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Topological degree theory and Caputo–Hadamard fractional boundary value problems

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Abstract

We study two hybrid and non-hybrid fractional boundary value problems via the Caputo–Hadamard type derivatives. We seek the existence criteria for these two problems separately. By utilizing the generalized Dhage's theorem, we derive desired results for an integral structure of solutions for the hybrid problems. Also by considering the special case as a non-hybrid boundary value problem (BVP), we establish other results based on the existing tools in the topological degree theory. In the end of the article, we examine our theoretical results by presenting some numerical examples to show the applicability of the analytical findings.

MSC: Primary 34A08; secondary 34A12

Keywords: Caputo–Hadamard fractional BVP; Condensing operator; Degree theory; The generalized Dhage's theorem

1 Introduction

The fractional calculus has always been one of the most widely used branches of mathematics in other applied and computational sciences. This degree of importance is due to the high flexibility of the tools and operators defined in this theory. On this basis, researchers have been using various powerful fractional operators in recent decades to model different types of existing natural processes in the world. In the meantime, because modeling based on fractional operators yields more accurate numerical results than modeling based on integer order operators, different generalizations of these fractional operators have been introduced by numerous mathematicians.

The fractional operators have developed over the years, and their importance has become apparent more and more to researchers today. Instances of the application of such fractional operators can be found in various sciences such as biomathematics, electrical circuits, medicine, etc. [1–15]. All of these reasons have led researchers to find many aspects of the structure of the fractional boundary value problems and the hereditary properties of their solutions. In this regard, many researchers have been investigating advanced fractional models [16–18] and related theoretical results and qualitative behaviors of such boundary value problems [19–28].

The fractional operators utilized in models of the current paper are the Hadamard and Caputo–Hadamard integration and differentiation operators, respectively. In this regard, one can point to some papers based on these operators; see, for example, [29–32]. In more recent decades, the attention of researchers has been focused on designing newer fractional hybrid BVPs subject to hybrid or non-hybrid conditions. For more details, see [33–39]. More precisely, this novel aspect of fractional modeling initiated with a research manuscript proposed by Dhage and Lakshmikantham in 2010 (see [40]). They turned to a new family of differential equation entitled hybrid differential equation and then established some useful existence criteria of extremal solutions by utilizing some basic inequalities [40]. Two years later, Zhao et al. extended their work to fractional type models and formulated a BVP relying on fractional hybrid differential equations [41]. Later, Ullah et al. continued this process and employed a new structure of hybrid fractional modeling in which both boundary conditions are presented in the hybrid framework by follows:

$$\begin{cases} {}^{\mathcal{R}}\mathcal{D}_{0^+}^{\kappa^*}[\frac{y(t)-p(t,y(t))}{q(t,y(t))}] = \psi(t,y(t)), \quad t\in[0,1], \\ [\frac{y(t)-p(t,y(t))}{q(t,y(t))}]|_{t=0} = 0, \qquad [\frac{y(t)-p(t,y(t))}{q(t,y(t))}]|_{t=1} = 0, \end{cases}$$

where $q \in \mathcal{C}_{\mathbb{R}\neq 0}([0,1]\times\mathbb{R})$ is nonzero, both p and ψ are continuous real-valued functions on $[0,1]\times\mathbb{R}$ and ${}^{\mathcal{R}}\mathcal{D}_{0^+}^{\kappa^*}$ represents the Riemann–Liouville derivative of order $\kappa^* \in (0,1]$ [42]. In 2020, Baleanu et al. presented a novel construction of a fractional hybrid model of a thermostat in which the thermostat controls the amount of heat based on the temperature detected by its sensors [16]. This hybrid model is described by

$$ {}^{\mathcal{C}}\mathcal{D}_{0^+}^{\kappa^*}\left[\frac{y(t)}{q(t,y(t))}\right] + \Phi\big(t,y(t)\big) = 0, \quad \kappa^*\in(1,2], t\in[0,1],$$

with the fractional hybrid boundary conditions

$$\begin{cases} \mathcal{D}[\frac{y(t)}{q(t,y(t))}]|_{t=0} = 0, \\ \lambda\,{}^{\mathcal{C}}\mathcal{D}_{0^+}^{\kappa^*-1}[\frac{y(t)}{q(t,y(t))}]|_{t=1} + [\frac{y(t)}{q(t,y(t))}]|_{t=\eta} = 0, \end{cases}$$

where $\lambda > 0$ denotes an arbitrary parameter, $\eta\in[0,1]$, and $\kappa^*-1\in(0,1]$. Moreover, $\mathcal{D} = \frac{\mathrm{d}}{\mathrm{d}t}$, ${}^{\mathcal{C}}\mathcal{D}_{0^+}^{\gamma}$ is the Caputo derivative of order $\gamma\in\{\kappa^*,\kappa^*-1\}$, $\Phi\in\mathcal{C}_{\mathbb{R}}([0,1]\times\mathbb{R})$, and $q\in\mathcal{C}_{\mathbb{R}\neq 0}([0,1]\times\mathbb{R})$ is nonzero [16]. By using the main ideas of these works, we are going to investigate the Caputo–Hadamard fractional hybrid differential equation

$$ {}^{\mathcal{CH}}\mathcal{D}_{1^+}^{\kappa^*}\left[\frac{y(t)-\Lambda(t,y(t),\lambda_1^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*}y(t))}{\Psi(t,y(t),\lambda_2^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*}y(t))}\right] = \hat{\Upsilon}\big(t,y(t)\big), \quad t\in[1,e], \tag{1}$$

with the mixed Hadamard integral hybrid boundary value conditions

$$\begin{cases} {}^{\mathcal{CH}}\mathcal{D}_{1^+}[\frac{y(t)-\Lambda(t,y(t),\lambda_1^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*}y(t))}{\Psi(t,y(t),\lambda_2^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*}y(t))}]|_{t=1} \\ \quad = \tilde{a}_1[\frac{y(t)-\Lambda(t,y(t),\lambda_1^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*}y(t))}{\Psi(t,y(t),\lambda_2^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*}y(t))}]|_{t=1}, \\ {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\theta^*}\,{}^{\mathcal{CH}}\mathcal{D}_{1^+}[\frac{y(t)-\Lambda(t,y(t),\lambda_1^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*}y(t))}{\Psi(t,y(t),\lambda_2^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*}y(t))}]|_{t=e} \\ \quad = \tilde{a}_2[\frac{y(t)-\Lambda(t,y(t),\lambda_1^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*}y(t))}{\Psi(t,y(t),\lambda_2^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*}y(t))}]|_{t=e}, \end{cases} \tag{2}$$

where $\kappa^* \in (1,2]$, $\gamma^*, \mu^*, \theta^* > 0$, and $\lambda_1^*, \lambda_2^*, \tilde{a}_1, \tilde{a}_2 \in \mathbb{R}$. Here, ${}^{\mathcal{CH}}\mathcal{D}_{1^+}^{\kappa^*}$ represents the Caputo–Hadamard fractional derivative of order κ^*, ${}^{\mathcal{H}}\mathcal{I}_{1^+}^{\eta}$ is the Hadamard fractional integral of order $\eta \in \{\gamma^*, \mu^*, \theta^*\}$, $\Psi : [1,e] \times \mathbb{R}^3 \to \mathbb{R} \setminus \{0\}$ is a nonzero continuous map, $\Lambda \in \mathcal{C}_{\mathbb{R}}([1,e] \times \mathbb{R}^3)$, and $\hat{\Upsilon} \in \mathcal{C}_{\mathbb{R}}([1,e] \times \mathbb{R})$.

By reviewing other papers published in recent years, we find that some researchers have combined the existence theory with the topological degree theory and studied different models using the existing analytical notions in this theory. For instance, one can point to some published papers in this regard, such as [43–48]. In the light of this, we address a special case of the Caputo–Hadamard hybrid BVP (1)–(2) in the sequel of the present paper. In other words, we set $\Lambda(t, y(t), \lambda_1^* \int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*} y(t)) = 0$ and $\Psi(t, y(t), \lambda_2^* \int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*} y(t)) = 1$. Then the Caputo–Hadamard fractional hybrid BVP (1)–(2) reduces to the following Caputo–Hadamard fractional non-hybrid BVP:

$$\begin{cases} {}^{\mathcal{CH}}\mathcal{D}_{1^+}^{\kappa^*} y(t) = \hat{\Upsilon}(t, y(t)), \\ {}^{\mathcal{CH}}\mathcal{D}_{1^+} y(1) = \tilde{a}_1 y(1), \qquad \frac{1}{\Gamma(\theta^*)} \int_1^e (\ln \frac{e}{\varpi})^{\theta^*-1} [{}^{\mathcal{CH}}\mathcal{D}_{1^+} y(\varpi)] \frac{\mathrm{d}\varpi}{\varpi} = \tilde{a}_2 y(e). \end{cases} \tag{3}$$

For this non-hybrid BVP, we will apply a new approach based on the topological degree theory. Note that both hybrid and non-hybrid BVPs (1)–(2) and (3) are novel in the sense that boundary conditions are written as mixed Hadamard integral and Caputo–Hadamard derivative simultaneously.

2 Preliminaries

First, some important and necessary preliminaries on the fractional calculus are recalled in this section. Assume that $\kappa^* \geq 0$. The Hadamard fractional integral of $y \in \mathcal{C}_{\mathbb{R}}([a,b])$ of order κ^* is given by ${}^{\mathcal{H}}\mathcal{I}_{a^+}^{0}(y(t)) = y(t)$ and ${}^{\mathcal{H}}\mathcal{I}_{a^+}^{\kappa^*}(y(t)) = \frac{1}{\Gamma(\kappa^*)} \int_a^t (\ln \frac{t}{\varpi})^{(\kappa^*-1)} y(\varpi) \frac{\mathrm{d}\varpi}{\varpi}$ whenever the RHS-integral has finite value [49, 50]. Note that for each $\kappa_1^*, \kappa_2^* \in \mathbb{R}^+$, we have ${}^{\mathcal{H}}\mathcal{I}_{a^+}^{\kappa_1^*}({}^{\mathcal{H}}\mathcal{I}_{a^+}^{\kappa_2^*} y(t)) = {}^{\mathcal{H}}\mathcal{I}_{a^+}^{\kappa_1^*+\kappa_2^*} y(t)$ and ${}^{\mathcal{H}}\mathcal{I}_{a^+}^{\kappa_1^*}(\ln \frac{t}{a})^{\kappa_2^*} = \frac{\Gamma(\kappa_2^*+1)}{\Gamma(\kappa_1^*+\kappa_2^*+1)} (\ln \frac{t}{a})^{\kappa_1^*+\kappa_2^*}$ for $t > a$ [50]. It is obvious that ${}^{\mathcal{H}}\mathcal{I}_{a^+}^{\kappa_1^*} 1 = \frac{1}{\Gamma(\kappa_1^*+1)} (\ln \frac{t}{a})^{\kappa_1^*}$ for any $t > a$ by setting $\kappa_2^* = 0$ [50]. Now, let $n = [\kappa^*] + 1$. The Hadamard fractional derivative of order κ^* for a function $y : (a,b) \to \mathbb{R}$ is introduced by ${}^{\mathcal{H}}\mathcal{D}_{a^+}^{\kappa^*}(y(t)) = \frac{1}{\Gamma(n-\kappa^*)} (t \frac{\mathrm{d}t}{t})^n \int_a^t (\ln \frac{t}{\varpi})^{(n-\kappa^*-1)} y(\varpi) \frac{\mathrm{d}\varpi}{\varpi}$ provided that the RHS-integral has finite value [49, 50]. The Caputo–Hadamard fractional derivative of order κ^* for $y \in \mathcal{AC}_{\mathbb{R}}^n([a,b])$ is represented by

$$ {}^{\mathcal{CH}}\mathcal{D}_{a^+}^{\kappa^*}\big(y(t)\big) = \frac{1}{\Gamma(n-\kappa^*)} \int_a^t \left(\ln \frac{t}{\varpi}\right)^{(n-\kappa^*-1)} \left(t \frac{\mathrm{d}t}{t}\right)^n y(\varpi) \frac{\mathrm{d}\varpi}{\varpi} $$

whenever the RHS-integral has finite value [49, 50]. Now assume that $y \in \mathcal{AC}_{\mathbb{R}}^n([a,b])$ and $n - 1 < \kappa^* \leq n$. In the monograph [50], it is verified that the general solution of the Caputo–Hadamard differential equation ${}^{\mathcal{CH}}\mathcal{D}_{a^+}^{\kappa^*}(y(t)) = 0$ is obtained of the form $y(t) = \sum_{j=0}^{n-1} m_j^* (\ln \frac{t}{a})^j$, and so we have

$$ {}^{\mathcal{H}}\mathcal{I}_{a^+}^{\kappa^*}\big({}^{\mathcal{CH}}\mathcal{D}_{a^+}^{\kappa^*} y(t)\big) = y(t) + m_0^* + m_1^* \left(\ln \frac{t}{a}\right) + m_2^* \left(\ln \frac{t}{a}\right)^2 + \cdots + m_{n-1}^* \left(\ln \frac{t}{a}\right)^{n-1} $$

for any $t > a$. In the following, we review some notions and results on the topological degree theory which are useful throughout the paper. Let $\mathbb{B}$ represent the collection of all bounded sets in a Banach space $\mathcal{X}$. The Kuratowski's measure of noncompactness $\mu : \mathbb{B} \to \mathbb{R}^+$ is defined by $\mu(\mathcal{B}) := \inf\{\epsilon > 0 : \mathcal{B} = \bigcup_{j=1}^{n} \mathcal{B}_j \text{ and } \operatorname{diam}(\mathcal{B}_i) \le \epsilon \text{ for } j = 1, \dots, n\}$, where $\operatorname{diam}(\mathcal{B}_j) = \sup\{|y - y'| : y, y' \in \mathcal{B}_j\}$ and $\mathcal{B}$ is a bounded element of $\mathbb{B}$. It is evident that $0 \le \mu(\mathcal{B}) \le \operatorname{diam}(\mathcal{B}) < +\infty$ [51, 52].

Lemma 1 ([51, 52]) *Let $\mathcal{X}$ be an arbitrary real Banach space and $\mathcal{B}, \mathcal{E} \in \mathbb{B}$ be bounded subsets of $\mathcal{X}$. Then the following statements are valid*:

(a1) *$\mathcal{B}$ is relatively compact if and only if $\mu(\mathcal{B}) = 0$*;

(a2) *$\mu(\mathcal{B}) = \mu(\overline{\mathcal{B}}) = \mu(\operatorname{cnvx}(\mathcal{B}))$, where $\overline{\mathcal{B}}$ and $\operatorname{cnvx}(\mathcal{B})$ represent the closure and convex hull of $\mathcal{B}$, respectively*;

(a3) *If $\mathcal{B} \subseteq \mathcal{E}$, then $\mu(\mathcal{B}) \le \mu(\mathcal{E})$*;

(a4) *$\mu(\lambda + \mathcal{E}) \le \mu(\mathcal{E})$ for each $\lambda \in \mathbb{R}$*;

(a5) *$\mu(\lambda\mathcal{B}) = |\lambda|\mu(\mathcal{B})$ for each $\lambda \in \mathbb{R}$*;

(a6) *$\mu(\mathcal{B} + \mathcal{E}) \le \mu(\mathcal{B}) + \mu(\mathcal{E})$ so that $\mathcal{B} + \mathcal{E} = \{y + y'; y \in \mathcal{B}, y' \in \mathcal{E}\}$*;

(a7) *$\mu(\mathcal{B} \cup \mathcal{E}) \le \max\{\mu(\mathcal{B}), \mu(\mathcal{E})\}$.*

Note that conditions (a5) and (a6) mean that μ is a seminorm. Let $\mathcal{B} \in \mathbb{B}$ be a bounded subset of a Banach space $\mathcal{X}$. We say that a continuous bounded map $\Phi : \mathcal{B} \to \mathcal{X}$ is μ-Lipschitz if there is a constant $\tilde{K}^* \ge 0$ such that $\mu(\Phi(\mathcal{B})) \le \tilde{K}^*\mu(\mathcal{B})$. Also, Φ is called a strict μ-contraction if $\tilde{K}^*$ is less than one [51]. A μ-condensing function Φ is supposed to satisfy $\mu(\Phi(\mathcal{B})) \le \mu(\mathcal{B})$ for each $\mathcal{B} \in \mathbb{B}$ with $\mu(\mathcal{B}) > 0$. Indeed, the inequality $\mu(\Phi(\mathcal{B})) \ge \mu(\mathcal{B})$ implies that $\mu(\mathcal{B}) = 0$ [51].

Proposition 2 ([53]) *Let $\Phi : \mathcal{B} \to \mathcal{X}$ be Lipschitz with constant $\tilde{K}^*$ where $\mathcal{B} \subset \mathcal{X}$. Then Φ is μ-Lipschitz with the same constant $\tilde{K}^*$.*

Proposition 3 ([53]) *For every $\mathcal{B} \subset \mathcal{X}$, if $\Phi : \mathcal{B} \to \mathcal{X}$ is compact, then Φ is μ-Lipschitz with constant $\tilde{K}^* = 0$.*

Proposition 4 ([53]) *For each $\mathcal{B} \subset \mathcal{X}$, assume that $\Phi_1, \Phi_2 : \mathcal{B} \to \mathcal{X}$ are two μ-Lipschitz operators with constant $\tilde{K}_1^*$ and $\tilde{K}_2^*$, respectively. Then $\Phi_1 + \Phi_2 : \mathcal{B} \to \mathcal{X}$ is μ-Lipschitz with constant $\tilde{K}_1^* + \tilde{K}_2^*$.*

The following theorem due to Dhage is utilized for our result related to the mixed Caputo–Hadamard hybrid BVP (1)–(2).

Theorem 5 ([54]) *Let $\mathcal{X}$ be a Banach algebra and $\mathcal{B}$ be a convex bounded closed nonempty subset of $\mathcal{X}$. Moreover, suppose that three operators $\Phi_1, \Phi_2 : \mathcal{X} \to \mathcal{X}$ and $\Phi_3 : \mathcal{B} \to \mathcal{X}$ satisfy the following three assumptions*:

(i) *Φ_1 and Φ_2 are Lipschitz with constants $\tilde{K}_1^*$ and $\tilde{K}_2^*$, respectively,*

(ii) *Φ_3 is compact and continuous,*

(iii) *$\tilde{K}_1^*\hat{\Delta} + \tilde{K}_2^* < 1$ so that $\hat{\Delta} = \|\Phi_3(\mathcal{B})\|_{\mathcal{X}} = \sup\{\|\Phi_3 y\|_{\mathcal{X}} : y \in \mathcal{B}\}$.*

Then either (a) the equation $(\Phi_1 y)(\Phi_3 y) + (\Phi_2 y) = y$ has a solution belonging to $\mathcal{B}$ or (b) for each $r > 0$, there is $v^* \in \mathcal{X}$ with $\|v^*\|_{\mathcal{X}} = r$ provided that $\alpha_0(\Phi_1 v^*)(\Phi_3 v^*) + \alpha_0(\Phi_2 v^*) = v^*$ for some $\alpha_0 \in (0, 1)$.

The following theorem due to Isaia is utilized for our result related to the mixed Caputo–Hadamard non-hybrid BVP (3).

Theorem 6 ([53]) *Let $\Phi : \mathcal{X} \to \mathcal{X}$ be a μ-condensing operator on the Banach space $\mathcal{X}$ and assume that*

$$\mathcal{B} = \{y \in \mathcal{X} : \text{there is } \lambda \in [0,1] \text{ such that } y = \lambda(\Phi y)\}.$$

If $\mathcal{B}$ is a bounded set in $\mathcal{X}$, so that there is a number $\rho > 0$ such that $\mathcal{B} \subset \overline{\mathcal{V}_\rho(0)}$, then we have $\deg(I - \lambda\Phi, \overline{\mathcal{V}_\rho(0)}, 0) = 1$. *Moreover, Φ has at least one fixed point, and the family of all fixed points of Φ belongs to $\overline{\mathcal{V}_\rho(0)}$.*

3 Main results

Now, we are ready to derive the desired analytical findings. For this reason, we build a new space as $\mathcal{X} = \{y(t) : y(t) \in \mathcal{C}_{\mathbb{R}}([1,e])\}$ supplemented with the sup-norm $\|y\|_{\mathcal{X}} = \sup_{t\in[1,e]} |y(t)|$ and the multiplication action on $\mathcal{X}$ by $(y \cdot y')(t) = y(t)y'(t)$ for each $y, y' \in \mathcal{X}$. Then it is easily verified that an ordered triple $(\mathcal{X}, \|\cdot\|_{\mathcal{X}}, \cdot)$ is a Banach algebra. In the following lemma, we derive an integral structure for the solution of the hybrid BVP (1)–(2).

Lemma 7 *Let $g \in \mathcal{X}$. Then a function $\tilde{y}_0$ is a solution for the Caputo–Hadamard hybrid equation*

$$ {}^{\mathcal{CH}}\mathcal{D}_{1^+}^{\kappa^*}\left[\frac{y(t) - \Lambda(t, y(t), \lambda_1^* \int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*} y(t))}{\Psi(t, y(t), \lambda_2^* \int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*} y(t))}\right] = g(t) \tag{4}$$

furnished with mixed Hadamard integral hybrid boundary value conditions

$$\begin{cases} {}^{\mathcal{CH}}\mathcal{D}_{1^+}\left[\frac{y(t)-\Lambda(t,y(t),\lambda_1^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*} y(t))}{\Psi(t,y(t),\lambda_2^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*} y(t))}\right]\Big|_{t=1} \\ \quad = \tilde{a}_1\left[\frac{y(t)-\Lambda(t,y(t),\lambda_1^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*} y(t))}{\Psi(t,y(t),\lambda_2^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*} y(t))}\right]\Big|_{t=1}, \\ {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\theta^*}\,{}^{\mathcal{CH}}\mathcal{D}_{1^+}\left[\frac{y(t)-\Lambda(t,y(t),\lambda_1^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*} y(t))}{\Psi(t,y(t),\lambda_2^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*} y(t))}\right]\Big|_{t=e} \\ \quad = \tilde{a}_2\left[\frac{y(t)-\Lambda(t,y(t),\lambda_1^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*} y(t))}{\Psi(t,y(t),\lambda_2^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*} y(t))}\right]\Big|_{t=e} \end{cases} \tag{5}$$

if and only if $\tilde{y}_0$ is a solution for the Hadamard integral equation

$$\begin{aligned} y(t) &= \Psi\left(t, y(t), \lambda_2^* \int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*} y(t)\right) \\ &\quad\times \left(\frac{1}{\Gamma(\kappa^*)}\int_1^t \left(\ln\frac{t}{\varpi}\right)^{\kappa^*-1} g(\varpi)\frac{\mathrm{d}\varpi}{\varpi}\right. \\ &\quad + \frac{\tilde{a}_2(1+\tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{\kappa^*-1} g(\varpi)\frac{\mathrm{d}\varpi}{\varpi} \\ &\quad \left. - \frac{(1+\tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*-1)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{\kappa^*+\theta^*-2} g(\varpi)\frac{\mathrm{d}\varpi}{\varpi}\right) \\ &\quad + \Lambda\left(t, y(t), \lambda_1^* \int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*} y(t)\right), \end{aligned} \tag{6}$$

where

$$\mathcal{Q}^* = \left| \frac{\tilde{a}_1 - (1 + \tilde{a}_1)\tilde{a}_2 \Gamma(\theta^* + 1)}{\Gamma(\theta^* + 1)} \right| \neq 0. \tag{7}$$

Proof As a first step, we assume that $\tilde{y_0}$ is a solution for the hybrid differential equation (4). Then, by properties of the κ^*th order Hadamard integral, we seek constants $m_0^*, m_1^* \in \mathbb{R}$ such that

$$\frac{\tilde{y_0}(t) - \Lambda(t, \tilde{y_0}(t), \lambda_1^* \int_1^e \tilde{y_0}(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*} \tilde{y_0}(t))}{\Psi(t, \tilde{y_0}(t), \lambda_2^* \int_1^e \tilde{y_0}(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*} \tilde{y_0}(t))}$$
$$= \frac{1}{\Gamma(\kappa^*)} \int_1^t \left(\ln \frac{t}{\varpi}\right)^{\kappa^*-1} g(\varpi) \frac{\mathrm{d}\varpi}{\varpi} + m_0^* + m_1^*(\ln t)$$

and so

$$\begin{aligned} \tilde{y_0}(t) = {} & \Lambda\left(t, \tilde{y_0}(t), \lambda_1^* \int_1^e \tilde{y_0}(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*} \tilde{y_0}(t)\right) \\ & + \Psi\left(t, \tilde{y_0}(t), \lambda_2^* \int_1^e \tilde{y_0}(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*} \tilde{y_0}(t)\right) \\ & \times \left(\frac{1}{\Gamma(\kappa^*)} \int_1^t \left(\ln \frac{t}{\varpi}\right)^{\kappa^*-1} g(\varpi) \frac{\mathrm{d}\varpi}{\varpi} + m_0^* + m_1^*(\ln t)\right). \end{aligned} \tag{8}$$

Thus,

$$\begin{aligned} & {}^{\mathcal{CH}}\mathcal{D}_{1^+}\left[\frac{\tilde{y_0}(t) - \Lambda(t, \tilde{y_0}(t), \lambda_1^* \int_1^e \tilde{y_0}(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*} \tilde{y_0}(t))}{\Psi(t, \tilde{y_0}(t), \lambda_2^* \int_1^e \tilde{y_0}(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*} \tilde{y_0}(t))}\right] \\ & = \frac{1}{\Gamma(\kappa^* - 1)} \int_1^t \left(\ln \frac{t}{\varpi}\right)^{\kappa^*-2} g(\varpi) \frac{\mathrm{d}\varpi}{\varpi} + m_1^*, \\ & {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\theta^*}\, {}^{\mathcal{CH}}\mathcal{D}_{1^+}\left[\frac{\tilde{y_0}(t) - \Lambda(t, \tilde{y_0}(t), \lambda_1^* \int_1^e \tilde{y_0}(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*} \tilde{y_0}(t))}{\Psi(t, \tilde{y_0}(t), \lambda_2^* \int_1^e \tilde{y_0}(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*} \tilde{y_0}(t))}\right] \\ & = m_1^* \frac{(\ln t)^{\theta^*}}{\Gamma(\theta^* + 1)} + \frac{1}{\Gamma(\kappa^* + \theta^* - 1)} \int_1^t \left(\ln \frac{t}{\varpi}\right)^{\kappa^*+\theta^*-2} g(\varpi) \frac{\mathrm{d}\varpi}{\varpi}. \end{aligned}$$

In the light of both mixed hybrid boundary conditions given in (5), we obtain

$$\begin{aligned} m_0^* = {} & \frac{\tilde{a}_2}{\mathcal{Q}^* \Gamma(\kappa^*)} \int_1^e \left(\ln \frac{e}{\varpi}\right)^{\kappa^*-1} g(\varpi) \frac{\mathrm{d}\varpi}{\varpi} \\ & - \frac{1}{\mathcal{Q}^* \Gamma(\kappa^* + \theta^* - 1)} \int_1^e \left(\ln \frac{e}{\varpi}\right)^{\kappa^*+\theta^*-2} g(\varpi) \frac{\mathrm{d}\varpi}{\varpi} \end{aligned}$$

and

$$\begin{aligned} m_1^* = {} & \frac{\tilde{a}_1 \tilde{a}_2}{\mathcal{Q}^* \Gamma(\kappa^*)} \int_1^e \left(\ln \frac{e}{\varpi}\right)^{\kappa^*-1} g(\varpi) \frac{\mathrm{d}\varpi}{\varpi} \\ & - \frac{\tilde{a}_1}{\mathcal{Q}^* \Gamma(\kappa^* + \theta^* - 1)} \int_1^e \left(\ln \frac{e}{\varpi}\right)^{\kappa^*+\theta^*-2} g(\varpi) \frac{\mathrm{d}\varpi}{\varpi}. \end{aligned}$$

By inserting the obtained values m_0^* and m_1^* into (8), we reach

$$\begin{aligned}
\tilde{y_0}(t) = {} & \Psi\left(t, \tilde{y_0}(t), \lambda_2^* \int_1^e \tilde{y_0}(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*}\tilde{y_0}(t)\right) \\
& \times \left(\frac{1}{\Gamma(\kappa^*)}\int_1^t \left(\ln\frac{t}{\varpi}\right)^{\kappa^*-1} g(\varpi)\frac{\mathrm{d}\varpi}{\varpi}\right. \\
& + \frac{\tilde{a}_2(1+\tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{\kappa^*-1} g(\varpi)\frac{\mathrm{d}\varpi}{\varpi} \\
& \left. - \frac{(1+\tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*-1)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{\kappa^*+\theta^*-2} g(\varpi)\frac{\mathrm{d}\varpi}{\varpi}\right) \\
& + \Lambda\left(t, \tilde{y_0}(t), \lambda_1^* \int_1^e \tilde{y_0}(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*}\tilde{y_0}(t)\right).
\end{aligned}$$

The last equation implies that $\tilde{y_0}$ satisfies the Hadamard integral equation (6), and so $\tilde{y_0}$ is the solution of the mentioned integral equation. In the opposite direction, we can easily confirm that $\tilde{y_0}$ is a solution for the two-point Caputo–Hadamard hybrid BVP (4)–(5) if $\tilde{y_0}$ is supposed to be a solution for the Hadamard integral equation (6). This completes the proof. □

Now, based on the obtained Hadamard integral equation in the above lemma, we provide an existence criterion for solutions of the mixed Caputo–Hadamard hybrid BVP (1)–(2).

Theorem 8 *Let $\Psi : [1,e]\times\mathcal{X}^3 \to \mathcal{X}\setminus\{0\}$ and $\Lambda : [1,e]\times\mathcal{X}^3 \to \mathcal{X}$ and $\hat{\Upsilon} : [1,e]\times\mathcal{X} \to \mathcal{X}$ be continuous. Moreover, consider the following hypotheses:*

($\mathcal{HP}1$) *There is a positive bounded mapping $\varrho : [1,e] \to \mathbb{R}^+$ so that for each $y_i, y_i' \in \mathcal{X}$,*

$$\left|\Psi\big(t, y_1(t), y_2(t), y_3(t)\big) - \Psi\big(t, y_1'(t), y_2'(t), y_3'(t)\big)\right| \leq \varrho(t)\sum_1^3 \left|y_i(t) - y_i'(t)\right|,$$

($\mathcal{HP}2$) *There is a positive bounded mapping $\sigma : [1,e] \to \mathbb{R}^+$ such that for each $y_i, y_i' \in \mathcal{X}$,*

$$\left|\Lambda\big(t, y_1(t), y_2(t), y_3(t)\big) - \Lambda\big(t, y_1'(t), y_2'(t), y_3'(t)\big)\right| \leq \sigma(t)\sum_1^3 \left|y_i(t) - y_i'(t)\right|,$$

($\mathcal{HP}3$) *There is a positive continuous function $\psi : [1,e] \to \mathbb{R}^+$ and a continuous non-decreasing map $\xi : [0,\infty) \to [0,\infty)$ such that $|\hat{\Upsilon}(t,y(t))| \leq \psi(t)\xi(\|y\|_{\mathcal{X}})$ for any $t \in [1,e]$ and $y \in \mathcal{X}$,*

($\mathcal{HP}4$) *There exists a number $\rho > 0$ such that*

$$\begin{aligned}
\rho > {} & \big(\Psi^*\tilde{M}\psi^*\xi\big(\|y\|_{\mathcal{X}}\big) + \Lambda^*\big) \\
& \Big/ \left(1 - \varrho^*\left[1 + \left|\lambda_2^*(e-1)\right| + \frac{1}{\Gamma(\mu^*+1)}\right]\tilde{M}\psi^*\xi\big(|y|_{\mathcal{X}}\big)\right. \\
& \left. - \sigma^*\left[1 + \left|\lambda_1^*(e-1)\right| + \frac{1}{\Gamma(\gamma^*+1)}\right]\right),
\end{aligned} \tag{9}$$

where we considered $\Psi^* = \sup_{t\in[1,e]} |\psi(t,0,0,0)|$, $\Lambda^* = \sup_{t\in[1,e]} |\Lambda(t,0,0,0)|$, $\psi^* = \sup_{t\in[1,e]} |\psi(t)|$, $\varrho^* = \sup_{t\in[1,e]} |\varrho(t)|$, $\sigma^* = \sup_{t\in[1,e]} |\sigma(t)|$ *and*

$$\tilde{M} = \frac{1}{\Gamma(\kappa^*+1)} + \left|\frac{\tilde{a}_2(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+1)}\right| + \left|\frac{(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*)}\right|.$$

Then the mixed Caputo–Hadamard hybrid BVP (1)–(2) *has at least one solution if*

$$\varrho^*\left[1+\left|\lambda_2^*(e-1)\right| + \frac{1}{\Gamma(\mu^*+1)}\right]\psi^*\xi\left(\|y\|_{\mathcal{X}}\right)\tilde{M} + \sigma^*\left[1+\left|\lambda_1^*(e-1)\right| + \frac{1}{\Gamma(\gamma^*+1)}\right] < 1.$$

Proof For every positive number $\rho \in \mathbb{R}$, we construct the ball $\overline{\mathcal{V}_\rho(0)} := \{y(t) \in \mathcal{X} : \|y\|_{\mathcal{X}} \leq \rho\}$ in the Banach algebra $\mathcal{X}$, where ρ satisfies (9). It is well known that $\overline{\mathcal{V}_\rho(0)}$ is a convex closed bounded subset of the Banach algebra $\mathcal{X}$. Based on Lemma 7, we introduce three operators $\Phi_1, \Phi_2 : \mathcal{X} \to \mathcal{X}$ and $\Phi_3 : \overline{\mathcal{V}_\rho(0)} \to \mathcal{X}$ by

$$(\Phi_1 y)(t) = \Psi\left(t, y(t), \lambda_2^* \int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*} y(t)\right),$$

$$(\Phi_2 y)(t) = \Lambda\left(t, y(t), \lambda_1^* \int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*} y(t)\right)$$

and

$$\begin{aligned}(\Phi_3 y)(t) = {}& \frac{1}{\Gamma(\kappa^*)}\int_1^t \left(\ln\frac{t}{\varpi}\right)^{\kappa^*-1} \hat{\Upsilon}\left(\varpi, y(\varpi)\right)\frac{\mathrm{d}\varpi}{\varpi} \\ & + \frac{\tilde{a}_2(1+\tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{\kappa^*-1} \hat{\Upsilon}\left(\varpi, y(\varpi)\right)\frac{\mathrm{d}\varpi}{\varpi} \\ & - \frac{(1+\tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*-1)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{\kappa^*+\theta^*-2} \hat{\Upsilon}\left(\varpi, y(\varpi)\right)\frac{\mathrm{d}\varpi}{\varpi}\end{aligned}$$

for any $t \in [1,e]$. It is evident that a function $y_0 \in \mathcal{X}$ is a solution for the mixed Caputo–Hadamard hybrid BVP (1)–(2) whenever y_0 satisfies the equation $(\Phi_1 y_0)(\Phi_3 y_0) + (\Phi_2 y_0) = y_0$. We intend to show that the three operators Φ_1, Φ_2 and Φ_3 satisfy all conditions of Theorem 5 and thus, by taking into account hypotheses of Theorem 5, we will find that there exists such a solution function. First of all, we verify that Φ_1 is Lipschitz. Let $y_1, y_2 \in \mathcal{X}$. By $(\mathcal{HP}1)$, we may write

$$\begin{aligned}&\left|(\Phi_1 y_1)(t) - (\Phi_1 y_2)(t)\right| \\ &\quad = \left|\Psi\left(t, y_1(t), \lambda_2^*\int_1^e y_1(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*} y_1(t)\right)\right. \\ &\qquad \left. - \Psi\left(t, y_2(t), \lambda_2^*\int_1^e y_2(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*} y_2(t)\right)\right| \\ &\quad \leq \varrho(t)\left[1+\left|\lambda_2^*(e-1)\right| + \frac{1}{\Gamma(\mu^*+1)}\right]\sup_{t\in[1,e]}\left|y_1(t)-y_2(t)\right|\end{aligned}$$

for any $t \in [1,e]$. Hence, we get

$$\|\Phi_1 y_1 - \Phi_1 y_2\|_{\mathcal{X}} \leq \varrho^*\left[1+\left|\lambda_2^*(e-1)\right| + \frac{1}{\Gamma(\mu^*+1)}\right]\|y_1 - y_2\|_{\mathcal{X}},$$

showing that Φ_1 is Lipschitz with constant $\varrho^*[1 + |\lambda_2^*(e-1)| + \frac{1}{\Gamma(\mu^*+1)}] > 0$ for each $y_1, y_2 \in \mathcal{X}$. Similarly, by using hypothesis $(\mathcal{HP}2)$, one can realize that Φ_1 is also Lipschitz on $\mathcal{X}$. The proof is straightforward as above. Indeed, we have

$$\begin{aligned}
&\bigl|(\Phi_2 y_1)(t) - (\Phi_2 y_2)(t)\bigr| \\
&\quad = \biggl| \Lambda\biggl(t, y_1(t), \lambda_1^* \int_1^e y_1(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*} y_1(t)\biggr) \\
&\qquad - \Lambda\biggl(t, y_2(t), \lambda_1^* \int_1^e y_2(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*} y_2(t)\biggr)\biggr| \\
&\quad \leq \sigma(t)\biggl[1 + \bigl|\lambda_1^*(e-1)\bigr| + \frac{1}{\Gamma(\gamma^*+1)}\biggr] \sup_{t\in[1,e]} \bigl|y_1(t) - y_2(t)\bigr|
\end{aligned}$$

for any $t \in [1,e]$. Thus, we get

$$\|\Phi_2 y_1 - \Phi_2 y_2\|_{\mathcal{X}} \leq \sigma^*\biggl[1 + \bigl|\lambda_1^*(e-1)\bigr| + \frac{1}{\Gamma(\gamma^*+1)}\biggr]\|y_1 - y_2\|_{\mathcal{X}},$$

demonstrating that Φ_2 is Lipschitz with constant $\sigma^*[1 + |\lambda_1^*(e-1)| + \frac{1}{\Gamma(\gamma^*+1)}] > 0$ for each $y_1, y_2 \in \mathcal{X}$. Therefore, the first condition of Theorem 5 is fulfilled for two operators Φ_1 and Φ_2. In the sequel, we establish the complete continuity of the operator Φ_3 on the given closed ball $\overline{\mathcal{V}_\rho(0)}$. We have to check that Φ_3 is continuous on $\overline{\mathcal{V}_\rho(0)}$. Thus, consider a convergent sequence $\{y_n\}$ in $\overline{\mathcal{V}_\rho(0)}$ so that $y_n \to y$, where $y \in \overline{\mathcal{V}_\rho(0)}$ is an arbitrary element. By assumption, we know that $\hat{\Upsilon}$ is continuous on $[1,e] \times \mathcal{X}$, so $\lim_{n\to\infty} \hat{\Upsilon}(t, y_n) = \hat{\Upsilon}(t, y)$. By Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}
\lim_{n\to\infty} (\Phi_3 y_n)(t) &= \frac{1}{\Gamma(\kappa^*)} \int_1^t \biggl(\ln\frac{t}{\varpi}\biggr)^{\kappa^*-1} \lim_{n\to\infty} \hat{\Upsilon}\bigl(\varpi, y_n(\varpi)\bigr)\frac{\mathrm{d}\varpi}{\varpi} \\
&\quad + \frac{\tilde{a}_2(1+\tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*)} \int_1^e \biggl(\ln\frac{e}{\varpi}\biggr)^{\kappa^*-1} \lim_{n\to\infty} \hat{\Upsilon}\bigl(\varpi, y_n(\varpi)\bigr)\frac{\mathrm{d}\varpi}{\varpi} \\
&\quad - \frac{(1+\tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*-1)} \int_1^e \biggl(\ln\frac{e}{\varpi}\biggr)^{\kappa^*+\theta^*-2} \lim_{n\to\infty} \hat{\Upsilon}\bigl(\varpi, y_n(\varpi)\bigr)\frac{\mathrm{d}\varpi}{\varpi} \\
&= \frac{1}{\Gamma(\kappa^*)} \int_1^t \biggl(\ln\frac{t}{\varpi}\biggr)^{\kappa^*-1} \hat{\Upsilon}\bigl(\varpi, y(\varpi)\bigr)\frac{\mathrm{d}\varpi}{\varpi} \\
&\quad + \frac{\tilde{a}_2(1+\tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*)} \int_1^e \biggl(\ln\frac{e}{\varpi}\biggr)^{\kappa^*-1} \hat{\Upsilon}\bigl(\varpi, y(\varpi)\bigr)\frac{\mathrm{d}\varpi}{\varpi} \\
&\quad - \frac{(1+\tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*-1)} \int_1^e \biggl(\ln\frac{e}{\varpi}\biggr)^{\kappa^*+\theta^*-2} \hat{\Upsilon}\bigl(\varpi, y(\varpi)\bigr)\frac{\mathrm{d}\varpi}{\varpi} = (\Phi_3 y)(t)
\end{aligned}$$

for any $t \in [1,e]$. Hence, we get $\Phi_3 y_n \to \Phi_3 y$ as $n \to \infty$, and this means that Φ_3 is continuous on $\overline{\mathcal{V}_\rho(0)}$. The next goal is to check the uniform boundedness of Φ_3 on the ball $\overline{\mathcal{V}_\rho(0)}$. Let us take $y \in \overline{\mathcal{V}_\rho(0)}$. Under hypothesis $(\mathcal{HP}3)$, the following estimate is obtained:

$$\begin{aligned}
\bigl|(\Phi_3 y)(t)\bigr| &\leq \frac{1}{\Gamma(\kappa^*)} \int_1^t \biggl(\ln\frac{t}{\varpi}\biggr)^{\kappa^*-1} \bigl|\hat{\Upsilon}\bigl(\varpi, y(\varpi)\bigr)\bigr|\frac{\mathrm{d}\varpi}{\varpi} \\
&\quad + \biggl|\frac{\tilde{a}_2(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*)}\biggr| \int_1^e \biggl(\ln\frac{e}{\varpi}\biggr)^{\kappa^*-1} \bigl|\hat{\Upsilon}\bigl(\varpi, y(\varpi)\bigr)\bigr|\frac{\mathrm{d}\varpi}{\varpi}
\end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*-1)}\right|\int_1^e\left(\ln\frac{e}{\varpi}\right)^{\kappa^*+\theta^*-2}\left|\hat{\Upsilon}\left(\varpi,y(\varpi)\right)\right|\frac{\mathrm{d}\varpi}{\varpi}\\
&\le\frac{\sup_{t\in[1,e]}\psi(t)\times\xi(\|y\|_{\mathcal{X}})}{\Gamma(\kappa^*+1)}+\left|\frac{\tilde{a}_2(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+1)}\right|\sup_{t\in[1,e]}\psi(t)\times\xi\left(\|y\|_{\mathcal{X}}\right)\\
&+\left|\frac{(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*)}\right|\sup_{t\in[1,e]}\psi(t)\times\xi\left(\|y\|_{\mathcal{X}}\right)
\end{aligned}
$$

for each $t\in[1,e]$. It follows that $\|(\Phi_3y)(t)\|_{\mathcal{X}}\le\psi^*\xi(\|y\|_{\mathcal{X}})$. Hence $\Phi_3(\overline{\mathcal{V}_\rho(0)})$ is a uniformly bounded subset of $\mathcal{X}$. To establish the complete continuity property of Φ_3 in the last step, it is enough to verify the equicontinuity of Φ_3. For this purpose, we take two arbitrary elements $t_1,t_2\in[1,e]$ so that $t_1<t_2$ and $y\in\overline{\mathcal{V}_\rho(0)}$. Then, under appropriate conditions, we can write

$$
\begin{aligned}
&\left|(\Phi_3y)(t_2)-(\Phi_3y)(t_1)\right|\\
&\le\frac{1}{\Gamma(\kappa^*)}\int_1^{t_1}\left[\left(\ln\frac{t_2}{\varpi}\right)^{\kappa^*-1}-\left(\ln\frac{t_1}{\varpi}\right)^{\kappa^*-1}\right]\left|\hat{\Upsilon}\left(\varpi,y(\varpi)\right)\right|\frac{\mathrm{d}\varpi}{\varpi}\\
&+\frac{1}{\Gamma(\kappa^*)}\int_{t_1}^{t_2}\left(\ln\frac{t_2}{\varpi}\right)^{\kappa^*-1}\left|\hat{\Upsilon}\left(\varpi,y(\varpi)\right)\right|\frac{\mathrm{d}\varpi}{\varpi}\\
&+\frac{|\tilde{a}_2\tilde{a}_1|(|\ln(t_2)-\ln(t_1)|)}{|\mathcal{Q}^*|\Gamma(\kappa^*)}\int_1^e\left(\ln\frac{e}{\varpi}\right)^{\kappa^*-1}\left|\hat{\Upsilon}\left(\varpi,y(\varpi)\right)\right|\frac{\mathrm{d}\varpi}{\varpi}\\
&+\frac{|\tilde{a}_1|(|\ln(t_2)-\ln(t_1)|)}{|\mathcal{Q}^*|\Gamma(\kappa^*+\theta^*-1)}\int_1^e\left(\ln\frac{e}{\varpi}\right)^{\kappa^*+\theta^*-2}\left|\hat{\Upsilon}\left(\varpi,y(\varpi)\right)\right|\frac{\mathrm{d}\varpi}{\varpi}\\
&\le\frac{((\ln(t_2))^{k^*}-(\ln(t_1))^{k^*})-(\ln(t_2)-\ln(t_1))^{k^*}}{\Gamma(\kappa^*+1)}\sup_{t\in[1,e]}\psi(t)\times\xi\left(\|y\|_{\mathcal{X}}\right)\\
&+\frac{(\ln(t_2)-\ln(t_1))^{k^*}}{\Gamma(\kappa^*+1)}\sup_{t\in[1,e]}\psi(t)\times\xi\left(\|y\|_{\mathcal{X}}\right)\\
&+\frac{|\tilde{a}_2\tilde{a}_1|(|\ln(t_2)-\ln(t_1)|)}{|\mathcal{Q}^*|\Gamma(\kappa^*)}\int_1^e\left(\ln\frac{e}{\varpi}\right)^{\kappa^*-1}\left|\hat{\Upsilon}\left(\varpi,y(\varpi)\right)\right|\frac{\mathrm{d}\varpi}{\varpi}\\
&+\frac{|\tilde{a}_1|(|\ln(t_2)-\ln(t_1)|)}{|\mathcal{Q}^*|\Gamma(\kappa^*+\theta^*-1)}\int_1^e\left(\ln\frac{e}{\varpi}\right)^{\kappa^*+\theta^*-2}\left|\hat{\Upsilon}\left(\varpi,y(\varpi)\right)\right|\frac{\mathrm{d}\varpi}{\varpi}.
\end{aligned}
$$

When we take the limit on both sides of the inequality as $t_1\to t_2$, then clearly the RHS of the inequality approaches 0 (regardless of $y\in\overline{\mathcal{V}_\rho(0)}$). Thus, $|(\Phi_3y)(t_2)-(\Phi_3y)(t_1)|\to0$ as $t_1\to t_2$, confirming the equicontinuity of the operator Φ_3. Here, by invoking the Arzela–Ascoli theorem, it is deduced that Φ_3 is completely continuous on $\overline{\mathcal{V}_\rho(0)}$. To fulfill the third condition of Theorem 5, we utilize hypothesis $(\mathcal{HP}3)$ and obtain

$$
\begin{aligned}
\hat{\Delta}&=\left\|\Phi_3\left(\overline{\mathcal{V}_\rho(0)}\right)\right\|_{\mathcal{X}}\\
&=\sup_{t\in[1,e]}\left\{\left|(\Phi_3y)(t)\right|:y\in\overline{\mathcal{V}_\rho(0)}\right\}\\
&\le\psi^*\xi\left(\|y\|_{\mathcal{X}}\right)\left(\frac{1}{\Gamma(\kappa^*+1)}+\left|\frac{\tilde{a}_2(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+1)}\right|+\left|\frac{(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*)}\right|\right)\\
&=\psi^*\xi\left(\|y\|_{\mathcal{X}}\right)\tilde{M}.
\end{aligned}
$$

Hence $\hat{\Delta} \leq \psi^*\xi(\|y\|_{\mathcal{X}})\tilde{M}$. Therefore we have

$$\begin{aligned}
&\varrho^*\left[1+\left|\lambda_2^*(e-1)\right|+\frac{1}{\Gamma(\mu^*+1)}\right]\hat{\Delta}+\sigma^*\left[1+\left|\lambda_1^*(e-1)\right|+\frac{1}{\Gamma(\gamma^*+1)}\right]\\
&\quad\leq \varrho^*\left[1+\left|\lambda_2^*(e-1)\right|+\frac{1}{\Gamma(\mu^*+1)}\right]\psi^*\xi\left(\|y\|_{\mathcal{X}}\right)\tilde{M}\\
&\qquad+\sigma^*\left[1+\left|\lambda_1^*(e-1)\right|+\frac{1}{\Gamma(\gamma^*+1)}\right]<1.
\end{aligned}$$

At this point, setting $\tilde{K}_1^* = \varrho^*[1+|\lambda_2^*(e-1)|+\frac{1}{\Gamma(\mu^*+1)}]$ and $\tilde{K}_2^* = \sigma^*[1+|\lambda_1^*(e-1)|+\frac{1}{\Gamma(\gamma^*+1)}]$, we reach $\tilde{K}_1^*\hat{\Delta}+\tilde{K}_2^* < 1$. So far, all three hypotheses of Theorem 5 are fulfilled. Thus in the following, we claim that one of the conditions (a) or (b) in Theorem 5 is possible. To begin, we check condition (b). Let $\alpha_0 \in (0,1)$ and suppose that there exists $y \in \mathcal{X}$ with $\|y\|_{\mathcal{X}} = \rho$ so that the equation $y = \alpha_0(\Phi_1 y)(\Phi_3 y) + \alpha_0(\Phi_2 y)$ holds. Then, we have

$$\begin{aligned}
\left|y(t)\right| &\leq \alpha_0\left|(\Phi_1 y)(t)\right|\left|(\Phi_3 y)(t)\right|+\alpha_0\left|(\Phi_2 y)(t)\right|\\
&\leq \alpha_0\left|\Psi\left(t,y(t),\lambda_2^*\int_1^e y(\varpi)\,\mathrm{d}\varpi,{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*}y(t)\right)\right|\\
&\quad\times\left(\left|\frac{1}{\Gamma(\kappa^*)}\int_1^t\left(\ln\frac{t}{\varpi}\right)^{\kappa^*-1}\hat{\Upsilon}\left(\varpi,y(\varpi)\right)\frac{\mathrm{d}\varpi}{\varpi}\right.\right.\\
&\quad+\frac{\tilde{a}_2(1+\tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*)}\int_1^e\left(\ln\frac{e}{\varpi}\right)^{\kappa^*-1}\hat{\Upsilon}\left(\varpi,y(\varpi)\right)\frac{\mathrm{d}\varpi}{\varpi}\\
&\quad-\frac{(1+\tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*-1)}\\
&\quad\left.\left.\times\int_1^e\left(\ln\frac{e}{\varpi}\right)^{\kappa^*+\theta^*-2}\hat{\Upsilon}\left(\varpi,y(\varpi)\right)\frac{\mathrm{d}\varpi}{\varpi}\right|\right)\\
&\quad+\alpha_0\left|\Lambda\left(t,y(t),\lambda_1^*\int_1^e y(\varpi)\,\mathrm{d}\varpi,{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*}y(t)\right)\right|\\
&\leq\left|\Psi\left(t,y(t),\lambda_2^*\int_1^e y(\varpi)\,\mathrm{d}\varpi,{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*}y(t)\right)-\Psi(t,0,0,0)+\Psi(t,0,0,0)\right|\\
&\quad\times\left(\frac{\sup_{t\in[1,e]}\psi(t)\times\xi(\|y\|_{\mathcal{X}})}{\Gamma(\kappa^*+1)}+\sup_{t\in[1,e]}\psi(t)\times\xi\left(\|y\|_{\mathcal{X}}\right)\left|\frac{\tilde{a}_2(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+1)}\right|\right.\\
&\quad\left.+\sup_{t\in[1,e]}\psi(t)\times\xi\left(\|y\|_{\mathcal{X}}\right)\left|\frac{(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*)}\right|\right)\\
&\quad+\left|\Lambda\left(t,y(t),\lambda_1^*\int_1^e y(\varpi)\,\mathrm{d}\varpi,{}^{\mathcal{H}}\mathcal{I}_{1^+}^{\gamma^*}y(t)-\Lambda(t,0,0,0)\right|+\left|\Lambda(t,0,0,0)\right)\right|\\
&\leq\left[\varrho^*\left(1+\left|\lambda_2^*(e-1)\right|+\frac{1}{\Gamma(\mu^*+1)}\right)\|y\|_{\mathcal{X}}+\Psi^*\right]\tilde{M}\psi^*\xi\left(\|y\|_{\mathcal{X}}\right)\\
&\quad+\sigma^*\left(1+\left|\lambda_1^*(e-1)\right|+\frac{1}{\Gamma(\gamma^*+1)}\right)\|y\|_{\mathcal{X}}+\Lambda^*.
\end{aligned}$$

So, we arrive at the following inequality:

$$\begin{aligned}\rho &= \|y(t)\|_{\mathcal{X}} \\ &\leq \left[\varrho^*\left(1 + \left|\lambda_2^*(e-1)\right| + \frac{1}{\Gamma(\mu^*+1)}\right)\rho + \tilde{M}\Psi^*\right]\psi^*\xi(\rho) \\ &\quad + \sigma^*\left(1 + \left|\lambda_1^*(e-1)\right| + \frac{1}{\Gamma(\gamma^*+1)}\right)\rho + \Lambda^*.\end{aligned}$$

This implies that

$$\rho \leq \frac{\tilde{M}\psi^*\xi(\rho)\Psi^* + \Lambda^*}{1 - \varrho^*(1 + |\lambda_2^*(e-1)| + \frac{1}{\Gamma(\mu^*+1)})\tilde{M}\psi^*\xi(\rho) - \sigma^*(1 + |\lambda_1^*(e-1)| + \frac{1}{\Gamma(\gamma^*+1)})}$$

which is impossible due to (9). Therefore, condition (b) stated in Theorem 5 is not fulfilled and so condition (a) in Theorem 5 holds. Consequently, the operator equation $(\Phi_1 y)(\Phi_3 y) + (\Phi_2 y) = y$ has a solution. This means that the mixed Caputo–Hadamard hybrid BVP (1)–(2) has at least one solution. □

3.1 Special cases

This subsection is devoted to deriving some analytical existence criteria for a special case formulated by mixed Caputo–Hadamard nonhybrid BVP (3). We state some hypotheses as follows:

($\mathcal{HP}5$) (Lipschitz property) There exists a constant $L_{\hat{\Upsilon}} > 0$ such that for each $y, y' \in \mathcal{X}$, we have $|\hat{\Upsilon}(t,y) - \hat{\Upsilon}(t,y')| \leq L_{\hat{\Upsilon}}|y - y'|$.

($\mathcal{HP}6$) (Boundedness property) There are constants $C_{\hat{\Upsilon}}$ and $M_{\hat{\Upsilon}}$ such that for each $y \in \mathbb{R}$ we have $|\hat{\Upsilon}(t,y)| \leq C_{\hat{\Upsilon}}|y| + M_{\hat{\Upsilon}}$.

($\mathcal{HP}7$) One has $\frac{L_{\hat{\Upsilon}}}{\Gamma(\kappa^*+1)} < 1$.

In the following lemma, an integral structure of the solution for the mixed Caputo–Hadamard BVP (3) is demonstrated.

Lemma 9 *Let $g \in \mathcal{X}$. Then a function $\tilde{y}_0$ is a solution for two-point Caputo–Hadamard fractional differential equation with mixed Hadamard integral boundary conditions*

$$\begin{cases} {}^{\mathcal{CH}}\mathcal{D}_{1^+}^{\kappa^*} y(t) = g(t), \\ {}^{\mathcal{CH}}\mathcal{D}_{1^+}^{\kappa^*} y(1) = \tilde{a}_1 y(1), \qquad \frac{1}{\Gamma(\theta^*)}\int_1^e (\ln\frac{e}{\varpi})^{\theta^*-1}[{}^{\mathcal{CH}}\mathcal{D}_{1^+}^{\kappa^*} y(\varpi)]\frac{\mathrm{d}\varpi}{\varpi} = \tilde{a}_2 y(e) \end{cases}$$

if and only if $\tilde{y}_0$ is a solution of the Hadamard fractional integral equation

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\kappa^*)}\int_1^t \left(\ln\frac{t}{\varpi}\right)^{\kappa^*-1} g(\varpi)\frac{\mathrm{d}\varpi}{\varpi} + \frac{\tilde{a}_2(1 + \tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{\kappa^*-1} g(\varpi)\frac{\mathrm{d}\varpi}{\varpi} \\ &\quad - \frac{(1 + \tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*-1)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{\kappa^*+\theta^*-2} g(\varpi)\frac{\mathrm{d}\varpi}{\varpi}, \end{aligned}$$

where $\mathcal{Q}^*$ is given by (7).

Proof The proof is similar to that of Lemma 7 and so is omitted. □

We define an operator $\Phi : \mathcal{X} \to \mathcal{X}$ by $\Phi y(t) = \Phi_1 y(t) + \Phi_2 y(t)$ which splits into two operators $\Phi_1 : \mathcal{X} \to \mathcal{X}$ and $\Phi_2 : \mathcal{X} \to \mathcal{X}$ as follows:

$$\Phi_1 y(t) = \frac{1}{\Gamma(\kappa^*)} \int_1^t \left(\ln \frac{t}{\varpi}\right)^{\kappa^*-1} \hat{\Upsilon}\big(\varpi, y(\varpi)\big) \frac{\mathrm{d}\varpi}{\varpi}, \tag{10}$$

$$\begin{aligned} \Phi_2 y(t) = {}& \frac{\tilde{a}_2(1 + \tilde{a}_1 \ln(t))}{\mathcal{Q}^* \Gamma(\kappa^*)} \int_1^e \left(\ln \frac{e}{\varpi}\right)^{\kappa^*-1} \hat{\Upsilon}\big(\varpi, y(\varpi)\big) \frac{\mathrm{d}\varpi}{\varpi} \\ & - \frac{(1 + \tilde{a}_1 \ln(t))}{\mathcal{Q}^* \Gamma(\kappa^* + \theta^* - 1)} \int_1^e \left(\ln \frac{e}{\varpi}\right)^{\kappa^*+\theta^*-2} \hat{\Upsilon}\big(\varpi, y(\varpi)\big) \frac{\mathrm{d}\varpi}{\varpi}, \end{aligned} \tag{11}$$

for each $y \in \mathcal{X}$ and $t \in [1, e]$. In this case, the equivalence of the existence of a solution for Caputo–Hadamard BVP (3) and the existence of a fixed point for operator Φ is obvious. Note that in all the following lemmas, we assume that $\mathcal{X}$ is a Banach space with sup-norm $\|\cdot\|_{\mathcal{X}}$ and two operators Φ_1 and Φ_2 are defined as in (10) and (11).

Lemma 10 *Under hypothesis* ($\mathcal{HP}5$), *the operator* Φ_1 *is Lipschitz with constant* $\tilde{K}_1^* = \frac{L_{\hat{\Upsilon}}}{\Gamma(\kappa^*+1)}$ *and the following growth condition holds*:

$$\|\Phi_1(y)\|_{\mathcal{X}} \leq \frac{C_{\hat{\Upsilon}}}{\Gamma(\kappa^* + 1)} \|y\|_{\mathcal{X}} + \frac{M_{\hat{\Upsilon}}}{\Gamma(\kappa^* + 1)},$$

for all $y \in \mathcal{X}$.

Proof By utilizing assumption ($\mathcal{HP}5$), we obtain

$$\begin{aligned} & \left|\Phi_1 y(t) - \Phi_1 y'(t)\right| \\ & = \frac{1}{\Gamma(\kappa^*)} \left| \int_1^t \left(\ln \frac{t}{\varpi}\right)^{\kappa^*-1} \hat{\Upsilon}\big(\varpi, y(\varpi)\big) \frac{\mathrm{d}\varpi}{\varpi} - \int_1^t \left(\ln \frac{t}{\varpi}\right)^{\kappa^*-1} \hat{\Upsilon}\big(\varpi, y'(\varpi)\big) \frac{\mathrm{d}\varpi}{\varpi} \right| \\ & \leq \frac{L_{\hat{\Upsilon}}}{\Gamma(\kappa^* + 1)} \|y - y'\|. \end{aligned}$$

This implies that Φ_1 is Lipschitz with constant $\tilde{K}_1^* = \frac{L_{\hat{\Upsilon}}}{\Gamma(\kappa^*+1)}$. Hence by Proposition 2, it is deduced that Φ_1 is also μ-Lipschitz with the same constant $\tilde{K}_1^* = \frac{L_{\hat{\Upsilon}}}{\Gamma(\kappa^*+1)}$, where μ is the Kuratowski's measure of noncompactness. Again, by considering ($\mathcal{HP}5$) for the growth condition, we get

$$\|\Phi_1(y)\|_{\mathcal{X}} \leq \frac{C_{\hat{\Upsilon}}}{\Gamma(\kappa^* + 1)} \|y\|_{\mathcal{X}} + \frac{M_{\hat{\Upsilon}}}{\Gamma(\kappa^* + 1)},$$

and the proof is concluded. □

Lemma 11 *Operator* Φ_2 *is continuous and also, in view of hypothesis* ($\mathcal{HP}6$), *we have the growth condition* $\|\Phi_2(y)\|_{\mathcal{X}} \leq \Delta_1 \|y\|_{\mathcal{X}} + \Delta_2$ *for every* $y \in \mathcal{X}$, *where* $\Delta_1 = C_{\hat{\Upsilon}}(|\frac{\tilde{a}_2(1+\tilde{a}_1)}{\mathcal{Q}^* \Gamma(\kappa^*+1)}| + |\frac{(1+\tilde{a}_1)}{\mathcal{Q}^* \Gamma(\kappa^*+\theta^*)}|)$ *and*

$$\Delta_2 = M_{\hat{\Upsilon}} \left(\left| \frac{\tilde{a}_2(1 + \tilde{a}_1)}{\mathcal{Q}^* \Gamma(\kappa^* + 1)} \right| + \left| \frac{(1 + \tilde{a}_1)}{\mathcal{Q}^* \Gamma(\kappa^* + \theta^*)} \right| \right).$$

Proof By assumption, we know that $\hat{\Upsilon}$ is continuous on $[1,e] \times \mathcal{X}$, and so we conclude that $\lim_{n\to\infty} \hat{\Upsilon}(t, y_n) = \hat{\Upsilon}(t, y)$. By invoking the Lebesgue's dominated convergence theorem, we obtain

$$\begin{aligned}
\lim_{n\to\infty} (\Phi_2 y_n)(t) &= \frac{\tilde{a}_2(1+\tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{\kappa^*-1} \lim_{n\to\infty}\hat{\Upsilon}\big(\varpi, y_n(\varpi)\big)\frac{\mathrm{d}\varpi}{\varpi} \\
&\quad - \frac{(1+\tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*-1)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{\kappa^*+\theta^*-2} \lim_{n\to\infty}\hat{\Upsilon}\big(\varpi, y_n(\varpi)\big)\frac{\mathrm{d}\varpi}{\varpi} \\
&= \frac{\tilde{a}_2(1+\tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{\kappa^*-1} \hat{\Upsilon}\big(\varpi, y(\varpi)\big)\frac{\mathrm{d}\varpi}{\varpi} \\
&\quad - \frac{(1+\tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*-1)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{\kappa^*+\theta^*-2} \hat{\Upsilon}\big(\varpi, y(\varpi)\big)\frac{\mathrm{d}\varpi}{\varpi} = (\Phi_2 y)(t)
\end{aligned}$$

for any $t \in [1,e]$. Hence, we see that $\Phi_2 y_n \to \Phi_2 y$ as $n \to \infty$, and so Φ_2 is continuous on $\overline{\mathcal{V}_\rho(0)}$. Now, for the sake of the investigation of the growth condition on Φ_2, we utilize hypothesis ($\mathcal{HP}6$) and obtain

$$\begin{aligned}
\left|\Phi_2 y(t)\right| &= \left|\frac{\tilde{a}_2(1+\tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{\kappa^*-1} \hat{\Upsilon}\big(\varpi, y(\varpi)\big)\frac{\mathrm{d}\varpi}{\varpi}\right. \\
&\quad \left. - \frac{(1+\tilde{a}_1\ln(t))}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*-1)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{\kappa^*+\theta^*-2} \hat{\Upsilon}\big(\varpi, y(\varpi)\big)\frac{\mathrm{d}\varpi}{\varpi}\right| \\
&\le \left|\frac{\tilde{a}_2(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{\kappa^*-1} \big(C_{\hat{\Upsilon}}\left|y(s)\right| + M_{\hat{\Upsilon}}\big)\frac{\mathrm{d}\varpi}{\varpi}\right| \\
&\quad + \left|\frac{(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*-1)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{\kappa^*+\theta^*-2} \big(C_{\hat{\Upsilon}}\left|y(s)\right| + M_{\hat{\Upsilon}}\big)\frac{\mathrm{d}\varpi}{\varpi}\right| \\
&\le C_{\hat{\Upsilon}}\left(\left|\frac{\tilde{a}_2(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+1)}\right| + \left|\frac{(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*)}\right|\right)\|y\|_{\mathcal{X}} \\
&\quad + M_{\hat{\Upsilon}}\left(\left|\frac{\tilde{a}_2(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+1)}\right| + \left|\frac{(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*)}\right|\right)
\end{aligned}$$

which is the desired conclusion. □

Lemma 12 *Operator $\Phi_2 : \mathcal{X} \to \mathcal{X}$ is compact. Moreover, Φ_2 is μ-Lipschitz with constant $\tilde{K}_2^* = 0$ where μ is the Kuratowski's measure of noncompactness.*

Proof Consider a bounded subset $\mathcal{B} \subset \overline{\mathcal{V}_\rho(0)}$ in $\mathcal{X}$ and take a sequence $\{y_n\}$ belonging to $\mathcal{B}$. Then, by Lemma 11, we have

$$\left\|\Phi_2(y_n)\right\|_{\mathcal{X}} \le \Delta_1\|y_n\|_{\mathcal{X}} + \Delta_2 < \infty$$

for each $y_n \in \mathcal{B}$ which yields that $\Phi_2(\mathcal{B})$ is a bounded set. Besides, we verify that $\{\Phi_2(y_n)\}$ is equicontinuous for each $y_n \in \mathcal{B}$. Take $t_1, t_2 \in [1,e]$ so that $t_1 < t_2$. Then, we obtain

$$\begin{aligned}
&\left|\Phi_2(y_n)(t_2) - \Phi_2(y_n)(t_1)\right| \\
&\quad \le \frac{|\tilde{a}_2\tilde{a}_1|(|\ln(t_2) - \ln(t_1)|)}{|\mathcal{Q}^*|\Gamma(\kappa^*)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{\kappa^*-1} \left|\hat{\Upsilon}\big(\varpi, y(\varpi)\big)\right|\frac{\mathrm{d}\varpi}{\varpi}
\end{aligned}$$

$$+\frac{|\tilde{a}_1|(|\ln(t_2)-\ln(t_1)|)}{|\mathcal{Q}^*|\Gamma(\kappa^*+\theta^*-1)}\int_1^e\left(\ln\frac{e}{\varpi}\right)^{\kappa^*+\theta^*-2}\big|\hat{\Upsilon}\big(\varpi,y(\varpi)\big)\big|\frac{\mathrm{d}\varpi}{\varpi}$$
$$\le\frac{|\tilde{a}_2\tilde{a}_1|(|\ln(t_2)-\ln(t_1)|)}{|\mathcal{Q}^*|\Gamma(\kappa^*+1)}\big(C_{\hat{\Upsilon}}\|y\|_{\mathcal{X}}+M_{\hat{\Upsilon}}\big)$$
$$+\frac{|\tilde{a}_1|(|\ln(t_2)-\ln(t_1)|)}{|\mathcal{Q}^*|\Gamma(\kappa^*+\theta^*)}\big(C_{\hat{\Upsilon}}\|y\|_{\mathcal{X}}+M_{\hat{\Upsilon}}\big).$$

Evidently, it is seen that the RHS of the inequality approaches 0 (regardless of the choice of $y_n\in\mathcal{B}$) whenever $t_1\to t_2$. Thus, letting $t_1\to t_2$, we get $|\Phi_2(y_n)(t_2)-\Phi_2(y_n)(t_1)|\to 0$ and so $\{\Phi_2(y_n)\}$ is equicontinuous. Taking into account the Arzela–Ascoli theorem, we obtain that $\Phi_2(\mathcal{B})$ is compact. In addition, in view of Proposition 3, Φ_2 is μ-Lipschitz with constant zero. □

In this position, we establish the main results for the mixed Caputo–Hadamard nonhybrid BVP (3) based on the above lemmas.

Theorem 13 *Under assumptions* $(\mathcal{HP}5)$ *and* $(\mathcal{HP}6)$, *the mixed Caputo–Hadamard nonhybrid BVP* (3) *has at least one solution* $y\in\mathcal{X}$ *provided* $\frac{C_{\hat{\Upsilon}}}{\Gamma(\kappa^*+1)}+\Delta_1<1$. *Further, the family of solutions of* (3) *is bounded in the space* $\mathcal{X}$.

Proof In view of the hypothesis $(\mathcal{HP}7)$ and Lemma 10, we deduce that $\Phi_1:\mathcal{X}\to\mathcal{X}$ defined in (10) is μ-Lipschitz with constant $\tilde{K}_1^*=\frac{L_{\hat{\Upsilon}}}{\Gamma(\kappa^*+1)}\in(0,1)$. Furthermore, we find that operator $\Phi_2:\mathcal{X}\to\mathcal{X}$ defined in (11) is μ-Lipschitz with $\tilde{K}_2^*=0$ according to Lemma 12. Here, Proposition 4 implies that the operator $\Phi:\mathcal{X}\to\mathcal{X}$ defined by $\Phi=\Phi_1+\Phi_2$ is a strict μ-contraction with constant $\tilde{K}^*=\tilde{K}_1^*+\tilde{K}_2^*=\tilde{K}_1^*$ and, since $\tilde{K}^*<1$, Φ is μ-condensing. Now, take

$$\mathcal{B}:=\big\{y\in\mathcal{X}:\text{there is }\lambda\in[0,1]\text{ so that }y=\lambda\Phi(y)\big\}.$$

In this step, it is enough to show that $\mathcal{B}$ is a bounded subset of $\mathcal{X}$. For this, select $y\in\mathcal{B}$. Then in the light of the growth conditions obtained in Lemmas 10 and 9, we may write

$$\|y\|_{\mathcal{X}}=\big\|\lambda\Phi(y)\big\|_{\mathcal{X}}=\lambda\big\|\Phi(y)\big\|_{\mathcal{X}}\le\lambda\big(\big\|\Phi_1(y)\big\|_{\mathcal{X}}+\big\|\Phi_2(y)\big\|_{\mathcal{X}}\big)$$
$$\le\lambda\left(\frac{C_{\hat{\Upsilon}}}{\Gamma(\kappa^*+1)}\|y\|_{\mathcal{X}}+\frac{M_{\hat{\Upsilon}}}{\Gamma(\kappa^*+1)}+\Delta_1\|y\|_{\mathcal{X}}+\Delta_2\right)$$
$$\le\lambda\left(\frac{C_{\hat{\Upsilon}}}{\Gamma(\kappa^*+1)}+\Delta_1\right)\|y\|_{\mathcal{X}}+\lambda\left(\frac{M_{\hat{\Upsilon}}}{\Gamma(\kappa^*+1)}+\Delta_2\right),$$

implying that the set $\mathcal{B}$ is bounded in $\mathcal{X}$. Thus there is a number $\rho>0$ such that $\mathcal{B}\subset\overline{\mathcal{V}_\rho(0)}$, and so we have $\deg(I-\lambda\Phi,\overline{\mathcal{V}_\rho(0)},0)=1$, by applying Theorem 6. Finally, under the hypotheses of Theorem 6 due to Isaia, the operator $\Phi=\Phi_1+\Phi_2$ has at least one fixed point and the family of fixed points of Φ is bounded in $\mathcal{X}$. This means that the mixed Caputo–Hadamard nonhybrid BVP (3) has at least one solution on $[1,e]$ and the family of solutions is bounded. The proof is finished. □

Eventually, we derive a uniqueness criterion for the mixed Caputo–Hadamard nonhybrid BVP (3) in the following theorem.

Theorem 14 *In addition to three hypotheses* $(\mathcal{HP}5)$, $(\mathcal{HP}6)$, *and* $(\mathcal{HP}7)$, *let us assume that*

$$L_{\hat{\Upsilon}}\left(\frac{1}{\Gamma(\kappa^*+1)}+\left|\frac{\tilde{a}_2(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+1)}\right|+\left|\frac{(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*)}\right|\right)<1.$$

Then the mixed Caputo–Hadamard nonhybrid BVP (3) *has a unique solution on* $[1,e]$.

Proof Let $y\in\mathcal{X}$ be arbitrary. By Lemma 10 and assumption $(\mathcal{HP}5)$, we obtain

$$\left|\Phi_1 y(t)-\Phi_1 y'(t)\right|\le\frac{L_{\hat{\Upsilon}}}{\Gamma(\kappa^*+1)}\|y-y'\|_{\mathcal{X}}, \tag{12}$$

where $\Phi_1:\mathcal{X}\to\mathcal{X}$ is defined in (10). Furthermore, we have the following estimate:

$$\begin{aligned}
&\left|\Phi_2 y(t)-\Phi_2 y'(t)\right|\\
&\quad\le\left|\frac{\tilde{a}_2(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*)}\right|\left|\left(\int_1^e\left(\ln\frac{e}{\varpi}\right)^{\kappa^*-1}\left[\hat{\Upsilon}(\varpi,y(\varpi))-\hat{\Upsilon}(\varpi,y'(\varpi))\frac{\mathrm{d}\varpi}{\varpi}\right]\right|\right.\\
&\qquad+\left|\frac{(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*-1)}\right|\left|\left(\int_1^e\left(\ln\frac{e}{\varpi}\right)^{\kappa^*+\theta^*-2}\left[\hat{\Upsilon}(\varpi,y(\varpi))\frac{\mathrm{d}\varpi}{\varpi}-\hat{\Upsilon}(\varpi,y'(\varpi))\right]\right|\right.\\
&\quad\le L_{\hat{\Upsilon}}\left(\left|\frac{\tilde{a}_2(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+1)}\right|+\left|\frac{(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*)}\right|\right)\|y-y'\|_{\mathcal{X}},
\end{aligned}\tag{13}$$

where $\Phi_2:\mathcal{X}\to\mathcal{X}$ is defined in (11). From (12) and (13), we have

$$\left|\Phi(y)\right|\le L_{\hat{\Upsilon}}\left(\frac{1}{\Gamma(\kappa^*+1)}+\left|\frac{\tilde{a}_2(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+1)}\right|+\left|\frac{(1+\tilde{a}_1)}{\mathcal{Q}^*\Gamma(\kappa^*+\theta^*)}\right|\right)\|y-y'\|_{\mathcal{X}},$$

which yields that $\Phi=\Phi_1+\Phi_2:\mathcal{X}\to\mathcal{X}$ is a contraction. By utilizing the Banach contraction principle, it is deduced that the mixed Caputo–Hadamard nonhybrid BVP (3) has a unique solution. □

4 Examples

In this part of the paper, we examine our theoretical results by presenting some numerical examples to show the applicability of the analytical findings.

Example 1 To illustrater the mixed Caputo-Hadamard hybrid BVP (1)–(2), we formulate the following hybrid equation:

$$\begin{aligned}
&{}^{\mathcal{CH}}\mathcal{D}_{1^+}^{1.78}\left[\frac{y(t)-\frac{1}{2+t}(y(t)+\cos(-\frac{1}{9}\int_1^e y(\varpi)\,\mathrm{d}\varpi)+\sin({}^{\mathcal{H}}\mathcal{I}_{1^+}^{0.33}y(t)))+0.2021}{\frac{t}{2020}(y(t)+\frac{\frac{1}{27}\int_1^e y(\varpi)\,\mathrm{d}\varpi+{}^{\mathcal{H}}\mathcal{I}_{1^+}^{2.11}y(t)}{1+\frac{1}{27}\int_1^e y(\varpi)\,\mathrm{d}\varpi+{}^{\mathcal{H}}\mathcal{I}_{1^+}^{2.11}y(t)})+0.11}\right]\\
&\quad=(1+t)^2\sin\bigl(y(t)\bigr)
\end{aligned}\tag{14}$$

furnished with mixed Hadamard integral hybrid boundary conditions

$$\begin{cases} {}^{\mathcal{CH}}\mathcal{D}_{1^+}\big[\frac{y(t)-\frac{1}{2+t}(y(t)+\cos(-\frac{1}{9}\int_1^e y(\varpi)\,\mathrm{d}\varpi)+\sin({}^{\mathcal{H}}\mathcal{I}_{1^+}^{0.33}y(t)))+0.2021}{\frac{t}{2020}(y(t)+\frac{\frac{1}{27}\int_1^e y(\varpi)\,\mathrm{d}\varpi+{}^{\mathcal{H}}\mathcal{I}_{1^+}^{2.11}y(t)}{1+\frac{1}{27}\int_1^e y(\varpi)\,\mathrm{d}\varpi+{}^{\mathcal{H}}\mathcal{I}_{1^+}^{2.11}y(t)})+0.11}\big]\big|_{t=1} \\ \quad = -0.66\big[\frac{y(t)-\frac{1}{2+t}(y(t)+\cos(-\frac{1}{9}\int_1^e y(\varpi)\,\mathrm{d}\varpi)+\sin({}^{\mathcal{H}}\mathcal{I}_{1^+}^{0.33}y(t)))+0.2021}{\frac{t}{2020}(y(t)+\frac{\frac{1}{27}\int_1^e y(\varpi)\,\mathrm{d}\varpi+{}^{\mathcal{H}}\mathcal{I}_{1^+}^{2.11}y(t)}{1+\frac{1}{27}\int_1^e y(\varpi)\,\mathrm{d}\varpi+{}^{\mathcal{H}}\mathcal{I}_{1^+}^{2.11}y(t)})+0.11}\big]\big|_{t=1}, \\ {}^{\mathcal{H}}\mathcal{I}_{1^+}^{1.44}\,{}^{\mathcal{CH}}\mathcal{D}_{1^+}\big[\frac{y(t)-\frac{1}{2+t}(y(t)+\cos(-\frac{1}{9}\int_1^e y(\varpi)\,\mathrm{d}\varpi)+\sin({}^{\mathcal{H}}\mathcal{I}_{1^+}^{0.33}y(t)))+0.2021}{\frac{t}{2020}(y(t)+\frac{\frac{1}{27}\int_1^e y(\varpi)\,\mathrm{d}\varpi+{}^{\mathcal{H}}\mathcal{I}_{1^+}^{2.11}y(t)}{1+\frac{1}{27}\int_1^e y(\varpi)\,\mathrm{d}\varpi+{}^{\mathcal{H}}\mathcal{I}_{1^+}^{2.11}y(t)})+0.11}\big]\big|_{t=e} \\ \quad = 0.89\big[\frac{y(t)-\frac{1}{2+t}(y(t)+\cos(-\frac{1}{9}\int_1^e y(\varpi)\,\mathrm{d}\varpi)+\sin({}^{\mathcal{H}}\mathcal{I}_{1^+}^{0.33}y(t)))+0.2021}{\frac{t}{2020}(y(t)+\frac{\frac{1}{27}\int_1^e y(\varpi)\,\mathrm{d}\varpi+{}^{\mathcal{H}}\mathcal{I}_{1^+}^{2.11}y(t)}{1+\frac{1}{27}\int_1^e y(\varpi)\,\mathrm{d}\varpi+{}^{\mathcal{H}}\mathcal{I}_{1^+}^{2.11}y(t)})+0.11}\big]\big|_{t=e} \end{cases} \tag{15}$$

so that $t \in [1,e]$, $\kappa^* = 1.78$, $\gamma^* = 0.33$, $\mu^* = 2.11$, $\theta^* = 1.44$, $\lambda_1^* = \frac{-1}{9}$, $\lambda_2^* = \frac{1}{27}$, $\tilde{a_1} = -0.66$, and $\tilde{a_2} = 0.89$. Define the function $\hat{\Upsilon} : [1,e] \times \mathbb{R} \to \mathbb{R}$ by $\hat{\Upsilon}(t, y(t)) = (1+t)^2 \sin(y(t))$. Obviously, $\hat{\Upsilon} \in \mathcal{C}_{\mathbb{R}}([1,e] \times \mathbb{R})$. Now, put $\psi(t) = (1+t)^2$ and $\xi(\|y\|) = 1$. Thus $\psi^* \approx 13.8256$. Further, define two continuous maps $\Lambda : [1,e] \times \mathbb{R}^3 \to \mathbb{R}$ and $\Psi : [1,e] \times \mathbb{R}^3 \to \mathbb{R} \setminus \{0\}$ by

$$\begin{aligned} &\Lambda\left(t, y(t), -\frac{1}{9}\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{0.33}y(t)\right) \\ &\quad = \frac{1}{2+t}\left(y(t) + \cos\left(-\frac{1}{9}\int_1^e y(\varpi)\,\mathrm{d}\varpi\right) + \sin\left({}^{\mathcal{H}}\mathcal{I}_{1^+}^{0.33}y(t)\right)\right) + 0.2021 \end{aligned}$$

and

$$\begin{aligned} &\Psi\left(t, y(t), \lambda_2^*\int_1^e y(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{\mu^*}y(t)\right) \\ &\quad = \frac{t}{2020}\left(y(t) + \frac{\frac{1}{27}\int_1^e y(\varpi)\,\mathrm{d}\varpi + {}^{\mathcal{H}}\mathcal{I}_{1^+}^{2.11}y(t)}{1 + \frac{1}{27}\int_1^e y(\varpi)\,\mathrm{d}\varpi + {}^{\mathcal{H}}\mathcal{I}_{1^+}^{2.11}y(t)}\right) + 0.11. \end{aligned}$$

Note that $\Lambda^* \approx 0.2021$ and $\Psi^* = 0.11$. We claim that function Λ is Lipschitz. To see this, for every $y, y' \in \mathbb{R}$, we have

$$\begin{aligned} &\left|\Lambda\left(t, y_1(t), -\frac{1}{9}\int_1^e y_1(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{0.33}y_1(t)\right)\right. \\ &\qquad \left. - \Lambda\left(t, y_2(t), -\frac{1}{9}\int_1^e y_2(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{0.33}y_2(t)\right)\right| \\ &\quad \le \frac{1}{2+t}\left[1 + \left|\frac{-1}{9}(e-1)\right| + \frac{1}{\Gamma(0.33+1)}\right] \sup_{t\in[1,e]} \left|y_1(t) - y_2(t)\right|. \end{aligned}$$

Letting $\sigma(t) = \frac{1}{2+t}$, we have $\sigma^*[1 + |\frac{-1}{9}(e-1)| + \frac{1}{\Gamma(0.33+1)}] \approx 0.8500$. In a similar manner, function Ψ is also Lipschitz. Indeed, for every $y, y' \in \mathbb{R}$, we have

$$\begin{aligned} &\left|\Psi\left(t, y_1(t), \frac{1}{27}\int_1^e y_1(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{2.11}y_1(t)\right)\right. \\ &\qquad \left. - \Psi\left(t, y_2(t), \frac{1}{27}\int_1^e y_2(\varpi)\,\mathrm{d}\varpi, {}^{\mathcal{H}}\mathcal{I}_{1^+}^{2.11}y_2(t)\right)\right| \end{aligned}$$

$$\leq \frac{t}{2020}\left[1+\left|\frac{1}{27}(e-1)\right|+\frac{1}{\Gamma(2.11+1)}\right]\sup_{t\in[1,e]}\left|y_1(t)-y_2(t)\right|$$
$$\leq \varrho(t)\left[1+\left|\lambda_2^*(e-1)\right|+\frac{1}{\Gamma(\mu^*+1)}\right]\sup_{t\in[1,e]}\left|y_1(t)-y_2(t)\right|$$

so that $\varrho^*[1+|\lambda_2^*(e-1)|+\frac{1}{\Gamma(\mu^*+1)}]=\frac{e}{2020}[1+|\frac{1}{27}(e-1)|+\frac{1}{\Gamma(2.11+1)}]=0.0020$. Eventually, we obtain $\tilde{M}\approx 0.9980$ and select $\rho > 0.8975 > 0$. In addition,

$$\varrho^*\left[1+\left|\lambda_2^*(e-1)\right|+\frac{1}{\Gamma(\mu^*+1)}\right]\psi^*\xi\left(\|y\|_{\mathcal{X}}\right)\tilde{M}+\sigma^*\left[1+\left|\lambda_1^*(e-1)\right|+\frac{1}{\Gamma(\gamma^*+1)}\right]$$
$$\approx 0.6057 < 1.$$

Hence, by invoking Theorem 8, it is realized that the mixed Caputo–Hadamard hybrid BVP (14)–(15) has a solution on $[1,e]$.

Example 2 To illustrate the mixed Caputo–Hadamard nonhybrid BVP (3), we formulate the following nonhybrid BVP:

$$\begin{cases} {}^{\mathrm{CH}}\mathcal{D}_{1^+}^{1.14}y(t)=\frac{1}{49+\exp(t^2-1)}\frac{|y(t)|}{(1+|y(t)|)}, \\ {}^{\mathrm{CH}}\mathcal{D}_{1^+}y(1)=-1.66y(1), \qquad \frac{1}{\Gamma(0.74)}\int_1^e(\ln\frac{e}{\varpi})^{-0.26}[{}^{\mathcal{CH}}\mathcal{D}_{1^+}y(\varpi)]\frac{\mathrm{d}\varpi}{\varpi}=0.56y(e), \end{cases} \tag{16}$$

so that $t\in[1,e]$, $\kappa^*=1.14$, $\theta^*=0.74$, $\tilde{a_1}=-1.66$, and $\tilde{a_2}=0.56$. Then, an integral structure of the solution for the mixed Caputo–Hadamard nonhybrid BVP (16) is represented by

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(1.14)}\int_1^t\left(\ln\frac{t}{\varpi}\right)^{0.14}\hat{\Upsilon}\left(t,y(t)\right)\frac{\mathrm{d}\varpi}{\varpi} \\ &\quad+\frac{0.56(1-1.66\ln(t))}{\mathcal{Q}^*\Gamma(1.14)}\int_1^e\left(\ln\frac{e}{\varpi}\right)^{0.14}\hat{\Upsilon}\left(t,y(t)\right)\frac{\mathrm{d}\varpi}{\varpi} \\ &\quad-\frac{(1-1.66\ln(t))}{\mathcal{Q}^*\Gamma(0.88)}\int_1^e\left(\ln\frac{e}{\varpi}\right)^{-0.12}\hat{\Upsilon}\left(t,y(t)\right)\frac{\mathrm{d}\varpi}{\varpi} \end{aligned} \tag{17}$$

for any $t\in[1,e]$, where the continuous function $\hat{\Upsilon}:[1,e]\times\mathbb{R}\to\mathbb{R}$ is defined by

$$\hat{\Upsilon}\left(t,y(t)\right)=\frac{1}{49+\exp(t^2-1)}\left(\frac{|y(t)|}{1+|y(t)|}\right).$$

Then, one can write

$$\left|\hat{\Upsilon}\left(t,y(t)\right)-\hat{\Upsilon}\left(t,y'(t)\right)\right|\leq\frac{1}{50}\left\|y(t)-y'(t)\right\|_{\mathbb{R}}$$

and

$$\left|\hat{\Upsilon}\left(t,y(t)\right)\right|\leq\frac{1}{50}\left|y(t)\right|$$

with $L_{\hat{\Upsilon}}=\frac{1}{50}$, $C_{\hat{\Upsilon}}=\frac{1}{50}$, and $M_{\hat{\Upsilon}}=0$. Now, define three operators $\Phi_1,\Phi_2,\Phi:\mathbb{R}\to\mathbb{R}$ as follows:

$$(\Phi_1 y)(t)=\frac{1}{\Gamma(1.14)}\int_1^t\left(\ln\frac{t}{\varpi}\right)^{0.14}\hat{\Upsilon}\left(t,y(t)\right)\frac{\mathrm{d}\varpi}{\varpi},$$

$$(\varPhi_2 y)(t) = \frac{0.56(1-1.66\ln(t))}{\mathcal{Q}^*\varGamma(1.14)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{0.14}\hat{\Upsilon}\big(t,y(t)\big)\frac{\mathrm{d}\varpi}{\varpi} - \frac{(1-1.66\ln(t))}{\mathcal{Q}^*\varGamma(0.88)}\int_1^e \left(\ln\frac{e}{\varpi}\right)^{-0.12}\hat{\Upsilon}\big(t,y(t)\big)\frac{\mathrm{d}\varpi}{\varpi}$$

and $(\varPhi y)(t) = (\varPhi_1 y)(t) + (\varPhi_2 y)(t)$. Since $\varPhi_1$ and $\varPhi_2$ are continuous and bounded, $\varPhi = \varPhi_1 + \varPhi_2$ is continuous and bounded, too. Further, we have

$$\left|\varPhi_1 y(t) - \varPhi_1 y'(t)\right| \le \frac{1}{50\times\varGamma(2.14)}\|y-y'\|_{\mathbb{R}}$$

which implies that $\varPhi_1$ is μ-Lipschitz with constant $\tilde{K}_1^* = \frac{1}{50\times\varGamma(2.14)}$ by Proposition 2. Also, by using hypothesis $(\mathcal{HP}6)$ for the growth condition, we get $\|\varPhi_1(y)\|_{\mathbb{R}} \le \frac{1}{50\times\varGamma(2.14)}\|y\|_{\mathbb{R}}$. Therefore, since $\varPhi_1$ is μ-Lipschitz with constant $\tilde{K}_1^* = \frac{1}{50\times\varGamma(2.14)}$ and $\varPhi_2$ is compact with constant $\tilde{K}_2^* = 0$, by Proposition 4, $\varPhi = \varPhi_1 + \varPhi_2$ is a strict μ-contraction with constant $\tilde{K}^* = \tilde{K}_1^* + \tilde{K}_2^* = \frac{1}{50\times\varGamma(2.14)} \simeq 0.0187 < 1$. Thus $\varPhi$ is a μ-condensing operator. Also, take

$$\mathcal{B} = \left\{y \in C_{\mathbb{R}}\big([1,e]\big) : \text{there is } \lambda \in [0,1] \text{ such that } y = \frac{1}{2}(\varPhi y)\right\}.$$

Then $\|y\|_{\mathbb{R}} \le \frac{1}{2}\|\varPhi y\|_{\mathbb{R}} \le 1$ implies that $\mathcal{B}$ is a bounded set and so, by Theorem 13, it is deduced that the mixed Caputo–Hadamard nonhybrid BVP (16) has at least one solution y in $C_{\mathbb{R}}([1,e])$. In addition,

$$L_{\hat{\Upsilon}}\left(\frac{1}{\varGamma(\kappa^*+1)} + \left|\frac{\tilde{a}_2(1+\tilde{a}_1)}{\mathcal{Q}^*\varGamma(\kappa^*+1)}\right| + \left|\frac{(1+\tilde{a}_1)}{\mathcal{Q}^*\varGamma(\kappa^*+\theta^*)}\right|\right) \approx 0.0331 < 1.$$

Therefore, Theorem 14 implies that the mixed Caputo–Hadamard nonhybrid BVP (16) has a unique solution.

5 Conclusion

The fractional calculus has always been one of the most widely used branches of mathematics in other applied and computational sciences. This degree of importance is due to the high flexibility of the tools and operators defined in this theory. On this basis, researchers have been using various powerful fractional operators in recent decades to model different types of existing natural processes in the world. In the current research article, two hybrid and nonhybrid fractional BVPs of Caputo–Hadamard type are addressed. We seek the existence criteria for these two problems separately. We first utilize the generalized Dhage's theorem to derive desired results for an integral structure of solutions for the proposed hybrid BVP (1)–(2). Next, we establish other results for nonhybrid BVP (3) based on some existing notions in the topological degree theory. At the end of the paper, we examine our theoretical results by presenting some numerical examples to show the applicability of the analytical findings.

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