Open access • Journal Article • DOI:10.1063/1.531238

Topological BF theories in 3 and 4 dimensions — Source link

Alberto S. Cattaneo, Paolo Cotta-Ramusino, Jürg Fröhlich, Maurizio Martellini

Published on: 01 Nov 1995 - Journal of Mathematical Physics (American Institute of PhysicsAIP)

Topics: BF model and Knot theory

Year: 1995

Topological BF theories in 3 and 4 dimensions

Cattaneo, A S; Cotta-Ramusino, P; Fröhlich, J; Martellini, M

Topological BF theories in 3 and 4 dimensions

Abstract

In this paper we discuss topological BF theories in 3 and 4 dimensions. Observables are associated to ordinary knots and links (in 3 dimensions) and to 2-knots (in 4 dimensions). The vacuum expectation values of such observables give a wide range of invariants. Here we consider mainly the 3 dimensional case, where these invariants include Alexander polynomials, HOMFLY polynomials and Kontsevich integrals. © 1995 American Institute of Physics.

Topological BF theories in 3 and 4 dimensions*

Alberto S. Cattaneo
Dipartimento di Fisica, Università di Milano
and I.N.F.N., Sezione di Milano
via Celoria 16, I-20133 Milano, Italy
E-mail: *cattaneo@vaxmi.mi.infn.it*

Paolo Cotta-Ramusino
Dipartimento di Matematica, Università di Milano
and I.N.F.N., Sezione di Milano
via Saldini 50, I-20133 Milano, Italy
E-mail: *cotta@vaxmi.mi.infn.it*

Jürg Fröhlich
Institut für Theoretische Physik, E.T.H.
Hönggerberg, CH-8093 Zürich, Switzerland
E-mail: *juerg@itp.ethz.ch*

Maurizio Martellini
Dipartimento di Fisica, Università di Milano
and I.N.F.N., Sezione di Pavia
via Celoria 16, I-20133 Milano, Italy
E-mail: *martellini@vaxmi.mi.infn.it*

May 3, 1995
hep-th/9505027, IFUM 503/FT
prepared for the special issue of *Journal of Math. Phys.* on *Quantum geometry and diffeomorphism-invariant quantum field theory*
P.A.C.S. 02.40, 11.15, 04.60

Abstract

In this paper we discuss topological BF theories in 3 and 4 dimensions. Observables are associated to ordinary knots and links (in 3 dimensions) and to 2-knots (in 4 dimensions). The vacuum expectation values of such observables give a wide range of invariants. Here we consider mainly the 3 dimensional case, where these invariants include Alexander polynomials, HOMFLY polynomials and Kontsevich integrals.

*Work supported by grants from MURST and by INFN

I. Introduction

This paper deals with a special kind of topological quantum field theories, the BF theories, that is the only known topological field theories that can, in principle, be defined on a a manifold M of any dimension. The symbol BF means that the action contains a term given by the wedge-product of an $(n-2)$-form B of the adjoint type times the curvature F of a connection A. Here we have set $n = \dim M$.

Topological field theories have been systematically considered by Witten [31], but somehow appeared in the literature much before [29].

In a celebrated paper of 1989, Witten [32] showed that it is possible to recover, via topological quantum field theories, the invariants of links and knots known as Jones and HOMFLY polynomials ([22, 21, 26, 18]). The key idea was to consider a special observable ("Wilson loop") associated to knots and links and compute its vacuum expectation value (v.e.v.) with respect to the Chern–Simons theory.

Let us now specifically consider BF theories (for an introduction to such theories see [8]) and ask ourselves what kind of topological invariants can we recover.

In an n-dimensional BF theory, it is natural to look for observables associated to imbedded (or immersed) manifolds in M of codimension 2. The v.e.v. with respect to BF theory, should then give topological invariants of these imbeddings (higher dimensional knots).

Even though the more interesting part of this program is related to the 4-dimensional case, we have only systematically developed, up to now, the 3-dimensional case, namely the case of ordinary knots and links. We have some hint and some preliminary computations about the 4-dimensional case (see [16]), but most of the work has yet to be done and, in this paper, the 4-dimensional case is only briefly sketched.

In the 3-dimensional case, we realized that, surprisingly enough, BF theory can significantly improve, in comparison with Chern–Simons theory, our understanding of the relation between quantum field theory and knot invariants. Moreover, it allows us to recover, as v.e.v.'s, some knot invariants that previously have not been associated to quantum field theories (Alexander–Conway polynomials)[13].

The original approach of Witten to Chern–Simons field theory put the main emphasis on the non-perturbative treatment. Here, instead, we stress the rôle of perturbation expansions in the construction of knot invariants. The use of perturbative methods is akin to the Vassil'ev approach to knot theory. More to the point: the coefficients of the perturbative series of topological field theories are precisely knot invariants of finite type.

We have essentially two kinds of BF theories. The first kind is what is called BF theory with a cosmological constant. The action is given by the difference of two Chern–Simons actions (computed for two different connections $A + \kappa B$ and $A - \kappa B$), where A is a connection, κ is a real parameter and B is a 1-form of the adjoint type. Pure BF theory is related to the Turaev-Viro [30] invariants just as pure Chern–Simons theory is related to Reshetikhin-Turaev [28] invariants.

In the BF theory with a cosmological constant, the observable to be associated to a knot is the trace of the holonomy of the connection $A + \kappa B$ expanded in a Taylor series in the variable κB, at $\kappa = 0$.

In BF theory, the fields that are canonically conjugate are A and B, instead of A being conjugate to itself (as in Chern–Simons). Hence only contractions between the fields A and B are to be considered. This feature of the BF theory is a very good one, since by considering, at the same time, the Taylor expansion (in κB) and the vertex insertions (each of them containing a factor multiplied by κ^2), one is able to keep track of the various contributions order by order in the variable κ. This provides a much better control of the perturbation series than in Chern–Simons theory.

The BF theory with a cosmological constant leads to the same knot invariants (HOMFLY and Jones polynomials) considered in Chern–Simons theory. But different choices of gauge produce different perturbation expansions, i.e., different sequences of Vassil'ev invariants of knots, but associated to the same knot polynomial. In this respect, let us point out that suitable normalization factors are to be taken into account, before one can show that different perturbation expansions lead to the same knot polynomials.

The scheme for BF theories is roughly as follows:

- *Covariant gauge.* In this gauge the terms of the perturbative expansion are multiple linking-integrals.
- *Holomorphic gauge.* In this gauge the terms of the perturbative expansion are Kontsevich integrals.
- *Axial gauge.* In this gauge the terms of the perturbative expansion are expressed as sums of "tensors" over the set of vertices of a given projection of the knot.

The second kind of 3-dimensional BF theory that we consider is the one *without* cosmological constant. The action here is given by the *derivative* of the Chern–Simons action computed for the connection $A+\kappa B$ at $\kappa=0$. The observable is an exponential function of the derivative of the holonomy.

The perturbative expansion of the BF theory without cosmological constant produces the coefficients of the Alexander–Conway polynomial. They cannot be recovered in the framework of Chern–Simons field theory.

Finally, concerning the higher dimensional BF theories, it is very likely that invariants of 2-knots as well as invariants of 4-manifolds can be recovered in the framework of such theory. In this respect BF theories can play a rôle in the loop-variables formulation of quantum gravity.

II. Geometry of BF theories

Topological BF theories are the only known topological quantum field theories that can be consistently defined in any dimension. Thus we consider a (compact, oriented, closed, Riemannian) manifold M of dimension n, with a G-principal bundle $P \longrightarrow M$. Here G is a compact simple Lie group with Lie algebra $\mathfrak{g}$. We will mainly consider $G=SU(N)$.

Let us denote by $\Omega^*(M)$ the space of differential forms on M and by $\Omega^*(M,\, \mathrm{ad}P)$ the space of differential forms on M with values in the adjoint bundle $\mathrm{ad}P \equiv P \times_{\mathrm{Ad}} \mathfrak{g}$ (locally $\mathfrak{g}$-valued forms on M).

On M we can consider a quantum field theory depending on two fields:

- the connection A (with curvature denoted by F_A, or simply by F, that is a form in $\Omega^2(M,\, \mathrm{ad}P)$)
- a form $B \in \Omega^{n-2}(M,\, \mathrm{ad}P)$

With the above ingredients we can construct an action

$$S_{BF} = \int_M \mathrm{Tr}\,(B \wedge F), \tag{1}$$

where the trace refers to an assigned representation of $\mathfrak{g}$. Most commonly, we will consider the fundamental representation. The corresponding Gibbs measure will be given by $\exp(ifS_{BF})$ where f denotes a coupling constant.

We denote by $\mathcal{G}$ the group of gauge transformations. For any $\psi \in \mathcal{G}$, locally given by a map $\psi : M \mapsto G$, the field B transforms as $B \longrightarrow \psi^{-1}B\psi$. The action (1) is then gauge invariant. Moreover it is invariant under diffeomorphisms (being given by the integral of a n-form) and it is independent of the metric in M. In other words, the action (1) defines, in principle, a *topological field theory in any dimension.*

In 3 and 4 dimensions we can study other types of BF action. Namely for any values of the parameter κ we can consider, in 3 dimensions, the action:

$$S_{BF,\kappa} = \int_M \mathrm{Tr}\,(B \wedge F) + \frac{\kappa^2}{3}\int_M \mathrm{Tr}\,(B \wedge B \wedge B), \tag{2}$$

and, in 4 dimensions, the action:

$$S_{BF,\kappa} = \int_M \mathrm{Tr}\,(B \wedge F) + \frac{\kappa}{2}\int_M \mathrm{Tr}\,(B \wedge B). \tag{3}$$

In order to understand the geometrical significance of the above actions, we recall that in 4 dimensions there is a topological invariant represented by the integral of the Chern–Weil form:

$$Q_2(F) \equiv \int_M \mathrm{Tr}\,(F \wedge F) \tag{4}$$

while, in 3 dimensions, we have the secondary topological invariant, locally represented by the integral of the Chern–Simons form:

$$S_{CS}(A) \equiv \int_M \mathrm{Tr}\,(A \wedge dA + \frac{2}{3}A \wedge A \wedge A). \tag{5}$$

The actions (1) (with $\dim M = 3, 4$), (2) and (3) are all *variants* of the above topological invariants.

More precisely the following simple relations hold in dimensions 3 and 4 respectively:

$$\begin{aligned} (1/2)[S_{CS}(A+\kappa B) - S_{CS}(A-\kappa B)] &= 2\kappa S_{BF,\kappa} \\ \frac{d}{d\kappa}S_{CS}(A+\kappa B)\Big|_{\kappa=0} &= 2S_{BF} \quad (\dim M = 3) \end{aligned} \tag{6}$$

and

$$\begin{aligned} Q_2(F+\kappa B) - Q_2(F) &= 2\kappa S_{BF,\kappa}, \\ \frac{d}{d\kappa}Q_2(F+\kappa B)\Big|_{\kappa=0} &= 2S_{BF} \quad (\dim M = 4) \end{aligned} \tag{7}$$

The action (2) (and sometimes also the action (3)) is called BF action *with a cosmological term.* The reason for this terminology is easily explained: let us consider in 3 dimensions, the frame bundle LM (with group $G = GL(3)$). The soldering form θ is a 1-form with values in $\mathbf{R}^3$ associated to the fundamental representation of $GL(3)$. When θ is restricted to the orthonormal frame bundle and is expressed in local coordinates we obtain the "dreibein" $\{e^i\}_{i=1,2,3}$.

In the so-called first-order formalism, the classical action for gravity is given by

$$\int_M \sum_{i,j,k} \epsilon_{ijk} e^i \wedge R^{jk} + \kappa^2 \int_M \sum_{i,j,k} \epsilon_{ijk} e^i \wedge e^j \wedge e^k \tag{8}$$

where the matrix R is the curvature 2-form and the second term in the integral is the cosmological term (κ^2 is the cosmological constant). In 3 dimensions we can consider the linear isomorphism $\mathbf{R}^3 \mapsto LieSO(3)$ given by $e_i \longrightarrow \sum_{j,k} \epsilon_{ijk}(E_k^j - E_j^k)$ where e_i are the elements of a canonical basis of $\mathbf{R}^3$ and E_k^j is the matrix whose (m,n)-entry is given by $\delta^{j,m}\delta_{k,n}$. Under this isomorphism, the soldering from is transformed into the B-field, and (8) becomes, up to a constant, the BF action (2).

Next we discuss the *symmetries* of the BF theories. First of all we have to consider the group of gauge transformations, whose infinitesimal action on the fields is given by:

$$A \longrightarrow A + d_A\xi; \quad B \longrightarrow B + [B,\xi]. \tag{9}$$

Here the infinitesimal gauge transformation ξ is an element of $\Omega^0(M, \mathrm{ad}P)$.

In BF theories there exists another important set of symmetries. In this regard, we have to distinguish between the 3-dimensional and the 4-dimensional case.

In 3 dimensions the action (2) is invariant also under the following infinitesimal transformations:

$$A \longrightarrow A + \kappa^2[B,\chi], \quad B \longrightarrow B + d_A\chi, \tag{10}$$

where $\chi \in \Omega^0(M, \mathrm{ad}P)$ is an infinitesimal gauge transformation (in general different from ξ). Instead in 4 dimensions (3) is invariant under the following infinitesimal transformations:

$$A \longrightarrow A + \kappa\eta, \quad B \longrightarrow B - d_A\eta, \tag{11}$$

where η is a form in $\Omega^1(M, \mathrm{ad}P)$, i.e. is the difference of two connections.

The geometrical meaning of the combination of transformations (9) and (10) is straightforward; when $\kappa \neq 0$, (9) and (10) are *equivalent* to the following infinitesimal gauge transformations:

$$\begin{array}{rl} A + \kappa B \longrightarrow & A + \kappa B + d_{A+\kappa B}(\xi + \kappa\chi) \\ A - \kappa B \longrightarrow & A - \kappa B + d_{A-\kappa B}(\xi - \kappa\chi) \end{array} \tag{12}$$

when $\kappa = 0$, (9) and (10) are equivalent to the two sets of transformations obtained by

1. evaluating both sides of (12) at $\kappa = 0$ and
2. applying the operator $\left.\frac{d}{d\kappa}\right|_{\kappa=0}$ to both sides of (12).

It is important to remark that in BF theory (with $\kappa = 0$ and with $\kappa \neq 0$) we have two distinct infinitesimal gauge transformations ξ and χ that generate the symmetries. In the corresponding quantum theory, this implies that there are two distinct set of ghosts that will produce cancellations in the perturbative expansion.

In 4 dimensions the invariance of (3) (with $\kappa \neq 0$) under (11) is nothing else but the independence of the BF action of the connection A. In this way we ensure that the 4-dimensional BF action has the same kind of symmetries as the Chern–Weil form.

When $\kappa = 0$ the 4-dimensional BF action is, in general, not independent of A anymore and the invariance under the transformation $B \longrightarrow B + d_A\eta$ is simply a consequence of the Bianchi identity.

In contrast to the 3-dimensional case, the two sets of ghosts generated by the invariance under (9) and (11) have a different nature (0-forms vs. 1-forms).

III. BF observables in 3 dimensions

The fundamental fields of our theory (in an n-dimensional manifold) are the connection 1-form A and the $(n-2)$-form B. This suggests that the right observables for the topological BF theories should be associated to collections of loops (links) in M (i.e. one-dimensional submanifolds) and to $(n-2)$-submanifolds.

Before discussing the precise definition of our observables, let us consider the case of an abelian BF theory in S^n that is both simple and instructive.

The action for such a theory is given by $S_{BF} = \int_{S^n} B \wedge dA$. This action is invariant under the transformations: $A \longrightarrow A + d\xi$, $\quad B \longrightarrow B + d\eta$, where $\xi \in \Omega^0(M)$ and $\eta \in \Omega^{n-2}(M)$.

In such a theory we can associate an observable to any imbedded oriented loop C and an observable to any imbedded oriented closed $n-2$ submanifold S.

These observables are given by $\mathcal{O}_1(C) \equiv \int_C A$ and $\mathcal{O}_{n-2}(S) \equiv \int_S B$. They are obviously invariant under the symmetries of BF theory.

The holonomy along an embedded circle C is given, in the abelian case, by

$$\mathrm{Hol}\,(A; C) = \exp\,(\mathcal{O}_1(C)).$$

Since only the kinetic term $B \wedge dA$ appears in the lagrangian, we only have to consider *vacuum expectation values* (v.e.v.) of the form:

$$\left\langle A_\nu(x) B_{\mu_1,\mu_2,\cdots,\mu_{n-2}}(y) \right\rangle = \frac{1}{if\Omega_n} \sum_k \epsilon_{\nu,\mu_1,\mu_2,\cdots,\mu_{n-2},k} \frac{x_k - y_k}{||x-y||^n}; \tag{13}$$

where Ω_n is the "volume" of the unit sphere in n dimensions, and x_k are the coordinates of x, Hence only observables with a number of A-fields equal to the number of B-fields will have non-vanishing expectation values.

In particular:

$$if < \mathcal{O}_1(C)\mathcal{O}_{n-2}(S) >= \int_C \int_S dx_\nu dy_{\mu_1} dy_{\mu_2} \cdots dy_{\mu_{n-2}} \left\langle A_\nu(x) B_{\mu_1,\cdots,\mu_{n-2}}(y) \right\rangle = lk(C, S)$$

where $lk(C, S)$ is the (higher-dimensional) linking number between C and S.

The v.e.v. of all the observables one can consider in the abelian theory are thus given by functions of linking numbers between loops and closed $(n-2)$-submanifolds.

The non-abelian theory is more complicated. To each loop C we can still associate the relevant (trace of the) holonomy of the connection A. As we will discuss below, to each $(n-2)$-dimensional imbedded submanifold S we can associate an observable closely related to the integral over S of the $(n-2)$-form B.

Since the kinetic part of the non-abelian theory is the same as the one of the abelian one, (13) still holds in the slightly modified form:

$$\left\langle A^a_\nu(x) B^b_{\mu_1,\mu_2,\cdots,\mu_{n-2}}(y) \right\rangle = \delta_{a,b} \frac{1}{if\Omega_n} \sum_k \epsilon_{\nu,\mu_1,\mu_2,\cdots,\mu_{n-2},k} \frac{x_k - y_k}{||x-y||^n}. \tag{14}$$

Moreover, in a non abelian theory, vertex terms are present; so we have other non trivial v.e.v.'s like

$$\left\langle A^a(x) B^b(y) B^c(z) \right\rangle ; \quad \left\langle A^a(x) A^b(y) A^c(z) \right\rangle$$

that will produce multiple integrals of (convolutions) of the same kernels that appear in (14) (iterated linking numbers). Here neither we have v.e.v.'s of the type:

$$\left\langle A^a(x)A^b(y)B^c(z)\right\rangle$$

nor we have "loops" since loops are cancelled by the corresponding diagrams involving ghosts.

In this way, non abelian gauge theories yield invariants associated to $(n-2)$-submanifolds and imbedded circles that are more sophisticated than the invariants related to the abelian theory.

In this paper we are mainly interested in the case $n = 3$, so imbedded $(n-2)$-submanifolds are knots. The 3-dimensional BF theory then becomes a theory of links in a 3-dimensional manifold.

We now consider the precise definition of our observables. In the framework of *3-dimensional BF theory with a cosmological constant*, the natural observables to be associated to a knot C are given by

$$\mathrm{Tr}\ \mathrm{Hol}\,(A \pm \kappa B; C) \tag{15}$$

while the natural observable to be associated to a knot C in a *3-dimensional BF theory without cosmological constant* is given by

$$\frac{d}{d\kappa}\Big|_{\kappa=0} \mathrm{Tr}\ \mathrm{Hol}\,(A+\kappa B; C) = \mathrm{Tr}\int_C \mathrm{Hol}\,{}_{x_0}^{y}(A;C)B(y)\,\mathrm{Hol}\,{}_{y}^{x_0}(A;C). \tag{16}$$

In this expression, $x_0 \in C$ is a fixed point on the knot, $\mathrm{Hol}\,{}_{x_0}^{y}(A;C) \equiv \mathcal{P}\exp\int_{x_0}^{y} A$, where $\mathcal{P}$ denotes path-ordering and the integral is meant to be computed along the arc of C joining x_0 to y in the direction prescribed by the orientation of the knot. Given a section $\sigma : M \longrightarrow P$, the group element $\mathrm{Hol}\,{}_{x_0}^{y}(A;C)$ can be equivalently described by the equation $\sigma(y)\,\mathrm{Hol}\,{}_{x_0}^{y}(A,C) = C^h(y)$ where C^h denotes the horizontal lift of C with starting point σx_0. Also, by the symbol $\mathrm{Hol}\,{}_{x_0}(A;C)$ we denote the holonomy along C with base point x_0.

We now consider the Taylor expansion of (15) at $\kappa = 0$. For this purpose we compute $\gamma_n(C, x_0) \equiv \dfrac{1}{n!}\dfrac{d^n}{d\kappa^n}\Big|_{\kappa=0} \mathrm{Hol}\,{}_{x_0}(A+\kappa B; C)$, obtaining

$$\begin{aligned}
\gamma_0(C,x_0) &= \mathrm{Hol}\,{}_{x_0}(A;C)\\
\gamma_1(C,x_0) &= \int_C \mathrm{Hol}\,{}_{x_0}^{y}B(y)\,\mathrm{Hol}\,{}_{y}^{x_0}\\
\gamma_2(C,x_0) &= \int_{y_1<y_2\in C} \mathrm{Hol}\,{}_{x_0}^{y_1}B(y_1)\,\mathrm{Hol}\,{}_{y_1}^{y_2}B(y_2)\,\mathrm{Hol}\,{}_{y_2}^{x_0}\\
&\cdots\cdots\\
\gamma_n(C,x_0) &= \int_{y_1<\cdots<y_n\in C} \mathrm{Hol}\,{}_{x_0}^{y_1}B(y_1)\,\mathrm{Hol}\,{}_{y_1}^{y_2}B(y_2)\cdots\,\mathrm{Hol}\,{}_{y_{n-1}}^{y_n}B(y_n)\,\mathrm{Hol}\,{}_{y_n}^{x_0}.
\end{aligned} \tag{17}$$

In our notation we do not write explicitly the dependence of $\gamma_n(C, x_0)$ on A and B.

The above formulas are *iterated Chen integrals*. In fact, let us define

$$\hat{B}(x) \equiv \mathrm{Hol}\,{}_{x_0}^{x}B(x)[\,\mathrm{Hol}\,{}_{x_0}^{x}]^{-1}. \tag{18}$$

This is a $\mathfrak{g}$-valued 1-form on C. The geometrical meaning of $\hat{B}$ is as follows: we can view the 1-form B equivalently as an element of $\Omega^1(M,\,\mathrm{ad}P)$ or as a $\mathfrak{g}$-valued 1-form on the total space of the principal bundle $P(M,G)$ which is tensorial under the adjoint action. Given a reference section $\sigma : M \longrightarrow P$, we can consider the horizontal lift C^h of C with starting point $\sigma(x_0)$. The integral of B (seen as a 1-form on P) along C^h is exactly the integral of $\hat{B}$ along the loop C.

The definitions (17) coincide with the following Chen integrals [15]:

$$\gamma_n(C, x_0) = \oint_{x_0} \underbrace{\hat{B} \cdot \hat{B} \cdots \hat{B} \cdot \hat{B}}_{n-1 \text{ times}} \cdot (\hat{B} \,\mathrm{Hol}_{x_0}(A; C)) \tag{19}$$

We recall that the iterated integral $\int_a^b \omega_1 \cdot \omega_2 \cdots \omega_n$ of n 1-forms $\{\omega_i\}_{i=1,\cdots n}$ (with values in any algebra) is given (in our notation) by the formula $\int_{a<x_1<\cdots<x_n<b} \omega_1(x_1) \wedge \omega_2(x_2) \wedge \cdots \wedge \omega_n(x_n)$.

Our Taylor expansion finally reads

$$\mathrm{Hol}_{x_0}(A + \kappa B; C) = \sum_n \kappa^n \gamma_n(C, x_0) \tag{20}$$

We may also try to consider as observables the quantities $\mathrm{Tr}\, \gamma_n(C, x_0)$. They are all gauge invariant, i.e. invariant under (9), but unfortunately they are not invariant under (10). In fact, under the transformations (10) we have the following transformation:

$$\gamma_n(C, x_0) \longrightarrow \gamma_n(C, x_0) + \kappa^2 \tilde{\gamma}_{n+1}(C, x_0) - \tilde{\gamma}_{n-1}(C, x_0) - [\chi(x_0), \gamma_{n-1}] \tag{21}$$

Here the map $\gamma \longrightarrow \tilde{\gamma}$ is meant to be the derivation that replaces in (17), the field B evaluated at a given set of points $\{y_i\}$ by the field $[B, \chi]$ evaluated at the same points y_i.

As a consequence of the above transformation laws, we conclude that only particular combinations of $\gamma_n(C, x_0)$ give rise to good observables.

Namely the observables that we can consider for the BF theory *with* a cosmological constant are only *the traces* of the following quantities

$$\begin{aligned}
&\mathrm{Hol}\,(A \pm \kappa B) \\
&\mathrm{Hol}_{x_0}^{\,even}(C) \equiv (1/2)[\,\mathrm{Hol}_{x_0}(A + \kappa B; C) + \,\mathrm{Hol}_{x_0}(A - \kappa B; C)] = \textstyle\sum_s \kappa^{2s} \gamma_{2s}(C, x_0) \\
&\mathrm{Hol}_{x_0}^{\,odd}(C) \equiv (1/2)[\,\mathrm{Hol}_{x_0}(A + \kappa B; C) - \,\mathrm{Hol}_{x_0}(A - \kappa B; C)] = \textstyle\sum_s \kappa^{2s+1} \gamma_{2s+1}(C, x_0).
\end{aligned} \tag{22}$$

Moreover, as expected, the BF theory *without* cosmological constant (i.e., with $\kappa = 0$), admits as observables, either $\mathrm{Tr}\,[\gamma_i(C; x_0)]$, $i = 0, 1$ or traces of products of $\gamma_i(C; x_0)$, $i = 0, 1$.

When we consider the last case, we have to allow only infinitesimal transformations (10) that satisfy the extra-condition $\chi(x_0) = 0$. In other words χ must belong to the Lie algebra of the group of gauge transformations, whose restriction to x_0 is the identity.

In particular, we will be interested in the following set of observables for the BF theory *without* cosmological constant:

$$\Gamma_n(C, x_0) = \frac{1}{n!} \underbrace{\oint_{x_0} \hat{B} \oint_{x_0} \hat{B} \cdots \oint_{x_0} \hat{B}}_{n-1 \text{ times}} \oint_{x_0} \hat{B} \,\mathrm{Hol}_{x_0}(A; C) \tag{23}$$

The observables γ_n and Γ_n do not coincide, since $\hat{B}(x)$ and $\hat{B}(y)$ are *not* commuting quantities. We now define

$$\mathcal{H}(C; \lambda) \equiv \sum_n \lambda^n \Gamma_n(C; x_0). \tag{24}$$

The quantity $\mathrm{Tr}\,\mathcal{H}(C;\lambda)$ replaces $\mathrm{Tr\ Hol}\,(A+\kappa B, C)$ as the *basic observable* for the BF theory with *zero cosmological constant*. The geometrical meaning of (24) is related to the action of the group G (or of its tangent bundle) on the tangent bundle of the total space P and will be discussed elsewhere.

IV. Formal relations between Chern–Simons and BF theories

Let us consider the Chern–Simons partition functions:

$$\begin{aligned} Z_{CS}(M,C;k) &\equiv \int \mathcal{D}A \exp(ikS_{CS}(A))\,\mathrm{Tr\ Hol}\,(A;C) \\ Z_{CS}(M;k) &\equiv \int \mathcal{D}A \exp(ikS_{CS}(A)) \end{aligned} \tag{25}$$

and the BF partition functions:

$$\begin{aligned} Z_{BF,\kappa}(M,C;f) &\equiv \int \mathcal{D}A\mathcal{D}B \exp(ifS_{BF,\kappa}(A,B))\,\mathrm{Tr\ Hol}\,(A+\kappa B) \\ Z_{BF,\kappa}(M;f) &\equiv \int \mathcal{D}A\mathcal{D}B \exp(ifS_{BF,\kappa}(A,B)) \end{aligned} \tag{26}$$

where k and f are coupling constants. The constant k is quantized, namely must be an integer multiple of $(4\pi)^{-1}$ in order to guarantee the invariance of the action S_{CS} under gauge transformations not connected to the identity. At the formal level we have:

$$Z_{CS}(M;k)\overline{Z_{CS}(M;k)} = Z_{BF,\kappa}(M;f). \tag{27}$$

In fact, the first term in the above equation is given by:

$$\int \mathcal{D}A_1\mathcal{D}A_2 \exp\left[ikS_{CS}(A_1) - ikS_{CS}(A_2)\right],$$

and this quantity is equal to $Z_{BF,\kappa}(M;f)$, provided we set:

$$2A = A_1 + A_2; \quad 2\kappa B = A_1 - A_2; \quad f = 4\kappa k. \tag{28}$$

Assuming that $Z_{CS}(M;k)$ represents the Reshetikhin–Turaev [28] invariant, then $Z_{BF,\kappa}(M,f)$ represents the Turaev–Viro [30] invariant.

Next we discuss the relations between the BF and the CS actions with *knots incorporated*. We require again relations (28). Then we have that

$$Z_{CS}(M,C;k)\overline{Z_{CS}(M;k)} = Z_{BF,\kappa}(M,C;f), \tag{29}$$

and hence

$$\frac{Z_{CS}(M,C;k)}{Z_{CS}(M;k)} = \frac{Z_{BF,\kappa}(M,C;f)}{Z_{BF,\kappa}(M;f)}. \tag{30}$$

When we choose $M = S^3$ and consider the fundamental representation of $SU(N)$, then the normalized partition function (30) *gives (a regular isotopy invariant version of) the HOMFLY polynomial* $P(l,m)$ evaluated at $l = \exp(-if^{-1}\kappa N)$, $\quad m = l^{1/N} - l^{-1/N}$. From now on we set $f = (2\pi)^{-1}$.

The polynomial $P(l,m)$ satisfies the skein relation: $lP(l,m)(C_+) - l^{-1}P(l,m)(C_-) = mP(l,m)C_0$, where $\{C_+, C_-, C_0\}$ is a Conway triple, and the normalization condition $P(l,m)(\emptyset) = 1$ for the empty knot $\emptyset$ is imposed.

V. Choice of gauge and v.e.v's

In order to quantize BF theory we first need to make a choice of gauge. The most natural choice of gauge is the *(background) covariant gauge*. Namely, we fix a background connection A_0 and we require that the fields A and B satisfy the following constraints:

$$d^*_{A_0}(A - A_0) = d^*_{A_0} B = 0, \tag{31}$$

where $d^*_{A_0}$ is the adjoint of the covariant derivative. This is a complete gauge condition, namely it provides us with a honest (local) section of the the bundle of gauge orbits $\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{G}$, where $\mathcal{A}$ denotes the space of all (irreducible) connections.

With this choice of gauge, we conclude that, for any equivalence class of connections $[A]$, $[B]$ represent a tangent vector in $T_A\left(\mathcal{A}/\mathcal{G}\right)$ the space of gauge orbits (or a cotangent vector if use the Hodge star operator to introduce an inner product in $T_A\left(\mathcal{A}/\mathcal{G}\right)$.

In physics one would like to choose the canonical flat connection, as a background connection, and hence replace the covariant derivative with the exterior derivative. This is always possible in 3-dimensions, when the group G is $SU(N)$. In this case, the *covariant gauge condition* read, in local coordinates,

$$\sum_\mu \partial^\mu A_\mu = \sum_\mu \partial^\mu B_\mu = 0.$$

When the 3-dimensional manifold is $\mathbf{R}^3$, or more generally $\Sigma \times \mathbf{R}$, for a given surface Σ, we can consider other gauges. These are not true complete gauge conditions, in the sense specified above, since, after imposing them, we are left with a residual freedom in the choice of gauge.

For $M = \Sigma \times \mathbf{R}$, we denote by t the coordinate of $\mathbf{R}$. We introduce a complex structure in Σ (with local coordinates $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$). This yields a decomposition of $\Omega^1(\Sigma,\, \mathrm{ad}P)$ into a holomorphic part $\Omega^{1,0}(\Sigma,\, \mathrm{ad}P)$ and an anti-holomorphic part $\Omega^{0,1}(\Sigma,\, \mathrm{ad}P)$ [1]. By saying that we choose the *light-cone gauge* in the holomorphic formulation, we mean that we assume that, for each $t \in \mathbf{R}$, both the connection $A(t)$ and the 1-form $B(t)$, restricted to Σ, are *holomorphic*. In other words, in local coordinates, A and B are expressed as:

$$A_z dz + A_0 dt, \quad B_z dz + B_0 dt.$$

This choice of gauge is equivalent to requiring that, in *real* coordinates x_1, x_2, t, we have $A_1 = A_2$ and $B_1 = B_2$. In this gauge the BF action becomes:

$$S_{BF,\kappa} = \int_M \mathrm{Tr}\,\left(B_z \wedge \bar{\partial} A_0 - B_0 \wedge \bar{\partial} A_z\right), \tag{32}$$

namely it is quadratic and independent of κ. The quantization of Chern–Simons theory in the light-cone gauge has been studied in [19].

Finally we consider the *axial gauge* $A_0 = B_0 = 0$. Here, again, the BF action is quadratic and independent of κ:

$$S_{BF,\kappa} = \int_M \mathrm{Tr}\,\left(B_2 \wedge d_0 A_1 - B_1 \wedge d_0 A_2\right). \tag{33}$$

Let us consider the vacuum expectation values of the BF theory in the three different gauges defined above. In the two singular gauges, we only have a quadratic kinetic term in the lagrangian; this implies that the two-point correlation functions determine all n-point correlations (by Wick's theorem). The v.e.v.'s in the different gauges are as follows

- Covariant gauge ($M = \mathbf{R}^3$):

$$\begin{array}{ll} \left\langle A^a_\mu(x) B^b_\nu(y) \right\rangle & = -4\pi i \delta^{ab} l(x,y) \\ \left\langle A^a_\mu(x) B^b_\nu(y) B^c_\rho(z) \right\rangle & = -(4\pi)^2 f^{abc} v_{\mu\nu\rho}(x,y,z) \\ \left\langle A^a_\mu(x) A^b_\nu(x) A^c_\rho(x) \right\rangle & = -(4\pi)^2 \kappa^2 f^{abc} v_{\mu\nu\rho}(x,y,z) \end{array} \tag{34}$$

where we have set:

$$\begin{array}{ll} l_{\mu\nu}(x,y) & \equiv \dfrac{1}{4\pi} \displaystyle\sum_\rho \epsilon_{\mu\nu\rho} \frac{x_\rho - y_\rho}{||x-y||^3} \\ v_{\mu\nu\rho}(x,y,z) & \equiv \displaystyle\int_{\mathbf{R}^3} d^3w \sum_{\alpha,\beta\gamma} \epsilon^{\alpha\beta\gamma} l_{\mu\alpha}(x,w) l_{\nu\beta}(y,w) l_{\rho\gamma}(z,w) \end{array} \tag{35}$$

- Holomorphic gauge ($M = \mathbf{C} \times \mathbf{R}$):

$$\left\langle A^a_z(z,t) B^b_0(w,s) \right\rangle = -2\delta^{ab} \frac{1}{(z-w)} \delta(t-s) \tag{36}$$

- Axial gauge ($M = \mathbf{R}^3$):

$$\left\langle A^a_1(x_1,x_2,x_0) B^b_2(y_1,y_2,y_0) \right\rangle = -(2\pi i)\delta^{ab} \,\mathrm{sgn}\,(x_0 - y_0)\delta(x_1-y_1)\delta(x_2-y_2). \tag{37}$$

VI. The framing of the knot and the skein relation

We require that v.e.v.'s involving the fields A and B are not computed at coincident points. This is equivalent to requiring that, in all the integrals of the perturbative expansion of v.e.v.'s of observables, the field A lives on a companion knot C_f of the original knot C where the field B is supposed to be integrated over. Thus we must consider a *framing* of the original knot.

We denote by ϵ the distance of the companion knot C_f from C. Eventually we will have to consider the limit $\epsilon \to 0$, in order to restore the diffeomorphism-invariance broken by the introduction of the framing.

Now we want to study the effect of a small deformation of the knot, concentrated around a given point x of C and, simultaneously, of its companion C_f.

These deformations will change the holonomies by a factor proportional to the curvature, i.e. they will modify the v.e.v. as follows:

$$\langle \,\mathrm{Tr}\ \mathrm{Hol}\,(A+\kappa B)(C) \rangle \longrightarrow \left\langle \,\mathrm{Tr}\, \{ \,\mathrm{Hol}\,^x_{x_0}(A+\kappa B) F_{A+\kappa B}(x) \,\mathrm{Hol}\,^{x_0}_x(A+\kappa B) \} \right\rangle .$$

Notice that $F_{A+\kappa B} = F_A + \kappa d_A B + \kappa^2 B \wedge B$. As in [17], we *assume* that we can perform an integration by parts. In order to do so, we first compute the functional derivatives of the BF action, obtaining:

$$\begin{array}{l} \dfrac{\delta S_{BF,\kappa}}{\delta A^a_\mu(x)} = \dfrac{1}{4} \displaystyle\sum_{\nu,\rho} \epsilon^{\mu\nu\rho} (d_A B)^a_{\nu\rho}(x) \\ \dfrac{\delta S_{BF,\kappa}}{\delta B^a_\mu(x)} = \dfrac{1}{4} \displaystyle\sum_{\nu,\rho} \epsilon^{\mu\nu\rho} \left(F^a_{\nu\rho}(x) + \kappa^2 f^{abc} B^b_\nu(x) B^c_\rho(x) \right). \end{array} \tag{38}$$

Assuming that an integration by part can be performed, we have, for any observable $\mathcal{O}$ classically represented by a G-valued function that transforms under conjugation (or a $\mathfrak{g}$-valued function that transforms under the adjoint action), that

$$\begin{gathered}\left\langle \operatorname{Tr}\left[(F+\frac{\kappa^2}{2}[B,B])^a_{\mu\nu}(x)+\kappa(d_AB)^a_{\mu\nu}(x)\right]\mathcal{O}\right\rangle = \\ 4\pi i\sum_\rho \epsilon_{\mu\nu\rho}\left\{\left\langle\frac{\delta\mathcal{O}}{\delta B^a_\rho(x)}+\frac{\delta\mathcal{O}}{\delta A^a_\rho(x)}\right\rangle\right\}.\end{gathered} \tag{39}$$

In other words we can replace the terms d_AB and $F+\kappa^2 B\wedge B$ by the functional derivatives with respect to A and, respectively, B.

When we are given a crossing point x in the (diagram of a) knot C we associate to it four configurations: $C_\pm$, C_0 and $C_\times$.

The first two configurations $C_\pm$ correspond to positive and negative crossing points in the diagram, C_0 corresponds to the link obtained by removing the crossing point in the only orientation-preserving way and, finally, $C_\times$ corresponds to a singular knot where the crossing point x is a *transversal* double point.

The observable $\operatorname{Tr}\operatorname{Hol}(A+\kappa B;C)$ can be extended to singular knots. In fact all the v.e.v.'s are regularized by the separation of the knot C from its companion C_f. Also the same observable can be easily extended to links as the product of the traces of the above holonomies evaluated along the various components of the link.

The framework of quantum field theory suggests that we should study the effect of two families of singular deformations applied to the knot C and its companion C_f:

1. a singular deformation of the knot C and its companion C_f, concentrated around a regular point x and characterized by the requirement that the surface element spanned by this deformation is transversal to the knot itself. In other words we are "twisting" the knot, or, in the terminology used by Kauffman, changing the writhe.

2. a singular deformation of a singular knot $C_\times$ (and of its companion $(C_\times)_f$) around a transversal double point x. The effect of this deformation will be to remove the double point and to create two different non-singular knots C_+ and C_-, depending on the direction of the deformation. In this case we are assuming that the surface element spanned by the deformation lies in the same plane with one of the two tangent vectors to the knot at x and is transversal to the other one (see [9] for a related approach).

We choose the fundamental representation of $SU(N)$. If we denote by R^a a basis of $Lie(SU(N))$, normalized so that $2\operatorname{Tr}R^aR^b=-\delta^{ab}$, then we can derive, as a consequence of the fact that $\{(1/\sqrt{N})\,\mathbf{I},(i\sqrt{2})R^a\}$ is an orthonormal basis in the space of complex $n\times n$ matrices, the well known Fierz identity, namely:

$$2\sum_a R^a\otimes R^a=\mathbf{P}-(1/N)\,\mathbf{I} \tag{40}$$

where $\mathbf{P}$ denotes the twist operator ($\mathbf{P}(x\otimes y)=(y\otimes x)$) and $\mathbf{I}$ is the identity. In components the Fierz identity reads:

$$2\sum_a R^a_{ij}R^a_{kl}=\delta_{il}\delta_{kj}-(1/N)\delta_{ij}\delta_{kl}.$$

We write the Casimir operator in the fundamental representation as: $\sum_a R^a R^a = c_2 \mathbf{I}$ with $c_2 = (2N)^{-1}(N^2 - 1)$.

First we consider a singular infinitesimal deformation of *type 1.* By integrating by parts we obtain that

$$\delta \langle \text{Tr Hol}\,(A + \kappa B)\rangle = \mp 4\pi i \kappa c_2 \langle \text{Tr Hol}\,(A + \kappa B)\rangle \tag{41}$$

where the sign $\mp$ depends on whether, by combining the orientation of the small surface bounded by the deformed loop and the orientation of the the knot, we obtain the given orientation of the ambient space or its opposite, respectively.

If we want to consider a finite deformation of type 1, as opposed to an infinitesimal one, we can use the non-abelian Stokes formula introduced in [3]. The holonomy of a loop bounding a rectangular surface Σ (with initial point x_0), is expressed in terms of a path ordered exponential of the surface integral

$$\mathcal{P} \exp \int_\Sigma dy \,\text{Hol}\,{}^y_{x_0}(\sigma) F(y) (\,\text{Hol}\,{}^y_{x_0})^{-1}(\sigma) \tag{42}$$

where σ is a path joining x_0 and $y \in \Sigma$ with a prescribed pattern.

In the above formula, we now replace the curvature $F_{\nu,\rho}$, computed w.r.t. the connection $A + \kappa B$, by the operator

$$\exp \left\{ 4\pi i \sum_\mu \epsilon_{\mu\nu\rho} \left(\frac{\delta}{\delta B^a_\mu(x)} + \frac{\delta}{\delta A^a_\mu(x)} \right) \right\}$$

and by a succession of integrations by part (see [11]), we can prove that a positive twisting of the given knot will multiply the v.e.v. by a factor $\alpha \equiv \exp(-4\pi i \kappa c_2)$. In other words, our v.e.v., computed over a knot-diagram C^w with a given writhe w, transforms, under a change of writhe, as follows:

$$\left\langle \text{Tr Hol}\,(A + \kappa B)(C^{w\pm1})\right\rangle = \alpha^{\pm 1} \langle \text{Tr Hol}\,(A + \kappa B)(C^w)\rangle\,. \tag{43}$$

The formula above follows from the fact that, thanks to integration by parts, the n-th order variation δ^n, inserts, into the v.e.v., a term

$$\sum_{a_1,\dots a_n} R^{a_1} R^{a_2} \cdots R^{a_n} R^{a_n} R^{a_{n-1}} \cdots R^{a_1} = (c_2)^n \,\mathbf{I}\,. \tag{44}$$

Next, we perform an infinitesimal deformation of *type 2* and use integration by parts again. Here the n-th order variation δ^n inserts, into the v.e.v., a matrix $S^{(n)}_{i,j,,k,l} \in End(\mathbf{C}^N \otimes \mathbf{C}^N)$, given by

$$S^{(n)}_{i,j,k,l} = \sum_{a_1,\dots a_n} (R^{a_1} R^{a_2} \cdots R^{a_n})_{i,j} \,(R^{a_1} R^{a_2} \cdots R^{a_n})_{k,l}\,. \tag{45}$$

By a repeated use of the Fierz identity we obtain that

$$S^{(n)} = a^{(n)}\,\mathbf{P} + b^{(n)}\,\mathbf{I}$$

where we have set

$$\begin{aligned} 2a^{(n)} &= (N-1)^n 2^{-n} N^{-n} - (-1)^n (N+1)^n 2^{-n} N^{-n} \\ 2b^{(n)} &= (N-1)^n 2^{-n} N^{-n} + (-1)^n (N+1)^n 2^{-n} N^{-n}. \end{aligned}$$

The infinitesimal variation is

$$\delta \langle \operatorname{Tr} \operatorname{Hol}(A+\kappa B; C_\times)\rangle = \mp 2\pi i \kappa c_2 \langle \operatorname{Tr} \operatorname{Hol}(A+\kappa B; C_0)\rangle + \frac{2\pi i\kappa}{N} \langle \operatorname{Tr} \operatorname{Hol}(A+\kappa B, C_\times)\rangle. \tag{46}$$

By considering the total variation, δ^{tot}, (sum of variations of all orders) we obtain

$$\langle \operatorname{Tr} \operatorname{Hol}(A+\kappa B; C_\pm)\rangle = a^\pm \langle \operatorname{Tr} \operatorname{Hol}(A+\kappa B; C_0)\rangle + b^\pm \langle \operatorname{Tr} \operatorname{Hol}(A+\kappa B, C_\times)\rangle, \tag{47}$$

where we have set

$$\begin{aligned} a^\pm &\equiv (1/2)\exp\{(\mp 2\pi i\kappa(N-1)/N\} - \exp\{\pm 2\pi i\kappa(N+1)/N\} \\ b^\pm &\equiv (1/2)\exp\{(\mp 2\pi i\kappa(N-1)/N\} + \exp\{\pm 2\pi i\kappa(N+1)/N\}. \end{aligned}$$

By defining $\beta^2 = b^-/b^+ = \exp(-4\pi i\kappa/N)$, we finally obtain a skein relation:

$$\begin{aligned} \beta \langle \operatorname{Tr} \operatorname{Hol}(A+\kappa B; C_+)\rangle - \; & \beta^{-1} \langle \operatorname{Tr} \operatorname{Hol}(A+\kappa B; C_-)\rangle \\ & = (\beta A^+ - \beta^{-1}A^-) \langle \operatorname{Tr} \operatorname{Hol}(A+\kappa B; C_0)\rangle, \end{aligned} \tag{48}$$

By combining (43) and (48), we conclude that $\langle \operatorname{Tr} \operatorname{Hol}(A+\kappa B; C)\rangle$ is the HOMFLY polynomial $P(l,m)$ evaluated at $l = \alpha\beta,\ m = l^{1/N} - l^{-1/N}$.

VII. Choice of gauge and link-invariants

In the previous section we showed that, by assuming that integration by parts is allowed in the functional integral, BF theory *with* a cosmological constant reproduces knot-invariants given by the HOMFLY polynomials evaluated at some specific values of the variables.

We expect that, by computing the perturbation expansion, we are able to recover these knot-invariants as power series. But in order to perform the perturbation expansion we need to make one of the (non-equivalent) choices of gauge.

Different gauge-choices produce different expansions that are recognized to be equal only after some global normalization factor (that may be given by a power series) is taken into account. Moreover, order by order in perturbation theory, one finds, in different gauges, different sets of Feynman diagrams to be summed over. In conclusion different gauge-choices may very well lead to the same invariant in ways that appear completely different.

Let us examine more closely the different choices of the gauge in perturbative BF theory (with cosmological constant).

- Covariant gauge

  In this gauge the knot-invariants are expressed as multiple integrals given by convolutions of kernels of type l and v ((35)). In fact, starting from (20), we can write the v.e.v. of the holonomy operator associated to a knot C as

$$\left\langle \sum_n \kappa^n \operatorname{Tr} \gamma_n(C, x_0) \right\rangle = \sum_n \kappa^n V_n(C)$$

  where the coefficients $V_n(C)$ are defined as follows: first we define $\langle \operatorname{Tr} \gamma_s(C)\rangle_j$ to be the terms in $\langle \operatorname{Tr} \gamma_s(C)\rangle$ that are obtained by inserting exactly j times the vertex term proportional to B^3 (this vertex term is multiplied by a factor κ^2). Then we define:

$$V_{2n}(C) \equiv \sum_{i=0}^{n} \langle \operatorname{Tr} \gamma_{2i}(C, x_0) \rangle_{n-i}$$
$$V_{2n+1}(C) \equiv \sum_{i=0}^{n} \langle \operatorname{Tr} \gamma_{2i+1}(C, x_0) \rangle_{n-i}. \tag{49}$$

Formally, we can prove that all the $V_n(C)$ [11] are knot-invariants. In fact let us consider a small deformation of the knot C and, simultaneously, of its companion C_f. When we study the effect of this deformation on the terms $V_{2n}(C)$, we collect the different contributions with the same order in κ and obtain:

$$\begin{aligned} \delta V_{2n} &= \textstyle\sum_{i=0}^{n-1} \left[\langle \operatorname{Tr}(\gamma_{2i} F) \rangle_{n-i} + \langle \operatorname{Tr}(\gamma_{2i} BB) \rangle_{n-i-1} \right] + \\ &+ \textstyle\sum_{i=0}^{n} \langle \operatorname{Tr}(\gamma_{2i-1} d_A B) \rangle_{n-i} + \langle \operatorname{Tr}(\gamma_{2n} F) \rangle_0 . \end{aligned} \tag{50}$$

Now (50) vanishes identically. In fact we have the following equations:

$$\begin{aligned} &\langle \operatorname{Tr}\ (\gamma_{2n} F) \rangle_0 = 0 \\ &\langle \operatorname{Tr}\ (\gamma_{2i} F) \rangle_{n-i} + \langle \operatorname{Tr}\ (\gamma_{2i} BB) \rangle = 0 \\ &\langle \operatorname{Tr}\ (\gamma_{2i-1} d_A B) \rangle = 0, \end{aligned}$$

that are, respectively, a direct consequence of the identities

$$\langle B(x)B(y)F(z) \rangle = 0 \tag{51}$$
$$\langle A(x)A(y)F(z) \rangle + \langle A(x)A(y)B(z)B(z) \rangle = 0 \tag{52}$$
$$\langle A(x)B(y)(d_A B)(z) \rangle = 0 \tag{53}$$

The proof of the above identities is straightforward: for instance the r.h.s. of (51) satisfies the following equation (where group factors have been omitted)

$$\langle B(x)B(y)dA(z) + B(x)B(y)A(z) \wedge A(z) \rangle = (d_3 v)(x, y, z) + l(x, z) \wedge l(y, z) = 0.$$

Here the kernels l and v are defined by (35) and can be interpreted as forms on $(\mathbf{R}^3)^2$ (of type (1,1)) and, respectively, on $(\mathbf{R}^3)^3$ (of type (1,1,1)). The operator d_3 acts on (1,1,1)-forms and produces (1,1,2)-forms (see [10, 6]). In other words the d_3-differential of the form v compensates the term $l \wedge l$ when the latter form is restricted to the part of the boundary of the configuration space $C_4(\mathbf{R}^3)$ characterized by 2 coincident points. The proof of (52) and (53) is completely similar. The variation of $V_{2n+1}(C)$ can be dealt with in a completely similar way.

Now each term $\langle \operatorname{Tr} \gamma_{2i}(C) \rangle_{n-i}$ gives rise to multiple integrals involving the kernels l and v (35). The kernel v corresponds to vertex contractions ($B \longleftrightarrow A \longleftrightarrow B$ and $A \longleftrightarrow A \longleftrightarrow A$), while the kernel l corresponds to a contraction $A \longleftrightarrow B$.

In conclusion, in $\langle \operatorname{Tr} \gamma_{2i}(C) \rangle_{n-i}$ we encounter the following contributions: an integral with n kernels of type v, an integral with $n-1$ kernels of type v and 2 kernels of type l, ..., and finally an integral with $2i$ kernels of type l and $n-i$ kernels of type v.

A similar computation yields V_{2n+1}. In this way we can represent the coefficients of the HOMFLY polynomial evaluated as in section VI, as sums of multiple integrals given by convolutions of the kernels (35). This representation of the coefficients of the HOMFLY polynomials is the one considered in the work of Bott and Taubes [10], that is in turn inspired by [20] and [6].

- Holomorphic gauge

  The holomorphic and the axial gauge involve a projection onto a plane. In these gauges one cannot expect that the coefficients of the perturbative expansion directly give knot invariants. The (3-dimensional) diffeomorphism invariance is broken and some correcting factor must be introduced in order to restore this invariance. Both the holomorphic and the axial gauge do not have vertex terms. This implies that, in contrast to the covariant gauge, the n-th term in the perturbative expansion in the variable κ is given by $\langle \operatorname{Tr} \gamma_n(C) \rangle$.

  The temporal delta function in (36) implies that we have to consider contractions between a B-field and an A-field only when these fields lie at the same height (in the t-variable). We take these level surfaces, to be transversal to the knot, for generic, i.e non critical, times. From (36) we see that the forms to be integrated are given by

$$\bigwedge_i \frac{-2(dz_i - dw_i)}{(z_i - w_i)}$$

  where the pairs of points (z_i, w_i) represent points of the knots C and on the companion C_f, where contractions occur [19].

  Perturbatively, the quantum theory is described by a family of Feynman diagrams D_P, depending on the set P of all possible contractions at a given order in κ. To each of these Feynman diagrams, and to each representation R of the group G, we associate a group factor $W_R(D_P)$. The v.e.v.'s of interest are given by

$$\left\langle \operatorname{Tr} \sum_n \kappa^n \gamma_n(C) \right\rangle = W_R(Z_\epsilon(C))$$

  where $Z_\epsilon(C)$ is a diagram-valued function (ϵ being the spacing between C and C_f). It is possible to let C approach C_f, by considering (see [2])

$$Z(C) = \lim_{\epsilon \to 0^+} e^{-2\kappa(n^+ - n^-)\Theta} Z_\epsilon(C), \tag{54}$$

  where $n^\pm$ are the number of critical points (positive and negative) of the height-function on the knot C, while Θ denotes the insertion of an isolated chord.

  The resulting diagram-valued partition function has been considered by Kontsevich [25, 7]:

$$Z(C) = \sum_{m=0}^{\infty} (-2\kappa)^m \int_{t_{\min} < t_1 < \cdots < t_m < t_{\max}} \sum_{P = \{(z_i, z_i')\}} (-1)^{\# P_\downarrow} D_P \bigwedge_{i=1}^{m} \frac{dz_i - dw_i}{z_i - w_i}, \tag{55}$$

  where $t_{\min}$ and $t_{\max}$ denote the lowest and highest height of C, respectively, (z_i, t_i) and (w_i, t_i) denote distinct points on C and $\# P_\downarrow$ denotes the number of points (z_i, t_i),(w_i, t_i) where the height is a decreasing function.

  In [7], it is shown that such integrals are well-defined knot invariants, provided that we use the normalization:

$$\hat{Z}(C) = \frac{Z(C)}{Z(\infty)^{\frac{c}{2} - 1}}, \tag{56}$$

  where c is the number of critical points and ∞ denotes the particular unknot with one crossing point whose diagram looks like the symbol ∞.

- Axial gauge

  The v.e.v.'s in this gauge (see eq. (37)) indicate that the interactions to be considered are localized only at the crossing points of the projected knot C (B-field line) with its framing C_f (A-field line). These crossing points occur either when C_f is "twisted" around C or when we have an actual crossing point of the projected knot C.

  There is an obvious invariance of v.e.v.'s under orientation-preserving transformations of the plane (or, more generally, of the surface Σ) onto which the knot is projected. Hence these interactions only depend on the type of the crossing (positive or negative) of the projected knot C with its companion C_f.

  We have first to take into consideration a change of the framing (i.e. a twisting of C_f). Once we have done this, we can then choose a specific framing (the "blackboard framing") where C_f is always at the right of C (with respect to the given orientation of C) and hence intersects C only near an actual vertex of the knot C. In this way, only actual vertices of C contribute to the interaction. After having chosen the framing as above, we explicitly compute $\left\langle \mathrm{Tr} \sum_n \kappa^n \gamma_n(C) \right\rangle$.

  At the n-th order of perturbation we have n interactions localized at the vertices. These interactions will be represented by (traces of) group factors. For the fundamental representation of $SU(N)$, they are given by functions of N.

  Moreover, among the n interactions that we are considering, n_1 of them can be localized at one vertex, n_2 at another vertex and n_j at a j-th vertex. The requirement here is that $\sum_j n_j = n$.

  In other words, we can write the n-th term in the perturbation expansion as a sum

$$\sum_{i_1,i_2,\cdots,i_n} \epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_n} D^{i_1,i_2,\cdots,i_n}_{n,n}(C) + \sum_{i_1,i_2,\cdots,i_{n-1}} \epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_{n-1}} D^{i_1,i_2,\cdots,i_{n-1}}_{n,n-1}(C) + \cdots + \sum_i \epsilon_i D^i_{n,1}(C), \tag{57}$$

  where the indices i_j label the vertices of the knot C and $D_{n,i}$ are (traces of) group factors corresponding to n interactions concentrated at i different vertices.

  We cannot hope to obtain the HOMFLY polynomial directly from (57), since we have had to make a particular choice of the framing (blackboard framing).

  But a good choice of the normalization factor for (57) will show that the perturbation theory in the axial gauge (with the fundamental representation of $SU(N)$) provides the expression of the coefficients of the HOMFLY polynomials in terms of tensors over the vertices of the knot (see the discussion in the appendix).

  The similarity between (78) and (57) is striking.

IX. The BF theory without cosmological constant and the Alexander–Conway polynomial

In this section we consider the BF theory *without* cosmological constant. The observable associated to a knot C is then given by $\mathrm{Tr}\,\mathcal{H}(C;\lambda)$, where λ is an expansion parameter and $\mathcal{H}$ is defined by (23) and (24).

Another observable that could be considered in BF theory without cosmological constant is $\mathrm{Tr}\,\exp(\lambda\Gamma_1(C;x_0))$, the difference between the v.e.v.'s computed for the two observables being given by powers of $lk(C_f,C)$. In [13] the latter definition of the observable was assumed; but the choice $\mathrm{Tr}\,\mathcal{H}(C;\lambda)$ is a more natural one if one wants to deal with arbitrary values of $lk(C_f,C)$.

For simplicity we assume that $lk(C_f,C)=0$ (standard framing); this makes the distinction between the two choices of observables irrelevant and allows us to use the results of [13].

The perturbation expansion in the covariant gauge reads

$$\left\langle \sum_n \lambda^n \,\mathrm{Tr}\,\Gamma_n(C,x_0)\right\rangle = \sum_n \lambda^n W_n(C)$$

where the coefficients $W_n(C)$ are defined as

$$W_n(C) = \langle\, \mathrm{Tr}\,\Gamma_n(C,x_0)\rangle\,. \tag{58}$$

In contrast to (49), we do not have to take into account the effect of vertex terms proportional to B^3 in eq. (58). Hence the structure of BF theory *without* cosmological constant is considerably simpler than the one of the BF theory *with* a cosmological constant.

We now formally prove that the terms $W_n(C)$ are knot-invariants.

In fact, the effect of a small deformation of the knot C is given by

$$\delta W_{2n} = \sum_{i=0}^{n} \langle\, \mathrm{Tr}\,(\Gamma_{2i}F)\rangle + \sum_{i=0}^{n} \langle\, \mathrm{Tr}\,(\Gamma_{2i-1}d_A B)\rangle = 0. \tag{59}$$

In order to prove (59), we have used again equations (51) and (53), while equation (52) did not play any rôle, since, here, there are no vertex terms proportional to B^3, and hence no terms like $\langle A^3\rangle$.

When we consider a knot C (*not* a link) and when the standard framing for C_f is selected, then it has been shown in [13] that

$$W_{2n+1} = 0 \tag{60}$$

holds for any n.

As far as integration by parts for BF theories without cosmological constant is concerned, formulas (38) still hold, provided that κ is set equal to zero. Formula (39) becomes

$$\begin{aligned} \left\langle\, \mathrm{Tr}\,\left[F^a_{\mu\nu}(x)\mathcal{O}\right]\right\rangle &= 4\pi i \sum_\rho \epsilon_{\mu\nu\rho}\left\langle \frac{\delta\mathcal{O}}{\delta B^a_\rho(x)}\right\rangle \\ \left\langle\, \mathrm{Tr}\,\left[(d_A B)^a_{\mu\nu}(x)\mathcal{O}\right]\right\rangle &= 4\pi i \sum_\rho \epsilon_{\mu\nu\rho}\left\langle \frac{\delta\mathcal{O}}{\delta A^a_\rho(x)}\right\rangle. \end{aligned} \tag{61}$$

We consider, once again, a deformation of the knot C and, simultaneously, of its framing C_f, while keeping $lk(C_f,C)=0$. We redo the computations of section VI by using integration by parts and the abelian Stokes formula.

We do not need to take into consideration deformations of type 1 (see section VI), since the imposition of the requirement $lk(C,C_f)=0$ will offset the effect of such deformations.

When we consider deformations of type 2 (see section VI) then the form of our observables shows that the B-field is not path-ordered any more. Hence, instead of (45), the n-th order variation inserts, at the selected crossing point, the matrix $U^{(n)}_{i,j,k,l} \in End(\mathbf{C}^N \otimes \mathbf{C}^N)$, given by

$$U^{(n)}_{i,j,k,l} = \frac{1}{n!} \sum_{a_1,\dots a_n;\sigma} \left(R^{a_1} R^{a_2} \cdots R^{a_n}\right)_{i,j} \left(R^{a_{\sigma(1)}} R^{a_{\sigma(2)}} \cdots R^{a_{\sigma(n)}}\right)_{k,l}, \tag{62}$$

where σ denotes a permutation of $\{1, 2, \cdots, n\}$ and the sum is extended over all permutations and over the indices $\{a_j\}$. It is possible to show that (45) is still a matrix of the form $\alpha_n \, \mathbf{P} + \beta_n \, \mathbf{I}$ [27]. This implies that we still obtain a relation like (47):

$$\langle \, \mathrm{Tr} \, \mathcal{H}(C_\pm; \lambda) \rangle = \alpha^\pm \, \langle \, \mathrm{Tr} \, \mathcal{H}(C_0; \lambda) \rangle + \beta^\pm \, \langle \, \mathrm{Tr} \, \mathcal{H}(C_\times; \lambda) \rangle$$

that, in turn, implies a skein relation that we write as:

$$q(\lambda) \sum \lambda^n W_n(C_+) - q^{-1}(\lambda) \sum \lambda^n W_n(C_-) = z(\lambda) \sum \lambda^n W_n(C_0). \tag{63}$$

In conclusion, the v.e.v. associated to a knot C (or a link) in the BF theory without cosmological constant assigns a skein polynomial

$$P(q(\lambda), z(\lambda))(C)$$

to C. In order to identify this polynomial $P(q(\lambda), z(\lambda))$ with the Alexander–Conway polynomial, we need one further observation: the transformation

$$\lambda \longrightarrow -\lambda$$

can be absorbed, in field theory, by the transformation $B \longrightarrow -B$ that, in turn, is equivalent to a change in the sign of the BF action, or to a change in the *orientation* of the manifold M $(= \mathbf{R}^3)$.

So we have that

$$P(q(-\lambda), z(-\lambda))(C) = P(q(\lambda), z(\lambda))(C^!) = P([q(\lambda)]^{-1}, -z(\lambda))(C) \tag{64}$$

where $C^!$ denotes the mirror image of C. The first identity in (64) is a consequence of the property (CT-symmetry) of field theory mentioned above,while the second identity is a consequence of the skein relation.

We now take a *knot* C, for which (60) implies that $P(q(\lambda), z(\lambda))(C) = P(q(\lambda), z(\lambda))(C^!)$.

Hence, for a knot C, we have $q(\lambda) = 1$ and, when we choose the normalization $P(1, z(\lambda))(\bigcirc) = 1$ for the unknot, then $P(1, z(\lambda))(C)$ must necessarily be the *Alexander–Conway polynomial*.

BF theory without cosmological constant yields the Alexander–Conway polynomial also in the case of links: but for the discussion of this case we refer to [13].

X. Observables for the four-dimensional BF theory

Our purpose, in this section, is only to sketch a few preliminary ideas on how to deal with 4-dimensional situations, leaving further developments to be carried out elsewhere.

As has been mentioned in Section III, the observables associated to a 4-dimensional BF theory must be associated to 2-dimensional surfaces Σ imbedded (or immersed) in the 4-manifold M. As in [16], we can associate to Σ (with a selected point x_0) the quantity:

$$\mathrm{Tr} \int_M \mathrm{Hol}\,(A)^y_{x_0} B(y) \, \mathrm{Hol}\,(A)^{x_0}_y . \tag{65}$$

In this formula the holonomies are meant to be computed along loops with base-point x_0, passing through the point $y \in \Sigma$.

What we would like to do is to associate to each point of the surface Σ a loop with base-point x_0. This can be a difficult task, if we want to preserve smoothness[1].

A simpler situation is encountered if we are given an oriented torus $\mathbf{T} = S^1 \times S^1$. The torus $\mathbf{T}$ is imbedded in M (or, more generally, generically immersed i.e. with only transversal double points).

We still denote by Σ the image of such imbedding (or immersion). Here we can define, as in [3], a special path joining x_0 to the generic point $y \in \mathbf{T}$. If the coordinates of the points x_0, y are $(s_0, t_0), (s, t) \in \mathbf{T}$, then we define a path σ_y by combining a (positively oriented) meridian arc joining (s_0, t_0) to (s, t_0) with a (positively oriented) longitudinal arc joining (s, t_0) to (s, t).

As in section III, we consider the $\mathfrak{g}$-valued 2-form of the adjoint type:

$$\hat{B}(y) \equiv \mathrm{Hol}\,(A, \sigma_y)^y_{x_0} B(y) \left[\mathrm{Hol}\,(A, \sigma_y)^y_{x_0}\right]^{-1} \tag{66}$$

and consider

$$\int_\Sigma \hat{B}(y)$$

Moreover, we can "complete" the above holonomies and obtain:

$$O(\Sigma) \equiv \int_\Sigma \hat{B}(y)\,\mathrm{Hol}\,(A)^{s,t_0}_{s_0,t_0}\,\mathrm{Hol}\,^{(l)}_{s,t_0}\,\mathrm{Hol}\,^{s_0,t_0}_{s,t_0} \tag{67}$$

where by $\mathrm{Hol}\,^{(l)}_{s,t_0}$ we mean the holonomy of the longitudinal circle with base point (s, t_0).

At this point we can exhibit the observable for the 4-dimensional BF theory, namely

$$\mathcal{O}(\Sigma, k) \equiv \mathrm{Tr}\,\exp\left(k\; O(\Sigma)\right). \tag{68}$$

This is an observable for the 4-dimensional BF theory without cosmological constant. We can now compute the relevant v.e.v.'s by perturbation expansion in the coupling constant k. The gauge-choices that we have here at our disposal are either the covariant gauge or the real axial gauge.

BF theory in 4 dimensions should provide the right framework for invariants of 2-knots (embedded surfaces) or of singular 2-knots (generally immersed surfaces).

Preliminary computations (see [16]) suggest that the expression of these invariants in the covariant gauge, should be given in terms of iterated integrals of kernels that are the higher dimensional generalization of (35).

Preliminary computations in another direction, show that the observable (68) could play a rôle in the recovery of some essential information concerning the differentiable structure of the four manifold M (Donaldson polynomial) in the framework of a pure Yang–Mills theory [14].

Finally let us point out that the observable (68) can be a relevant object in the approach to quantum gravity based on loop-variables (see [4] and reference therein). In this framework, imbedded (or generically immersed) surfaces (or tori) represent the time-evolution of the loop variables of [4]. An elementary consideration shows moreover that, once we are given a background metric g (and so a corresponding *-operator) the B-field can provide a fluctuation of the background metric.

[1] We acknowledge a useful discussion with John Baez, in this respect.

In fact one can construct a symmetric tensor

$$h_{\mu,\nu} \equiv f_{a,b,c} B^a_{\mu,\rho} {}^*(B^b)^{\rho,\sigma} B^c_{\sigma,\nu}$$

where $f^{a,b,c}$ denote the structure constant and a sum over repeated indices is understood (see [12]).

We will discuss more about the 4-dimensional BF theory in future work.

Acknowledgements

We thank John Baez and Maurizio Rinaldi for very useful discussions. Two of us (A.S.C. & M.M.) thank E.T.H. Zürich for hospitality.

Appendix: On the coefficients of the skein-polynomials

In this appendix we want to review the representation of the coefficients of the skein-polynomials in terms of suitably defined "tensors" with coefficients in **Z**.

The motivation for this appendix is that this representation is the one obtained from the quantization of the BF theories in the (real) axial gauge, with the fundamental representation of the group $SU(N)$.

First we consider a link diagram L with $|L|$ oriented components, and we denote by $V(L)$ the set of vertices of L (crossing points). We order the set $V(L)$ by ordering arbitrarily the components of L and by choosing arbitrarily a starting point in each component of L. We denote the sign (writhe) of the i-th vertex by $\epsilon_i = \pm 1$. A k-tensor $T \equiv T^{i_1,i_2,\cdots,i_k}$ is defined as a map

$$\underbrace{V(L) \times \cdots \times V(L)}_{k \text{ times}} \longrightarrow \mathbf{Z}\,.$$

Once we are given a k-tensor T, we can "saturate" it with the writhe of the vertices in $V(L)$

$$\sum_{V(L)\times\cdots\times V(L)} \epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k} T^{i_1,i_2,\cdots,i_k}. \tag{69}$$

Here we want to show that the coefficients of the skein polynomials are given by sums of expressions (69).

We denote by S_j the operation of switching the vertex $v_j \in V(L)$ (i.e. changing the writhe) and by E_j the operation of eliminating the vertex v_j in the only orientation-preserving way. Let σ be any sequence of the above operations. Given a k-tensor T on σL we can pull it back to a k-tensor on L by defining:

$$[\sigma^*(T)]^{i_1,i_2,\cdots,i_k} = \begin{cases} 0 & \text{if one of the vertices } v_{i_r} \\ & \text{has been eliminated,} \\ \rho^\sigma(v_{i_1}, v_{i_2}, \cdots, v_{i_k}) T^{\sigma(i_1),\sigma(i_2),\cdots,\sigma(i_k)} & \text{if none of the vertices } v_{i_r} \\ & \text{has been eliminated,} \end{cases}$$

where $\rho^\sigma(v_{i_1}, v_{i_2}, \cdots, v_{i_k})$ is defined as 1 if an even number of the vertices $v_{i_1}, \cdots, v_{i_k}$ has been switched by σ and is defined as -1 otherwise. The pulled-back k-tensor satisfies the relation:

$$\sum_{i_1,i_2,\cdots,i_k} \epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_k} [\sigma^*(T)]^{i_1,i_2,\cdots,i_k}(L) = \sum_{j_1,j_2,\cdots,j_k} \epsilon_{j_1}\epsilon_{j_2}\cdots\epsilon_{j_k} T_n^{j_1,j_2,\cdots,j_n}(\sigma L), \tag{70}$$

where the first sum is extended over all the k-uples of vertices of L, while the second sum is extended over all the k-uples of vertices of $\sigma(L)$.

We now consider the Alexander–Conway polynomial $\Delta(L)(z) \equiv \sum_n a_n(L) z^n$, with the standard normalization conditions.

Let $\{v_{j_1}, v_{j_2}, \cdots, v_{j_s}\}$ be any sequence of vertices of L with the property that when we switch all these vertices then the link diagram L is transformed into the diagram of the unlink $U_{|L|}$ (with $|L|$ components). We have ([5, 23]):

$$\begin{array}{l} a_n(L) - a_n(S_{j_1}L) = \epsilon_{j_1} a_{n-1}(E_{j_1}L) \\ a_n(S_{j_1}L) - a_n(S_{j_2}S_{j_1}L) = \epsilon_{j_2} a_{n-1}(E_{j_2}S_{j_1}L) \\ \cdots\cdots \\ a_n(S_{j_{s-1}}\cdots S_{j_1}L) - a_n(S_{j_s}S_{j_{s-1}}\cdots S_{j_1}L) = \epsilon_{j_s} a_{n-1}(E_{j_s}S_{j_{s-1}}\cdots S_{j_1}L). \end{array} \tag{71}$$

By assumption we have that $S_{j_s}S_{j_{s-1}}\cdots S_{j_1}L = U_{|L|}$ and hence

$$a_n(S_{j_s}S_{j_{s-1}}\cdots S_{j_1}L) = \begin{cases} 1 & \text{if } |L| = 1 \text{ and } n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

When $n > 1$ we obtain:

$$a_n(L) = \sum_{l=1}^{l=s} \epsilon_{j_l} a_{n-1}(E_{j_l}S_{j_{l-1}}\cdots S_{j_1}L). \tag{72}$$

We are now ready to prove, by induction, that for any link L, the n-th coefficient of the Alexander polynomial is given by an expression like:

$$a_n(L) = \sum_{i_1, i_2, \cdots, i_n} \epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_n} A_n^{i_1, i_2, \cdots, i_n}(L), \tag{73}$$

where $A_n(L)$ is a suitable n-tensor with integer entries and the sum is extended over all the n-uples of vertices in L.

For $n = 1$, we have $a_1(L) \equiv \sum_j \epsilon_j A^j(L)$, where the 1-tensor A^j is defined as

$$A^j(L) = \begin{cases} 1 & \text{if } |L| = 2 \text{ and the first component passes } \textit{over} \\ & \text{the second one at the } j\text{-th vertex} \\ 0 & \text{otherwise} \end{cases}. \tag{74}$$

Equation (72) can be rewritten as a sum extended over *all* the vertices of L:

$$a_n(L) = \sum_i \epsilon_i \tilde{a}^i_{n-1}, \tag{75}$$

where $\tilde{a}^i_{n-1}$ is defined as either 0 (when the i-th vertex is not one of vertices v_{j_l} that we need to switch in order to transform L into the unlink) or is given by (72). We now assume that $a_{n-1}(E_{j_l}S_{j_{l-1}}\cdots S_{j_1})$ can be expressed in terms of an $(n-1)$-tensor of the link $(E_{j_l}S_{j_{l-1}}\cdots S_{j_1})$. This implies that it can also be expressed in terms of an $(n-1)$-tensor of the link L, and so eq. (75) directly gives (73).

We finally consider the 1-variable HOMFLY Polynomial $P(\exp(hN), 2\sin(h))$. We represent this polynomial as a power series in the variable h, namely as $\sum_{n=0}^{\infty} a_n h^n$, where it is understood that the coefficients a_n depend on the link L and on the integer N. We choose the following normalization condition for the unlink with k components

$$P(U_k) = \left(\frac{\exp(hN) - \exp(-hN)}{\exp(h) - \exp(-h)}\right)^k. \tag{76}$$

The skein relation and eq. (76) imply that for, any link L, we have $a_0(L) = N^{|L|}$. We also have $a_1(U_k) = 0$, and, for any vertex v_j,

$$a_1(L) - a_1(S_j L) = \epsilon_j 2(N^{|E_j L|} - N^{|L|+1}).$$

This implies that if L is a link with zero linking numbers between its components, then $a_1(L) = 0$. More generally, for a link L with $k > 1$ components, we have:

$$a_1(L) = \sum_s \epsilon_{j_s} 2(N^{k-1} - N^{k+1})$$

where $\{v_{j_s}\}$ is a set of vertices, where different components of L meet and whose switching separates the components.

For the generic coefficient a_n we now have a set of equations that is more complicated than (71)

$$\begin{array}{ll} \sum_{k=0}^{n}(n-k)! & \left[N^{n-k}a_k(L) - (-N)^{n-k}a_k(S_{j_1}L)\right] = \epsilon_{j_1}\sum_{k=0}^{n-1}(n-k)![1-(-1)^{n-k}]a_k(E_{j_1}L) \\ \cdots\cdots & \\ \sum_{k=0}^{n}(n-k)! & \left[(N^{n-k}a_k(S_{j_{s-1}}\cdots S_{j_1}L) - (-N)^{n-k}a_k(S_{j_s}S_{j_{s-1}}\cdots S_{j_1}L)\right] = \\ & \epsilon_{j_s}\sum_{k=0}^{n-1}(n-k)![1-(-1)^{n-k}]a_k(E_{j_s}S_{j_{s-1}}\cdots S_{j_1}L). \end{array} \tag{77}$$

Here $\{v_{j_s}\}$ is a set of vertices of L whose switching transforms L into the unlink $U_{|L|}$.

By an argument similar to the one considered before, instead of (73) we now have equations like:

$$\begin{array}{ll} a_n(L) = \sum_{i_1,i_2,\cdots,i_n}\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_n}A_{n,n}^{i_1,i_2,\cdots,i_n}(L) + & \sum_{i_1,i_2,\cdots,i_{n-1}}\epsilon_{i_1}\epsilon_{i_2}\cdots\epsilon_{i_{n-1}}A_{n,n-1}^{i_1,i_2,\cdots,i_{n-1}}(L) \\ & +\cdots+\sum_i \epsilon_i A_{n,1}^{i}(L), \end{array} \tag{78}$$

where to each index n we associate tensors $A_{n,i}$ of order i for $i = 1, \cdots, n$. In (78) the sums are extended over the set $V(L)$.

References

[1] M.F.Atiyah, R.Bott *The Yang-Mills Equations over Riemann Surfaces* Phil. Trans. R. Soc. Lond. **A** (1982) 523-615

[2] D.Altschuler, L.Freidel *On Universal Vassiliev Invariants* hep-th/9403053

[3] I.Ya.Aref'eva *Non-abelian Stokes formula* Teor. Math. Fiz. **43** 111-116

[4] A.Ashtekar, C.Rovelli, L.Smolin *Weaving a Classical Geometry with Quantum Threads* Phys. Rev. Lett. **69** (1992) 237-240

[5] R.Ball, M.L.Mehta *Sequence of Invariants for knots and Links* J. Physique **42** (1981) 1103-1199

[6] D.Bar-Natan *Perturbative aspects of the Chern–Simons field theory* PhD Thesis, Princeton University (1991)

[7] D.Bar-Natan *On Vassiliev's Knot Invariants*, Topology (to appear)

[8] D.Birmingham, M.Blau, M.Rakowski, G.Thompson *Topological Field theories*, Phys. Rep. **209** (1991) p.129-340

[9] B. Brügmann, *On a geometric derivation of Witten's identity for Chern–Simons theory*, preprint MPI-PH-93-107, hep-th/9401055.

[10] R.Bott, C.Taubes *On the self-linking of knots* J. Math. Phys. **35** (1994) 5247-5287

[11] A.S.Cattaneo *Teorie topologiche di tipo BF ed invarianti dei nodi* PhD Thesis, University of Milan (February 1995)

[12] R.Capovilla, T.Jacobson, J.Dell *General relativity without the metric* Phys. Rev. Lett. **63** (1989) 2325-2328

[13] A.S.Cattaneo, P.Cotta-Ramusino, M.Martellini, *Three-dimensional BF Theories and the Alexander–Conway Invariant of Knots* Nucl. Phys. **B 436**, 355-384

[14] A.S. Cattaneo, P.Cotta-Ramusino, A. Gamba, M.Martellini *The Donaldson-Witten Invariants and Pure QCD with Order and Disorder 't Hooft-like Operators* Milan University preprint (1995) IFUM 493/FT, hep-th/9502110

[15] K.T.Chen *Iterated Path Integrals* Bull. Am. Math. Soc. **83** (1977) 831-879

[16] P.Cotta-Ramusino, M.Martellini, *BF-theories and 2-knots* in *Knots and Quantum Gravity*, J.Baez (ed.), Oxford University Press, Oxford, New York (1994) 169-189

[17] P.Cotta-Ramusino, E.Guadagnini, M.Martellini, M.Mintchev, *Quantum field theory and link invariants*, Nucl. Phys. **B330** p. 557-574 (1990)

[18] P.De La Harpe, M.Kervaire, C.Weber *On the Jones Polynomials*, L'Ens. Math. **32** (1986) 271-335

[19] J.Fröhlich, C.King *The Chern–Simons Theory and Knot-Polynomials* Comm. Math. **126** (1989) 167-199

[20] E.Guadagnini, M.Martellini, M.Mintchev, *Chern–Simons model and new relations between the HOMFLY coefficients*, Phys. Lett. **B228**, p. 489-494 (1989)

[21] P. Freyd, D. Yetter, J. Hoste, W.B.R. Lickorish, K.C. Millet, A. Ocneanu *A new polynomial invariants of Knots and Links* Bull. Am. Math. Soc. **12** (1985) 239-246

[22] V.F.R. Jones *A polynomial invariant for links via Von Neumann algebras* Bull. Am. Math. Soc. **12** (1985) 103-112

[23] L.H.Kauffman *On Knots* Princeton University Press (1987), Princeton N.J.

[24] L.H. Kauffman *The Conway Polynomial*, Topology **20** (1981) 101-108

[25] M.Kontsevich *Vassil'ev Knot Invariants* Adv. Sov. Math. (1992) **16** 137-150

[26] W.B.R. Lickorish, K.C. Millet *A polynomial invariant of Oriented links*, Topology **26**, p.107-141 (1987)

[27] M.Rinaldi, private communication

[28] N.Reshetikhin, V.G.Turaev *Invariants of 3-Manifolds via Link Polynomials and Quantum Groups* Invent. Math. **103** (1991) 547-598

[29] A.S. Schwarz *The Partition Function of a degenerate quadratic functional and the Ray-Singer invariants* Lett. Math. Phys. **2** (1978) 247-252

[30] V.G.Turaev, O.Yu.Viro *State Sum Invariants of 3-Manifolds and Quantum 6j-symbols* Topology **31** (1992) 865-902

[31] E.Witten *Topological Quantum Field Theory* Comm. Math. Phys. **117** (1988) 353-386

[32] E.Witten *Quantum Field Theory and the Jones polynomials* Comm. Math. Phys. **121** (1989) 351-399