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TOPOLOGICAL CLASSIFICATION OF MORSE–SMALE DIFFEOMORPHISMS WITHOUT HETEROCLINIC INTERSECTIONS

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We study the class $G(M^n)$ of orientation-preserving Morse–Smale diffeomorfisms on a connected closed smooth manifold M^n of dimension $n \geqslant 4$ which is defined by the following condition: for any $f \in G(M^n)$ the invariant manifolds of saddle periodic points have dimension 1 *and* $(n-1)$ *and contain no heteroclinic intersections. For diffeomorfisms in $G(M^n)$ we establish the topoligical type of the supporting manifold which is determined by the relation between the numbers of saddle and node periodic orbits and obtain necessary and sufficient conditions for topological conjugacy. Bibliography:* 14 *titles.*

1 Introduction and Formulation of the Results

This paper is a continuation of [1, 2] and is based on the approach developed by the authors in the works (cf., for example, [3]) on the topological classification of Morse–Smale diffeomorfisms on three-dimensional manifolds.

We consider the class $G(M^n)$ of orientation-preserving Morse–Smale diffeomorfisms on a connected closed smooth orientable manifold M^n of dimension $n \geqslant 4$ such that invariant manifolds of any saddle point of a cascade $f \in G(M^n)$ have dimension 1 and $n-1$; moreover, invariant manifolds of distinct saddle points do not intersect. We introduce the notation: Ω_f is the non-

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wandering set of diffeomorphism $f \in G(M^n)$, $\Omega^i_f = \{p \in \Omega_f |\, \dim W^u_p = i\}$, $i \in \{0, 1, n-1, n\}$, and $|P|$ is the cardinality of a set P.

Theorem 1.1. *Let $f \in G(M^n)$. Then $g_f = \big[|\Omega_1 \cup \Omega_{n-1}| - |\Omega_0 \cup \Omega_n| + 2\big]/2$ is a nonnegative integer and the following assertions hold.*

1. If $g_f = 0$, then M^n is a sphere S^n.

2. If $g_f > 0$, then M^n is homeomorphic to the connected sum of g_f copies of the manifold $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

For understanding the dynamics of a diffeomorphism $f \in G(M^n)$ we represent the manifold M^n as the union of the connected attractor

$$A_f = \Big(\bigcup_{\sigma \in \Omega^1_f} W^u_\sigma\Big) \cup \Big(\bigcup_{\omega \in \Omega^0_f} \omega\Big),$$

repeller

$$R_f = \Big(\bigcup_{\sigma \in \Omega^{n-1}_f} W^s_\sigma\Big) \cup \Big(\bigcup_{\alpha \in \Omega^n_f} \alpha\Big),$$

and the set $V_f = M^n \setminus (A_f \cup R_f)$ of wandering orbits of the diffeomorphism f going from A_f to R_f. We denote by $\widehat{V}_f = V_f/f$ the space of orbits of the action of f on V_f, by $p_f : V_f \to \widehat{V}_f$ the natural projection, and by $\eta_f : \pi_1(\widehat{V}_f) \to \mathbb{Z}$ the epimorphism induced by the map p_f (necessary definitions are given in Subsection 2.1). We set

$$\widehat{L}^s_f = \bigcup_{\sigma \in \Omega^1_f} p_f(W^s(\sigma) \setminus \sigma), \quad \widehat{L}^u_f = \bigcup_{\sigma \in \Omega^{n-1}_f} p_f(W^u(\sigma) \setminus \sigma).$$

Definition 1.1. $S_f = (\widehat{V}_f, \eta_f, \widehat{L}^s_f, \widehat{L}^u_f)$ is called the *scheme* of a diffeomorphism $f \in G(M^n)$.

Definition 1.2. The schemes S_f and $S_{f'}$ of diffeomorphisms $f, f' \in G(M^n)$ are *equivalent* if there exists a homeomorphism $\widehat{\varphi} : \widehat{V}_f \to \widehat{V}_{f'}$ such that

1) $\eta_f = \eta_{f'} \widehat{\varphi}_*$,

2) $\widehat{\varphi}(\widehat{L}^s_f) = \widehat{L}^s_{f'}$ and $\widehat{\varphi}(\widehat{L}^u_f) = \widehat{L}^u_{f'}$.

Theorem 1.2. *A necessary and sufficient condition of the topological conjugacy of diffeomorphisms $f, f' \in G(M^n)$ is the equivalence of their schemes S_f and $S_{f'}$.*

It is known (cf., for example, [4, Theorem 2.2.2]) that the space $\widehat{V}_f$ is a smooth connected n-manifold. If the ambient manifold M^n is a sphere, then the dynamics of a diffeomorphism $f \in G(S^n)$ and the topology of $\widehat{V}_f$ can be specified as follows.

Theorem 1.3. *The class $G(S^n)$ exhausts the set of Morse–Smale diffeomorfisms without heteroclinic intersections on S^n and for any diffeomorphism $f \in G(S^n)$ the manifold $\widehat{V}_f$ is homeomorphic to $S^{n-1} \times S^1$.*

The last assertion of Theorem 1.3 is a consequence of the fact that the attractor A_f and the repeller R_f of any diffeomorphism $f \in G(S^n)$ are separated by a sphere S^{n-1} for $n > 3$. Generally speaking, this is not so in the case $n = 3$. A diffeomorphism $f \in G(S^3)$ such that A_f and R_f are not separated by the sphere S^2 and the space of orbits $\widehat{V}_f$ is not homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ was described in [5].

2 Canonical Manifolds and Maps

2.1. Discontinuous actions of transformation groups. We recall some properties of a transformation group $\{g^n, n \in \mathbb{Z}\}$ that is an infinite cyclic group discontinuously acting on some smooth (in general, not compact) manifold X and is generated by the diffeomorphism $g : X \to X$. Such transformation groups naturally appear in the study of restrictions of the original Morse–Smale diffeomorfism to some subset of wandering points and generate topological invariants used for solving problems of topological classification.

Remark 2.1. A group $\mathscr{G}$ *acts* on a manifold X if there is a map $\zeta : \mathscr{G} \times X \to X$ possessing the following properties:

1) $\zeta(e, x) = x$ for all $x \in X$, where e is a neutral (identity) element of the group $\mathscr{G}$,

2) $\zeta(g, \zeta(h, x)) = \zeta(gh, x)$ for all $x \in X$ and $g, h \in \mathscr{G}$.

A group $\mathscr{G}$ *discontinuously acts* on a manifold X if for every compact subset $K \subset X$ the set of elements $g \in \mathscr{G}$ such that $\zeta(g, K) \cap K \neq \varnothing$ is finite.

We denote by X/g the space of orbits of the action of the group $\{g^n, n \in \mathbb{Z}\}$ and by $p_{_{X/g}} : X \to X/g$ the natural projection. By [6, Theorem 3.5.7]), the natural projection $p_{_{X/g}} : X \to X/g$ is a covering map and the space X/g is a manifold. We introduce a homomorphism $\eta_{_{X/g}} : \pi_1(X/g) \to \mathbb{Z}$ as follows. Let $\widehat{c} \subset X/g$ be a loop that is not homotopic to zero in X/g, and let $[\widehat{c}] \in \pi_1(X/g)$ be the homotopic equivalence class of the loop c. We choose an arbitrary point $\widehat{x} \in c$, denote by $p^{-1}_{_{X/g}}(\widehat{x})$ the complete preimage of $\widehat{x}$, and fix the point $\widetilde{x} \in p^{-1}_{_{X/g}}(\widehat{x})$. Since $p_{_{X/g}}$ is a covering, there exists a unique path $\widetilde{c}(t)$ started at the point $\widetilde{x}$ (i.e., $\widetilde{c}(0) = \widetilde{x}$) that covers the loop c (i.e., $p_{_{X/g}}(\widetilde{c}(t)) = \widehat{c}$). Therefore, there exists $n \in \mathbb{Z}$ such that $\widetilde{c}(1) = f^n(\widetilde{x})$. We set $\eta_{_{X/g}}([\widehat{c}]) = n$. From [7, Chapter 18] it follows that the homomorphism $\eta_{_{X/g}}$ is an epimorphism.

The following assertion is proved in [8].

Proposition 2.1. *Let X and Y be connected smooth manifolds, and let $g : X \to X$ and $h : Y \to Y$ be diffeomorphisms such that the groups $\{g^n, n \in \mathbb{Z}\}$ and $\{h^n, n \in \mathbb{Z}\}$ discontinuously act on X and Y respectively. Then the following assertions hold.*

1. *If $\varphi : X \to Y$ is a homeomorphism (diffeomorphism) conjugating the diffeomorphisms f and g, then the map $\widehat{\varphi} : X/g \to Y/h$ given by $\widehat{\varphi} = p_{_{Y/h}} \varphi p^{-1}_{_{X/g}}$ is a homeomorphism (diffeomorphism). Moreover, $\eta_{_{X/g}} = \eta_{_{Y/h}} \varphi_*$, where $\varphi_* : \pi_1(X/g) \to \pi_1(Y/h)$ is the homomorphism induced by the map φ.*

2. *If $\widehat{\varphi} : X/g \to Y/h$ is a homeomorphism (diffeomorphism) such that $\eta_{_{X/g}} = \eta_{_{Y/h}} \varphi_*$, $\widehat{x} \in X/g$, $\widetilde{x} \in p^{-1}_{_{X/g}}(x)$, and $y = \widehat{\varphi}(x)$, $\widetilde{y} \in p^{-1}{}_{Y/h}(y)$, then there exists a unique homeomorphism (diffeomorphism) $\varphi : X \to Y$ conjugating the diffeomorphisms g and h and such that $\varphi(\widetilde{x}) = \widetilde{y}$.*

2.2. Canonical manifolds connected with hyperbolic periodic points. By an *n-ball* (*n-disk*) we mean a manifold homeomorphic to the standard ball $\mathbb{B}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \ldots + x_n^2 \leqslant 1\}$. By an *open n-ball* (*$(n-1)$-sphere*) we mean a manifold homeomorphic to the interior $\operatorname{int} \mathbb{B}^n$ (the boundary $\partial \mathbb{B}^n = \mathbb{S}^{n-1}$) of the ball $\mathbb{B}^n$. We recall that a *locally trivial bundle* is a quadruple $\xi = \{E, B, Y, \pi\}$ where E, B, Y are topological spaces, $\pi : E \to B$ is a continuous map such that the manifold B admits an open covering $\{U\}$ such that for every $U \in \{U\}$ there exists a homeomorphism $\varphi : \pi^{-1}(U) \to U \times Y$ possessing the following property: if $p_1 : U \times Y \to U$ is the projection onto the first factor (i.e., $p_1(x, y) = x$), then

$\pi|_{\pi^{-1}(U)} = p_1\varphi|_{\pi^{-1}(U)}$. The spaces E, B, and Y are called the *space*, *base*, and *fibre* of the locally trivial bundle respectively. The pair $(U, \varphi : \pi^{-1}(U) \to U \times Y)$ is called a *chart*, $\{(U, \varphi)\}$ is referred to as the *atlas* of the locally trivial bundle, and the maximal atlas is called the *structure*. With each closed path $\lambda \subset B$ started and ended at a point x it is associated the homotopic class of homeomorphisms $T_\lambda : \xi_x \to \xi_x$ induced by the coordinate transformation while moving along the loop, called the *monodromy transformation*. The *vector bundle of dimension* n is a locally trivial bundle $\xi = \{E, B, \mathbb{R}^n, \pi\}$ such that for any two charts (U, φ) and (V, ψ) with U and V intersecting at the point x the following condition holds: if $\varphi_x = p_2\varphi|_{\pi^{-1}(x)}$ and $\psi_x = p_2\psi|_{\pi^{-1}(x)}$, then the map $\psi_x^{-1}\varphi_x : \mathbb{R}^n \to \mathbb{R}^n$ is linear (here, $p_2 : U \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection onto the second factor). The fibre $\xi_x = \pi^{-1}(x)$ over a point $x \in B$ is equipped with the structure of vector space relative to which the map $\psi_x : \xi_x \to \mathbb{R}^n$ is an isomorphism of vector spaces. The zero section of a vector bundle is the image $\zeta(B) \subset E$ under the map $\zeta : B \to E$ associating with a point $x \in B$ the zero of the space ξ_x.

Let $a_\nu : \mathbb{R}^n \to \mathbb{R}^n$, $\nu \in \{-1, +1\}$ be a linear map of Euclidean spaces given by

$$a_\nu(x_1, \ldots, x_n) = \Big(\nu\frac{1}{2}x_1, \frac{1}{2}x_2, \ldots, \frac{1}{2}x_n\Big).$$

For each hyperbolic periodic point p of the Morse–Smale diffeomorfism $f : M^n \to M^n$ we denote by $\mathscr{O}_p$ its orbit, by m_p the period, by q_p the dimension of the unstable manifold, and by ν_p the orientation type, i.e., the number equal to $+1$ if $f^{m_p}|_{W_p^u}$ preserves orientation or -1 otherwise.

Proposition 2.2 (cf. [4, Proposition 2.1.1])**.** *Let $f : M^n \to M^n$ be a Morse–Smale diffeomorfism with a periodic hyperbolic point p. Then there exists a homeomorphism $\Psi : W_p^s \setminus p \to \mathbb{R}^{n-q_p} \setminus \{O\}$ such that $f^{m_p}|_{W_p^s \setminus p} = \Psi^{-1} a_{\nu_p} \Psi|_{W_p^s \setminus p}$.*

We set $\mathbb{R}_0^n = \mathbb{R}^n \setminus \{O\}$. The quotient space $\mathbb{K}_{+1}^n = \mathbb{R}_0^n / a_{+1}$ is diffeomorphic to the direct product $\mathbb{S}^{n-1} \times \mathbb{S}^1$. We identify $\mathbb{S}^{n-1} \times \mathbb{S}^1$ with $\mathbb{K}_{+1}^n$. The quotient space $\mathbb{K}_{-1}^n = \mathbb{R}_0^n / a_{-1}$ is called the *standard generalized n-dimensional Klein bottle*, and a manifold homeomorphic to $\mathbb{K}_{-1}^n$ is referred to as the *generalized Klein bottle*. The canonical projection $p_{a_{-1}} : \mathbb{R}_0^n \to \mathbb{K}_{-1}^n$ induces on $\mathbb{K}_{-1}^n$ the structure of locally trivial bundle over $\mathbb{S}^1$ with fibre $\mathbb{S}^{n-1}$. This bundle is nonorientable since the monodromy transformation corresponding to the loop $p_{a_\nu}(l_\nu)$, where l_ν is the segment of the $0x_n$-axis joining the points $(0, \ldots, 0, 1)$ and $(0, \ldots, 1/2)$, changes orientation. Hence K_{-1}^n is a nonorientable mandifold.

Since $\mathbb{R}_0^n$ is a universal covering of $\mathbb{K}_\nu^n$, the fundamental group $\pi_1(\mathbb{K}_\nu^n)$ is isomorphic to the group $\mathbb{Z}$ (cf. [7, Corollary 19.4]).

We set $V_p^s = W_p^s \setminus p$ and $\widehat{V}_p^s = V_p^s / f$. From Propositions 2.1 and 2.2 we obtain the following assertion.

Corollary 2.1. *The quotient space $\widehat{V}_p^s$ is homeomorphic to $\mathbb{K}_{\nu_p}^{n-q_p}$.*

Let $b_\nu : \mathbb{R}^n \to \mathbb{R}^n$, $\nu \in \{+1, -1\}$, be the linear automorphism of Euclidean spaces defined by

$$b_\nu(x_1, x_2, \ldots, x_n) = \Big(\nu\frac{1}{2}x_1, \frac{1}{2}x_2, \ldots, \frac{1}{2}x_{n-1}, 2\nu x_n\Big).$$

We set $U_\tau = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n^2(x_1^2 + \ldots + x_{n-1}^2) \leqslant \tau^2\}$, $\tau \in (0, 1]$, $U = U_1$, $U_0 = \{(x_1, \ldots, x_n) \mid x_n = 0\}$, and $U^u = U \setminus U_0$, $U^s = U \setminus Ox_n$. The origin O is a unique fixed point of the automorphism b_ν; moreover, O is a hyperbolic saddle fixed point such that its stable manifold W_O^s coincides with the hyperplane $x_n = 0$, whereas the unstable manifold W_O^u coincides with the Ox_n-axis.

Proposition 2.3 (cf. [4, Corollary 4.3.2])**.** *Let $\theta : U_\tau \setminus Ox_n \to U^s$ be a topological embedding that is identical on U_0 and satisfies the condition $\theta b_\nu|_{U_\tau} = b_\nu \theta|_{U_\tau}$, $\nu \in \{+1, -1\}$. Let $0 < \tau_1 < \tau_2 < \tau$ be chosen in such a way that $U_{\tau_2} \subset \theta(U_\tau)$, $\theta(U_{\tau_1}) \subset \operatorname{int} U_{\tau_2}$. Then there exists a homeomorphism $\Theta : U \to U$ such that $\Theta b_\nu|_U = b_\nu \Theta|_U$ and $\Theta|_{U_{\tau_1}} = \theta|_{U_{\tau_1}}$, $\Theta|_{U \setminus \operatorname{int} U_{\tau_2}} = \operatorname{id}|_{U \setminus \operatorname{int} U_{\tau_2}}$.*

On U_τ, we introduce two b_ν-invariant foliations T^s and T^u as follows: each fibre $T^s(x_n)$ of T^s is the intersection of the hyperplane parallel to the coordinate plane $x_n = 0$ and passing through the point $(0, \dots, 0, x_n)$ with the set U_τ, whereas each fibre $T^u(x_1, \dots, x_{n-1})$ of T^u is the intersection of the line parallel to the Ox_n-axis and passing through the point $(x_1, \dots, x_{n-1}, 0)$ with U_τ. We denote by $\pi_u : U_\tau \to W^s_O$ and $\pi_s : U_\tau \to W^u_O$ the projections along fibres of the foliations T^u and T^s respectively ($\pi_u(x_1, x_2, \dots, x_{n-1}, x_n) = (0, 0, \dots, 0, x_n)$ and $\pi_s(x_1, x_2, \dots, x_{n-1}, x_n) = (x_1, x_2, \dots, x_{n-1}, 0)$).

Proposition 2.4 (cf. [4, Theorem 2.1.2])**.** *Suppose that $f \in G$ and $\sigma \in \Omega_i(f)$, $i \in \{1, n-1\}$. Then there exists a neighborhood v_σ of σ and a homeomorphism $\chi_\sigma : v_\sigma \to U^1$ such that*

1) $\chi_\sigma f^{m_\sigma}|_{v_\sigma} = a_{\nu_\sigma} \chi_\sigma|_{v_\sigma}$ *if* $i = 1$,
2) $\chi_\sigma f^{m_\sigma}|_{v_\sigma} = a^{-1}_{\nu_\sigma} \chi_\sigma|_{v_\sigma}$ *if* $i = n - 1$.

We set $v^\tau_\sigma = \chi^{-1}_\sigma(U^\tau)$, $T^s_\sigma = \chi^{-1}_\sigma(T^s)$, and $T^u_\sigma = \chi^{-1}_\sigma(T^u)$. Let $\widehat{\mathbb{N}}_\nu = U^s/b_\nu$. The space $\widehat{\mathbb{N}}_\nu$ is called the *canonical neighborhood* of the manifold $\mathbb{K}^{n-1}_\nu$. From definitions we obtain the following assertion.

Proposition 2.5. 1. *$\widehat{\mathbb{N}}_{+1}$ is diffeomorphic to the direct product $\mathbb{K}^{n-1}_{+1} \times [-1, 1]$.*

2. *$\widehat{\mathbb{N}}_-$ is a tubular neighborhood of the zero section of a nonorientable one-dimensional vector bundle over $\mathbb{K}^{n-1}_-$; the boundary $\partial\widehat{\mathbb{N}}_-$ is diffeomorphic to $\mathbb{K}^{n-1}_+$; and $\eta_{b_{-1}}(i_*(\pi_1(\partial\widehat{\mathbb{N}}_-))) = 2\mathbb{Z}$, where $i_* : \pi_1(\partial\widehat{\mathbb{N}}_-) \to \pi_1(\widehat{\mathbb{N}}_-)$ is the homomorphism induced by inclusion.*

3 Dynamics of a Morse–Smale Diffeomorfism $f : M^n \to M^n$ and Topology of Manifold M^n

In this section, we prove Theorems 1.1 and 1.3.

3.1. Dynamics of diffeomorphisms of class $G(M^n)$. If σ is a saddle periodic point of a diffeomorphism $f \in G(M^n)$ of index 1 ($(n-1)$), then we denote by l^s_σ (l^u_σ) the stable (unstable) separatrice of the point σ, i.e., the connection component of the set $W^s_\sigma \setminus \sigma$ ($W^u_\sigma \setminus \sigma$).

The following assertion directly follows from the results of [9] (cf. also [4] for more details).

Proposition 3.1. *The set $\overline{l^u_\sigma} \setminus (l^u_\sigma \cup \sigma)$ consists of a periodic sink point. The set $\overline{l^s_\sigma} \setminus (l^s_\sigma \cup \sigma)$ consists of a periodic source point.*

Corollary 3.1. *For any saddle point σ the closure of its one-dimensional separatrice is a compact arc and the closure of its j-dimensional separatrice, $j > 1$, is a j-sphere.*

We recall that a sphere $S^{n-1} \subset M^n$ is said to be *bicollared* to M^n if there exists a topological embedding $h : \mathbb{S}^{n-1} \times [-1; +1] \to M^n$ such that $h(\mathbb{S}^{n-1} \times \{0\}) = S^{n-1}$.

The following important assertion follows from the results of [10, 11]. For a detailed proof we refer to [1, Lemma 3.2].

Proposition 3.2. *Suppose that $\sigma \in \Omega^1_f$ ($\sigma \in \Omega^{n-1}_f$) and $n \geqslant 4$. Then the sphere $\overline{l^s_\sigma}$ ($\overline{l^u_\sigma}$) is bicollared.*

3.2. Proof of Theorem 1.1. Note that the idea of the proof of Theorem 1.1 is similar to that of Theorem 1 in [12].

Up to a consideration of power, we can assume that Ω_f consists only of fixed points and all the separatrices of saddle points are invariant under the diffeomorphism f. We prove the lemma by induction on the number $r = |\Omega^1_f \cup \Omega^{n-1}_f|$ of saddle points of the diffeomorphism f. We set $l = |\Omega^0_f \cup \Omega^n_f|$.

Let $r = 0$. Then Ω_f consists of exactly two points, source and sink, and the manifold M^n is homeomorphic to the sphere $\mathbb{S}^n$ (cf., for example, [4, Theorem 2.2.1]), so that the required assertion is valid.

Let $r > 0$. We assume that the required assertion is proved for $r' < r$. For the sake of definiteness, we assume that the set Ω^{n-1}_f is nonempty (otherwise, we can proceed with the diffeomorphism f^{-1}). Let $\sigma \in \Omega^{n-1}_f$. By Corollary 3.1 and Proposition 3.2, the manifold $\overline{l}^u_\sigma$ is a cylndrically embedded sphere. Consequently, there exists a closed neighborhood $W_\sigma \subset M^n$ of the sphere $\overline{l}^u_\sigma$ homeomorphic to the direct product $\mathbb{S}^{n-1} \times [-1, 1]$ by means of a homeomorphism ξ such that $\xi(\overline{l}^u_\sigma) = \mathbb{S}^{n-1} \times \{0\}$. Let $S_1 \subset \xi^{-1}(\mathbb{S}^{n-1} \times (-1, 0))$ ($S_2 \subset \xi^{-1}(\mathbb{S}^{n-1} \times (0, 1))$) be an $(n-1)$-sphere that is a smooth submanifold of M^n and such that $\xi^{-1}(\mathbb{S}^{n-1} \times (-1, 0)) \setminus S_1$ ($\xi^{-1}(\mathbb{S}^{n-1} \times (0, 1)) \setminus S_2$) is the union of two disjoint open annuli. We denote by K a closed neighborhood of the sphere $\overline{l}^u_\sigma$ bounded by the spheres S_1 and S_2. Since $\overline{l}^u_\sigma$ is an attractor, without loss of generality we can assume that $f(K) \subset \operatorname{int} K$ (otherwise, it is possible to pass to a suitable power of the diffeomorphism f). Removing the domain $\operatorname{int} K$ from M^n we obtain a compact manifold with two boundary components S_1 and S_2. We denote by M_1 the compact manifold without boundary obtained from $M^n \setminus \operatorname{int} K$ by gluing two closed 3-balls B_1 and B_2 together along the boundary components S_1 and S_2 and introduce a Morse–Smale diffeomorfism $f_1 \colon M_1 \to M_1$ such that f_1 coincides with f on $M^n \setminus K$, has two attracting fixed points $\omega_1 \in B_1$, $\omega_2 \in B_2$ and no other periodic points in $B_1 \cup B_2$. Then f_1 has the same number of fixed points as f and the number of its fixed saddle points is equal to $r - 1$, whereas the number of sinks and sources is equal to $l + 1$. We consider two cases.

Case (a). $M^n \setminus K$ is not connected. In this case, M_1 is the disjoint union of two manifolds $\widetilde{M}_1$ and $\check{M}_1$ and M^n is the connected sum $\widetilde{M}_1 \# \check{M}_1$. Denote by $\widetilde{f}_1$ and $\check{f}_1$ the restrictions of f_1 to the manifolds $\widetilde{M}_1$ and $\check{M}_1$ respectively, by $r_1 = \widetilde{r}_1 + \check{r}_1 = r - 1$ the number of saddle points and by $l_1 = \widetilde{l}_1 + \check{l}_1 = l + 1$ the number of sinks and sources of the diffeomorphism f_1. Since $\widetilde{r}_1$ and $\check{r}_1$ are strictly less than r, from the induction assumption it follows that the manifolds $\widetilde{M}_1$ and $\check{M}_1$ are the connected of $\widetilde{m}_1 = (\widetilde{r}_1 - \widetilde{l}_1)/2 + 1$ and $\check{m}_1 = (\check{k}_1 - \widehat{l}_1)/2 + 1$ copies of $\mathbb{S}^{n-1} \times \mathbb{S}^1$ respectively (by a manifold of 0 copies of $\mathbb{S}^{n-1} \times \mathbb{S}^1$ we understand the manifold $\mathbb{S}^n$). Consequently, M^n is the connected sum of

$$\frac{\widetilde{r}_1 - \widetilde{l}_1}{2} + 1 + \frac{\widehat{r}_1 - \widehat{l}_1}{2} + 1 = \frac{r_1 - l_1}{2} + 2 = \frac{r - l}{2} + 1$$

copies of $\mathbb{S}^{n-1} \times \mathbb{S}^1$. Thus, the theorem is valid in case (a).

Case (b). $M^n \setminus K$ is connected. In this case, M_1 is connected and $M^n = M_1 \# M_*$, where M_* is diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$ (cf., for example, [13, Lemma 7]). We again denote by r_1 the number of saddles and by l_1 the number of sinks and sources of f_1. Since $r_1 = r - 1$, from the induction assumption it follows that M_1 is the sphere S^n if $\frac{r_1 - l_1}{2} + 1 = 0$ or the connected sum of $\frac{r_1-l_1}{2}+1$ copies of $\mathbb{S}^{n-1}\times\mathbb{S}^1$. Since $(r-l)/2+1=((r_1-l_1)/2+1)+1$, we find that M^n is the connected sum of $(r-l)/2+1$ copies of $\mathbb{S}^{n-1}\times\mathbb{S}^1$. Thus, the theorem holds in case (b).

3.3. Proof of Theorem 1.3 consists of three lemmas.

Lemma 3.1. *Let $f: S^n\to S^n$ be a orientation-preserving Morse–Smale diffeomorfism without heteroclinic intersections, and let $n>3$. Then $f\in G(S^n)$.*

Proof. It suffices to prove that for a diffeomorphism f the set $\Omega_f^j=\{p\in\Omega_f\mid \dim W_p^u=j\}$ is empty if $1<j<n-1$. Assume the contrary. Let $1<j<(n-1)$, $\Omega_f^j\neq\varnothing$, and let $\sigma\in\Omega_f^j$. By Corollary 3.1, the closures $\overline{W_\sigma^u}$ and $\overline{W_\sigma^s}$ of the stable and unstable manifolds of the point σ are spheres of dimension j and $n-j$ respectively. We set $S^j=\overline{W_\sigma^u}, S^{n-j}=\overline{W_\sigma^s}$. By the conditions defining the class G, the spheres S^j, S^{n-j} transversally intersect at a single point σ. Hence the index of the intersection of S^j, S^{n-j} is equal to ±1 (the sign depends on the choice of orientation of the spheres S^j, S^{n-j}, and S^n). From [14] it follows that the index of the intersection of any closed submanifolds of the sphere S^n vanishes. The obtained contradiction proves that $\Omega_f^j=\varnothing$. □

Proposition 3.3 proved in [1, Lemma 4.1] plays a key role in the proof of Lemma 3.2.

Proposition 3.3. *Suppose that ω is a sink periodic point of a diffeomorphism f with period m_ω and $\gamma_\omega^1,\dots,\gamma_\omega^k$ are all the one-dimensional separatrices of the saddle points $\sigma_1,\dots,\sigma_{k_\omega}$ lying in W_ω^s. Then there exists a bicollared $(n-1)$-sphere $S_\omega\subset W_\omega^s$ bounding an open n-ball $B_\omega\subset W_\omega^s$, $B_\omega\supset\omega$ and such that*

1) $f^{m_\omega}(S_\omega)\subset B_\omega$,

2) *for any $i\in\{1,\dots,k\}$ the intersection $\gamma_\omega^i\cap S_\omega$ consists of the single point z_ω^i,*

3) *the sphere S_ω is smooth in some neighborhood $V_{z_\omega^i}$ of the point z_ω^i.*

Lemma 3.2. *For any diffeomorphism $f\in G(S^n)$ there exists a bicollared sphere $S^{n-1}\subset V_f$ bounding an open ball B^n, $A_f\subset B^n\subset S^n\setminus R_f$, and such that $f(S^{n-1})\subset B^n$.*

Proof. Let $B_{\omega,0},B_{\omega,1},\dots,B_{\omega,m_\omega-1}$ be a sequence of balls that are bounded by pairwise disjoint spheres $S_{\omega,0},S_{\omega,1},\dots,S_{\omega,m_\omega-1}$ respectively, possess the properties described in Proposition 3.3, and $B_{\omega,0}\subset B_{\omega,1}\subset\dots\subset B_{\omega,m_\omega-1}\subset f^{-m_\omega}(B_{\omega,0})$. We choose exactly one point in each sink periodic orbit and denote by $\widetilde{\Omega}_f^0$ the obtained set. For each point $\omega\in\widetilde{\Omega}_f^0$ we set

$$B_\omega=\bigcup_{j=0}^{m_\omega-1} f^j(\overline{B_{\omega,j}}).$$

One can directly verify that $f(B_\omega)\subset \operatorname{int} B_\omega$. We set $B=\bigcup\limits_{\omega\in\widetilde{\Omega}_f^0} B_\omega$.

Let $\mathscr{O}_\sigma$ be a saddle periodic orbit of period m_σ and index 1. By the hyperbolicity of a point $\sigma\in\mathscr{O}_\sigma$, there exists a neighborhood U_σ of the orbit $\mathscr{O}_\sigma$ where the so-called local Morse–Lyapunov function is defined, i.e., a smooth function $\psi_\sigma: U_\sigma\to\mathbb{R}$ such that

1) $\psi_\sigma(f(x))<\psi_\sigma(x)$ for any $x\in f^{-1}(U_\sigma)\setminus\mathscr{O}_\sigma$ and $\psi_\sigma(f(\sigma))=\psi_\sigma(\sigma)=0$ for any $\sigma\in\mathscr{O}_\sigma$,

2) the set of critical points of ψ_σ coincides with $\mathscr{O}_\sigma$ and every critical point has index 1,

3) for any point $\sigma \in \mathscr{O}_\sigma$ there exist local coordinates $(x_1, \dots, x_n)$ such that $W^u_\sigma \cap U_\sigma \subset Ox_n$, $W^s_\sigma \cap U_\sigma \subset Ox_1 \dots x_{n-1}$ and the function ψ_σ has the form $\psi_\sigma(x_1, \dots, x_n) = x_1^2 + \dots + x_{n-1}^2 - x_n^2$.

The construction of such a function can be found in [4, Lemma 2.2.1].

We choose exactly one saddle periodic point in each saddle orbit of index 1 and denote by $\widetilde{\Omega}^1_f$ the obtained set. We choose smooth $(n-1)$-disks $D_+, D_- \subset \partial B$ containing the points $z_+ = \partial B \cap W^u_\sigma$ and $z_- = \partial B \cap W^u_\sigma$ respectively. By the λ-lemma (cf., for example, [4, Lemma 1.2.1]), for any $\varepsilon > 0$ there exists a natural number k_σ such that the connected component K_+ (K_-) of the set $f^{-km_\sigma}(D_+) \cap U_\sigma$ $(f^{-km_\sigma}(D_-) \cap U_\sigma)$ containing the point $f^{-km_\sigma}(z_+)$ $(f^{-km_\sigma}(z_-))$ and the set $W^s_\sigma \cap U_\sigma$ are ε-C^1-close for any $k > k_\sigma$. Consequently, there exists $c_\sigma > 0$ such that the set $H_{\sigma,c} = \{(x_1, \dots, x_n) \in U_\sigma : x_1^2 + \dots + x_{n-1}^2 - x_n^2 \leqslant c\}$ transversally intersects K_+ (K_-) along the $(n-1)$-disk for all $c < c_\sigma$.

We set $\mathbf{k} = \max\limits_{\sigma \in \widetilde{\Omega}^1_f} k_\sigma$, $\mathbf{c} = \min\limits_{\sigma \in \widetilde{\Omega}^1_f} c_\sigma$, and $H_\sigma = \bigcup\limits_{i=0}^{m_\sigma - 1} f^i(H_{\sigma,\mathbf{c}})$. By the definition of the Morse–Lyapunov function, $f(H_\sigma) \subset \operatorname{int} H_\sigma$. We set $H = \bigcup\limits_{\sigma \in \widetilde{\Omega}^1_f} H_\sigma$.

Since the supporting manifold is the sphere S^n and the closures of stable manifolds of saddle periodic points in Ω^1_f are the spheres S^{n-1}, it follows that the spheres S^{n-1} divide S^n into $|\Omega^1_f| + 1$ open balls that are the sink basins. Then $\Omega^n_f = |\Omega^1_f| + 1$ and the attractor A_f does not contain subsets homeomorphic to a circle. Then $f^{-\mathbf{k}}(B) \cup H$ is the ball B^n and its boundary ∂B^n is the sought sphere. $\square$

Lemma 3.3. *Let $f \in G(S^n)$, $n > 3$. Then $\widehat{V}_f$ is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$.*

Proof. By the annulus theorem, the set $B^n \setminus \operatorname{int} f(B^n)$ is homeomorphic to the annulus $\mathbb{S}^{n-1} \times [0,1]$. By the definition of the ball B^n, the space of orbits $\widehat{V}_f$ is homeomorphic to the manifold obtained from $B^n \setminus \operatorname{int} f(B^n)$ by gluing together the connected components of its boundary by the diffeomorphism f. Thus, $\widehat{V}_f$ is homeomorphic to $\mathbb{S}^{n-1} \times \mathbb{S}^1$. $\square$

4 Necessary and Sufficient Conditions for Topological Conjugacy of $G(M^n)$

In Section 1, we defined the scheme $S_f = (\widehat{V}_f, \eta_f, \widehat{L}^s_f, \widehat{L}^u_f)$ of a diffeomorphism $f \in G(M^n)$, where $\widehat{V}_f$ is the space of orbits of the action of the diffeomorphism f on the manifold V_f, $\eta_f : \pi_1(\widehat{V}_f) \to \mathbb{Z}$ is an epimorphism, $\widehat{L}^s_f$, $\widehat{L}^u_f$ are the projections of the $(n-1)$-dimensional separatrices of saddle periodic points of f onto the manifold $\widehat{V}_f$. We denote by $\gamma^s_\sigma(\gamma^u_\sigma)$ the connected component of $\widehat{L}^s_f$ $(\widehat{L}^u_f)$.

From Subsection 2.1 it follows that the scheme S_f possesses the following properties.

Proposition 4.1. *Let $f \in G(M^n)$. Then the following assertions hold.*

1. *The space $\widehat{V}_f$ is a smooth connected n-manifold.*

2. *$\gamma^\delta_\sigma, \delta \in \{s, u\}$ is a smooth submanifold of $\widehat{V}_f$ diffeomorphic to $\mathbb{K}^{n-1}_{\nu_\sigma}$ and $\eta_f(i_*(\pi_1(\gamma^\delta_\sigma))) = m_\sigma \mathbb{Z}$, where $i : \gamma^\delta_\sigma \to \widehat{V}_f$ is the inclusion map.*

Proof of Theorem 1.2. *Necessity* follows from Proposition 2.1.

Sufficiency. Assume that the schemes S_f and $S_{f'}$ are equivalent via the homeomorphism $\widehat{\varphi} : \widehat{V}_f \to \widehat{V}_{f'}$. We construct step by step a homeomorphism $h : M^n \to M^n$ conjugating the diffeomorphisms f and f'.

Step 1. By Proposition 2.1, there exists a lifting $\varphi : V_f \to V_{f'}$ of the homeomorphism $\widehat{\varphi}$ that is a homeomorphism conjugating the diffeomorphisms $f|_{V_f}$ and $f'|_{V_{f'}}$ such that for any saddle point $\sigma \in \Omega^1_f$ ($\sigma \in \Omega^{n-1}_f$) there is a point $\sigma' \in \Omega^1_{f'}$ ($\sigma' \in \Omega^{n-1}_{f'}$) such that $\varphi(W^s_\sigma \setminus \sigma) = W^s_{\sigma'} \setminus \sigma'$ ($\varphi(W^u_\sigma \setminus \sigma) = W^u_{\sigma'} \setminus \sigma'$). Thus, the homeomorphism φ is uniquely extended to saddle points.

Step 2. We choose exactly one point in each saddle orbit of index 1 and denote by $\widetilde{\Omega}^1_f$ the obtained set. Proposition 2.4 and the absence of heteroclinic intersections imply the existence of a family of pairwise disjoint neighborhoods $\{v_\sigma\}(\{v'_\sigma\})$ of saddle points in $\widetilde{\Omega}^1_f$ ($\widetilde{\Omega}^1_{f'}$) and maps $\chi_\sigma : v_\sigma \to U^1$ ($\chi_{\sigma'} : v_{\sigma'} \to U^1$) conjugating the restriction of the diffeomorphism f^{m_σ} ($f'^{m_{\sigma'}}$) on v_σ ($v_{\sigma'}$) and the diffeomorphism $a_\nu|_{U^1}$. We set $\varphi^u_\sigma = \chi^{-1}_{\sigma'}\chi_\sigma|_{W^u_\sigma}$. We choose $\tau \in (0,1]$ such that the topological embedding $\psi : v^\tau_\sigma \to v_{\sigma'}$ is well defined on the set v^τ_σ by the formula

$$\psi(x) = T^s_{\sigma'}(\varphi(\pi^s_\sigma(x))) \cap T^u_{\sigma'}(\varphi^u_\sigma(\pi^u_\sigma(x)))$$

and $\psi(v^\tau_\sigma \setminus W^u_\sigma) \subset \varphi(v^\tau_\sigma \setminus W^u_\sigma)$. We define the topological embedding $\theta_\sigma : v^\tau_\sigma \to v_\sigma$ by the equality $\theta = \varphi^{-1}\psi$. By Proposition 2.3, there exists a number $0 < \tau_1 < \tau$ and a homeomorphism $\Theta : v_\sigma \to v_\sigma$ coinciding with θ on $v^{\tau_1}_\sigma$ and identical on ∂v_σ. We define homeomorphisms

$$h_{\sigma,\sigma'} : v_\sigma \to v'_\sigma, \quad h_{O(\sigma),O(\sigma')} : \bigcup_{i=0}^{m_\sigma - 1} V_{f^i(\sigma)} \to \bigcup_{i=0}^{m_\sigma - 1} V_{f'^i(\sigma')}$$

by the equalities

$$h_{\sigma,\sigma'} = \varphi\Theta, \quad h_{O(\sigma),O(\sigma')} = f'^i h_{\sigma,\sigma'} f^{-i}(x), \quad x \in V_{f^i(\sigma)}$$

and denote by

$$H_1 : \bigcup_{\sigma \in \Omega^1_f} v_\sigma \to \bigcup_{\sigma' \in \Omega^1_{f'}} v_{\sigma'}$$

a homeomorphism coinciding with $h_{O(\sigma),O(\sigma')}$ at each point $\sigma \in \Omega^1_f$.

Step 3. For points of Ω^{n-1}_f we repeat the constructions of Step 2 with s and a_ν replaced by u a^{-1}_ν. The obtained homeomorphism is denoted by

$$H_{n-1} : \bigcup_{\sigma \in \Omega^{n-1}_f} v_\sigma \to \bigcup_{\sigma' \in \Omega^{n-1}_{f'}} v_{\sigma'}.$$

Step 4. We define the homeomorphism $H : M^n \setminus (\Omega^0_f \cup \Omega^{n-1}_f) \to M^n \setminus (\Omega^0_{f'} \cup \Omega^{n-1}_{f'})$ by

$$H(x) = \begin{cases} \varphi(x), & x \in M^n \setminus \bigcup\limits_{\sigma \in \Omega^1_f \cup \Omega^{n-1}_f} v_\sigma, \\ H_\delta(x), & x \in v_\sigma, \end{cases}$$

where $\sigma \in \Omega^\delta_f$, $\delta \in \{1, n-1\}$, and extend the homeomorphism H to the set $\Omega^0_f, \Omega^{n-1}_f$ in such a way that the obtained homeomorphism $\mathbf{H} : M^n \to M^n$ satisfies the condition $f' = \mathbf{H}^{-1} f \mathbf{H}$. $\square$

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