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Topological Complexity of Motion Planning*

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Abstract. In this paper we study a notion of topological complexity $\mathbf{TC}(X)$ for the motion planning problem. $\mathbf{TC}(X)$ is a number which measures discontinuity of the process of motion planning in the configuration space X. More precisely, $\mathbf{TC}(X)$ is the minimal number k such that there are k different "motion planning rules," each defined on an open subset of $X \times X$, so that each rule is continuous in the source and target configurations. We use methods of algebraic topology (the Lusternik–Schnirelman theory) to study the topological complexity $\mathbf{TC}(X)$. We give an upper bound for $\mathbf{TC}(X)$ (in terms of the dimension of the configuration space X) and also a lower bound (in terms of the structure of the cohomology algebra of X). We explicitly compute the topological complexity of motion planning for a number of configuration spaces: spheres, two-dimensional surfaces, products of spheres. In particular, we completely calculate the topological complexity of the problem of motion planning for a robot arm in the absence of obstacles.

1. Definition of Topological Complexity

Let X be the space of all possible configurations of a mechanical system. In most applications the configuration space X comes equipped with a structure of topological space. The motion planning problem consists of constructing a program or a device, which takes pairs of configurations $(A, B) \in X \times X$ as an input and produces as an output a continuous path in X, which starts at A and ends at B, see [4], [6], and [7]. Here A is the initial configuration, and B is the final (desired) configuration of the system.

We assume below that the configuration space X is path-connected, which means that for any pair of points of X there exists a continuous path in X connecting them.

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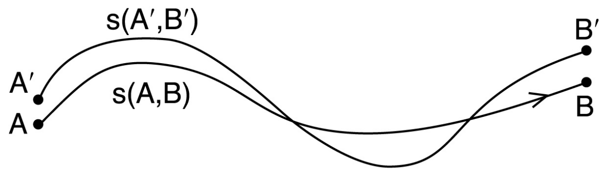


Fig. 1. Continuity of motion planning: close initial-final pairs (A, B) and (A', B') produce close movements $s(A, B)$ and $s(A', B')$.

Otherwise, the motion planner has first to decide whether the given points A and B belong to the same path-connected component of X.

The motion planning problem can be formalized as follows. Let PX denote the space of all continuous paths $\gamma\colon [0,1] \to X$ in X. We denote by $\pi\colon PX \to X \times X$ the map associating to any path $\gamma \in PX$ the pair of its initial and end points $\pi(\gamma) = (\gamma(0), \gamma(1))$. Equip the path space PX with compact-open topology. Rephrasing the above definition we see that the problem of motion planning in X consists of finding a function $s\colon X \times X \to PX$ such that the composition $\pi \circ s = \mathrm{id}$ is the identity map. In other words, s must be a section of π.

Does there exist a continuous motion planning in X? Equivalently, we ask whether it is possible to construct a motion planning in the configuration space X so that the continuous path $s(A, B)$ in X, which describes the movement of the system from the initial configuration A to the final configuration B, depends continuously on the pair of points (A, B)? (See Fig. 1.) In other words, does there exist a motion planning in X such that the section $s\colon X \times X \to PX$ is continuous?

Continuity of motion planning is an important natural requirement. Absence of continuity will result in the instability of behavior: there will exist arbitrarily close pairs (A, B) and (A', B') of initial-desired configurations such that the corresponding paths $s(A, B)$ and $s(A', B')$ are not close.

Unfortunately, as the following theorem states, a continuous motion planning exists only in very simple situations.

Theorem 1. *A continuous motion planning $s\colon X \times X \to PX$ exists if and only if the configuration space X is contractible.*

Proof. Suppose that a continuous section $s\colon X \times X \to PX$ exists. Fix a point $A_0 \in X$ and consider the homotopy

$$h_t\colon X \to X, \qquad h_t(B) = s(A_0, B)(t),$$

where $B \in X$ and $t \in [0, 1]$. We have $h_1(B) = B$ and $h_0(B) = A_0$. Thus h_t gives a contraction of the space X into the point $A_0 \in X$.

Conversely, assume that there is a continuous homotopy $h_t\colon X \to X$ such that $h_0(A) = A$ and $h_1(A) = A_0$ for any $A \in X$. Given a pair $(A, B) \in X \times X$, we may compose the path $t \mapsto h_t(A)$ with the inverse of $t \mapsto h_t(B)$, which gives a continuous motion planning in X.

Thus, we get a motion planning in a contractible space X by first moving A into the base point A_0 along the contraction, and then following the inverse of the path, which brings B to A_0. □

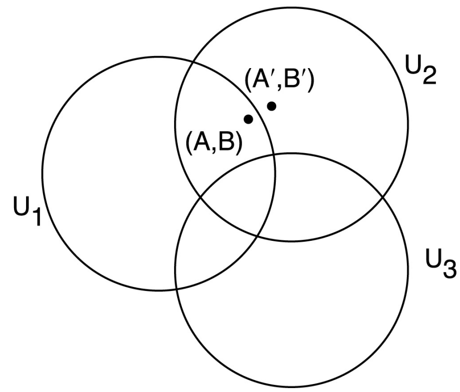


Fig. 2. Discontinuity of the motion planner corresponding to a covering $\{U_i\}$.

Definition 2. Given a path-connected topological space X, we define the *topological complexity of the motion planning* in X as the minimal number $\mathbf{TC}(X) = k$, such that the Cartesian product $X \times X$ may be covered by k open subsets

$$X \times X = U_1 \cup U_2 \cup \cdots \cup U_k \tag{1}$$

such that for any $i = 1, 2, \ldots, k$ there exists a continuous motion planning $s_i \colon U_i \to PX$, $\pi \circ s_i = \text{id}$ over U_i. If no such k exists we will set $\mathbf{TC}(X) = \infty$.

Intuitively, the topological complexity $\mathbf{TC}(X)$ is the measure of discontinuity of any motion planner in X.

Given an open cover (1) and sections s_i as above, one may organize a motion planning algorithm as follows. Given a pair of initial-desired configurations (A, B), we first find the subset U_i with the smallest index i such that $(A, B) \in U_i$ and then we give the path $s_i(A, B)$ as an output. Discontinuity of the output $s_i(A, B)$ as a function of the input (A, B) is obvious: suppose that (A, B) is close to the boundary of U_1 (see Fig. 2) and to a pair $(A', B') \in U_2 - U_1$; then the output $s_1(A, B)$ compared with $s_2(A', B')$ may be completely different, since the sections $s_1|_{U_1 \cap U_2}$ and $s_2|_{U_1 \cap U_2}$ are in general distinct.

According to Theorem 1, we have $\mathbf{TC}(X) = 1$ if and only if the space X is contractible.

Example. Suppose that X is a convex subset of a Euclidean space $\mathbf{R}^n$. Given a pair of initial-desired configurations (A, B), we may move with constant velocity along the straight line segment connecting A and B. This clearly produces a continuous algorithm for the motion planning problem in X. This is consistent with Theorem 1: we have $\mathbf{TC}(X) = 1$ since X is contractible.

Example. Consider the case when $X = S^1$ is a circle. Since S^1 is not contractible, we know that $\mathbf{TC}(S^1) > 1$. Let us show that $\mathbf{TC}(S^1) = 2$. Define $U_1 \subset S^1 \times S^1$ as $U_1 = \{(A, B); A \neq -B\}$. A continuous motion planning over U_1 is given by the map $s_1 \colon U_1 \to PS^1$ which moves A towards B with constant velocity along the unique shortest arc connecting A to B. This map s_1 cannot be extended to a continuous map on the pairs of antipodal points $A = -B$. Now define $U_2 = \{(A, B); A \neq B\}$. Fix an orientation of the circle S^1. A continuous motion planning over U_2 is given by the map $s_2\colon U_2 \to PS^1$ which moves A towards B with constant velocity in the positive direction along the circle. Again, s_2 cannot be extended to a continuous map on the whole $S^1 \times S^1$.

Remark. Our definition of the topological complexity $\mathbf{TC}(X)$ is motivated by the notion of a genus of a fiber space, introduced by Schwarz [5]. In fact $\mathbf{TC}(X)$ is the Schwarz genus of the path space fibration $PX \to X \times X$.

The theory of Schwarz genus was used by Smale [8] and Vassiliev [10], [11] to define the topological complexity of algorithms of finding roots of polynomial equations.

2. Homotopy Invariance

The following property of homotopy invariance often allows us to simplify the configuration space X without changing the topological complexity $\mathbf{TC}(X)$.

Theorem 3. $\mathbf{TC}(X)$ *depends only on the homotopy type of* X.

Proof. Suppose that X *dominates* Y, i.e., there exist continuous maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$. We show that then $\mathbf{TC}(Y) \leq \mathbf{TC}(X)$. Assume that $U \subset X \times X$ is an open subset such that there exists a continuous motion planning $s\colon U \to PX$ over U. Define $V = (g \times g)^{-1}(U) \subset Y \times Y$. We construct a continuous motion planning $\sigma\colon V \to PY$ over V explicitly. Fix a homotopy $h_t\colon Y \to Y$ with $h_0 = \mathrm{id}_Y$ and $h_1 = f \circ g$; here $t \in [0, 1]$. For $(A, B) \in V$ and $\tau \in [0, 1]$ set

$$\sigma(A, B)(\tau) = \begin{cases} h_{3\tau}(A), & \text{for} \quad 0 \leq \tau \leq \frac{1}{3}, \\ f(s(gA, gB)(3\tau - 1)), & \text{for} \quad \frac{1}{3} \leq \tau \leq \frac{2}{3}, \\ h_{3(1-\tau)}(B), & \text{for} \quad \frac{2}{3} \leq \tau \leq 1. \end{cases}$$

Thus we obtain that for $k = \mathbf{TC}(X)$ any open cover $U_1 \cup \cdots \cup U_k = X \times X$ with a continuous motion planning over each U_i defines an open cover $V_1 \cup \cdots \cup V_k$ of $Y \times Y$ with similar properties. This proves that $\mathbf{TC}(Y) \leq \mathbf{TC}(X)$, and obviously implies the statement of the theorem. □

3. An Upper Bound for $\mathbf{TC}(X)$

Theorem 4. *For any path-connected paracompact locally contractible topological space* X, *we have*

$$\mathbf{TC}(X) \leq 2 \cdot \dim X + 1. \tag{2}$$

In particular, if X *is a connected polyhedral subset of* $\mathbf{R}^n$, *then the topological complexity* $\mathbf{TC}(X)$ *can be estimated from above as follows*:

$$\mathbf{TC}(X) \leq 2n - 1. \tag{3}$$

Here $\dim(X)$ denotes the covering dimension of paracompact X. Recall that $\dim(X) \leq n$ iff any open cover of X has a locally finite open refinement such that no point of X belongs to more than $n+1$ open sets of the refinement. If X is a polyhedron, then $\dim(X)$ coincides with the maximum of the dimensions of the simplices of X.

A topological space X is called *locally contractible* if any point of X has an open neighborhood $U \subset X$ such that the inclusion $U \to X$ is null-homotopic.

We use a relation between $\mathbf{TC}(X)$ and the Lusternik–Schnirelman category $\mathrm{cat}(X)$. Recall that $\mathrm{cat}(X)$ is defined as the smallest integer k such that X may be covered by k open subsets $V_1 \cup \cdots \cup V_k = X$ with each inclusion $V_i \to X$ null-homotopic.

Theorem 5. *If X is path-connected and paracompact, then*

$$\mathrm{cat}(X) \leq \mathbf{TC}(X) \leq 2 \cdot \mathrm{cat}(X) - 1. \tag{4}$$

Proof. Let $U \subset X \times X$ be an open subset such that there exists a continuous motion planning $s\colon U \to PX$ over U. Let $A_0 \in X$ be a fixed point. Denote by $V \subset X$ the set of all points $B \in X$ such that (A_0, B) belongs to U. Then clearly the set V is open and is contractible in X.

If $\mathbf{TC}(X) = k$ and $U_1 \cup \cdots \cup U_k$ is a covering of $X \times X$ with a continuous motion planning over each U_i, then the sets V_i, where $A_0 \times V_i = U_i \cap (A_0 \times X)$ form a categorical open cover of X. This shows that $\mathbf{TC}(X) \geq \mathrm{cat}(X)$.

The second inequality follows from the obvious inequality

$$\mathbf{TC}(X) \leq \mathrm{cat}(X \times X)$$

combined with $\mathrm{cat}(X \times X) \leq 2 \cdot \mathrm{cat}(X) - 1$, see Proposition 2.3 of [3]. □

Proof of Theorem 4. It is well known that under the above assumptions $\mathrm{cat}(X) \leq \dim(X) + 1$, see Proposition 2.1 of [3]. Together with the right-hand inequality in (4) this gives (2).

If $X \subset \mathbf{R}^n$ is a connected polyhedral subset, then X has a homotopy type of an $(n-1)$-dimensional polyhedron Y. Using the homotopy invariance (Theorem 3) we find $\mathbf{TC}(X) = \mathbf{TC}(Y) \leq 2(n-1) + 1 = 2n - 1$. □

Remark. Consider the following example. Let $X \subset \mathbf{R}^2$ be the union of circles C_n, where $n = 1, 2, \ldots$ and the center of C_n is at point $(1/n, 0)$ and the radius of C_n equals $1/n$. The point $(0, 0) \in X$ has no neighborhoods which are contractbible in X. Hence $\mathrm{cat}(X) = +\infty$ although $\dim X = 1$. This example shows that Proposition 2.1 from [3] is false without assuming local contractibility of X.

4. A Lower Bound for $\mathbf{TC}(X)$

Let $\mathbf{k}$ be a field. The cohomology $H^*(X; \mathbf{k})$ is a graded $\mathbf{k}$-algebra with the multiplication

$$\cup\colon\ H^*(X; \mathbf{k}) \otimes H^*(X; \mathbf{k}) \to H^*(X; \mathbf{k}) \tag{5}$$

given by the cup-product, see [1] and [9]. The tensor product $H^*(X;\mathbf{k}) \otimes H^*(X;\mathbf{k})$ is also a graded $\mathbf{k}$-algebra with the multiplication

$$(u_1 \otimes v_1) \cdot (u_2 \otimes v_2) = (-1)^{|v_1|\cdot|u_2|}\, u_1u_2 \otimes v_1v_2. \tag{6}$$

Here $|v_1|$ and $|u_2|$ denote the degrees of cohomology classes v_1 and u_2 correspondingly. The cup-product (5) is an algebra homomorphism.

Definition 6. The kernel of homomorphism (5) is called *the ideal of the zero-divisors* of $H^*(X;\mathbf{k})$. The *zero-divisors-cup-length* of $H^*(X;\mathbf{k})$ is the length of the longest nontrivial product in the ideal of the zero-divisors of $H^*(X;\mathbf{k})$.

Example. Let $X = S^n$. Let $u \in H^n(S^n;\mathbf{k})$ be the fundamental class, and let $1 \in H^0(S^n;\mathbf{k})$ be the unit. Then $a = 1 \otimes u - u \otimes 1 \in H^*(S^n;\mathbf{k}) \otimes H^*(S^n;\mathbf{k})$ is a zero-divisor, since applying homomorphism (5) to a we obtain $1 \cdot u - u \cdot 1 = 0$. Another zero-divisor is $b = u \otimes u$, since $u^2 = 0$. Computing $a^2 = a \cdot a$ by means of rule (6) we find

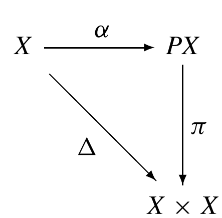
$$a^2 = ((-1)^{n-1} - 1) \cdot u \otimes u.$$

Hence $a^2 = -2b$ for n even and $a^2 = 0$ for n odd; the product ab vanishes for any n. We conclude that *the zero-divisors-cup-length of $H^*(S^n;\mathbf{Q})$ equals* 1 *for n odd and* 2 *for n even.*

Theorem 7. *The topological complexity of motion planning* $\mathbf{TC}(X)$ *is greater than the zero-divisors-cup-length of* $H^*(X;\mathbf{k})$.

To illustrate this theorem, consider the special case $X = S^n$. Using the computation of the zero-divisors-cup-length for S^n (see the example above) and applying Theorem 7 we find that $\mathbf{TC}(S^n) > 1$ for n odd and $\mathbf{TC}(S^n) > 2$ for n even. This means that any motion planner on the sphere S^n must have at least two open sets U_i; moreover, any motion planner on the sphere S^n must have at least three open sets U_i if n is even.

Proof. Consider the following commutative diagram:



Here α associates to any point $x \in X$ the constant path $[0,1] \to X$ at this point. $\Delta\colon X \to X \times X$ is the diagonal map $\Delta(x) = (x, x)$. Note that α is a homotopy equivalence. The composition

$$H^*(X;\mathbf{k}) \otimes H^*(X;\mathbf{k}) \simeq H^*(X \times X;\mathbf{k}) \xrightarrow{\pi^*} H^*(PX;\mathbf{k}) \underset{\simeq}{\xrightarrow{\alpha^*}} H^*(X;\mathbf{k}) \qquad (7)$$

coincides with the cup-product homomorphism (5). Here the homomorphism on the left is the Künneth isomorphism.

As we mentioned above, the topological complexity of motion planning $\mathbf{TC}(X)$ is the Schwarz genus (see [5]) of the fibration $\pi\colon PX \to X \times X$. The statement of Theorem 7 follows from our remarks above concerning homomorphism (7) and from the cohomological lower bound for the Schwarz genus, see Theorem 4 of [5]. ☐

5. Motion Planning on Spheres

Theorem 8. *The topological complexity of motion planning on the n-dimensional sphere S^n is given by*

$$\mathbf{TC}(S^n) = \begin{cases} 2, & \text{for } n \text{ odd},\\ 3, & \text{for } n \text{ even}.\end{cases}$$

Proof. First we show that $\mathbf{TC}(S^n) \leq 2$ for n odd. Let $U_1 \subset S^n \times S^n$ be the set of all pairs (A, B) where $A \neq -B$. Then there is a unique shortest arc of S^n connecting A and B and we construct a continuous motion planning $s_1\colon U_1 \to PS^n$ by setting $s_1(A, B) \in PS^n$ to be this shortest arc passed with a constant velocity. The second open set will be defined as $U_2 = \{(A, B);\, A \neq B\} \subset S^n \times S^n$. A continuous motion planning over U_2 will be constructed in two steps. In the first step we move the initial point A to the antipodal point $-B$ along the shortest arc as above. In the second step we move the antipodal point $-B$ to B. For this purpose fix a continuous tangent vector field v on S^n, which is nonzero at every point; here we use the assumption that the dimension n is odd. We may move $-B$ to B along the spherical arc

$$-\cos \pi t \cdot B + \sin \pi t \cdot \frac{v(B)}{|v(B)|}, \qquad t \in [0, 1].$$

This proves that $\mathbf{TC}(S^n) \leq 2$ for n odd; hence by Theorem 1, $\mathbf{TC}(S^n) = 2$ for n odd.

Assume now that n is even. Let us show that then $\mathbf{TC}(S^n) \leq 3$. We define a continuous motion planning over the set $U_1 \subset S^n \times S^n$ as above. For n even we may construct a continuous tangent vector field v on S^n, which vanishes at a single point $B_0 \in S^n$ and is nonzero for any $B \in S^n$, $B \neq B_0$. We define the second set $U_2 \subset S^n \times S^n$ as $\{(A, B);\, A \neq B \,\&\, B \neq B_0\}$. We may define $s_2\colon U_2 \to PS^n$ as above. Now, $U_1 \cup U_2$ covers everything except the pair of points $(-B_0, B_0)$. Choose a point $C \in S^n$, distinct from $B_0, -B_0$, and set $Y = S^n - C$. Note that Y is diffeomorphic to $\mathbf{R}^n$ and so there exists a continuous motion planning over Y. This means that we may take $U_3 = Y \times Y$. This proves that $\mathbf{TC}(S^n) \leq 3$. On the other hand, using Theorem 7 and the preceding Example, we find $\mathbf{TC}(S^n) \geq 3$ for n even. This completes the proof. ☐

6. More Examples

Theorem 9. *Let $X = \Sigma_g$ be a compact orientable two-dimensional surface of genus g. Then*

$$\mathbf{TC}(X) = \begin{cases} 3, & \textit{if} \quad g \leq 1, \\ 5, & \textit{if} \quad g > 1. \end{cases}$$

Consider first the case $g \geq 2$. Then we may find cohomology classes $u_1, v_1, u_2, v_2 \in H^1(X; \mathbf{Q})$ forming a symplectic system, i.e., $u_i^2 = 0$, $v_i^2 = 0$, and $u_1 v_1 = u_2 v_2 = A \neq 0$, where $A \in H^2(\Sigma_g; \mathbf{Q})$ is the fundamental class; moreover, $v_i u_j = v_i v_j = u_i u_j = 0$ for $i \neq j$. Then it holds in the algebra $H^*(X; \mathbf{Q}) \otimes H^*(X; \mathbf{Q})$ that

$$\prod_{i=1}^{2} (1 \otimes u_i - u_i \otimes 1)(1 \otimes v_i - v_i \otimes 1) = 2A \otimes A \neq 0$$

and hence we obtain, using Theorem 7, that $\mathbf{TC}(X) \geq 5$. The opposite inequality follows from Theorem 4.

The case $g = 0$ follows from Theorem 8 since then $X = S^2$. The case $g = 1$, which corresponds to the two-dimensional torus T^2, is considered later in Theorem 13.

Theorem 10. *Let $X = \mathbf{CP}^n$ be the n-dimensional complex projective space. Then* $\mathbf{TC}(X) \geq 2n + 1$.

Proof. If $u \in H^2(X; \mathbf{Q})$ is a generator, then

$$(1 \otimes u - u \otimes 1)^{2n} = (-1)^n \binom{2n}{n} u^n \otimes u^n \neq 0.$$

Hence Theorem 7 gives $\mathbf{TC}(X) \geq 2n + 1$. □

7. Product Inequality

Theorem 11. *For any path-connected metric spaces X and Y,*

$$\mathbf{TC}(X \times Y) \leq \mathbf{TC}(X) + \mathbf{TC}(Y) - 1. \tag{8}$$

Proof. Denote $\mathbf{TC}(X) = n$, $\mathbf{TC}(Y) = m$. Let $U_1, \ldots, U_n$ be an open cover of $X \times X$ with a continuous motion planning $s_i\colon U_i \to PX$ for $i = 1, \ldots, n$. Let $f_i\colon X \times X \to \mathbf{R}$, where $i = 1, \ldots, n$, be a partition of unity subordinate to the cover $\{U_i\}$. Similarly, let $V_1, \ldots, V_m$ be an open cover of $Y \times Y$ with a continuous motion planning $\sigma_j\colon V_j \to PY$ for $j = 1, \ldots, m$, and let $g_j\colon Y \times Y \to \mathbf{R}$, where $j = 1, \ldots, m$, be a partition of unity subordinate to the cover $\{V_j\}$.

For any pair of nonempty subsets $S \subset \{1, \ldots, n\}$ and $T \subset \{1, \ldots, m\}$, let

$$W(S, T) \subset (X \times Y) \times (X \times Y)$$

denote the set of all 4-tuples $(A, B, C, D) \in (X \times Y) \times (X \times Y)$, such that for any $(i, j) \in S \times T$ and for any $(i', j') \notin S \times T$ it holds that

$$f_i(A, C) \cdot g_j(B, D) > f_{i'}(A, C) \cdot g_{j'}(B, D).$$

One easily checks that:

(a) *each set* $W(S, T) \subset (X \times Y) \times (X \times Y)$ *is open*;
(b) $W(S, T)$ *and* $W(S', T')$ *are disjoint if neither* $S \times T \subset S' \times T'$ *nor* $S' \times T' \subset S \times T$;
(c) *if* $(i, j) \in S \times T$, *then* $W(S, T)$ *is contained in* $U_i \times V_j$; *therefore there exists a continuous motion planning over each* $W(S, T)$ (*it can be described explicitly in terms of* s_i *and* σ_j);
(d) *the sets* $W(S, T)$ (*with all possible nonempty* S *and* T) *form a cover of* $(X \times Y) \times (X \times Y)$.

Let us prove (d). Suppose that $(A, B, C, D) \in (X \times Y) \times (X \times Y)$. Let S be the set of all indices $i \in \{1, \dots, n\}$, such that $f_i(A, C)$ equals the maximum of $f_k(A, C)$, where $k = 1, 2, \dots, n$. Similarly, let T be the set of all $j \in \{1, \dots, m\}$, such that $g_j(B, D)$ equals the maximum of $g_\ell(B, C)$, where $\ell = 1, \dots, m$. Then clearly (A, B, C, D) belongs to $W(S, T)$.

Let $W_k \subset (X \times Y) \times (X \times Y)$ denote the union of all sets $W(S, T)$, where $|S| + |T| = k$. Here $k = 2, 3, \dots, n + m$. The sets $W_2, \dots, W_{n+m}$ form an open cover of $(X \times Y) \times (X \times Y)$. If $|S| + |T| = |S'| + |T| = k$, then the corresponding sets $W(S, T)$ and $W(S', T')$ either coincide (if $S = S'$ and $T = T'$) or are disjoint. Hence we see (using (c)) that there exists a continuous motion planning over each open set W_k.

This completes the proof. □

Remark. The above proof represents a modification of the arguments of the proof of the product inequality for the Lusternik–Schnirelman category, see page 333 of [3].

8. Motion Planning for a Robot Arm

Consider a robot arm consisting of n bars $L_1, \dots, L_n$, such that L_i and L_{i+1} are connected by flexible joins. We assume that the initial point of L_1 is fixed. In the planar case, a configuration of the arm is determined by n angles $\alpha_1, \dots, \alpha_n$, where α_i is the angle between L_i and the x-axis (Fig. 3). Thus, in the planar case, the configuration space of the robot arm (when no obstacles are present) is the n-dimensional torus

$$T^n = S^1 \times S^1 \times \cdots \times S^1.$$

Similarly, the configuration space of a robot arm in the three-dimensional space $\mathbf{R}^3$ is the Cartesian product of n copies of the two-dimensional sphere S^2.

Theorem 12. *The topological complexity of the motion planning problem of a plane* n*-bar robot arm equals* $n+1$. *The topological complexity of the motion planning problem of a spacial* n*-bar robot arm equals* $2n + 1$.

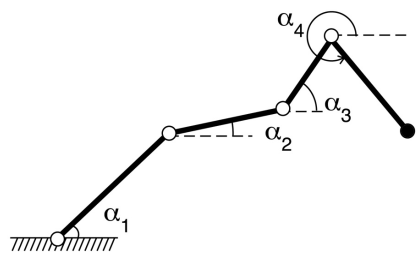


Fig. 3. Planar robot arm.

Remark. It is not difficult to construct motion planners explicitly for the planar and spacial robot arms, which have the minimal possible topological complexity. Such algorithms could be based on the ideas used in the proof of the product inequality (Theorem 11).

Theorem 12 automatically follows from the next statement:

Theorem 13. *Let $X = S^m \times S^m \times \cdots \times S^m$ be a Cartesian product of n copies of the m-dimensional sphere S^m. Then*

$$\mathbf{TC}(X) = \begin{cases} n+1, & \text{if } m \text{ is odd},\\ 2n+1, & \text{if } m \text{ is even}. \end{cases} \tag{9}$$

Proof. Using the product inequality (Theorem 11) and the calculation for spheres (Theorem 8) we find that $\mathbf{TC}(X)$ is less than or equal to the right-hand side of (9). To establish the inverse inequality we use Theorem 7. Let $a_i \in H^m(X;\mathbf{Q})$ denote the cohomology class which is the pull-back of the fundamental class of S^m under the projection $X \to S^m$ onto the ith factor; here $i = 1, 2, \dots, n$. We see that

$$\prod_{i=1}^{n}(1 \otimes a_i - a_i \otimes 1) \neq 0 \in H^*(X \times X;\mathbf{Q}).$$

This shows that the zero-divisors-cup-length of X is at least n. If m is even, then

$$\prod_{i=1}^{n}(1 \otimes a_i - a_i \otimes 1)^2 \neq 0 \in H^*(X \times X;\mathbf{Q}).$$

Hence for m even, the zero-divisors-cup-length of X is at least $2n$. Application of Theorem 7 completes the proof. ☐

Further results developing the notion of topological complexity of configuration spaces and applications to specific motion planning problems can be found in my preprint [2].

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