ	0	$\{a,b\}$	$\{a,b\}$	ϕ	1
$\{a,c\}$	ϕ	$\{a,c\}$	$\{a,c\}$	0	ϕ	$\{a,c\}$	$\{a,c\}$	0	$\{a,c\}$	$\{a,c\}$	ϕ	1
$\{a,d\}$	$\{d\}$	$\{a,c,d\}$	$\{a,c\}$	$\frac{1}{3}$	$\{a,d\}$	$\{a,d\}$	ϕ	1	$\{a,d\}$	$\{a,d\}$	ϕ	1
$\{b,c\}$	$\{b\}$	$\{a,b,c\}$	$\{a,c\}$	$\frac{1}{3}$	$\{b,c\}$	$\{b,c\}$	ϕ	1	$\{b,c\}$	$\{b,c\}$	ϕ	1
$\{b,d\}$	$\{b\}$	X	$\{a,c,d\}$	$\frac{1}{4}$	$\{b,d\}$	X	$\{a,c\}$	$\frac{1}{2}$	$\{b,d\}$	$\{b,d\}$	ϕ	1
$\{c,d\}$	$\{d\}$	$\{a,c,d\}$	$\{a,c\}$	$\frac{1}{3}$	$\{c,d\}$	$\{c,d\}$	ϕ	1	$\{c,d\}$	$\{c,d\}$	ϕ	1
$\{a,b,c\}$	$\{b\}$	$\{a,b,c\}$	$\{a,c\}$	$\frac{1}{3}$	$\{a,b,c\}$	$\{a,b,c\}$	ϕ	1	$\{a,b,c\}$	$\{a,b,c\}$	ϕ	1
$\{a,b,d\}$	$\{b,d\}$	X	$\{a,c\}$	$\frac{1}{2}$	$\{a,b,d\}$	X	$\{c\}$	$\frac{3}{4}$	$\{a,b,d\}$	$\{a,b,d\}$	ϕ	1
$\{a,c,d\}$	$\{d\}$	$\{a,c,d\}$	$\{a,c\}$	$\frac{1}{3}$	$\{a,c,d\}$	X	$\{b\}$	0	$\{a,c,d\}$	$\{a,c,d\}$	ϕ	1
$\{b,c,d\}$	$\{b,d\}$	X	$\{a,c\}$	$\frac{1}{2}$	$\{b,c,d\}$	X	$\{a\}$	$\frac{3}{4}$	$\{b,c,d\}$	$\{b,c,d\}$	ϕ	1
X	X	X	ϕ	1	X	X	ϕ	1	X	X	ϕ	1

For example, take $A = \{b, d\}$, then the boundary and accuracy by the present method in Definition 4.1 are ϕ and 1 respectively. Whereas, the boundary and accuracy by using Abd El-Monsef et al.'s method 2.4 [2] are $\{a, c, d\}$ and $\frac{1}{4}$ respectively and by using Amer et al.'s method 2.7 [6] are $\{a, c\}$ and $\frac{1}{2}$ respectively.

5. *J*-nearly rough membership functions via ideals

This section concentrates on generalization the concept of rough membership function by introducing the concept of J-nearly rough membership functions via ideals.

Definition 5.1. *Let (X, ρ, ξ_J) be a J-nbdS, $\mathcal{I}$ be an ideal on $X, x \in X$ and $A \subseteq X$:*

1. *if $x \in \underline{\rho}_J^{\mathcal{I}-\zeta}(A)$, then x is J-nearly surely with respect to $\mathcal{I}$ ($\mathcal{I} - \zeta_J$-surely) belongs to A, denoted by $x \;\underline{\in}_J^{\mathcal{I}-\zeta} A$.*
2. *if $x \in \overline{\rho}_J^{\mathcal{I}-\zeta}(A)$, then x is J-nearly possibly with respect to $\mathcal{I}$ (briefly $\mathcal{I} - \zeta_J$-possibly) belongs to A, denoted by $x \;\overline{\in}_J^{\mathcal{I}-\zeta} A$.*

It is called J-nearly strong and J-nearly weak membership relations with respect to $\mathcal{I}$ respectively.

Remark 5.1. *According to Definition 5.1 the J-nearly approximations via ideal for any $A \subseteq X$ can be written as:*

1. $\underline{\rho}_J^{\mathcal{I}-\zeta}(A) = \{x \in X : x \;\underline{\in}_J^{\mathcal{I}-\zeta} A\}$.
2. $\overline{\rho}_J^{\mathcal{I}-\zeta}(A) = \{x \in X : x \;\overline{\in}_J^{\mathcal{I}-\zeta} A\}$.

Lemma 5.1. *Let (X, ρ, ξ_J) be a J-nbdS, $\mathcal{I}$ be an ideal on X and $A \subseteq X$. Then*

1. *if $x \;\underline{\in}_J^{\mathcal{I}-\zeta} A$, then $x \in A$.*
2. *if $x \in A$, then $x \;\overline{\in}_J^{\mathcal{I}-\zeta} A$.*

Proof. Straightforward.

Remark 5.2. *The converse of Lemma 5.1 is not true in general, as it is shown in Example 3.3 that if $A = \{a\}$:*

1. *$a \in A$, but $a \underline{\notin}_L^{\mathcal{I}-p} A$.*
2. *$a \;\overline{\in}_L^{\mathcal{I}-p} A$, but $a \notin A$.*

Proposition 5.1. *Let (X, ρ, ξ_J) be a J-nbdS, $\mathcal{I}$ be an ideal on X and $A \subseteq X$. Then*

1. *if* $x \underline{\in}_J A \Rightarrow x \underline{\in}_J^{\zeta} A \Rightarrow x \underline{\in}_J^{\mathcal{I}-\zeta} A$.
2. *if* $x \overline{\in}_J^{\mathcal{I}-\zeta} A \Rightarrow x \overline{\in}_J^{\zeta} A \Rightarrow x \overline{\in}_J A$.

Proof. We prove (1) and the other similarly. $x \underline{\in}_J A \Rightarrow x \underline{\in}_J^{\zeta} A$ by Proposition 2.2. Let $x \underline{\in}_J^{\zeta} A$. Then, $x \in \underline{\rho}_J^{\zeta}(A) \Rightarrow x \in \underline{\rho}_J^{\mathcal{I}-\zeta}(A) \Rightarrow x \underline{\in}_J^{\mathcal{I}-\zeta} A$.

Remark 5.3. *The converse of Proposition 5.1 is not true in general, as it is shown in Example 3.3 that if $A = \{c\}$,*

1. *then* $c \underline{\in}_L^{\mathcal{I}-\beta} A$, *but* $c \underline{\notin}_L^{\beta} A$ *and* $c \underline{\notin}_L A$.
2. *then* $a \overline{\in}_L A$, *but* $a \overline{\in}_L^{\beta} A$ *and* $a \overline{\in}_L^{\mathcal{I}-\beta} A$.

Definition 5.2. *Let (X, ρ, ξ_J) be a J-nbdS, $\mathcal{I}$ be an ideal on $X, A \subseteq X$ and $x \in X$. The J-nearly rough membership functions of a J-nbdS on X for a A are defines by $\mu_A^{\mathcal{I}-\zeta_J} \rightarrow [0, 1]$, where*

$$\mu_A^{\mathcal{I}-\zeta_J}(x) = \begin{cases} 1 & if\ 1 \in \psi_A^{\mathcal{I}-\zeta_J}(x). \\ min(\psi_A^{\mathcal{I}-\zeta_J}(x)) & otherwise. \end{cases}$$

$$and\ \psi_A^{\mathcal{I}-\zeta_J}(x) = \frac{|\mathcal{I}-\zeta_J(x) \cap A|}{|\mathcal{I}-\zeta_J(x)|}, x \in \mathcal{I} - \zeta_J(x), \mathcal{I} - \zeta_J(x) \in \mathcal{I}\text{-}\zeta_J O(X).$$

Remark 5.4. *The $\mathcal{I}$-J-nearly rough membership functions are used to define the $\mathcal{I}$-J-nearly lower (upper) approximations as follows:*

1. $\underline{\rho}_J^{\mathcal{I}-\zeta}(A) = \{x \in X : \mu_A^{\mathcal{I}-\zeta_J}(x) = 1\}$.
2. $\overline{\rho}_J^{\mathcal{I}-\zeta}(A) = \{x \in X : \mu_A^{\mathcal{I}-\zeta_J}(x) > 0\}$.
3. $B_J^{\mathcal{I}-\zeta}(A) = \{x \in X : 0 < \mu_A^{\mathcal{I}-\zeta_J}(x) < 1\}$.

Proposition 5.2. *Let (X, ρ, ξ_J) be a J-nbdS, $\mathcal{I}$ be an ideal on X and $A, B \subseteq X$. Then*

1. *if* $\mu_A^{\mathcal{I}-\zeta_J}(x) = 1 \Leftrightarrow x \in \underline{\in}_J^{\mathcal{I}-\zeta_J} A$.
2. *if* $\mu_A^{\mathcal{I}-\zeta_J}(x) = 0 \Leftrightarrow x \in X - \overline{\rho}_J^{\mathcal{I}-\zeta}(A)$.
3. *if* $0 < \mu_A^{\mathcal{I}-\zeta_J}(x) < 1 \Leftrightarrow x \in B_J^{\mathcal{I}-\zeta}(A)$.
4. *if* $\mu_{A'}^{\mathcal{I}-\zeta_J}(x) = 1 - \mu_A^{\mathcal{I}-\zeta_J}(x), \forall\ x \in X$.
5. *if* $\mu_{A\cup B}^{\mathcal{I}-\zeta_J}(x) \geq max(\mu_A^{\mathcal{I}-\zeta_J}(x), \mu_B^{\mathcal{I}-\zeta_J}(x)), \forall\ x \in X$.
6. *if* $\mu_{A\cap B}^{\mathcal{I}-\zeta_J}(x) \leq min(\mu_A^{\mathcal{I}-\zeta_J}(x), \mu_B^{\mathcal{I}-\zeta_J}(x)), \forall\ x \in X$.

Proof. I prove (1), and the others similarly. $x \underline{\in}_J^{\mathcal{I}-\zeta_J} A \Leftrightarrow x \underline{\rho}_J^{\mathcal{I}-\zeta}(A)$. Since $\underline{\rho}_J^{\mathcal{I}-\zeta}(A)$ is $\mathcal{I}$-J-nearly open contained in A, thus
$\frac{|\underline{\rho}_J^{\mathcal{I}-\zeta}(A) \cap A|}{|\underline{\rho}_J^{\mathcal{I}-\zeta}(A)|} = \frac{|\underline{\rho}_J^{\mathcal{I}-\zeta}(A)|}{|\underline{\rho}_J^{\mathcal{I}-\zeta}(A)|} = 1$. Then, $1 \in \psi_A^{\mathcal{I}-\zeta_J}(x)$ and accordingly $\mu_A^{\mathcal{I}-\zeta_J}(x) = 1$.

Lemma 5.2. *Let (X, ρ, ξ_J) be a J-nbdS, $\mathcal{I}$ be an ideal on X and $A \subseteq X$. Then*

1. $\mu_A^J(x) = 1 \Rightarrow \mu_A^{\zeta_J}(x) = 1 \Rightarrow \mu_A^{\mathcal{I}-\zeta_J}(x) = 1, \forall\ x \in X$.
2. $\mu_A^J(x) = 0 \Rightarrow \mu_A^{\zeta_J}(x) = 0 \Rightarrow \mu_A^{\mathcal{I}-\zeta_J}(x) = 0, \forall\ x \in X$.

Proof.

1. $\mu_A^J(x) = 1 \Rightarrow \mu_A^{\zeta_J}(x) = 1$ directly from Lemma 2.1. Let $\mu_A^{\zeta_J}(x) = 1$, then $x \in \underline{\rho}_J^{\zeta}(A) \Rightarrow x \in \underline{\rho}_J^{\mathcal{I}-\zeta}(A) \Rightarrow \mu_A^{\mathcal{I}-\zeta_J}(x) = 1, \forall\ x \in X$.
2. $\mu_A^J(x) = 0 \Rightarrow \mu_A^{\zeta_J}(x) = 0$ directly from Lemma 2.1. Let $\mu_A^{\zeta_J}(x) = 0$, then $x \in X - \overline{\rho}_J^{\zeta}(A) \Rightarrow x \in X - \overline{\rho}_J^{\mathcal{I}-\zeta}(A) \Rightarrow \mu_A^{\mathcal{I}-\zeta_J}(x) = 0, \forall\ x \in X$.

Remark 5.5. *The converse of the previous lemma is not true in general, as it is shown in Example 3.3.*

6. Conclusions

General topology has been considered the entrance to understand topology science, moreover the base of general topology is the topological space, which has been considered a representation of universal space in general, and geometric shape in special, also the mathematical analysis concepts. The purpose of the present work is to apply topological concepts into rough set. In classical rough set model approximations are based on equivalence relations, but this condition does not always hold in many practical problems and also this restriction limits the wide applications of this theory. For the mentioned reasoning we redefine the approximations by using topological concepts. The topology induced by binary relations, ideals, and nearly open sets generalize rough set. The properties of suggested methods are studied. The importance of the current approximations is not only that it is reducing or deleting the boundary regions, but also it's satisfying all properties of Pawlak's rough sets without any restrictions. Also, we defined the concept of rough membership function using our approach. The results in this paper show that topology can be a powerful method to study rough set models. We conclude that construction of topological properties on rough sets will help us to find rough measure which will enable to find the missing attribute values. Topological properties of rough sets gives useful tool for creating the consistency base for Knowledge Discovery in Database (KDD). According to this connection between rough set concepts and topological notions, we can say that the improvements in abstract topology results help in some way for modifications of rough sets theory and consequently in its real-life applications. Finally, topological structure of this theory brings various techniques for Data Processing and Bio-mathematics.

The following points will be studied in the future:

1. New connections between rough set theory, rough multiset, soft set theory, fuzzy set theory based on $\mathcal{I}$-J-nearly open sets.
2. Applications of these new approximations in various real-life fields.

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