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# TOPOLOGICAL ASPECTS OF CHOW QUOTIENTS

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## Abstract

This paper studies the canonical Chow quotient of a smooth projective variety by a reductive algebraic group. The main purpose is to introduce the Perturbation–Translation–Specialization relation that gives a computable characterization of the Chow cycles of the Chow quotient. Also, we provide, in the languages that are familiar to topologists and differential geometers, many topological interpretations of Chow quotient that have the advantage to be more intuitive and geometric. More precisely, over the field of complex numbers, these interpretations are, symplectically, the moduli spaces of stable orbits with prescribed momentum charges; and topologically, the moduli space of stable action-manifolds.

## 1. Introduction

Moduli spaces of geometric objects (e.g., vector bundles, algebraic curves, etc.) have played central roles in many theories in algebraic geometry and in its neighboring fields. The constructions of moduli spaces are frequently done by expressing them as quotients of schemes by reductive algebraic groups.

However, taking quotients in algebraic geometry is much subtler than it may appear. Mumford, based upon Hilbert's invariant theory, developed a systematic method, the Geometric Invariant Theory ([**22**]), to deal with projective quotients. There are several other quotient theories, among them are [**18**] and [**16**] which construct quotients as algebraic spaces.

It has become well known now that, for a reductive algebraic group action on a smooth projective variety, Mumford's quotients depend, in a flip-flop fashion, on choices of linearized line bundles ([**4**] and [**26**]). Nevertheless, it is a drawback that none of Mumford's quotients is in general canonical. Besides this, the closures of the orbits parameterized by a GIT quotient almost always belong to different cohomology classes, and this, among other reasons, oftentimes makes GIT quotients rather

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misbehaved compactifications. This is unsatisfactory from the viewpoint of moduli problem, where a moduli space always parameterizes geometric objects of same topological type, and awkward to use for purpose of some geometric computations. To overcome these drawbacks, we are led to consider a canonical quotient, the Chow quotient. There is another canonical quotient, the Hilbert quotient. Despite the fact that the Hilbert quotient, derived from a Hilbert scheme, enjoys more functorial properties, the Chow quotient, as it parameterizes cycles, is more geometrically friendly and approachable.

The Chow quotient, $X//^{ch}G$, of a projective variety $X$ by a reductive group $G$ is introduced by Kapranov–Sturmfels–Zelevinsky [**14**] for toric varieties and by Kapranov [**13**] in general. The definition of a general Chow quotient is very obscure — it is defined as the closure of some Zariski open subset in an ambient compact variety (Chow variety, to be precise, another obscure space).

The main purpose of this paper is to give some topological interpretations and characterization of Chow quotient which have the advantage to be more intuitive and geometric. This is to be done over the field of complex numbers and in the languages that are familiar to topologists and differential geometers. Here, we are content to focus on torus actions even though some of our results remain true for more general reductive groups. (The case of general group actions will be treated elsewhere.)

To give the reader some good ideas about Chow quotient as well as its various topological characterization and interpretations that we will formally introduce in the main body of the paper, let us *informally* consider a simple, yet quite informative example.

Let $G = \mathbb{C}^*$ act on $\mathbb{P}^2$ by

$$\lambda \cdot [x : y : z] = [\lambda x : \lambda^{-1} y : z].$$

Consider a map $\Phi : \mathbb{P}^2 \to \mathbb{R}$ given by

$$\Phi([x : y : z]) = \frac{|x|^2 - |y|^2}{|x|^2 + |y|^2 + |z|^2}.$$

This is the moment map for the induced symplectic $S^1$-action with respect to the Fubibi-Study metric. Its image is the interval $[-1, 1]$.

The $\mathbb{C}^*$ orbits are classified as follows. (See Figure 1 for an illustration.)

- Generic $\mathbb{C}^*$-orbits are conics $XY = aZ^2$ minus two points [1:0:0] and [0:1:0] for $a \neq 0, \infty$. (In this introduction, we will always assume that $a \neq 0, \infty$.) We denote these orbits by $(XY = aZ^2)$. The moment map image of the orbit $(XY = aZ^2)$ is $(-1, 1)$.

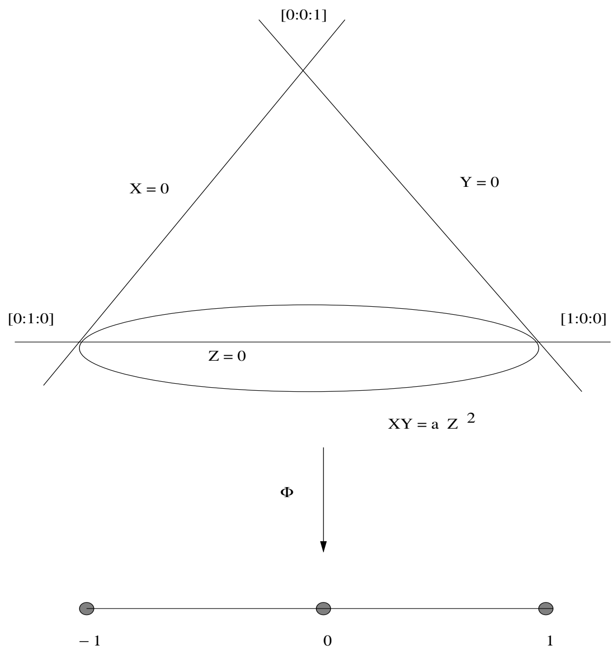


**Figure 1.** Conics.

- Other 1-dimensional orbits are the three coordinate lines $X = 0$, $Y = 0$, and $Z = 0$ minus the coordinate points on them. We denote these orbits by $(X = 0)$, $(Y = 0)$, and $(Z = 0)$. The moment map images of the orbits $(X = 0)$, $(Y = 0)$, and $(Z = 0)$ are $(-1, 0), (0, 1)$, and $(-1, 1)$, respectively.
- Finally, the fixed points are the three coordinate points, [1:0:0], [0:1:0], and [0:0:1], and their moment map images are $1, -1$, and 0, respectively.

The moment map has three critical values $-1, 0$, and 1 which divide the interval into two top chambers $[-1, 0]$ and $[0, 1]$, and three 0-dimensional chambers $\{-1\}, \{0\}, \{1\}$. Each chamber $C$ defines a GIT stability: a point $[x : y : z]$ is semi-stable with respect to the chamber $C$ if $C \subset \Phi(\overline{\mathbb{C}^* \cdot [x : y : z]})$, and it is stable if the (relative) interior $C^\circ \subset \Phi(\mathbb{C}^* \cdot [x : y : z])$ and $\dim \mathbb{C}^* \cdot [x : y : z] = 1$. (For a reference for this, see for example, [9].) Thus, the orbit $(X = 0)$ is stable with respect to $[-1, 0]$, unstable with respect to $[0, 1]$; while the orbit $(Y = 0)$ is stable with respect to $[1, 0]$, unstable with respect to $[-1, 0]$. But, $(X = 0)$, $(Y = 0)$, and $[0 : 0 : 1]$ are all semi-stable with respect to the chamber $\{0\}$.

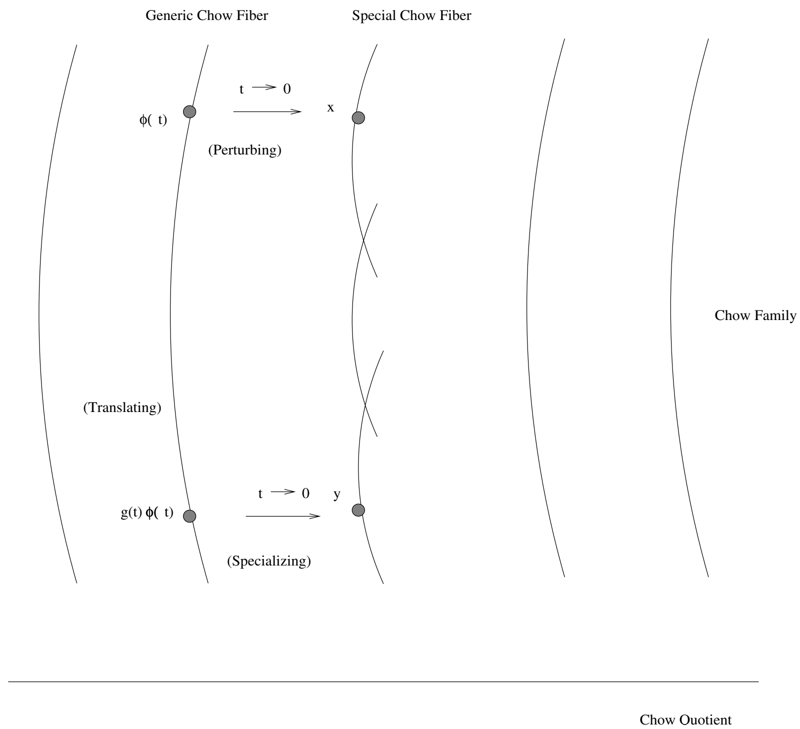


**Figure 2.** Perturbing-Translating-Specializing.

A general feature of GIT quotient is that it parameterizes orbits that are *closed* in the semi-stable locus. Hence, the GIT quotient $X_{[-1,0]}$ defined by the chamber $[-1, 0]$ parameterizes $(XY = aZ^2)$, $(Z = 0)$, and $(X = 0)$. The GIT quotient $X_{[0,1]}$ defined by the chamber $[0, 1]$ parameterizes $(XY = aZ^2)$, $(Z = 0)$, and $(Y = 0)$. And, the GIT quotient $X_{\{0\}}$ defined by the chamber $\{0\}$ parameterizes $(XY = aZ^2)$, $(Z = 0)$, and $[0 : 0 : 1]$. Now, observe that the lines $X = 0$ and $Y = 0$, which are of degree 1, have different homology classes than the conic orbits $XY = aZ^2$, which are of degree 2. This is undesirable. Even worst, the two orbits $(X = 0)$ and $(Y = 0)$ are all identified with the closed orbit $[0 : 0 : 1]$ in the quotient $X_{\{0\}}$, and the unique closed orbit $[0 : 0 : 1]$ even has different dimension than those of the conic orbits.

However, Chow quotient takes a completely different approach. It first considers the closures of the generic $\mathbb{C}^*$-orbits, $XY = aZ^2$ $(a \neq 0, \infty)$, and then looks at all their possible degenerations. When $a = 0$, we get the degenerated conic $XY = 0$, two crossing lines; and when $a = \infty$, we obtain $Z^2 = 0$, a double line. They all have the same homology classes (degree 2). And the Chow quotient is *the space of all* $\mathbb{C}^*$*-invariant conics*. (See Figure 1.) Each point of the Chow quotient represents an invariant algebraic cycle. In this case, these cycles are $[XY = aZ^2]$ $(a \neq 0, \infty)$, $[X = 0] + [Y = 0]$, and $2[Z = 0]$.

From this example, we see that Chow quotient parameterizes cycles of generic orbit closures and their toric degenerations which are *certain sums of orbit closures of dimension* $\dim G$. We will call these cycles *Chow fibers*. So, when do two arbitrary points belong to the same Chow fiber? Consider the example again. We have that $[0 : y : z]$ and $[x : 0 : z]$ $(xyz \neq 0)$ belong to the same Chow fiber $XY = 0$. To get $[x : 0 : z]$ from $[0 : y : z]$, we first *perturb* $[0 : y : z]$ to a general position $\varphi(t) = [tx : y : z]$ $(t \neq 0)$, then *translate* it by $g(t) = t^{-1} \in \mathbb{C}^*$ to $g(t) \cdot \varphi(t) = [x : ty : z]$, and then $g(t) \cdot \varphi(t)$ *specializes* to $[x : 0 : z]$ when $t$ goes to 0. We will call this process *perturbing–translating–specializing* (PTS). It turns out this simple relation holds true in general. That is, we prove in general that two points $x$ and $y$ of $X$, with $\dim G \cdot x = \dim G \cdot y = \dim G$, belong to the same Chow fiber if and only if $x$ can be perturbed (to general positions), translated along $G$-orbits (to positions close to $y$), and then specialized to the point $y$. See Figure 2 for an illustration. This is our Theorem 3.13.

An upshot of PTS relation is that, comparing to the non-descriptive definition of special Chow fibers, it is computable, and thus provides some much needed information on boundary cycles of the Chow quotient. As an application, we apply Theorem 3.13 to the case of point configurations on $\mathbb{P}^n$ $(n > 1)$, and propose a *geometric interpretation* of the Chow quotients of $(\mathbb{P}^n)^m$ (equivalently, the Chow quotients of higher Grassmannians). It was relayed to me that Lafforgue's space constructed in [**20**] is in general reducible, one component of which is the Chow quotient, and, Theorem 3.13 may be used to give a way of telling Lafforgue's components apart. (We will pursue the above and related issues in a forthcoming paper.)

Back to our example, let

$$G = \mathbb{C}^* = S^1 \times \mathbb{R}_{>0} = K \times A$$

be the polar decomposition. Then, one checks easily that the moment map $\Phi$ is $S^1$-invariant and $\Phi(r \cdot [x : y : z])$ is a strictly increasing function of $r$ for $r \in \mathbb{R}_{>0}$ and for any given non-fixed point $[x : y : z]$. This implies, for example, that in each closed orbit parameterized by a point of the GIT quotient $X_{[-1,0]}$ there is a unique $S^1$-orbit whose moment map image is, say, $-\frac{1}{2}$ (any other point in $(-1, 0)$ will work equally well).

This in turn implies that $X_{[-1,0]}$ is homeomorphic to $\Phi^{-1}(-\frac{1}{2})/S^1$. Similarly, we have $X_{\{}0\} \cong \Phi^{-1}(0)/S^1$ and $X_{[0,1]} \cong \Phi^{-1}(\frac{1}{2})/S^1$. In particular, $\Phi^{-1}(-\frac{1}{2})/S^1$ has a unique point parameterizing the orbit $(X = 0)$, $\Phi^{-1}(\frac{1}{2})/S^1$ instead has a point parameterizing the orbit $(Y = 0)$, while $\Phi^{-1}(0)/S^1$ has a point parameterizing the smaller orbit $(X = 0 = Y)$ $(= [0 : 0 : 1])$. As it turns out, these are just examples of the very general correspondence between GIT quotients and symplectic quotients. Put it more formally (cf. [**15**], [**22**]), symplectic quotients endow various symplectic structures, possibly singular, on GIT quotients. Pushing this circle of ideas further, it is natural to ask whether a Chow quotient admits its own symplectic counterparts.

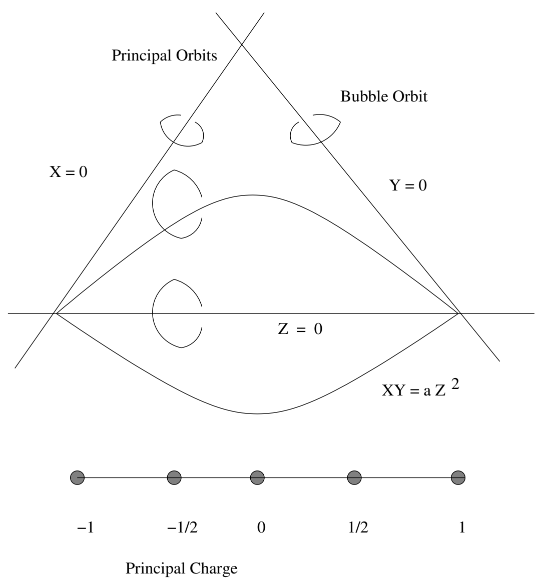


**Figure 3.** Stable $S^1$-Orbits.

To this end, we introduce the so-called *stable K-orbits* with a fixed set of *momentum charges*, $\gamma$. Consider again the Chow quotient of our previous example, the space of all invariant conics in $\mathbb{P}^2$. So, here $K = S^1$. We first fix a general point, say $-\frac{1}{2}$, in $[-1, 1]$, and call it the *principal momentum charge*. Then, each generic conic orbit contains a unique $S^1$-orbit whose moment map image is $-\frac{1}{2}$. These $S^1$-orbits are the generic stable $K$-orbits with the principal momentum charge $-\frac{1}{2}$. The same can be done for the orbit $(Z = 0)$. That is, there is a unique $S^1$-orbit in $(Z = 0)$ whose moment map image is principal momentum charge $-\frac{1}{2}$. The point here is that $\Phi(XY = aZ^2) = \Phi(Z = 0) = [-1, 1]$

($a \neq 0, \infty$), hence $\Phi(XY = aZ^2)$ and $\Phi(Z = 0)$ do not subdivide $[-1, 1]$. For the cycle $[X = 0] + [Y = 0]$, the moment map image $\Phi(X = 0)$ and $\Phi(Y = 0)$ form a subdivision of $[-1, 1] = [-1, 0] \cup [0, 1]$. In this case, in $[-1, 0]$, we (have to) stick with the principal momentum charge $-\frac{1}{2}$, but in $[0, 1]$, we may choose any interior point, say, $\frac{1}{2}$. Then, we have two unique $S^1$-orbits $S^1 \cdot [0 : 1 : 1]$ and $S^1 \cdot [1 : 0 : 1]$ in the cycle $[X = 0] + [Y = 0]$ over the prescribed momentum charges $\{-\frac{1}{2}, \frac{1}{2}\}$. The union of these two orbits $S^1 \cdot [0 : 1 : 1] \cup S^1 \cdot [1 : 0 : 1]$ is what we call a special stable $S^1$-orbit with the prescribed set of momentum charges. (See Figure 3[1].) Then, we see that the Chow quotient also parameterizes stable $S^1$-orbits with a fixed set of momentum charges $\gamma$ (in this example, $\gamma$ consists of $-\frac{1}{2}$ and $\frac{1}{2}$). We put all stable $S^1$-orbits together, denote it by $\mathfrak{M}_\gamma$, and call it the moduli space of stable $K$-orbits with prescribed momentum charges $\gamma$. We prove in general (see Assumption 3.4) that the space $\mathfrak{M}_\gamma$ is always homeomorphic to the Chow quotient $X/\!/^{ch}G$, regardless of the choice of $\gamma$ (Theorem 4.15).

Now, we insert here a favorable property enjoyed by the Chow quotient: it dominates all GIT quotients. This was proved by Kapranov, and the dominating morphisms were also discovered in [**9**]. In fact, the Chow quotient , under Assumption 3.4, is the *least common refinement* of all GIT quotients, in a *strict* sense. That is, $X/\!/^{ch}G$ is homeomorphic to the limit quotient $X/\!/^{lim}G$, the distinguished irreducible component of the inverse limit of all GIT quotient (Theorem 3.8). When $X$ is a toric variety, this is proved in [**14**].

Then, the morphism from the Chow quotient to a GIT quotient, in terms of stable $K$-orbits, corresponds to a map from $\mathfrak{M}_\gamma$ to the symplectic quotient $\Phi^{-1}(\mathbf{r})/K$ where $\mathbf{r}$ is the principal momentum charge in $\gamma$, and this last map is quite transparent. Every stable orbit contains a unique principal orbit, the one with the principal momentum charge, all the rest will be called "bubble" orbits of the principal one. Then, the map from $\mathfrak{M}_\gamma$ to $\Phi^{-1}(\mathbf{r})/K$ simply forgets all the bubble orbits and send a stable orbit to its principal part. See Figure 3.

The whole picture of the stable $K$-orbits resembles that of stable polygons in [**10**] where we give a symplectic construction of $\overline{M}_{0,n}$, the moduli space of stable $n$-pointed rational curves. In fact, stable polygon is the source of inspiration for the introduction of stable $K$-orbit, and,

[1] A few words on Figure 3. Note that each $G$-orbit closure is a piecewise fibration over the moment map image with the fibers the compact group orbits. In Figure 3, we depict the circle fibers over the principal charge $-\frac{1}{2}$ and the charge $\frac{1}{2}$. A circle winding around a curve or a line means that the circle is contained in the cycle represented by the curve or the line.

[10] may be viewed as an (interesting and long) example of the general theory described in this paper.

The space $\mathfrak{M}_\gamma$ reflects the Hamiltonian aspects of the $G$-action on $X$. Transversal to Hamiltonian flows are the gradient flows of various 1-dimensional projections of the moment map. Indeed, let $G = K \cdot A$ be the polar decomposition of $G$. Then, Hamiltonian flows are tangential to $K$-orbits, while gradient flows are tangential to $A$-orbits. This shift of viewpoint leads us to find another topological approach to the Chow quotient. Here, to make things work coherently, we introduce the so-called *stable action-manifolds*: generic stable action-manifolds are simply the closures of $A$-orbits through generic points; special ones are certain configurations of closures of $A$-orbits that are resulted as the limits of generic ones. In the example, generic stable action-manifolds are $\overline{\mathbb{R}_> \cdot [a:1:1]}$ ($a \in \mathbb{R}, \neq 0, \infty$). The special action-manifolds are $\overline{\mathbb{R}_> \cdot [1:1:0]}$ and $\overline{\mathbb{R}_> \cdot [0:1:1]} \cup \overline{\mathbb{R}_> \cdot [1:0:1]}$. Up to $K$-action, these are all the piece-wise gradient flow lines of the map $\Phi$ connecting $[1:0:0]$ and $[0:1:0]$. See Figure 4. In general, stable action-manifolds are piece-wise smooth manifolds with corners.

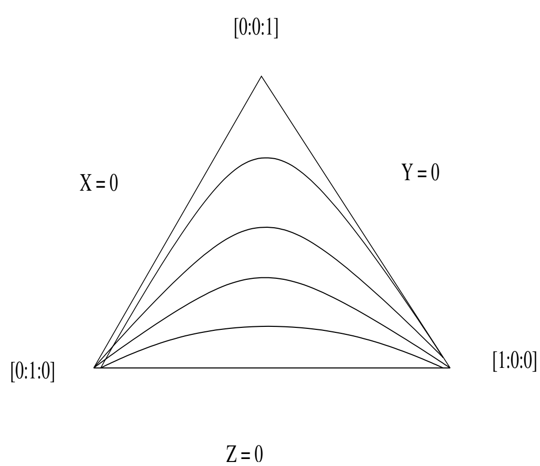


**Figure 4.** Stable Action-Manifolds.

To rigorously construct the special configurations, we apply a real version of our theorem on PTS, which is formulated (for the ad hoc purpose), and proved in Section 5.2. Two stable action-manifolds are said to be equivalent if one can be transferred to the other by the action of an element of the compact group $K$. Then we prove, again under Assumption 3.4, that the moduli space $\mathfrak{M}$ of equivalence classes of stable action-manifolds exists as a separated complex variety, and is homeomorphic to the Chow quotient. Stable action-manifolds are orthogonal to that of stable $K$-orbits. Hence in this sense, the space $\mathfrak{M}_\gamma$ parameterizes Hamiltonian slices of Chow fibers, while the moduli of stable action manifolds $\mathfrak{M}$ parameterizes gradient slices (with respect to the moment map) of Chow fibers which are orthogonal to Hamiltonian slices.

In the previous example, we worked out all sort of the quotients considered in this paper (GIT and Chow), but they are all isomorphic to $\mathbb{P}^1$, the unique compactification of $\mathbb{C}^*$, because we insist an example that are very simple to describe. Here, it should be fair to at least point out to the reader a workable example where a non-trivial wall crossing phenomenon and different quotients do occur. Hence, we take the liberty to include the following example with details left to the reader. So, consider the action of $\mathbb{C}^*$ on $\mathbb{P}^3$ by

$$\lambda \cdot [x : y : z : w] = [\lambda x : \lambda y : \lambda^{-1} z : w].$$

The moment map is

$$\Phi([x : y : z : w]) = \frac{|x|^2 + |y|^2 - |z|^2}{|x|^2 + |y|^2 + |z|^2 + |w|^2}.$$

The image $\Phi(X)$ is $[-1, 1]$ with three critical values $-1, 0, 1$. So, we consider the level sets $\Phi^{-1}(-\frac{1}{2}), \Phi^{-1}(0), \Phi^{-1}(\frac{1}{2})$. In Figure 5, we illustrate the *real parts* $\Phi^{-1}(-\frac{1}{2})_\mathbb{R}, \Phi^{-1}(0)_\mathbb{R}, \Phi^{-1}(\frac{1}{2})_\mathbb{R}$ of the level sets restricted to $\mathbb{C}^3 \subset \mathbb{P}^3$ (the $\mathbb{C}^3$ is defined by setting $w = 1$). It turns out this real picture preserves all the topological information we need.

To understand this picture, note that the real points of $S^1$ are $\mathbb{Z}_2 = \{-1, 1\}$. So, the real parts of

$$\Phi^{-1}\left(-\frac{1}{2}\right) \Big/ S^1, \Phi^{-1}(0)/S^1, \Phi^{-1}\left(\frac{1}{2}\right) \Big/ S^1$$

are

$$\Phi^{-1}\left(-\frac{1}{2}\right)_\mathbb{R} \Big/ \mathbb{Z}_2, \Phi^{-1}(0)_\mathbb{R}/\mathbb{Z}_2, \Phi^{-1}\left(\frac{1}{2}\right)_\mathbb{R} \Big/ \mathbb{Z}_2.$$

Note that $\mathbb{Z}_2$ acts on each level set by identifying the lower part with the upper part. Hence, the quotient can be naturally identified with the upper part. Now, observe that the left map happens to be an isomorphism, but the right map is a (real) blowup along the origin so that the special fiber is $\mathbb{RP}^1$. Now, complexifying this picture and compactifying the results, we obtain three quotients: $X_{[-1,0]} \cong \mathbb{P}^2$, $X_{\{0\}} \cong \mathbb{P}^2$, and $X_{[0,1]}$ isomorphic to the blowup of $\mathbb{P}^2$ along a point. The reader may try to classify the generic $\mathbb{C}^*$-orbits and study their degenerations. He can verify that the Chow quotient is also isomorphic to the blowup of $\mathbb{P}^2$ along a point.

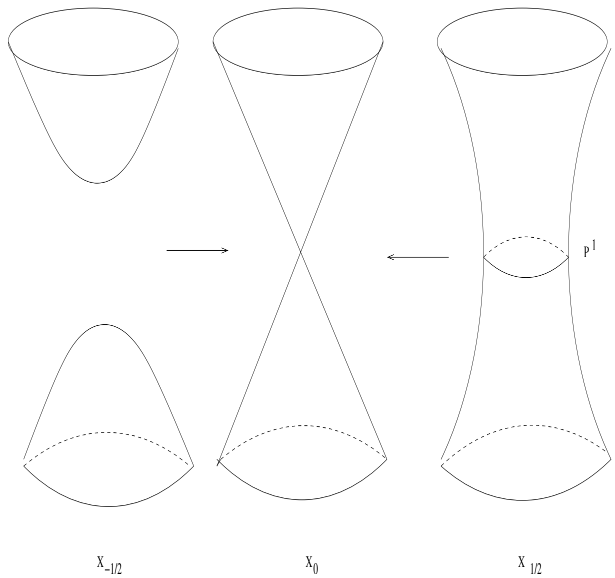


**Figure 5.** Wall-Crossing Maps.

I learned from Igor Dolgachev, during the Arizona conference on Geometry and Topology of Quotients (Dec. 5–8, 2002), that Yuri Neretin first considered the PTS relation (Neretin did not use this terminology. Keel is partially responsible for our choice of the term.) ([**23**], [**24**]), and conjectured that it may relate to some quotient construction. Soon after the conference, I realized that the relation actually characterizes Chow fibers (Theorem 3.13), and its real topological modification can be used to geometrically compactify the moduli space of generic action-manifolds (Theorem 5.11), improving our original approach. It seems from this paper that Neretin's quotient and "hinges" for symmetric spaces ([**23**], [**24**]) are analogs of Chow quotient and Chow fibers in the topological situation (This was observed through a communication with Dolgachev after posting this paper on ArXiv.).

## 2. GIT quotient and Symplectic Reduction

**2.1. GIT quotient.** Let the torus $G = \mathbb{C}^n$ act on a smooth projective variety $X$ over the field of complex numbers. Throughout the paper, we will assume that the action is generically free, that is, the isotropy subgroups of generic points are the identity subgroup.

Let $K = (S^1)^n$ be the compact form of $G$. Fix an ample linearized line bundle $L$ on $X$. Now, pick a $K$-invariant Hermitian metric on $L$;

equivalently, a $K$-invariant symplectic form in $[c_1(L)]$. Then, there is an uniquely associated moment map

$$\Phi_L : X \to \mathfrak{k}^*$$

where $\mathfrak{k}^*$ is the linear dual of the Lie algebra $\mathfrak{k}$ of $K$.

The moment map image $P_L = \Phi_L(X)$ is a compact polytope ([**1**], [**7**]). Atiyah also shows that $\Phi_L(\overline{G \cdot x})$ is a subpolytope of $P_L$ for any $x \in X$, and $\Phi_L(\overline{G \cdot x}) = P_L$ for generic $x$. For simplicity, we often simply write $P$ instead of $P_L$. This polytope admits a natural chamber decomposition, $\mathcal{C}_L$, by the common refinement of all the subpolytopes of the form

$$\Phi_L(\overline{G \cdot x}), x \in X.$$

For any point $\mathbf{r} \in P$, we will use $[\mathbf{r}]$ to denote the minimal chamber that $\mathbf{r}$ belongs to.

Given any rational point $\mathbf{r} \in P$, an integral multiple $m\mathbf{r}$ can be identified with a character $\chi$ of $G$. Let $L^m(\chi)$ be the linearized line bundle $L^m$ twisted by $-\chi$. In the symplectic terms, this means we replace the moment map $\Phi_L$ by $\Phi' = m\Phi_L - m\mathbf{r}$ (this is equivalent to $\Phi_L - \mathbf{r}$ as far as stability is concerned). Then, we will obtain a Zariski open subset $X^{ss}(L^m(\chi))$, the semi-stable locus with respect to the linearization $L^m(\chi)$. By Theorem 8.3 of [**22**], a point $x \in X$ is semi-stable with respect to $L^m(\chi)$ if $0 \in \Phi'(\overline{G \cdot x})$ and it is stable if $0 \in \Phi'(G \cdot x)$ and $\dim G \cdot x = \dim G$. It follows that $X^{ss}(L^m(\chi))$ does not depend on the choice of the multiple $m$, hence, we will denote it by

$$X^{ss}(\mathbf{r}),$$

and call it the set of semi-stable points with respect to $\mathbf{r}$. Observe that

$$x \in X^{ss}(\mathbf{r}) \ \Leftrightarrow \ \mathbf{r} \in \Phi_L(\overline{G \cdot x}).$$

The GIT quotient $X^{ss}(\mathbf{r})//G$ exists as a separated projective variety. Topologically, $X^{ss}(\mathbf{r})//G$ is obtained from $X^{ss}(\mathbf{r})$ by identifying points with the following equivalence relation:

$$x \in X^{ss}(\mathbf{r}) \sim y \in X^{ss}(\mathbf{r}) \text{ iff } \overline{G \cdot x} \cap \overline{G \cdot y} \cap X^{ss}(\mathbf{r}) \neq \emptyset.$$

It follows that when all semi-stable points are actually stable, two different orbits will never be identified. And, in such a case, $X^{ss}(\mathbf{r})//G$ is (topologically) the ordinary orbit space.

Moreover, one can easily deduce that

$$X^{ss}(\mathbf{r}) = X^{ss}(\mathbf{r}')$$

if and only if $\mathbf{r}$ and $\mathbf{r}'$ are in the relative interior of the same chamber $C$ for some $C \in \mathcal{C}_L$. Hence, we may write $X^{ss}(C)$ for $X^{ss}(\mathbf{r})$ for all interior rational point $\mathbf{r} \in C$. Observe that in this case

$$x \in X^{ss}(C) \ \Leftrightarrow\ C \subset \Phi_L(\overline{G \cdot x}).$$

In particular, if $C$ is of top dimension, all semi-stable points are stable. To make our notation concise, we will use $X_{[\mathbf{r}]}$ or $X_C$ to denote the GIT quotient

$$X^{ss}(\mathbf{r})//G = X^{ss}(C)//G.$$

For any chamber $C$, if $D \subset C$ is a face of $C$, then one can check from the definition that, we have the inclusion

$$X^{ss}(C) \subset X^{ss}(D),$$

and this inclusion induces a canonical birational projective morphism

$$f_{CD} : X_C \to X_D.$$

They all together form an inverse system

$$\{X_C, f_{CD} | D \subset C \in \mathcal{C}_L\}.$$

For a reference of the above, one may consult [**9**].

**2.2. Symplectic reduction.** For any point $\mathbf{r} \in P$, rational or not, we have an inclusion

$$G \cdot \Phi_L^{-1}(\mathbf{r}) \subset X^{ss}(C),$$

where $C$ is the unique minimal chamber containing $\mathbf{r}$. In fact, $G \cdot \Phi_L^{-1}(\mathbf{r})$ is exactly the set of closed orbits of $X^{ss}(C)$. Hence, the above inclusion induces a natural homeomorphism

$$\Phi^{-1}(\mathbf{r})/K \to X_C,$$

from the *symplectic reduction* $\Phi^{-1}(\mathbf{r})/K$ to the GIT quotient $X_C$, thanks to a theorem of Kirwan (cf. [**22**]). This basically says that the GIT quotient $X_C$ carries a family of symplectic structures, possibly singular, parameterized by the interior points of the chamber $C$.

Again, to be concise, we will use $X_{\mathbf{r}}$ to denote the symplectic reduction $\Phi^{-1}(\mathbf{r})/K$. Note that when $\mathbf{r}$ is rational, we have used $X_{[\mathbf{r}]}$ to denote the GIT quotient defined by $\mathbf{r}$; the subscript $[\mathbf{r}]$ emphasizes the fact that the GIT quotient only depends on the minimal chamber that $\mathbf{r}$ belongs to, but not the individual point $\mathbf{r}$.

## 3. Chow Quotient: algebro-geometric approach

**3.1. Chow Variety.** We recall briefly the Chow variety. The reference is [**6**]. For a full account, see [**19**].

A $k$-dimensional algebraic cycle in $\mathbb{P}^n$ is a formal finite linear combination $C = \sum_i m_i C_i$ with non-negative integer coefficients, where $C_i$ are $k$-dimensional irreducible closed subvarieties in $\mathbb{P}^n$. The degree of $C$

is $\sum_i m_i \deg(C_i)$. Assume $Y \subset \mathbb{P}^n$ is a $k$-dimensional irreducible subvariety of degree $d$. Consider the set $Z(Y)$ of all $(n-k-1)$-dimensional linear subspaces of $\mathbb{P}^n$ that intersect $Y$. Then, $Z(Y)$ is an irreducible hypersurface of degree $d$ in the Grassmannian $\mathrm{Gr}(n-k-1, \mathbb{P}^n)$. Let

$$B = \oplus_m B_m = \oplus_m H^0(\mathrm{Gr}(n-k-1, \mathbb{P}^n), \mathcal{O}(m))$$

be the coordinate ring of $\mathrm{Gr}(n-k-1, \mathbb{P}^n)$. Then, $Z(X)$ is defined by the vanishing of some element $R_Y \in B_d$ which is unique up to homothety (multiplication by a non-zero scalar). This element will be called the Chow form of $Y$. Now, let $C = \sum_i m_i C_i$ be an algebraic cycle of degree $d$. We define the Chow form of $C$ as

$$R_C = \prod_i R_{C_i}^{m_i} \in B_d.$$

Let $\mathrm{Chow}_d(\mathbb{P}^n)$ be the set of all $k$-dimensional algebraic cycles of degree $d$ in $\mathbb{P}^n$. Then, the map

$$\mathrm{Chow}_d(\mathbb{P}^n) \to \mathrm{Proj}(B_d), \quad C \to R_C$$

defines an embedding of $\mathrm{Chow}_d(\mathbb{P}^n)$ into the projective space $\mathrm{Proj}(B_d)$. The variety $\mathrm{Chow}_d(\mathbb{P}^n)$ with the projective algebraic structure defined by the above embedding is called the Chow variety of $k$-dimensional algebraic cycles of degree $d$ in $\mathbb{P}^n$.

For a general projective variety $X$, an algebraic cycle is defined in the same way as a non-negative linear combination of irreducible subvarieties. Each irreducible subvariety represents a homology class. The homology class of an algebraic cycle is defined as the induced sum of homology classes of the irreducible components. Now, fix a homology class $\delta$. Let $\mathrm{Chow}_\delta(X)$ be the set of all algebraic cycles of $X$ representing the homology class $\delta$. We can embed $X$ into some projective space $\mathbb{P}^n$, and then cycles of $X$ of homology class $\delta$ become cycles of $\mathbb{P}^n$ of degree $d$ for some $d$. Hence, $\mathrm{Chow}_\delta(X)$ is embedded into $\mathrm{Chow}_d(\mathbb{P}^n)$, and the variety $\mathrm{Chow}_\delta(X)$ with the induced projective algebraic structure is called Chow variety of algebraic cycles of $X$ of homology class $\delta$. (See [**19**] for further details.)

**3.2. Definition of Chow quotient.** Consider a reductive algebraic group action on a projective variety over the field of complex numbers

$$G \times X \to X,$$

besides Mumford's GIT quotient, Kapranov–Sturmfels–Zelevensky ([**14**]) for toric varieties and Kapranov ([**13**]) in general, introduced the canonical Chow quotient.

**Definition 3.1** (Kapranov [**13**])**.** Let $x_0$ be a fixed *generic* point of $X$ and $\delta$ be the induced homology class represented by the cycle $\overline{G\cdot x_0}$. There is a Zariski open subset, $U \subset X$, containing the point $x_0$, such that $\overline{G\cdot x}$ represents the same homology class $\delta$ for all $x \in U$. Let $\mathrm{Chow}_\delta(X)$ be the component of the Chow variety of $X$ containing cycles of homology class $\delta$. Then, there is an embedding

$$\iota : U/G \to \mathrm{Chow}_\delta(X)$$
$$[G\cdot x] \to [\overline{G\cdot x}] \in \mathrm{Chow}_\delta(X).$$

The Chow quotient, denoted by $X/\!/^{ch}G$, is defined to be the closure of $\iota(U/G)$.

This definition is independent of the choice of the Zariski open subset $U$. Note that the group $G$ acts on the Chow variety $\mathrm{Chow}_\delta(X)$ by moving the Chow cycles, and under this action, the Chow quotient $\overline{\iota(U/G)}$ is contained in the fixed point set.

**Remark 3.2.** By our assumption of the action, there is a Zariski open subset $U_0$ such that points of $U_0$ are isotropy-free. Let $U$ be the Zariski open subset in Definition 3.1. Now, we point out that, in this paper, unless otherwise indicated, by generic points, we shall always mean points in the Zariski open subsets $U_0 \cap U$.

**3.3. The Chow family.** Let

$$F \subset X \times (X/\!/^{ch}G)$$

be the family of algebraic cycles over the Chow quotient $X/\!/^{ch}G$ defined by

$$F = \{(x, Z) \in X \times (X/\!/^{ch}G) | x \in Z\}.$$

Then, we have a diagram

$$\begin{array}{ccc} F & \xrightarrow{\mathrm{ev}} & X \\ {\scriptstyle f}\big\downarrow & & \\ X/\!/^{ch}G & & \end{array}$$

where ev and $f$ are the projections to the first and second factor, respectively. For any point $q \in X/\!/^{ch}G$, we will call the fiber, $f^{-1}(q)$, the Chow fiber over the point $q$, and sometimes denote it by $F(q)$.

By [**1**], for a generic point $x \in X$, we have

$$\Phi_L(\overline{G\cdot x}) = \Phi_L(X).$$

That is, for a generic point $q \in X/\!/^{ch}G$, we have

$$\Phi_L(\mathrm{ev}(F(q))) = \Phi_L(X).$$

In fact, this is true for any point $p \in X/\!/^{ch}G$.

**Proposition 3.3.** *For any point $p \in X//^{ch}G$, we have $\Phi_L(\mathrm{ev}(F(p))) = \Phi_L(X)$.*

*Proof.* This is proved in Theorem (0.5.1) of [**2**]. We give slightly different arguments. By replacing $L$ by a large tensor power, we may assume that all polytopes of the form $\Phi_L(\overline{G \cdot x})$ (for some $x \in X$) are lattice polytopes (see Mumford's Appendix to [**25**]). Let $p \in X//^{ch}G$ be an arbitrary point and $q \in X//^{ch}G$ be a nearby generic point. Since $p$ and $q$ are nearby points, we have that $\mathrm{ev}(F(p))$ and $\mathrm{ev}(F(q))$ are nearby in the Hausdorff topology on the compact subsets of $X$. Hence, by the continuity of $\Phi_L$, the two *compact lattice* polytopes $\Phi_L(\mathrm{ev}(F(p)))$ and $\Phi_L(\mathrm{ev}(F(q)))$ are also nearby. Therefore, they must be equal. q.e.d.

Each Chow fiber $F(q)$ as an algebraic cycle is of the form $\sum_i m_i \overline{G \cdot x_i}$ where $m_i$ are non-negative integers and $G \cdot x_i$ are orbits of the top dimension. We will call $\cup_i \overline{G \cdot x_i}$ the support of $F(q)$ and denote it by $|F(q)|$. Note that $\Phi_L(\mathrm{ev}(F(q))) = \Phi_L(|F(q)|)$.

Observe from Proposition 3.3 that $\Phi_L(|F(q)|)$ gives a subdivision of the polytope $P$ by the subpolytopes $\Phi_L(\overline{G \cdot x_i})$, any two of which either do not meet or intersect along a proper face (see Theorem (0.5.1) of [**2**]).

We will make the following technical assumption which says that there are no two Chow fibers with exactly the same support but different multiplicities. This assumption is needed for most of the paper except Section 3.6 (Perturbing, translating, and specializing) which is not affected.

**Assumption 3.4.** If $p \neq q$, then $|F(p)| \neq |F(q)|$.

We do not know whether this assumption always holds, nor do we know an example violating the assumption. When $X = \mathbb{P}^n$ and $G$ is a subtorus of the dense open torus $(\mathbb{C}^*)^n \subset \mathbb{P}^n$, if a Chow fiber $F(q) = \sum_i m_i \overline{G \cdot x_i}$ is a minimal toric degeneration of the generic orbit closures (also called extreme toric degeneration), then $m_i$ is *uniquely* computed by the volume of certain simplex of $\dim G$ (Theorem 3.2, Chapter 8, [**6**]. See also [**14**]). Hence, it seems that the assumption holds in this case.

It has been known that the Chow quotient dominates all GIT quotients ([**13**]). For any GIT quotient $X_C$ (Section 2.1), we shall use

$$\pi_C : X//^{ch}G \to X_C$$

to denote the corresponding canonical projection.

**Remark 3.5.** In fact, let $U$ be any invariant open subset such that the compact geometric quotient $U/G$ exists. Then, for any $q \in X//^{ch}G$,

$F(q) \cap U$ is a single orbit in $U$ (Theorem 0.4, [3]). In particular, let $\pi : X//^{ch}G \to U/G$ be the projection, then $\pi(q) = [F(q) \cap U] \in U/G$.

Before we proceed further, two conventions are needed.

(1) The symbols $[\overline{G \cdot x}]$ or $[\sum_i \overline{G \cdot x_i}]$ will be used to denote a point in the Chow quotient $X//^{ch}G$;

(2) while $[G \cdot x]$ will be used to denote a point in a given GIT quotient $X^{ss}//G$.

**3.4. Relation with the limit quotient.** Recall that for a fixed ample linearized line bundle $L$, the moment map image $P = \Phi_L(X)$ has a natural wall and chamber structure. The set of GIT quotients associated to the moment map $\Phi_L$ is indexed by the set $\mathcal{C}_L$ of all chambers. They form a finite inverse system $\{X_C, f_{CD} | D \subset C \in \mathcal{C}_L\}$, that is, a finite set of projective varieties, together with canonical projective morphisms $f_{CD}$ among them.

**Definition 3.6.** Let $\lim_{C\in\mathcal{C}_L} X_C$ be the inverse limit of the system $\{X_C, f_{CD} | D \subset C \in \mathcal{C}_L\}$. ($\lim_{C\in\mathcal{C}_L} X_C$ may be reducible, in general.) The unique irreducible component of $\lim_{C\in\mathcal{C}_L} X_C$ that contains the open subset $U/G$ is called the *limit quotient* of $X$ by $G$ (associated with $L$), and is denoted by $X//^{\lim}_L G$.

It has been expected for quite a while that the Chow quotient should in some way be related to the inverse limit of GIT quotients. Since it lacks a reference, we provide the necessary details below.

**Lemma 3.7.** *Suppose that Assumption 3.4 holds. Let $p$ and $q$ be two points in $X//^{ch}G$. Then, $p = q$ if and only if $\pi_C(p) = \pi_C(q)$ for all maximal (hence all) chambers $C$.*

*Proof.* Since all the projections, $\pi_C : X//^{ch}G \to X_C$, factor through GIT quotients defined by maximal chambers, we only need to consider maximal chambers $C$. The necessary direction is trivial. For sufficient direction, assume that $p \neq q$. Then, by Assumption 3.4, $|F(p)| \neq |F(q)|$. By Proposition 3.3, we always have

$$\Phi_L(|F(p)|) = \Phi_L(|F(q)|) = \Phi_L(X).$$

Hence, we can find $x \in |F(p)|$ and $y \in |F(q)|$ with

$$\dim G \cdot x = \dim G \cdot y = \dim G$$

such that

$$G \cdot x \neq G \cdot y$$

and

$$\Phi(G \cdot x) \cap \Phi(G \cdot y) \neq \emptyset.$$

Now, choose any maximal chamber $C$ such that

$$C \subset \Phi(\overline{G \cdot x}) \cap \Phi(\overline{G \cdot y}).$$

(Here, we have used the fact that $x$ is a regular point of the moment map $\Phi$ if and only if $\dim G \cdot x = \dim G$.) It follows that $x, y \in X^{ss}(C)$.

Then, by Remark 3.5, $\pi_C(p) = [G \cdot x]$ and $\pi_C(q) = [G \cdot y]$. But since $C$ is a top chamber, all points in $X^{ss}(C)$ are stable. Hence, the two different orbits $G \cdot x$ and $G \cdot y$ will not be identified in the geometric quotient $X_C$. Thus, we obtain

$$\pi_C(p) = [G \cdot x] \neq [G \cdot y] = \pi_C(q).$$

q.e.d.

**Theorem 3.8.** *Suppose that Assumption 3.4 holds. Then, there is a natural birational projective morphism $\ell$ from $X//^{ch}G$ to $X//_L^{\lim}G$ which is also bijective. In particular, $X//^{ch}G$ is homeomorphic to $X//_L^{\lim}G$.*

*Proof.* First of all, since $X//^{ch}G$ maps naturally to all GIT quotients $X_C$, we have that $X//^{ch}G$ maps naturally to $\lim_{C \in \mathcal{C}} X_C$, and the map is an isomorphism when restricted to $U/G$. Hence, the image is contained in $X//_L^{\lim}G$. This gives a birational surjective projective morphism

$$\ell : X//^{ch}G \to X//_L^{\lim}G.$$

It suffices to show that $\ell$ is injective. For any two points $p, q \in X//^{ch}G$, if $\ell(p) = \ell(q)$, then $\pi_C(p) = \pi_C(q)$ for all $C \in \mathcal{C}$. By Lemma 3.7, we have $p = q$.

$\ell$ is homeomorphic because a bijective continuous map between two compact Hausdorff spaces is a homeomorphism. q.e.d.

Consequently, we have that $X//_L^{\lim}G$ is independent of the ample line bundle $L$, at least up to homeomorphism. This may look surprising. But when $G$ is a torus – and this is what we deal with in this paper, changing the line bundle $L$ will deform the moment map image; under this deformation, a very few GIT quotients change slightly: a few older GIT quotients, disappear, while a few new ones emerge, but the limit quotient remains homeomorphic to each other. The same cannot be said when $G$ is a general reductive group. In fact, in the general case, even if we take limit over all ample line bundles, we suspect that Theorem 3.8 should be false.

**3.5. Ample line bundles over the Chow quotient.** Fix a very ample linearized line bundle $L$. Replacing $L$ by a large tensor power, if necessary, we may assume that $L$ descends to a very ample line bundle $L_C$ over the GIT quotient $X_C$, simultaneously for all chambers $C \in \mathcal{C}_L$.

Define a line bundle $L_{ch}$ over $X//^{ch}G$ by setting

$$L_{ch} = \otimes_{\text{maximal } C\in\mathcal{C}_L} \pi_C^* L_C.$$

We will show that this is an ample line bundle over $X//^{ch}G$.

**Lemma 3.9.** *For any curve $Z \subset X//^{ch}G$, there exists a maximal chamber $C$ such that $\pi_C(Z)$ is a non-trivial curve in $X_C$.*

*Proof.* Pick two distinct points $q_1$ and $q_2$ in $Z \subset X//^{ch}G$. If $\pi_C(q_1) = \pi_C(q_2)$ for all chambers $C$, then $q_1 = q_2$ by Lemma 3.7. Hence there is a chamber $C$ such that $\pi_C(q_1) \neq \pi_C(q_2)$. This shows that $\pi_C(Z)$ is a non-trivial curve in the GIT quotient $X_C$. q.e.d.

**Proposition 3.10.** *$L_{ch}$ is ample over $X//^{ch}G$.*

*Proof.* Take any curve $Z$ in $X//^{ch}G$. By the previous lemma, there is a maximal chamber $C_0$ such that $\pi_{C_0}(Z)$ is a non-trivial curve in $X_{C_0}$. We have

$$\begin{aligned} L_{ch}\cdot[Z] &= \sum_{\text{maximal } C\in\mathcal{C}_L} \pi_C^* L_C \cdot [Z] \\ &= \sum_{\text{maximal } C\in\mathcal{C}_L} L_C \cdot [\pi_C(Z)]. \end{aligned}$$

All $L_C \cdot [\pi_C(Z)] \geq 0$, and $L_{C_0} \cdot [\pi_{C_0}(Z)] > 0$. Hence, $L_{ch} \cdot [Z] > 0$. Now, observe that $L_{ch}$ is semi-ample by the definition, hence the above positivity implies that it can not contract any curve. Therefore, it is ample. q.e.d.

**Remark 3.11.** If we define $L'_{ch}$ by setting

$$L'_{ch} = \sum_{\text{all } C\in\mathcal{C}_L} \pi_C^* L_C,$$

then, this is also an ample line bundle. The proof is the same.

**Remark 3.12.** Choose a $K$-invariant symplectic form $\omega$ in $[c_1(L)]$. For each top chamber $C$, choose an interior point $\mathbf{r}_C \in C$. Then, via symplectic reduction, we obtain a symplectic form $\omega_{\mathbf{r}_C}$ on the quotient $X_C$. Let $\pi_C^*\omega_{\mathbf{r}_C}$ be the pullback of the form on $X//^{ch}G$. Then, add them up for all $C$, we obtain

$$\sum_C \pi_C^*\omega_{\mathbf{r}_C}$$

which is a symplectic form on (the smooth locus of) $X//^{ch}G$ because the map $X//^{ch}G \to X//^{\lim}G$ does not contract anything.

**3.6. Perturbing, translating and specializing.** From this subsection till Section 3.8, we do not suppose that Assumption 3.4 holds.

A point of $X$ is said to be isotropy-free if its isotropy subgroup is the identity subgroup.

**Theorem 3.13.** *Let $x$ and $y$ be two points in $X$ such that $\dim G \cdot x = \dim G \cdot y = \dim G$. Then, the points $x$ and $y$ belong to the same Chow fiber $F(q)$ for some $q \in X//^{ch}G$ if and only if there is a generic holomorphic map from the complex unit disk $\Delta = \{z||z| < 1\}$ to $X$*

$$\varphi : \Delta \to X$$

*with $\varphi(0) = x$ and a holomorphic map from the punctured disk $\Delta^* = \Delta \setminus \{0\}$ to $G$*

$$g : \Delta^* \to G$$

*such that*

$$y = \lim_{t \to 0} g(t) \cdot \varphi(t).$$

*Proof.* (For an illustration of this theorem, see Figure 2.)

To prove the sufficient part, we can assume that, for $t \in \Delta^* = \Delta \setminus \{0\}$, the orbits $G \cdot \varphi(t)$ are generic (see Remark 3.2), hence we obtain a well-defined holomorphic map

$$\tilde{\varphi} : \Delta^* \to X//^{ch}G$$

with

$$\tilde{\varphi}(t) = [\overline{G \cdot \varphi(t)}].$$

Now, because $\varphi(0) = x$, we have

$$\lim_{t \to 0} \tilde{\varphi}(t) = F_x,$$

where $F_x$ is a Chow fiber that contains $G \cdot x$. On the other hand, we have that

$$\tilde{\varphi}(t) = [\overline{G \cdot \varphi(t)}] = [\overline{G \cdot g(t) \cdot \varphi(t)}].$$

Now, take the limit $t \to 0$, since $y = \lim_{t \to 0} g(t) \cdot \varphi(t)$, we obtain that

$$\lim_{t \to 0} \tilde{\varphi}(t) = F_y,$$

where $F_y$ is a Chow fiber that contains $G \cdot y$. Hence, $F_x = F_y$.

Conversely, assume that the two points $x$ and $y$ belong to the same Chow fiber, $F(q)$, for some $q \in X//^{ch}G$.

First, since $U_0 \cap U$ is Zariski open (see Remark 3.2), using Luna's étale slice around the point $x$ (see [**21**]), we can choose a holomorphic map

$$\varphi : \Delta \to X$$

with $\varphi(0) = x$ and $\varphi(t) \in U_0 \cap U$ for all $t \neq 0$ such that $[\overline{G \cdot \varphi(t)}] \in X//^{ch}G$ approaches the Chow point $q$ as $t$ goes to 0. Note in particular, that the orbit $G \cdot \varphi(t)$ is an isotropy-free generic point for $t \neq 0$.

Next, for the point $y$, there is an invariant open subset $U_y$ such that $U_y$ contains $y$ and the compact geometric GIT quotient $U_y/G$ exists. For example, take any top chamber $C$ that is contained in the polytope $\Phi_L(\overline{G \cdot y})$ (which is also top dimensional), and let $U_y = X^{ss}(C)$.

Now, for $t \neq 0$, consider the orbits

$$[G \cdot \varphi(t)]_y \in U_y/G$$

as points in the geometric quotient $U_y/G$, and

$$[\overline{G \cdot \varphi(t)}] \in X//^{ch}G$$

as points in the Chow quotient. We have the following diagram

$$\begin{array}{ccc} [\overline{G \cdot \varphi(t)}] & \xrightarrow{t \to 0} & q \\ \pi \big\downarrow & & \pi \big\downarrow \\ [G \cdot \varphi(t)]_y & \xrightarrow{t \to 0} & [F(q) \cap U_y]_y \end{array}$$

where the horizonal arrows are taking limits and the down arrows are the projection morphism from Chow quotient $X//^{ch}G$ to the GIT quotient $U_y/G$. The top horizontal arrow $\lim_{t \to 0}[\overline{G \cdot \varphi(t)}] = q$ holds because of the choice of $\varphi$. The down arrows hold because of Remark 3.5. Now, by the continuity of $\pi$, we obtain the bottom horizontal arrow

$$\lim_{t \to 0}[G \cdot \varphi(t)]_y = [F(q) \cap U_y]_y.$$

But $G \cdot y \subset F(q) \cap U_y$. Hence, $G \cdot y = F(q) \cap U_y$ (Remark 3.5). Therefore, we obtain

$$(*) \qquad \lim_{t \to 0}[G \cdot \varphi(t)]_y = [G \cdot y]_y \in U_y/G.$$

For the point $y$, there is an analytic slice by Luna's étale slice Theorem ([**21**]), $y \in S_y \subset U_y$, such that $S_y$ meets transversely, at a unique point, every $G$-orbit in the open subset $W_y = G \cdot S_y$. Hence, there is $\delta > 0$ such that

$$\{G \cdot \varphi(t) \cap S_y | 0 < |t| < \delta\}$$

is a holomorphic curve in $S_y$. Let $G \cdot \varphi(t) \cap S_y = g(t) \cdot \varphi(t) \in S_y$ for $0 < |t| < \delta$. Because $\varphi(t)$ is isotropy-free and the orbit $G \cdot \varphi(t)$ meets the slice $S_y$ transversely at a unique point, $g(t)$ is holomorphically uniquely determined for each $0 < |t| < \delta$. In other words, $g(t)$ is equivalent to the holomorphic map

$$\{t | 0 < |t| < \delta\} \to S_y$$

$$t \to g(t) \cdot \varphi(t).$$

Hence, we have that

$$g : \{z | 0 < |z| < \delta\} \to G$$

is a holomorphic map. Let $y' \in S_y$ be the limit of $g(t) \cdot \varphi(t)$ as $t$ approaches 0. Since $[G \cdot \varphi(t)]_y = [G \cdot g(t) \cdot \varphi(t)]_y$, we obtain, by the identity (*), that

$$[G \cdot y]_y = \lim_{t \to 0} [G \cdot g(t) \cdot \varphi(t)]_y = [G \cdot y']_y.$$

This implies that $y' = y$ because $y, y' \in G \cdot y \cap S_y$. Now, by a suitable parameter change, we may assume $\delta = 1$.

This completes the proof. q.e.d.

**Remark 3.14.** In the proof, we require that $\varphi(t)$ has the trivial isotropy subgroups. This is not necessary. But it simplifies proofs and also some applications as we will see in Section 5.2.

**Remark 3.15.** This theorem is *topological* in nature. In other words, if we replace all the words "holomorphic (maps)" by "continuous (maps)", the theorem and its proof remain unchanged. The theorem also holds when we replace "Chow" by "Hilbert". This, again, is due to (shows) the topological nature of the theorem.

**Remark 3.16.** Now, a few words on the terminologies. The effect of the map $\varphi : \Delta \to X$ is to push the point $x = \varphi(0)$ to general positions, $\varphi(t)$, $t \neq 0$; then the group elements $g(t)$ translate the points $\varphi(t)$ along the group orbits to positions close to $y$, allowing the final desired specialization. This motivates us to use the descriptive terms "perturbing, translating and specializing". We sometimes abbreviate it as "P.T.S".

**Definition 3.17.** We say that a point $x$ of $X$ can be perturbed (to general positions), translated (along $G$-orbits), and specialized to a point $y$ of $X$ if they have the relation as described in Theorem 3.13. In this case, we will write $x \to_G y$.

**Remark 3.18.** Let $X_{(0)}$ be the set of all points of $X$ whose isotropy subgroups are finite. Then, P.T.S. formulation defines a relation on $X_{(0)}$. By the above theorem, this relation is equivalent to the relation defined by: $x$ and $y$ "belong to the same Chow fiber", which is obviously symmetric. This can also be seen directly. Assume that we have $x \to_G y$, that is, there is a generic holomorphic map from the unit disk to $X$

$$\varphi : \Delta \to X$$

with $\varphi(0) = x$, and a holomorphic map from the punctured disk $\Delta^* = \Delta \setminus \{0\}$ to $G$

$$g : \Delta^* \to G,$$

such that

$$y = \lim_{t \to 0} g(t) \cdot \varphi(t).$$

Define $\psi(t) = g(t)\varphi(t)$. Then, this can be extended to a holomorphic map from the unit disk to $X$ with $\psi(0) = y$. Let $h(t) = g^{-1}(t)$. Then, this defines a holomorphic map from the punctured disk $\Delta^* = \Delta \setminus \{0\}$ to $G$. Clearly, we have

$$x = \lim_{t \to 0} h(t) \cdot \psi(t).$$

That is, $y \to_G x$

However, an orbit of points of $X_{(0)}$ may belong to several different Chow fibers. (This may happen only when the orbit is not generic). Hence, P.T.S. fails to be transitive, and thus is not an equivalent relation on $X_{(0)}$.

**3.7. Point configurations on $\mathbb{P}^n$.** In this section, we draw some special consequences for the diagonal action of $\mathrm{PGL}(n+1, \mathbb{C})$ on $(\mathbb{P}^n)^m$.

Theorem 3.13 holds for all torus action, and in particular, holds for the action of the maximal torus $H = (\mathbb{C}^*)^m/\Delta$ on $\mathrm{Gr}(n+1, \mathbb{C}^m)$, the Grassmannian of $n+1$-dimensional planes in $\mathbb{C}^m$ $(n+1 < m)$, where $\Delta$ is the diagonal subgroup of $(\mathbb{C}^*)^m$ which acts trivially on $\mathrm{Gr}(n+1, \mathbb{C}^m)$. Using the Gelfand–MacPherson correspondence, we can transform the properties of the $H$-action on $\mathrm{Gr}(n+1, \mathbb{C}^m)$ to the corresponding properties of the $G$-action on $(\mathbb{P}^n)^m$ where $G = \mathrm{PGL}(n+1)$.

The GM correspondence can be seen as follows. We represent a point of $\mathrm{Gr}(n+1, \mathbb{C}^m)$ as a matrix $A$ of size $(n+1) \times m$. Write $A$ in column vectors, $A = (a_1, \ldots, a_m)$, with $a_i \in \mathbb{C}^{n+1}$. $A$ is of full rank and we assume that all $a_i \neq 0$. Let $U$ be the set of all such matrices. The group $\mathrm{GL}(n+1)$ acts on $A \in U$ from the left. The group $(\mathbb{C}^*)^m$ acts on $A$ from the right by multiplying the column vectors componentwise. Take the quotient of $U$ by $\mathrm{GL}(n+1)$ first, we obtain the Grassmannian $\mathrm{Gr}(n+1, \mathbb{C}^m)$ with the residual $H = (\mathbb{C}^*)^m/\Delta$- action. Take the quotient of $U$ by $(\mathbb{C}^*)^m$ first, we obtain $(\mathbb{P}^n)^m$ with the residual diagonal action of $G = \mathrm{PGL}(n+1)$. This establishes a correspondence between $H$-orbits on $\mathrm{Gr}(n+1, \mathbb{C}^m)$ and $G$-orbits on $(\mathbb{P}^n)^m$: they all correspond to $\mathrm{GL}(n+1) \times_{\mathbb{C}^*} (\mathbb{C}^*)^m$-orbits on $U$. Then, taking quotient in stages, we get a natural correspondence between GIT quotients as well as Chow quotients (see [**13**] for details. See also [**12**] for a generalization of GM correspondence to the product of Grassmannians).

Thus, by the GM correspondence and in terms of point configurations on $X = (\mathbb{P}^n)^m$, Theorem 3.13 reads

**Theorem 3.19.** *Let $\underline{x}$ and $\underline{y}$ be two points in $X = (\mathbb{P}^n)^m$ such that $\dim G \cdot \underline{x} = \dim G \cdot \underline{y} = \dim G$. Then, the points $\underline{x}$ and $\underline{y}$ belong to the same Chow fiber, $F(q)$, for some $q \in X//^{ch}G$ if and only if there is a generic holomorphic map from the complex unit disk $\Delta = \{z||z| < 1\}$ to $X$*

$$\varphi : \Delta \to X$$

*with $\varphi(0) = \underline{x}$ and a holomorphic map from the punctured disk $\Delta^* = \Delta \setminus \{0\}$ to $G$*

$$g : \Delta^* \to G$$

*such that*

$$\lim_{t\to 0} g(t) \cdot \varphi(t) = \underline{y}.$$

The theorem bears an interesting corollary in the case of $(\mathbb{P}^1)^m$. Let $\{1, \ldots, m\} = J \cup J^c$, where $J$ is a subset containing at least two elements and $J^c$ is the complement.

**Theorem 3.20.** *Let $\underline{x} = (x_1, \ldots, x_m)$ be a point in $(\mathbb{P}^1)^m$ such that its isotropy subgroup is trivial, that is, at least three points are distinct. Assume that $\underline{x}_J$, the points with indexes in $J$, coincide, but $\underline{x}_{J^c}$ are all distinct. Let $\underline{y} = \lim_{t\to 0} g(t) \cdot \varphi(t)$ such that $\underline{y}_J$ are all distinct. Then, $\underline{y}_{J^c}$ must coincide.*

*Proof.* By Theorem 3.13, the algebraic cycle

$$[\overline{G \cdot \underline{x}}] + [\overline{G \cdot \underline{y}}]$$

lies in a Chow fiber $F(q)$. Via Kapranov's isomorphism ([**13**]) between $(\mathbb{P}^1)^m//^{ch}G$ and $\overline{M}_{0,n}$ (the moduli space of stable $n$-pointed rational curves), $F(q)$ corresponds to a stable $n$-pointed rational curve $C$. The curve $C$ contains two components, one component, corresponding to the cycle $[\overline{G \cdot \underline{x}}]$, contains the distinct points $\underline{x}_{J^c}$; and another component, corresponding to the cycle $[\overline{G \cdot \underline{y}}]$, contains the distinct point $\underline{y}_J$. Since $\underline{x}_{J^c}$ and $\underline{y}_J$ together give $n$ distinct points on the stable $n$-pointed rational curve $C$, we see that there should be no other components in $C$. Therefore, $\underline{y}_{J^c}$ must coincide, and the two components are glued by joining the point $\underline{x}_J$ with the point $\underline{y}_{J^c}$. q.e.d.

This elementary and interesting "new" phenomenon seems elusive before the discovery of Theorem 3.13. It can, indeed, be explained by the elementary complex analysis. We think of $\mathbb{P}^1$ as the extended plane. As earlier, using $\varphi(t)$, we can separate $\underline{x}_J$ to distinct points $\underline{x}_J(t)$ $(t \neq 0)$

around $x_j$ ($j \in J$). For simplicity, we may assume that $x_j = 0$. Now, we want to apply a one-parameter curve $g(t)$ in PGL(2) such that in the limit, $g(t) \cdot \underline{x}_J(t)$ get separated. Recall that linear transformations are made of translations, rotations, homotheties, and inversions. Among the four kinds, only inversion will do the work. Hence, we may assume that $g(t)$ are inversions. Take a small neighborhood $D$ of $x_j = 0$, inversions $g(t)$ will amplify $D$ to a large neighborhood at infinity, and simultaneously shrink the complement of $D$ into a small neighborhood around 0. Hence, after taking limit, the neighborhood $D$ expands to the whole extended plane, and in the mean time, the complement of $D$ collapses to the single point 0 — and this explains why the points $x_{J^c}$ collide into a single point in the end.

**Example 3.21.** As a concrete example, take a set of points of $(\mathbb{P}^1)^m$ represented by a $2 \times m$ matrix

$$\begin{pmatrix} a & a & \cdots & a & a_{j+1} & \cdots & a_m \\ b & b & \cdots & b & b_{j+1} & \cdots & b_m \end{pmatrix} \in (\mathbb{P}^1)^m$$

with $b \neq 0$ such that the first $j$ points coincide, and the rest

$$\begin{pmatrix} a_{j+1} & \cdots & a_m \\ b_{j+1} & \cdots & b_m \end{pmatrix}$$

is sufficiently general. Perturb

$$\begin{pmatrix} a & a & \cdots & a & a_{j+1} & \cdots & a_m \\ b & b & \cdots & b & b_{j+1} & \cdots & b_m \end{pmatrix}$$

to a general position $\varphi(t)$ as

$$\begin{pmatrix} e^t a & e^{2t} a & \cdots & e^{jt} a & a_{j+1} & \cdots & a_m \\ b & b & \cdots & b & b_{j+1} & \cdots & b_m \end{pmatrix}.$$

Let $g(t)$ be given by

$$\begin{pmatrix} \frac{1}{t} & -\frac{a}{b}\frac{1}{t} \\ 0 & 1 \end{pmatrix}.$$

Then, $g(t) \cdot \varphi(t)$ is

$$\begin{pmatrix} \frac{e^t-1}{t} a & \frac{e^{2t}-1}{t} a & \cdots & \frac{e^{jt}-1}{t} a & \frac{1}{t}(a_{j+1} - \frac{a}{b} b_{j+1}) & \cdots & \frac{1}{t}(a_m - \frac{a}{b} b_m) \\ b & b & \cdots & b & b_{j+1} & \cdots & b_m \end{pmatrix}.$$

Let $t$ go to 0, we obtain a new set of points

$$\begin{pmatrix} a & 2a & \cdots & ja & 1 & \cdots & 1 \\ b & b & \cdots & b & 0 & \cdots & 0 \end{pmatrix}$$

where, as predicted in the theorem, the first $j$ points get separated, and the rest collide at a single point.

**Example 3.22.** We can also think of

$$\begin{pmatrix} a & a & \cdots & a & a_{j+1} & \cdots & a_m \\ b & b & \cdots & b & b_{j+1} & \cdots & b_m \end{pmatrix} \in \mathrm{Gr}(2,m)$$

as a point in the Grassmannian $\mathrm{Gr}(2,m)$. We can have the same perturbation

$$\varphi(t) = \begin{pmatrix} e^t a & e^{2t} a & \cdots & e^{jt} a & a_{j+1} & \cdots & a_m \\ b & b & \cdots & b & b_{j+1} & \cdots & b_m \end{pmatrix}.$$

But $\varphi(t)$, as a point in $\mathrm{Gr}(2,m)$, is the same as $g(t)\cdot\varphi(t)$, which, in terms of matrix, is

$$\begin{pmatrix} \frac{e^t-1}{t} a & \frac{e^{2t}-1}{t} a & \cdots & \frac{e^{jt}-1}{t} a & \frac{1}{t}(a_{j+1} - \frac{a}{b} b_{j+1}) & \cdots & \frac{1}{t}(a_m - \frac{a}{b} b_m) \\ b & b & \cdots & b & b_{j+1} & \cdots & b_m \end{pmatrix}.$$

Now, let $h(t) \in (\mathbb{C}^*)^m$ be given by

$$(1, \ldots, 1, t\frac{1}{a_{j+1} - \frac{a}{b} b_{j+1}}, \ldots, t\frac{1}{a_m - \frac{a}{b} b_m}).$$

Then, $h(t)\cdot\varphi(t)$ becomes

$$\begin{pmatrix} \frac{e^t-1}{t} a & \frac{e^{2t}-1}{t} a & \cdots & \frac{e^{jt}-1}{t} a & 1 & \cdots & 1 \\ b & b & \cdots & b & t\frac{1}{a_{j+1} - \frac{a}{b} b_{j+1}} b_{j+1} & \cdots & t\frac{1}{a_m - \frac{a}{b} b_m} b_m \end{pmatrix}.$$

Let $t$ go to zero, we again obtain

$$\begin{pmatrix} a & 2a & \cdots & ja & 1 & \cdots & 1 \\ b & b & \cdots & b & 0 & \cdots & 0 \end{pmatrix}.$$

As one can see, although equivalent, it is sometimes easier to describe the properties of a matrix as a configuration of points on $\mathbb{P}^1$ than as a 2-plane in $\mathbb{C}^m$.

**3.8. A geometric interpretation of Chow quotients of higher Grassmannians.** As far as moduli spaces of point configurations on $\mathbb{P}^n$ are concerned, the case of $n = 1$ is very special. Here, by higher Grassmannians, we mean $\mathrm{Gr}(n,m)$ with $n > 2$, $m > 5$. Again, the Chow quotients of higher Grassmannians correspond to Chow quotients of $(\mathbb{P}^n)^m$ with $n > 1, m > 5$.

Take a point configuration, $\underline{x} = (x_1, \ldots, x_m)$, in $\mathbb{P}^n$, such that its automorphism group is trivial. By Theorem 3.19, up to projective transformations, there are only finitely many points, including $\underline{x}$ itself,

$$\underline{x}_1, \ldots, \underline{x}_l,$$

which can be obtained from $\underline{x}$ by perturbing, translating and specializing.

**Definition 3.23.** We will call $\underline{x}_1, \ldots, \underline{x}_l$, or the union of their $G$-orbits, a stable configuration of $m$-points in $\mathbb{P}^n$.

Then, by Theorem 3.19, we have

**Theorem 3.24.** *The Chow quotient of* $(\mathbb{P}^n)^m$ *by the group* $\mathrm{PGL}(n+1, \mathbb{C})$ *parameterizes stable configurations of* $m$*-points in* $\mathbb{P}^n$.

**Remark 3.25.** Note that from the original definition, the Chow quotient is defined by taking the closure of $\iota(U/G)$. Taking closure usually does not provide further information on the boundary points. Definition 3.23 and Theorem 3.24, relying on the computable *perturbing–translating–specializing* formulation, fill the gap to a certain degree.

Ideally, it would be desirable to specify how the orbit closures, $\overline{G \cdot \underline{x}_i}$, are glued together like the case of $n = 1$ (cf. Theorem 3.20 and the proof). Despite the intimidating combinatorial complexity, we hope that the ideas presented here will lead to, in a forthcoming paper, a much better understanding of the Chow quotients of higher Grassmannians (After posting this paper on ArXiv, we saw Paul Hacking's paper [**8**] where he, geometrically interprets the Chow quotient as the moduli space of stable log pairs. Later, Keel and Tevelev posted [**17**] which contains some results similar to Hacking's.).

## 4. Chow Quotient: symplectic approach

Throughout the rest of the paper, we suppose that Assumption 3.4 holds.

We begin with discussions on two questions (Sections 4.1, 4.2) that motivate the topic of this section.

**4.1. Symplectic reductions for the Chow quotient?** We have known that a GIT quotient can be identified with various symplectic reductions. Put it differently, a GIT quotient carries many (stratified) symplectic structures. The connection is established by the theory of moment map. In the same vein, we may ask: *what are "symplectic reductions" for the Chow quotient?* This is a natural question. It admits inspiring approximations in the following interesting cases.

Consider the diagonal action of $\mathrm{PGL}(2)$ on $(\mathbf{P}^1)^{n+3}$. Kapranov proved that the Chow quotient of this action is $\overline{M}_{0,n}$, the moduli space of stable $n$-pointed rational curves. In [**10**], we give a family of "symplectic" constructions of $\overline{M}_{0,n}$ using *stable polygons with prescribed side lengths*. To say it differently, moduli spaces of stable polygons are in some sense "symplectic reductions" for the Chow quotient $\overline{M}_{0,n}$.

This case is rather special, in that (stable) polygons play indispensable roles. But, for the general case, it does inspire us to introduce the following new notion, *stable orbits with fixed momentum charges*, to take the role of stable polygons with fixed side lengths. To further motivate the precise description of these new objects, we next explore intuitively what they should mean geometrically.

**Remark 4.1.** In what follows, the word "orbit" will always refer to "$K$-orbit". When another group is involved, we will specify the group, e.g., $G$-orbits.

**4.2. Geometrically meaningful compactification.** Take a compact form $K$ of $G$. Let $\mathfrak{k}$ be the Lie algebra of $K$. As mentioned earlier, we will focus on torus actions only. Let

$$\Phi : X \to \mathfrak{k}^*$$

be a moment map for the $K$-action on $X$. Pick any point $\mathbf{r} \in \Phi(X)$. Orbits in $\Phi^{-1}(\mathbf{r})$ will be said to have the momentum charge $\mathbf{r}$. To keep with the theme of the rest of the paper, we will, from now and on, frequently call symplectic reduction,

$$X_{\mathbf{r}} = \Phi^{-1}(\mathbf{r})/K,$$

the moduli space of orbits with momentum charge $\mathbf{r}$. Let $X_{\mathbf{r}}^{\circ} \subset X_{\mathbf{r}}$ be the moduli space of *generic orbits* with momentum charge $\mathbf{r}$. Here, an orbit $O = K \cdot x$ is said to be generic if $\Phi(\overline{G \cdot x})$ equals the whole polytope $\Phi(X)$. For example, orbits through the points in the open subset $U$ of Definition 3.1 are generic. Thus, $X_{\mathbf{r}}^{\circ}$ contains an open subset that is homeomorphic to $U/G$, and is itself an open variety in $X_{\mathbf{r}}$. From the definition, the orbits in the complement $X_{\mathbf{r}} \setminus X_{\mathbf{r}}^{\circ}$, measured by moment map image, are of smaller size than those of generic ones. In the spirit of geometric moduli problem, it is natural to ask for compactifications of $X_{\mathbf{r}}^{\circ}$ with the following two desirable characteristics:

(1) the added boundary points should have natural geometric meanings and;
(2) the limiting geometric objects should be of the same size as the generic ones.

To this end, we have proposed to add "stable orbits" as boundary points. So, what are ***stable K-orbits***? First, $K$-orbits through generic points in $\Phi^{-1}(\mathbf{r})$ are automatically considered to be *stable*. When a family of generic orbits degenerate to a special orbit in $X_{\mathbf{r}}$, we can imagine it as some kind of collision occurs, resulting orbits of smaller sizes. In the case of spatial polygons, this means some edges become positively parallel (pointing to the same direction). To get stable polygons, we introduce "bubble" polygons *with certain fixed side lengths* ([**10**]). In our current situation, what is needed is to introduce "bubble" orbits *with* *certain fixed momentum charges*. To know what momentum charges to work with, some choices are to be made, just like in the case of stable polygons, where we have to make choices of side lengths. The detail is to be explicitly spelled out in the subsequent section.

**Remark 4.2.** The moduli space $\mathfrak{M}_\gamma$ of stable $K$-orbits to be constructed below answers the question of Section 4.2 quite successfully. It is the interpretation of $\mathfrak{M}_\gamma$ as a "symplectic reduction" for the Chow quotient still unsatisfactory, although it is obviously related. But, to find a "symplectic reduction" for the Chow quotient is one of the motivations that get this project started.

**4.3. Momentum charges.** Recall that we have the Chow family

$$F \subset X \times (X//^{ch}G)$$

with the following diagram

$$\begin{array}{ccc} F & \xrightarrow{\text{ev}} & X \\ f\big\downarrow & & \\ X//^{ch}G & & \end{array}$$

where ev and $f$ are the first and second projection, respectively.

For each $q \in X//^{ch}G$, the support of the Chow fiber $F(q)$ is a union, $\cup_i \overline{G \cdot x_i}$, of orbit closures with $\dim G \cdot x_i = \dim G$ for all $i$. The moment map image of each orbit closure in $F(q)$ is a subpolytope of $\Phi(X)$, and by the virtue of Proposition 3.3, they all together form a subdivision of $\Phi(F(q)) = \Phi_L(X) = P_L$.

**Definition 4.3.** A coherent subdivision of $P_L$ is the collection of top dimensional subpolytopes $\Phi_L(\overline{G \cdot x_i})$ where $\cup_i \overline{G \cdot x_i} = |F(q)|$ for some $q \in X//^{ch}G$.

There are only finitely many such polytopal subdivisions. We will use letter $S$ to denote such a subdivision. We point out here that in this paper we will always consider the subdivision $S$ as the *collection of the top dimensional subpolytopes that occur in the subdivision.* Also, in this paper, only coherent subdivisions will be considered, so we will drop the word "coherent".

**Definition 4.4.** The set of all subdivisions of the form,

$$\Phi(F(q)), q \in X//^{ch}G,$$

will be denoted by $\mathcal{S}$. There is a partial order on the set $\mathcal{S}$. For any two elements $S, S' \in \mathcal{S}$, we say that $S < S'$ if $S$ is refined by $S'$. Under this partial order, the poset $\mathcal{S}$ has a unique minimal element, namely the (non-subdivided) polytope $P = \Phi(X)$.

Fix a general point $\mathbf{r}$ in $\Phi(X)$. For every polytopal subdivision $S \in \mathcal{S}$, we choose a set of points,

$$\{\mathbf{r}_D \in \text{ the interior of } D | D \in S\}.$$

(Recall here that $D$ is of top dimension.) In other words, we have an injective function,

$$\gamma_S : S \to \Phi(X),$$

from the set of subpolytopes of $S$ to $\Phi(X)$, by sending a polytope $D \in S$ to a point $\mathbf{r}_D$ in the interior $D^\circ$ of $D$. Let $\gamma$ denote the collection of all the above choices $\{\gamma_S : S \in \mathcal{S}\}$.

**Definition 4.5.** $\gamma$ is called an admissible set of momentum charges with the principal charge $\mathbf{r}$ if the following conditions are satisfied.

(1) (principal main charge) $\gamma_P(P) = \mathbf{r}$;
(2) (local main charge) Let a subdivision $S \in \mathcal{S}$ refine another subdivision $S' \in \mathcal{S}$. Let $D \in S$ be contained in $D' \in S'$, and $D$ contains $\gamma_{S'}(D')$. Then, $\gamma_S(D) = \gamma_{S'}(D')$. In particular, for any subdivision $S$, we must have $\gamma_S(D_\mathbf{r}) = \mathbf{r}$, where $D_\mathbf{r}$ is the unique subpolytope in $S$ that contains the original charge $\mathbf{r}$;
(3) (compatibility) For any two subdivisions $S$ and $S'$, if $D$ appears in both $S$ and $S'$, then

$$\gamma_S(D) = \gamma_{S'}(D).$$

**Remark 4.6.** Notice that (2) implies that if the subdivision $S \in \mathcal{S}$ is refined by another subdivision $S' \in \mathcal{S}$, then $\text{Image}(\gamma_S) \subset \text{Image}(\gamma_{S'})$. Note also that in the sense of (2), every given polytope $D$ admits a (local) main charge $\gamma_S(D)$, and in particular, $\mathbf{r} = \gamma_P(P)$ is the global main charge.

In this paper, only admissible set of momentum charges will be considered. So, we will drop the word “admissible”. It worths to point out that $\gamma$ is analogous to the choices $(\mathbf{r}, \{\epsilon\})$ of side lengths in the case of stable polygons ([**10**]).

**Remark 4.7.** ***How do we choose momentum charges?*** In practice, momentum charges in $\gamma$ may be chosen as follows by hierarchy. First, list $P$ at the top. Then, we choose a general interior point $\mathbf{r}$, which serves as the principal charge. On the next level, we list all coherent subdivisions that only refine $P$, but no others. For any polytope $D$ occurring in the subdivisions of this level, if $D$ contained $\mathbf{r}$, we have to stick with it and set $\gamma_S(D) = \mathbf{r}$. Otherwise, we may choose freely in the interior of $D$ as long as the compatibility condition (3) is satisfied. Then, we move on to the third level and list all subdivisions that only refine the subdivisions from the previous level. Again, for any polytope $D$ in this level, if it contains a charge from the previous level, we stick with it. Otherwise, we choose freely in the interior of $D$ as long as Condition (3) is satisfied. We can go on with this process until all subdivisions are addressed.

### 4.4. Stable orbits with prescribed momentum charges.

**Definition 4.8.** Fixed a set $\gamma$ of momentum charges. A finite collection of $K$-orbits, $\mathbf{O} = \{O_i\}_i$, is called a stable orbit with momentum charges $\gamma$ if

(1) there is a point $q \in X//^{ch}G$ such that $\cup_i \overline{G \cdot O_i}$ equals to the support $|F(q)|$;
(2) for each polytope $D$ in the subdivision $S = \Phi(|F(q))|$, there is a unique orbit $O_i$ in $\Phi^{-1}(\gamma_S(D))$. (We will often denote this orbit by $O_D$.)

In this case, we will say that the stable orbit $\mathbf{O}$ is of type $S = \Phi(|F(q)|)$.

Observe that each set $G \cdot O_i$ is a single $G$-orbit. Since $D$ is of top dimensional, and $\gamma_S(D)$ is in the interior of $D$, it follows from (2) that $\cup_i G \cdot O_i = \cup_i G \cdot x_i$ where $|F(q)| = \cup_i \overline{G \cdot x_i}$ for some $x_i \in X$. In fact, if we need, we can even pick $x_i \in O_i$. For a depiction of stable $K$-orbits, consult Figure 3.

**Remark 4.9.** From the definition, for any stable orbit $\mathbf{O}$, there must be an orbit $O_i$ with the principal momentum charge $\mathbf{r}$. We will denote it by $O_{\mathbf{r}}$, and name it as the principal or main orbit. All other orbits will be referred as *bubble* orbits of $O_{\mathbf{r}}$. Moreover, for every $D$ and a subdivision $D = \cup_i D_i$, the orbits $O_{D_i}$ (if any) will be called *bubbles* of $O_D$ (if any).

**Remark 4.10.** The definition of a particular stable orbit $\mathbf{O}$ with momentum charges $\gamma$ utilizes (or depends on) only the values of $\gamma$ on a single subdivision $\Phi(f^{-1}(q))$. It appears that, we may as well define a stable orbit using only the values of $\gamma$ on the subpolytopes of $\Phi(f^{-1}(q))$, without referring to the whole $\gamma$. However, to form a meaningful global moduli space, various stable orbits must have compatible momentum charges. Hence, we insist to associate $\mathbf{O}$ with the whole $\gamma$, even if it only depends on the values of $\gamma$ on just one particular subdivision.

**Remark 4.11.** In the definition of stable orbits, we may allow some orbits to occur with multiplicities in the same way as orbit closures may occur in the Chow family. This would be useful if one wishes to construct and utilizes universal families. Since our approach and application are topological, the multiplicity issue will be suppressed in this paper.

**4.5. Local moduli and correspondence varieties.** Let $P$ be the minimal element in $\mathcal{S}$, and $U_P$ the set of all stable orbits of type $P$. This is the local moduli space associated to (the non-subdivision) $P$.

**Definition 4.12.** In general, given any polytopal subdivision $S \in \mathcal{S}$, let $Z_S$ be the set of all stable orbits of type $S$.

This is a subset in

$$\Pi_{D \in S} X_{\gamma_S(D)}.$$

We now describe a neighborhood of $Z_S$, which is to be an incident variety in a product space.

Let $\widetilde{S} = \bigcup_{S' \leq S} S'$. We will think $\widetilde{S}$ as a collection of subpolytopes. Consider the product space,

$$\Pi_{C \in \widetilde{S}} X_{\gamma_{S'}(C)},$$

where $S'$ is any member of $\widetilde{S}$ that contains $C$. Note that the expression does not depend on the choice of $S'$, for, if $S''$ is another one that contains $C$, then by the compatibility of the set of momentum charges, $\gamma_{S'}(C) = \gamma_{S''}(C)$. We define an analytic correspondence variety

$$U_S \subset \Pi_{C \in \widetilde{S}} X_{\gamma_{S'}(C)}$$

as follows.

**Definition 4.13.** A point $\mathbf{O} = \{O_C\}$ of $\Pi_{C \in \widetilde{S}} X_{\gamma_{S'}(C)}$ belongs to $U_S$ if both of the following are true:

(1) there is a unique $S' \leq S$ such that the components

$$\mathbf{O}_{S'} = \{O_C | C \in S'\}$$

form a stable orbit of type $S'$;

(2) the rest of the components are uniquely determined by

$$\mathbf{O}_{S'} = \{O_C | C \in S'\}$$

by the means as specified below. For any $D \in S$ (note that $D$ is of top dimension, cf. Definition 4.3), $D$ is contained in a unique $C \in S'$ since $S$ refines $S'$. In this case, we set

$$O_D = (G \cdot O_C) \cap \Phi^{-1}(\gamma_S(D)).$$

For any other polytope $C'' \in S'' \subset \widetilde{S} \setminus (S \cup S')$, since $S''$ is refined by $S$, there must be a polytope $D$ of $S$ such that $D \subset C''$ and $\gamma_{S''}(C'') \in D$. Hence, by Definition 4.5 (2), $\gamma_{S''}(C'') = \gamma_S(D)$. Then, in this case, we simply require $O_{C''}$ to equal to $O_D$.

Observe that the relation used in this definition is analytic.

Recall that $Z_S$ be the set of all stable orbits of type $S$. From the above, we see that for any $S' \leq S$, there is an injective map

$$Z_{S'} \hookrightarrow U_S$$

because the components in $Z_{S'}$ completely determine the rest in $U_S$ as in Definition 4.13 (2). After identifying $Z_{S'}$ with its image in $U_S$, we see that

$$U_S = \bigcup_{S' \leq S} Z_{S'} \subset \Pi_{C \in \widetilde{S}} X_{\gamma_{S'}(C)}$$

From here, we immediately have

$$U_{S'} \subset U_S$$

whenever $S' < S$. Now, from Definition 4.13, we see that the incident relation is analytic and the inclusion $U_{S'} \hookrightarrow U_S$ is an analytic open embedding. That is, we have

**Proposition 4.14.** *$U_S$ is an analytic subset of $\Pi_{C \in \widetilde{S}} X_{\gamma_{S'}(C)}$ (not closed in general). Furthermore, $U_{S'}$ is an open analytic subvariety of $U_S$ whenever $S' < S$.*

**4.6. Global moduli of stable orbits.** Now, let $\mathfrak{M}_\gamma$ be the set of all stale orbits of type $\gamma$. Then, $\mathfrak{M}_\gamma = \cup_S U_S$. It follows from the construction that the complex structures on $U_S$ all agree with each other on the overlaps, and it induces a Hausdorff topology on $\mathfrak{M}_\gamma$. That is, $\mathfrak{M}_\gamma$ is the inverse limit $\lim_S U_S$ of the system $\{U_S, U_{S'} \hookrightarrow U_S | S' < S\}$. Furthermore, we have

**Theorem 4.15.** *The moduli space $\mathfrak{M}_\gamma$ exists as a separated complex variety, and is homeomorphic to the Chow quotient $X//^{ch}G$.*

*Proof.* We only need to prove the second statement.

Locally on $U_S$, we define a map

$$\alpha_S : U_S \to X//^{ch}G$$

as follows. For any point $\mathbf{O} = \{O_C\} \in U_S \subset \Pi_{C \in \widetilde{S}} X_{\gamma_{S'}(C)}$, there is a unique $S' \leq S$ such that $\mathbf{O}_{S'} = \{O_C | C \in S'\}$ is a stable orbit of type $S'$, and the rest components are uniquely determined by $\mathbf{O}_{S'}$. Let $q \in X//^{ch}G$ be the point in the Chow quotient such that $\mathbf{O}_{S'} \subset |F(q)|$. By Definition 4.8 and Assumption 3.4, $q$ is unique. Then, we can define

$$\alpha_S(\mathbf{O}) = q.$$

All those locally defined maps apparently agree with each other on the overlaps, thus they glue together to give a globally defined map

$$\alpha : \mathfrak{M}_\gamma \to X//^{ch}G.$$

This map has the inverse

$$\beta : X//^{ch}G \to \mathfrak{M}_\gamma$$

by sending a point $q \in X//^{ch}G$ to the stable orbit

$$\mathbf{O} = \{O_D | D \in S = \Phi(f^{-1}(q))\}$$

where $O_D = F(q) \cap \Phi^{-1}(\gamma_S(D))$. (One verifies from the definition that $\mathbf{O}$ is indeed a stable orbit of type $S$.)

Note that the Chow quotient, $X//^{ch}G$, can be stratified according to the subdivision type of $\Phi(f^{-1}(q))$. That is,

$$X//^{ch}G = \bigcup_{S \in \mathcal{S}} Y_S,$$

where $Y_S = \{q \in X//^{ch}G | \Phi(f^{-1}(q)) = S\}$. Then

$$V_S = \bigcup_{S' \leq S} Y_{S'}$$

is an open neighborhood of $Y_S$. The restriction of $\beta$ to $V_S$ has the image in $U_S$. And, the map

$$\beta|_{V_S} : V_S \to U_S \subset \Pi_{C \in \widetilde{S}} X_{\gamma_{S'}(C)}$$

is defined component-wise by the projections

$$X//^{ch}G \to X_{\gamma_{S'}(C)},$$

for all $S \in \mathcal{S}$. Hence $\beta$ is analytic, in particularly, continuous. Since $\beta$ is a continuous bijection between two compact Hausdorff spaces, it must be homeomorphism. So is the inverse map $\alpha$. q.e.d.

For a stable orbit $\mathbf{O} = \cup_i O_i$, we have realized an open neighborhood $U_{\mathbf{O}}$ around it with an incident variety in the product space $\prod_{D \in S} X_{\gamma_S(D)}$ so that it admits an induced symplectic form on the smooth locus of the neighborhood. To put these pieces together to get a global symplectic or Poisson structure is a task worth pursuing. For, if we compare with the role of $\mathbf{r}$ in the symplectic quotient $\Phi^{-1}(\mathbf{r})/K$, it vividly suggests that, just like what $\mathbf{r}$ does for $\Phi^{-1}(\mathbf{r})/K$, $\gamma$ should lead toward symplectic/Poisson structures on the Chow quotient. This would make $\mathfrak{M}_\gamma$ a *genuine* symplectic reduction for the Chow quotient, adding a new correspondence to the usual "GIT=Reduction" picture. This calls for further investigation.

### 4.7. Blowup along arrangement of subvarieties.

**Theorem 4.16.** *Let the notation be as before. Then, there is a holomorphic projection*

$$\mathfrak{M}_\gamma \to X_{\mathbf{r}}, \ \mathbf{O} \to O_{\mathbf{r}}$$

*defined by sending a stable orbit* $\mathbf{O}$ *to its principal orbit* $O_{\mathbf{r}}$. *This map restricts to an isomorphism on the open subset* $X_{\mathbf{r}}^\circ$.

*Proof.* This follows immediately from the construction of $\mathfrak{M}_\gamma$. q.e.d.

Obviously, under the enlarged KN correspondence, this map corresponds to the algebraic map

$$X//^{ch}G \to X_{[\mathbf{r}]}.$$

Every symplectic reduction $X_{\mathbf{r}}$ has a decomposition

$$X_{\mathbf{r}} = \bigcup_D M_{\mathbf{r},D},$$

where $D$ is a subpolytope of $P$, and an orbit $O \in X_{\mathbf{r}}$ belongs to $M_{\mathbf{r},D}$ if $\Phi(\overline{G \cdot O}) = D$. We point out that $M_{\mathbf{r},D} = \emptyset$ unless $D$ contains $\mathbf{r}$. Hence, we may write

$$X_{\mathbf{r}} = \bigcup_{\mathbf{r} \in D} M_{\mathbf{r},D}.$$

When $D$ is the whole polytope $P$, $M_{\mathbf{r},P}$ is the open subset of generic points in $X_{\mathbf{r}}$. Set

$$N_{\mathbf{r},D} = \bigcup_{C \subset D} M_{\mathbf{r},C}.$$

This is closed in $X_{\mathbf{r}}$. The complement of $M_{\mathbf{r},P}$ is a union of closed subvarieties,

$$\bigcup_{D \neq P} N_{\mathbf{r},D}.$$

If the group action is quasi-free (An action is quasi-free if all the isotropy subgroups are connected. Using the orbifold/stack language, this assumption may be removed.) and for some general $\mathbf{r}$,

$$\{N_{\mathbf{r},D} | D \neq P\}$$

form an arrangement of smooth subvarieties (see Definition 1.2 of [11]), then, we expect that the projection map

$$\mathfrak{M}_\gamma \to X_{\mathbf{r}}$$

is a *blowup along the arrangement of smooth subvarieties*, in the sense of Theorem 1.1 of [11]. For example, this is the case for the maximal torus action on the Grassmannian $\mathrm{Gr}(2, \mathbf{C}^n)$ (Theorem 6.5 [10]).

**Remark 4.17.** The above seems to provide a (rare) criterion for the smoothness of the Chow quotient, that is, assuming quasi-free, it is smooth if $\{M_{\mathbf{r},D}|D\}$ is an arrangement of smooth subvarieties of $X_{\mathbf{r}}$ for some general $\mathbf{r}$. This line of approach may be applied to the Chow quotients of higher Grassmannians $\mathrm{Gr}(n, \mathbf{C}^m)$ $(n > 2, m > 5)$, but the combinatorics involved seems too intimidating at the moment.

## 5. Chow Quotient: topological approach

There is yet another topological approach to the Chow quotient, which is somewhat "orthogonal" to the approach of stable $K$-orbits.

**5.1. Action-manifolds.** Instead of $K$-orbits, we can also consider the following infinitesimal action. For any $\xi \in \mathfrak{k}$, set

$$\sqrt{-1}\xi_{X,x} := \frac{d}{dt}_{|t=0} \exp\left(t\sqrt{-1}\xi\right) \cdot x.$$

Treating $\sqrt{-1}\mathfrak{k}$ as a distribution of vector fields on $X$, we obtain its integral manifolds through points of $X$. In this case, the integral manifolds are not closed. Hence, we take the closures of these integral manifolds. We will see that the closures are homeomorphic, via the moment map, to subpolytopes of $\Phi(X)$, and hence are manifolds with corners, in general. Thus, we may call the above integral manifolds *open action-manifolds* and their closures *action-manifolds* (with corners), because they come from the group action.

Let $G = K \cdot A$ be the polar decomposition. Then, it can be verified that

**Proposition 5.1** ([1], [7]). *The open action–manifold through a point $x$ is $A \cdot x$. The action-manifold through the point $x$ is $\overline{A \cdot x}$. Moreover, the moment map $\Phi$ induces a homeomorphism between $\overline{A \cdot x}$ and $\Phi(\overline{A \cdot x})$.*

For action-manifolds through *generic points* (e.g., points of the open subset $U$ in Definition 3.1.), we will call them generic action-manifolds. Two generic action-manifolds are equivalent if one can be obtained from the other by the action of an element of $K$. Let $\mathfrak{M}^\circ$ be the moduli space of equivalence classes of generic action-manifolds. Then, $\mathfrak{M}^\circ$ contains $U/G$ as an open subset and is itself an open variety. We would like to describe a geometrically meaningful compactification $\mathfrak{M}$ of $\mathfrak{M}^\circ$ by providing natural geometric meanings of the boundary points. These boundary points will be called stable action-manifolds. Generic action-manifolds, as generic points of $\mathfrak{M}$, are automatically stable. So, what are the rest stable action-manifolds? To answer this question, we need some preparation.

**5.2. Perturb, translate and specialize: topological version.** Let $x$ and $y$ be two points with

$$\dim A \cdot x = \dim A \cdot y = \dim A.$$

Let $\mathbf{r} = \Phi(x)$. Take a real analytic slice (topological slice will suffice. cf. Remark 5.3 and also Remark 3.15.),

$$R_x \subset \Phi^{-1}(\mathbf{r}),$$

around the point $x$, transversal to $K$-orbits.

Recall that a point of $X$ is said to be isotropy-free if its isotropy subgroup is the identity.

**Definition 5.2.** We say $x$ can be perturbed (to general positions), translated (along $A$-orbits), and specialized to $y$, which is denoted by

$$x \rightarrow_A y,$$

if there is a generic real holomorphic map from the interval $I = [-1, 1]$ to the slice $R_x$,

$$\varphi : I \rightarrow R_x,$$

with $\varphi(0) = x$ such that $\varphi(t)$ is a generic isotropy-free point for all $t \neq 0$, and in addition, there is a real holomorphic map from the punctured interval $I^* = I \setminus \{0\}$ to the group $A$

$$a : I^* \rightarrow A$$

such that $\psi(t) = a(t) \cdot \varphi(t) \in \Phi^{-1}(\Phi(y))$, and

$$y = \lim_{t \rightarrow 0} a(t) \cdot \varphi(t).$$

Just as the original P.T.S, the above relation is symmetric. To prove this, we can choose a real analytic slice $R_y$ containing $\psi(t) = a(t) \cdot \varphi(t)$, and then repeat the arguments of Remark 3.18. That is, if $x \rightarrow_A y$, then $y \rightarrow_A x$. Because of this, we may write $x \sim_A y$.

**Remark 5.3.** In the above definition, we can replace "real holomorphic map" by "continuous map", all the statements and proofs, which are all topological in nature, remain unchanged.

In Definition 5.2, after the choice of the map $\varphi$ is made, $a(t)$ is uniquely determined.

**Lemma 5.4.** *For $t \neq 0$, $a(t)$ is the unique point $a$ in $A$ such that $\Phi(a \cdot \varphi(t)) = \Phi(y)$.*

*Proof.* For any $t \neq 0$, since $\varphi(t)$ is isotropy-free and generic, we have a homeomorphism

$$A \xrightarrow{\cong} A \cdot \varphi(t) \xrightarrow{\Phi} P^\circ$$

where $P^\circ$ is the interior of $P$. Hence, $a(t)$ is the unique point $a$ in $A$ such that $\Phi(a \cdot \varphi(t)) = \Phi(y) \in P^\circ$. q.e.d.

### 5.3. Stable action-manifolds.

**Definition 5.5.** A finite union of action-manifolds, $\cup_i \overline{A \cdot x_i}$, is called a stable action-manifold if

(1) $\dim A \cdot x_i = \dim A$ for all $i$;
(2) $x_i \sim_A x_j$ for all $i$ and $j$;
(3) the moment map $\Phi$ induces a homeomorphism between $\cup_i \overline{A \cdot x_i}$ and $\Phi(X)$.

Note that the condition (3) implies that $\Phi(\cup_i \overline{A \cdot x_i}) = \Phi(X)$. It also implies that $\cup_i \overline{A \cdot x_i}$ is connected subset of $X$. In particular, the indexes can be re-arranged so that $\overline{A \cdot x_i} \cap \overline{A \cdot x_{i+1}} \neq \emptyset$. (Consult Figure 4 for an illustration.)

**Proposition 5.6.** *If $x \sim_A y$, then $x$ and $y$ are in the same Chow fiber. In particular, every given stable action manifold is contained in a single Chow fiber.*

*Proof.* After modulo the action of $K$, we can treat orbit $A \cdot x$ as point $[G \cdot x]$ in GIT quotient. Note also that the proof of Theorem 3.13 only use the continuity of the maps involved, but not the holomorphic properties (cf. Remark 3.15). Henceforth, the proofs of Theorem 3.13 (the sufficient part) can be repeated almost word by word to conclude the statement of this proposition. Further details are omitted. q.e.d.

Thus, if a stable action-manifold, $\mathbf{M} = \cup_i \overline{A \cdot x_i}$, is contained in a Chow fiber $f^{-1}(q)$, then $S = \cup_i \Phi(\overline{A \cdot x_i}) = \Phi(f^{-1}(q))$ is a subdivision of $P = \Phi(X)$. In this case, we will say that $\mathbf{M}$ corresponds to the subdivision $S$ or is of type $S$.

**Lemma 5.7.** *Suppose that we have $x \sim_A x'$ and $y \sim_A y'$. Assume further that $y = k_1 \cdot x$ and $y' = k_2 \cdot x'$ for some $k_1, k_2 \in K$. Then, $k_1 = k_2$.*

*Proof.* Assume that we have

$$\lim_{t \to 0} a(t) \cdot \varphi(t) = x'$$

with $\varphi(0) = x$. Then, $\psi(t) = k_1 \cdot \varphi(t)$ defines a generic real holomorphic map from the interval $I$ to a slice $R_y$ near $y$ with $\psi(0) = k_1 \cdot x = y$. Let $b(t)$ be as in Definition 5.2 such that

$$\lim_{t \to 0} b(t) \cdot \psi(t) = y'.$$

Then, because $\psi(t) = k_1 \cdot \varphi(t)$ and $y' = k_2 \cdot x'$, we obtain

$$\lim_{t \to 0} b(t) k_2^{-1} k_1 \cdot \varphi(t) = x'.$$

By Lemma 5.4, for generic $t$, we must have $b(t) k_2^{-1} k_1 = a(t)$, that is,

$$b(t) a(t)^{-1} = k_2 k_1^{-1}.$$

Since $A \cap K = \mathrm{id}$, we have $k_1 = k_2$. q.e.d.

**Definition 5.8.** Two stable action-manifolds are said to be equivalent if there is an element $k \in K$ such that the action of $k$ sends one stable action-manifold to the other.

It is immediate that two equivalent action manifolds must lie in the same Chow fiber because any Chow fiber is $G$-invariant. The following proves the converse.

**Proposition 5.9.** *If two stable action manifolds $\mathbf{M}_1$ and $\mathbf{M}_2$ are in the same Chow fiber, then they are equivalent.*

*Proof.* Assume that $\mathbf{M}_1 = \cup_i \overline{A \cdot x_i}$ and $\mathbf{M}_2 = \cup_i \overline{A \cdot y_i}$ are in the same Chow fiber. Then, we have $\cup_i \overline{G \cdot x_i} = \cup_i \overline{G \cdot y_i}$. By Definition 5.5 and Proposition 5.6, we can re-arrange so that there is a one-to-one correspondence between $\{x_i\}$ and $\{y_i\}$, and $G \cdot x_i = G \cdot y_i$. By choosing different representatives of $A \cdot x_i$ (for all $i$) if necessary, we may assume that

$$y_i = k_i \cdot x_i$$

for some $k_i \in K$ for all $i$. Now apply Lemma 5.7. q.e.d.

**Remark 5.10.** This proposition shows that a stable action-manifold is not just an arbitrary union $\cup_i \overline{A \cdot x_i}$ of $A$-orbits even if we require that $\cup_i \overline{G \cdot x_i}$ is a Chow fiber. Geometrically, stable action-manifolds occur as the limiting configurations of families of generic $\overline{A \cdot x}$.

**5.4. Moduli of stable action-manifolds.** We will use $\mathfrak{M}$ to denote the set of equivalence classes of all stable action-manifolds. (Note that the definition of $\mathfrak{M}$ involves no choices; while the moduli space of stable $K$-orbits with momentum charges $\gamma$, as it depends on $\gamma$, is always denoted by $\mathfrak{M}_\gamma$.)

**Theorem 5.11.** *The moduli space $\mathfrak{M}^\circ$ of generic stable action-manifolds admits a natural compactification $\mathfrak{M}$ by adding stable action-manifolds. The resulting space is analytic, and is homeomorphic to the Chow quotient $X//^{ch}G$.*

*Proof.* The approach to this theorem, although somewhat "orthogonal" to that of Theorem 4.15, is in spirit related and similar to it. We will only give a sketch.

Let $G = A \cdot K$ be the polar decomposition. First, recall that every piece $M$ in a stable action-manifold $\mathbf{M}$ is of the form $\overline{A \cdot x}$. Since $K$ is compact, there is one-to-one correspondence between (the closures of) $G$-orbits and (the closures of) $A$-orbits modulo $K$, and hence, we may identify the two kinds of orbits in GIT quotient $X^{ss}//G$. In other words, we may write $[A \cdot x]$ for $[G \cdot x]$ in $X^{ss}//G$.

The moduli space $\mathfrak{M}$ is canonically defined (depends on no choices). However, to prove this theorem, we have to make some auxiliary choices. That is, we will fix a set $\gamma$ of momentum charges. By a small perturbation, we may require that all the charges are rational. (This technical maneuver is only needed to allow us to use GIT quotients.)

By Proposition 5.6, any stable action manifold $\mathbf{M}'$ is contained in some Chow fiber $f^{-1}(q)$. Hence, it corresponds to some subdivision $S = \Phi(f^{-1}(q))$. Using the convention mentioned in the beginning of the proof, we will embed $\mathbf{M}'$ in the product space of some GIT quotients,

$$\Pi_{C \in \widetilde{S}} X_{[\gamma_{S'}(C)]},$$

and then define an open neighborhood $W_S$ of $\mathbf{M}'$ as an incident analytic subvariety in $\Pi_{C \in \widetilde{S}} X_{[\gamma_{S'}(C)]}$. Here, $\widetilde{S} = \bigcup_{S' \leq S} S'$, and $S'$ is any member of $\widetilde{S}$ that contains $C$. The product space does not depend on the choice of $S'$ (cf. the remark in the paragraph immediately before Definition 4.13). As remarked in the beginning of the proof, we will represent a point of $X_{[\gamma_{S'}(C)]}$ by an $A$-orbit closure.

A point $\mathbf{M} = \{[M_C]\}$ of $\Pi_{C \in \widetilde{S}} X_{[\gamma_{S'}(C)]}$ belongs to $W_S$ if both of the following are true:

(1) there is a unique $S' \leq S$ such that the components

$$\mathbf{M}_{S'} = \{[M_C] | C \in S'\}$$

is a stable action-manifold corresponding to the subdivision $S'$;

(2) the rest of the components are completely determined by $\mathbf{M}_{S'}$ as follows. For any $D \in S$, $D$ is contained in a unique $C \in S'$ since $S$ refines $S'$. In particular, $C$ contains the Chamber $[\gamma_S(D)] \subset D$. In this case, we require

$$[M_D] = [M_C] \in X_{[\gamma_S(D)]}.$$

Here, using the remark in the beginning of the proof, we may treat $M_C$, originally an orbit (closure) of type $C$ (i.e., $\Phi(M_C) = C$), as an orbit in $X_{[\gamma_S(D)]}$ as well. For any other polytope $C'' \in S'' \subset \widetilde{S} \setminus (S \cup S')$, since $S''$ is refined by $S$, there must be a polytope $D$ of $S$ such that $\gamma_{S''}(C'') = \gamma_S(D)$ (cf. Definition 4.13 (2)), and in this case, we require that $[M_{C''}]$ equals $[M_D]$.

This makes $W_S$ an analytic subvariety of $\Pi_{C \in \widetilde{S}} X_{[\gamma_{S'}(C)]}$. As in the case of stable $K$-orbits, after the obvious identifications, we have that $W_{S'} \subset W_S$ whenever $S' < S$. In particular, all these complex structures agree with each other on the overlaps. Consequently, we obtain that the moduli space $\mathfrak{M}$ is a separated complex analytic variety.

Now, using basically the same argument as in the proof of Theorem 4.15, we can define a map

$$\theta : \mathfrak{M} \to X//^{ch}G$$

and its inverse

$$\theta^{-1} : X//^{ch}G \to \mathfrak{M},$$

and prove that $\mathfrak{M}$ is homeomorphic to the Chow quotient $X//^{ch}G$. Further details are omitted. q.e.d.

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