Tôhoku Math. J.
49 (1997), 527–535

TOPOLOGICAL DEGREE OF SOLUTION MAPPINGS IN FUNCTIONAL AND ORDINARY DIFFERENTIAL EQUATIONS

Dedicated to Professor Junji Kato on his sixtieth birthday

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(Received March 26, 1996, revised January 10, 1997)

Abstract. We shall define a topological degree for multi-valued solution mappings of functional differential equations with finite delay including ordinary differential equations. Earlier, we introduced a degree of ordinary differential equations which is different from that given in this article. We show that these two definitions are equivalent for ordinary differential equations.

1. Introduction. We denote by $C[a, b]$ the Banach space of all $\boldsymbol{R}^n$-valued continuous functions defined on a compact interval $[a, b]$ with supremum norm $|\cdot|$. We shall use the same symbol $|\cdot|$ as a norm in $\boldsymbol{R}^n$ without any fear of confusion. Let $r>0$ be a given constant and denote by X the Banach space $C[-r, 0]$. For any continuous function $x: [a-r, b]\to\boldsymbol{R}^n$, let x_t be the function defined by $x_t(s)=x(t+s)$ for $s\in[-r, 0]$. Then x_t belongs to X and is continuous in t for $t\in[a, b]$.

Suppose that $T>0$ is a given constant and $f: [0, T]\times X\to\boldsymbol{R}^n$ is a continuous mapping. We consider an initial value problem

$$x'(t)=f(t, x_t)\,, \qquad x_0=\varphi\,, \tag{1}$$

where a prime denotes differentiation with respect to t, φ belongs to X and f is assumed to satisfy the following condition:

(A) There exist two positive constants a and b such that $|f(t, \psi)|\le a+b|\psi|$ for $(t, \psi)\in[0, T]\times X$.

This assumption ensures that every solution of (1) is extendable to be continuous up to $t=T$ for any $\varphi\in X$ (see Remark 2.1). Let $\Gamma(\varphi)$ be the cross section with the hyperplane $t=T$ of the set of solution curves to (1) in $[0, T]\times X$, namely,

$$\Gamma(\varphi):=\{x_T;\ x \text{ is a solution of (1) defined on } [-r, T]\}\,. \tag{2}$$

Then we obtain a multi-valued mapping Γ from X into itself. Our purpose is to define a topological degree for the mapping Γ.

When $r=0$, (1) is reduced to an initial value problem for an ordinary differential equation. In this case, we have already defined the topological degree for Γ in [4] by constructing a sequence $\{\gamma_k\}$ of single-valued continuous mappings which approximates Γ in some sense. In order to construct such a sequence, we approximated f by a sequence

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1991 *Mathematics Subject Classification.* Primary 34K15; Secondary 34A45, 34C99.

$\{f_k\}$ of continuous functions which are Lipschitz continuous with respect to the second variable, and defined γ_k by using solutions of (1) in which f is replaced by f_k. This method is also applicable to a semilinear parabolic partial differential equation (see [5]). However, when $r>0$, it is impossible to find such a sequence $\{f_k\}$ because the domain of f is an infinite dimensional space. In Section 2, we shall introduce an approximate sequence of Γ different from that in [4]. Further in Section 3, the topological degree of Γ will be defined. The result given in Section 3 is also valid in the case where $r=0$, and consequently, we have two different definitions of the degree for Γ if we restrict ourselves to ordinary differential equations. In Section 4, we shall show that these two definitions are equivalent.

2. Construction of approximate solutions. For any $0<\varepsilon\leq 1$ and $\varphi\in X$, we define two functions $\tilde{\varphi}$ and ξ by

$$\tilde{\varphi}(t)=\begin{cases}\varphi(-r) & \text{for} \quad -r-1\leq t\leq -r\,,\\ \varphi(t) & \text{for} \quad -r\leq t\leq 0\end{cases}$$

and

$$\xi(t)=\begin{cases}\tilde{\varphi}(t) & \text{for} \quad -r-1\leq t\leq 0\,,\\ \varphi(0)+\displaystyle\int_0^t f(s,\xi_{s-\varepsilon})ds & \text{for} \quad 0\leq t\leq T\,.\end{cases} \tag{3}$$

LEMMA 2.1. *The function ξ given by* (3) *belongs to $C[-r-1, T]$ and satisfies an inequality $|\xi_t|\leq\{(1+\varepsilon b)|\varphi|+aT\}e^{bT}$ for $0\leq t\leq T$.*

PROOF. It follows from (A) that, for $0\leq t<T$,

$$\begin{aligned}|\xi(t)|&\leq|\varphi|+at+\int_0^t b|\xi_{s-\varepsilon}|ds\\ &<|\varphi|+aT+\int_0^\varepsilon b|\xi_{s-\varepsilon}|ds+\int_\varepsilon^{t+\varepsilon} b|\xi_{s-\varepsilon}|ds\\ &\leq|\varphi|+aT+\varepsilon b|\varphi|+b\int_0^t|\xi_s|ds=:\Psi(t)\,.\end{aligned}$$

Since $\Psi(t)$ is nondecreasing, we can easily prove that $|\xi_t|<\Psi(t)$ for $0\leq t<T$, and hence the assertion follows from Gronwall's inequality (see, e.g., [7, p. 82]). □

REMARK 2.1. For any solution x of (1), the same argument as in the proof of Lemma 2.1 gives an estimate $|x(t)|\leq(|\varphi|+at)e^{bt}$ for $t\geq 0$ as long as it exists. Therefore, by the fundamental theorems for solutions of (1) (see, e.g., [2]), every solution of (1) is extendable to be continuous up to $t=T$ and satisfies the above inequality on $[0, T]$.

We shall denote by $S(\varepsilon, \varphi)$ and Y the function ξ given in (3) and the space $C[-r-1, T]$, respectively. Then we obtain a mapping $S: (0, 1] \times X \to Y$.

LEMMA 2.2. *The mapping* $S: (0, 1] \times X \to Y$ *is continuous.*

PROOF. Let $\{(\varepsilon_k, \varphi^k)\}$ be a sequence converging to (ε, φ) in $(0, 1] \times X$, and denote $S(\varepsilon_k, \varphi^k)$ and $S(\varepsilon, \varphi)$ by ξ^k and ξ, respectively. It suffices to show that $\{\xi^k\}$ converges to ξ uniformly on $[-r-1, T]$. Since $\{\varphi^k\}$ converges to φ in X, the set $\Phi := \{\tilde{\varphi}^k; k \in N\} \cup \{\tilde{\varphi}\}$ is compact in $C[-r-1, 0]$ and $\{\xi^k\}$ converges to ξ uniformly on $[-r-1, 0]$. It follows from Lemma 2.1 that $\{\xi_t^k; k \in N, t \in [-1, T]\}$ is bounded in X, which implies the existence of a constant $M>0$ satisfying $|f(t, \xi_{t-\varepsilon})| \leq M$ and $|f(t, \xi_{t-\varepsilon_k}^k)| \leq M$ for any $k \in N$ and $t \in [0, T]$. It is clear from Ascoli-Arzelà's theorem that

$$L = \{\eta \in Y; \eta|_{[-r-1,0]} \in \Phi \text{ and } \eta \text{ is } M\text{-Lipschitz continuous on } [0, T]\}$$

is compact in Y, where $\eta|_{[-r-1,0]}$ denotes the restriction of η to the interval $[-r-1, 0]$. Since the mapping $Y \times [-1, T] \ni (\eta, t) \mapsto \eta_t \in X$ is continuous, the set $E = \{\eta_t; \eta \in L, t \in [-1, T]\}$ is also compact in X, and hence f is uniformly continuous on $[0, T] \times E$.

For $0 \leq t \leq T$, we have

$$\begin{aligned} |\xi(t) - \xi^k(t)| &\leq |\varphi - \varphi^k| + \int_0^t |f(s, \xi_{s-\varepsilon}) - f(s, \xi_{s-\varepsilon}^k)| ds + \int_0^T |f(s, \xi_{s-\varepsilon}^k) - f(s, \xi_{s-\varepsilon_k}^k)| ds \\ &=: |\varphi - \varphi^k| + F_k(t) + a_k . \end{aligned}$$

It is clear that $|\varphi - \varphi^k| \to 0$ as $k \to \infty$. Since $\xi^k \in L$, the family $\{\xi^k\}$ is equicontinuous on $[-r-1, T]$, which ensures that $\lim_{|\sigma-\tau| \to 0} |\xi_\sigma^k - \xi_\tau^k| = 0$ uniformly for $k \in N$ and for $\sigma, \tau \in [-1, T]$. It then follows from the uniform continuity of f on $[0, T] \times E$ that $a_k \to 0$ as $k \to \infty$.

Now we shall show that $\{F_k\}$ converges to 0 uniformly on $[0, T]$. For $0 \leq t \leq \varepsilon$, we have an estimate

$$0 \leq F_k(t) \leq \int_{-\varepsilon}^0 |f(s+\varepsilon, \xi_s) - f(s+\varepsilon, \xi_s^k)| ds .$$

Since $\{\xi^k\}$ converges to ξ uniformly on $[-r-1, 0]$, it follows from the above estimate that $\{F_k(t)\}$ converges to 0 uniformly on $[0, \varepsilon]$, which implies that $\{\xi^k\}$ converges to ξ uniformly on $[-r-1, \varepsilon]$. Similarly, for $\varepsilon \leq t \leq 2\varepsilon$, we have

$$0 \leq F_k(t) \leq \int_{-\varepsilon}^\varepsilon |f(s+\varepsilon, \xi_s) - f(s+\varepsilon, \xi_s^k)| ds .$$

This estimate and the uniform convergence of $\{\xi^k\}$ on $[-r-1, \varepsilon]$ imply that $\{F_k(t)\}$ converges to 0 uniformly on $[0, 2\varepsilon]$, and consequently, $\{\xi^k\}$ converges to ξ uniformly on $[-r-1, 2\varepsilon]$. Repeating this process, we arrive at the desired assertion. □

LEMMA 2.3. *Let* $\{\varepsilon_k\}$ *be a sequence in* $(0, 1]$ *converging to* 0, *and let* $\{\varphi^k\}$ *be a* *sequence converging to* φ *in* X. *Then* $\{S(\varepsilon_k, \varphi^k)\}$ *contains a subsequence which converges to some* ξ *in* Y, *and furthermore*, $\xi|_{[-r,T]}$ *is a solution of* (1).

PROOF. For each $k \in N$, we denote the function $S(\varepsilon_k, \varphi^k)$ by ξ^k. By the same argument as in the proof of Lemma 2.2, it follows that $\{\xi_k\}$ is equicontinuous on $[-r-1, T]$, and hence we may assume that $\{\xi_k\}$ converges to some ξ in Y by taking a subsequence if necessary. Since ξ_k satisfies $\xi_0^k = \varphi^k$ and

$$\xi^k(t) = \varphi^k(0) + \int_0^t f(s, \xi^k_{s-\varepsilon_k})ds \qquad \text{for} \quad 0 \le t \le T,$$

we obtain $\xi_0 = \varphi$ and

$$\xi(t) = \varphi(0) + \int_0^t f(s, \xi_s)ds \qquad \text{for} \quad 0 \le t \le T,$$

because $|\xi^k_{s-\varepsilon_k} - \xi_s| \le |\xi^k_{s-\varepsilon_k} - \xi^k_s| + |\xi^k_s - \xi_s| \to 0$ as $k \to \infty$ uniformly for $s \in [0, T]$. □

3. Topological degree of solution mappings. Throughout this section, we assume that $r \le T$. Let $J = [0, 1]$ and let I denote the identity mapping on $X = C([-r, 0])$.

It is easy to see that $\Gamma(\varphi)$ is a compact subset of X for each $\varphi \in X$, where $\Gamma(\varphi)$ is the set given in (2). Hence we obtain a mapping $\Gamma: X \to K(X)$, where $K(X)$ denotes the family of all nonempty compact subsets of X. For a point $\psi \in X$ and a subset G of X, let $\psi - G := \{\psi - \eta; \eta \in G\}$. We define a mapping $I - \Gamma$ by $(I - \Gamma)(\psi) = \psi - \Gamma(\psi)$, while we let $(I - \Gamma)(G) := \bigcup_{\psi \in G}(I - \Gamma)(\psi)$.

Suppose that D is a bounded and open subset of X and choose a point

$$p \in X \setminus (I - \Gamma)(\partial D), \tag{4}$$

where ∂D denotes the boundary of D. For the above D and p, we shall define the topological degree as $\deg(I - \Gamma, D, p)$. It is well known that the degree can be defined for a compact and convex set-valued, completely continuous and upper semicontinuous mapping (see, e.g., [6]). We can readily verify that Γ is completely continuous and upper semicontinuous. Although the set $\Gamma(\varphi)$ is connected (see [3]), it is not always convex. Therefore, known results are not applicable to Γ.

Let $\{\varepsilon_k\}$ be any fixed sequence in $(0, 1]$ converging to 0, and let $\gamma_k: X \to X$ be a mapping defined by

$$\gamma_k(\varphi) := S(\varepsilon_k, \varphi)_T \qquad \text{for} \quad \varphi \in X.$$

By virtue of Lemma 2.2, γ_k is a continuous mapping. Moreover, it follows from Lemma 2.1 and the argument used in the proof of Lemma 2.2 that γ_k is completely continuous because $r \le T$.

We now show that $\deg(I - \gamma_k, D, p)$ is defined and is independent of k for large k, where $\deg(I - \gamma_k, D, p)$ denotes the Leray-Schauder degree (see [1] or [6]). For any

$(k, l, \theta) \in \boldsymbol{N} \times \boldsymbol{N} \times J$, let $\gamma_{k,l,\theta}: X \to X$ denote the mapping defined by

$$\gamma_{k,l,\theta}(\varphi) = S((1-\theta)\varepsilon_k + \theta\varepsilon_l, \varphi)_T$$

for $\varphi \in X$. Then, by Lemma 2.2, $\gamma_{k,l,\theta}(\varphi)$ is continuous in $(\theta, \varphi) \in J \times X$ for fixed k and l. Similarly to γ_k, we find that $\gamma_{k,l,\theta}$ is continuous and completely continuous, and furthermore, we have $\gamma_{k,l,0} = \gamma_k$ and $\gamma_{k,l,1} = \gamma_l$.

LEMMA 3.1. *There exists an integer n_0 such that $p \notin (I - \gamma_{k,l,\theta})(\partial D)$ holds for any pair of integers k, $l \geq n_0$ and any $\theta \in J$.*

PROOF. Suppose the contrary. Then, for each $m \in \boldsymbol{N}$, there exist integers $k_m \geq m$, $l_m \geq m$, $\theta_m \in J$ and $\varphi^m \in \partial D$ such that

$$\varphi^m - \gamma_{k_m, l_m, \theta_m}(\varphi^m) = p \qquad \text{for} \quad m \in \boldsymbol{N}. \tag{5}$$

Denoting $S((1-\theta_m)\varepsilon_{k_m} + \theta_m \varepsilon_{l_m}, \varphi^m)$ by ξ^m, we have $\gamma_{k_m, l_m, \theta_m}(\varphi^m) = \xi_T^m$. Since $\{\varphi^m\}$ is bounded and $r \leq T$, the sequence $\{\xi_T^m\}$ is bounded and equicontinuous in X, and hence we may assume that $\{\xi_T^m\}$ converges to some η in X. It then follows from (5) that $\{\varphi^m\}$ converges to $\eta + p =: \varphi$ which belongs to ∂D because ∂D is closed.

Since $(1-\theta_m)\varepsilon_{k_m} + \theta_m \varepsilon_{l_m} \to 0$ and $\varphi^m \to \varphi$ as $m \to \infty$, we may assume, by Lemma 2.3, that $\{\xi^m\}$ converges to some ξ in Y and that $\xi|_{[-r,T]}$ is a solution of (1). Therefore, $\Gamma(\varphi) \ni \xi_T = \eta = \varphi - p$, which yields $p \in \varphi - \Gamma(\varphi) = (I - \Gamma)(\varphi)$. This contradicts (4), since $\varphi \in \partial D$. □

Using this lemma, we can conclude that $\lim_{k \to \infty} \deg(I - \gamma_k, D, p)$ exists. Furthermore, it is easy to see that the limit does not depend on the choice of the sequence $\{\varepsilon_k\}$. Thus, we can define $\deg(I - \Gamma, D, p)$ by

$$\deg(I - \Gamma, D, p) = \lim_{k \to \infty} \deg(I - \gamma_k, D, p). \tag{6}$$

THEOREM 3.1. *If $\deg(I - \Gamma, D, p) \neq 0$, then there exists a point $\varphi \in D$ such that $(I - \Gamma)(\varphi) \ni p$.*

PROOF. Let $\{\varepsilon_k\}$ be any sequence in $(0, 1]$ which converges to 0. We may assume that $\deg(I - \gamma_k, D, p) \neq 0$ for each $k \in \boldsymbol{N}$, and hence it follows that D contains a φ^k satisfying $\varphi^k - \gamma_k(\varphi^k) = p$ such that $\gamma_k(\varphi^k) = \xi_T^k$, where ξ^k denotes the function $S(\varepsilon_k, \varphi^k)$. By using the same argument as in the proof of Lemma 3.1, the sequences $\{\varphi^k\}$ and $\{\xi^k\}$ contain subsequences which converge to some $\varphi \in \bar{D}$ and $\xi \in Y$, respectively. Moreover, by Lemma 2.3, we see that $\varphi - p = \xi_T \in \Gamma(\varphi)$, namely, $p \in (I - \Gamma)(\varphi)$. By this and (4), we are done. □

4. Topological degree in ordinary differential equations. In this section, we compare two definitions of the topological degree for a solution mapping to an initial value problem

$$x' = f(t, x), \qquad x(0) = u, \tag{7}$$

where $u \in \boldsymbol{R}^n$ and $f: [0, T] \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ is a continuous mapping satisfying the following assumption:

(A_0) There exist two positive constants a and b such that $|f(t, x)| \leq a + b|x|$ for $(t, x) \in [0, T] \times \boldsymbol{R}^n$.

For every $u \in \boldsymbol{R}^n$, similarly to (2), we denote $\Gamma(u) := \{x(T);\ x \text{ is a solution of (7)}\}$. Then we obtain a mapping $\Gamma: \boldsymbol{R}^n \to K(\boldsymbol{R}^n)$, where $K(\boldsymbol{R}^n)$ denotes the family of all nonempty compact subsets of $\boldsymbol{R}^n$.

Let D be a bounded and open subset of $\boldsymbol{R}^n$ and let

$$p \in \boldsymbol{R}^n \setminus \Gamma(\partial D). \tag{8}$$

As a special case of Section 3, we can define $\deg(\Gamma, D, p)$ in the sense of (6). On the other hand, we have another definition of the degree introduced in [4]. Although the assumption imposed on p in [4] is somewhat stronger than (8), it can be relaxed to (8) by the same consideration as in [5]. In this connection, we shall improve the result given in [4] along the line of [5], and, as a consequence, prove that these two definitions are equivalent. In order to do so, we first summarize the two definitions.

The result in Section 3 is reduced to the following. For any $u \in \boldsymbol{R}^n$ and any $k \in \boldsymbol{N}$, let $v(t) = v_k(t; u)$ denote the function

$$v(t) = \begin{cases} u & \text{for } t \in [-1, 0], \\ u + \displaystyle\int_0^t f\left(s, v\left(s - \frac{1}{k}\right)\right) ds & \text{for } t \in [0, T], \end{cases}$$

and define a mapping $\gamma_k: \boldsymbol{R}^n \to \boldsymbol{R}^n$ by $\gamma_k(u) = v_k(T; u)$. Then the arguments used in Section 3 ensure that $\deg(\gamma_k, D, p)$ is defined and is independent of k for large k, where $\deg(\gamma_k, D, p)$ is the Brouwer degree (see [1], [6]). We denote $\lim_{k \to \infty} \deg(\gamma_k, D, p)$ by $d(\Gamma, D, p)$ instead of $\deg(\Gamma, D, p)$, namely,

$$d(\Gamma, D, p) = \lim_{k \to \infty} \deg(\gamma_k, D, p). \tag{9}$$

Now, we improve the result given in [4]. For the function f satisfying (A_0), there exists a sequence $\{f_k(t, x)\}$ of continuous functions on $[0, T] \times \boldsymbol{R}^n$ with the following two properties:

(P_1) $\{f_k\}$ converges to f uniformly on every compact set in $[0, T] \times \boldsymbol{R}^n$, and every f_k satisfies (A_0) in which f is replaced by f_k.

(P_2) For each $k \in \boldsymbol{N}$, $f_k(t, x)$ is locally Lipschitz continuous in x, namely, for any $M > 0$, there exists a positive constant $L(k, M)$ such that

$$|f_k(t, x) - f_k(t, y)| \leq L(k, M)|x - y|$$

for $t \in [0, T]$ and $x, y \in \boldsymbol{R}^n$ with $|x| \leq M$ and $|y| \leq M$.

Although the function f is assumed to be bounded in [4], we can easily prove the existence of such a sequence $\{f_k\}$ by using mollifier under the assumption (A_0).

Let $w_k(t; u)$ be the unique solution of

$$x' = f_k(t, x), \qquad x(0) = u,$$

and let $\lambda_k : \boldsymbol{R}^n \to \boldsymbol{R}^n$ be the mapping defined by $\lambda_k(u) := w_k(T; u)$. In [4], we showed that $\deg(\lambda_k, D, p)$ is defined and is independent of k for large k when p satisfies

$$p \in \boldsymbol{R}^n \setminus \Gamma^*(\partial D) \tag{10}$$

instead of (8), where $\Gamma^*(u) = \operatorname{co} \Gamma(u)$ denotes the convex hull of $\Gamma(u)$. The reason why we assumed (10) is due to the construction of a homotopy connecting λ_k and λ_l. The homotopy used in [4] is given by $(1-\theta)\lambda_k + \theta\lambda_l$ for $\theta \in J$. Instead, we consider a homotopy $\lambda_{k,l} : J \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ defined by

$$\lambda_{k,l}(\theta, u) = w_{k,l}(T; \theta, u),$$

where $w_{k,l}(t; \theta, u)$ denotes the unique solution of

$$x' = (1-\theta) f_k(t, x) + \theta f_l(t, x), \qquad x(0) = u.$$

Then even if the condition (10) is weakened to (8), we can obtain the following lemma which is easily proved in a manner similar to Lemma 3.1 (or see [5; Lemma 3.2]).

LEMMA 4.1. *If p is a point satisfying* (8), *then there exists an integer n_1 such that $\lambda_{k,l}(J, \partial D)$ does not contain p for all $k, l \geq n_1$.*

Since $\lambda_{k,l}(0, \cdot) = \lambda_k$ and $\lambda_{k,l}(1, \cdot) = \lambda_l$ hold, Lemma 4.1 ensures that $\deg(\lambda_k, D, p)$ is defined and is independent of k for large k. Let us denote

$$\rho(\Gamma, D, p) = \lim_{k \to \infty} \deg(\lambda_k, D, p). \tag{11}$$

In what follows, we shall prove

$$d(\Gamma, D, p) = \rho(\Gamma, D, p). \tag{12}$$

For any $(k, l) \in \boldsymbol{N} \times \boldsymbol{N}$, $\theta \in J$ and $u \in \boldsymbol{R}^n$, let $y(t) = y_{k,l}(t; \theta, u)$ denote the unique solution of

$$\begin{cases} y(t) = u & \text{for} \quad t \in [-1, 0], \\ y'(t) = \theta f\left(t, y\left(t - \dfrac{1}{l}\right)\right) + (1-\theta) f_k(t, y(t)) & \text{for} \quad t \in [0, T], \end{cases}$$

or equivalently,

$$(13)\quad y(t)=\begin{cases} u & \text{for } t\in[-1,0]\,,\\ u+\displaystyle\int_0^t\left(\theta f\left(s, y\left(s-\frac{1}{l}\right)\right)+(1-\theta)f_k(s, y(s))\right)ds & \text{for } t\in[0, T]\,.\end{cases}$$

Here, we notice that $y(t)$ is uniquely determined for u and that relations $y_{k,l}(t; 0, u)=w_k(t; u)$ and $y_{k,l}(t; 1, u)=v_l(t; u)$ hold.

LEMMA 4.2. *Let k and l be fixed positive integers. Then $y_{k,l}(t; \theta, u)$ is continuous in $(\theta, u)\in J\times \boldsymbol{R}^n$ for each $t\in[0, T]$.*

PROOF. Let $\{(\theta_m, u_m)\}$ be a sequence converging to (θ, u) in $J\times \boldsymbol{R}^n$. We denote $y_{k,l}(t; \theta_m, u_m)$ and $y_{k,l}(t; \theta, u)$ by $y_m(t)$ and $y(t)$, respectively. It suffices to show that $y_m(t)\to y(t)$ as $m\to\infty$ for each $t\in[0, T]$. Since $\{u_m\}$ is a convergent sequence, there exists an $M_1>0$ such that $|u_m|\le M_1$ for all $m\in \boldsymbol{N}$. It then follows from (A_0), (P_1) and the argument used in the proof of Lemma 2.1 that there exists an $M_2>0$ satisfying $|y(t)|\le M_2$ and $|y_m(t)|\le M_2$ for $m\in \boldsymbol{N}$ and $t\in[-1, T]$.

Using (13), we have, for $0\le t\le T$,

$$\begin{aligned}|y_m(t)-y(t)|\le |u_m-u|&+\int_0^T\left|\theta_m f\left(s, y_m\left(s-\frac{1}{l}\right)\right)-\theta f\left(s, y_m\left(s-\frac{1}{l}\right)\right)\right|ds\\ &+\int_0^t\left|\theta f\left(s, y_m\left(s-\frac{1}{l}\right)\right)-\theta f\left(s, y\left(s-\frac{1}{l}\right)\right)\right|ds\\ &+\int_0^T|(1-\theta_m)f_k(s, y_m(s))-(1-\theta)f_k(s, y_m(s))|ds\\ &+\int_0^t|(1-\theta)f_k(s, y_m(s))-(1-\theta)f_k(s, y(s))|ds\\ &\le |u_m-u|+2|\theta_m-\theta|MT+\int_0^t\left|f\left(s, y_m\left(s-\frac{1}{l}\right)\right)-f\left(s, y\left(s-\frac{1}{l}\right)\right)\right|ds\\ &+\int_0^t L(k, M_2)|y_m(s)-y(s)|ds\,,\end{aligned}$$

where $M=a+bM_2$. Here, we notice that $\int_0^t|f(s, y_m(s-1/l))-f(s, y(s-1/l))|ds$ is non-decreasing in $t\in[0, T]$, and hence Gronwall's inequality [7, p. 82] gives the following estimate:

$$(14)\qquad |y_m(t)-y(t)|\le\left(c_m+\int_0^t\left|f\left(s, y_m\left(s-\frac{1}{l}\right)\right)-f\left(s, y\left(s-\frac{1}{l}\right)\right)\right|ds\right)e^{L(k,M_2)T}\,,$$

where $c_m=|u_m-u|+2|\theta_m-\theta|MT$. It is obvious that $c_m\to 0$ as $m\to\infty$.

For $0\le t\le l^{-1}$, (14) shows that $|y_m(t)-y(t)|\to 0$ uniformly on $[0, l^{-1}]$ as $m\to\infty$ because $y_m(s-l^{-1})=u_m$ and $y(s-l^{-1})=u$ hold for $0\le s\le t\le l^{-1}$. From this, we see that

$y_m(t-l^{-1}) \to y(t-l^{-1})$ uniformly for $t \in [l^{-1}, 2l^{-1}]$, and hence (14) implies that $y_m(t) \to y(t)$ uniformly on $[0, 2l^{-1}]$ as $m \to \infty$. We reach the required assertion by repeating the above process. □

Let $\mu_{k,l}: J \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ be the mapping defined by

$$\mu_{k,l}(\theta, u) = y_{k,l}(T; \theta, u) \qquad \text{for} \quad (\theta, u) \in J \times \boldsymbol{R}^n .$$

By virtue of Lemma 4.2, $\mu_{k,l}$ is a continuous mapping for each $(k, l) \in \boldsymbol{N} \times \boldsymbol{N}$. Since $\mu_{k,l}(0, u) = y_{k,l}(T; 0, u) = w_k(T, u) = \lambda_k(u)$ and $\mu_{k,l}(1, u) = y_{k,l}(T; 1, u) = v_l(T, u) = \gamma_l(u)$ hold, the mapping $\mu_{k,l}$ is a homotopy connecting λ_k and γ_l.

LEMMA 4.3. *If p is a point satisfying* (8), *then there exists an integer n_2 such that $\mu_{k,l}(J, \partial D)$ does not contain p for all k, $l \geq n_2$.*

This lemma is easily proved by the same argument as in the proof of Lemma 3.1. By Lemma 4.3, $\deg(\lambda_k, D, p) = \deg(\gamma_l, D, p)$ for k, $l \geq n_2$, which, together with (9) and (11), implies (12).

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