TOPOLOGICAL ENTROPY BOUNDS MEASURE-THEORETIC ENTROPY

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Let T be a homeomorphism from a compact space X onto itself and let μ be a T-invariant probability measure on the Borel sets of X. It was conjectured in [1] that the measure-theoretic entropy of T with respect to μ is less than or equal to the topological entropy of T. The purpose of this paper is to show, under the assumption that X is metric, that the inequality holds when T is assumed only to be a continuous map from X into itself.

We shall first prove the inequality under the assumption that X is a closed subset of the Hilbert cube which is invaraint under a certain type of shift operator, and T is the restriction of the operator to X. The generalization will be obtained by considering representations of T as such shifts.

By a *flow* we mean a pair (X, T), where X is a compact metric space and T is a continuous map from X into itself. Throughout the paper, (X, T) and (Y, S) will denote arbitrary flows. A continuous map $\phi: X \to Y$ will be called a *homomorphism* from (X, T) into (Y, S) if $\phi \circ T = S \circ \phi$. If α is any finite cover of X, we let $N(\alpha)$ be the number of members in a subcover of α of minimal cardinality. As in [1], we write $\alpha \vee \beta = \{U \cap V: U \in \alpha,\ V \in \beta\}$ and we write $\alpha > \beta$ to mean that α is a refinement of β, though this is contrary to the notation of many authors. As in [1], it follows from the fact that $N(\alpha \vee \beta) \leqq N(\alpha) \cdot N(\beta)$, that the limit exists in the following definition:

$$h(\alpha, T) = \lim_{n \to \infty} \frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha \right),$$

for any finite cover α of X. Finally, we note that if $\alpha > \beta$, then $N(\alpha) \geqq N(\beta)$, and $h(\alpha, T) \geqq h(\beta, T)$. The *topological entropy of* T is defined as

$$h(T) = \sup h(\alpha, T),$$

where the supremum is taken over all finite open covers of X.

It is easily seen that if ϕ is a homomorphism from (X, T) onto (Y, S) and if α is a finite cover of Y, then $h(\phi^{-1}(\alpha), T) = h(\alpha, S)$. It follows that $h(S) \leqq h(T)$.

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Let $M(X, T)$ denote the set of all T-invariant probability measures on the Borel sets of X. For a finite measurable partition α of X, we write, as in [2],

$$H_\mu(\alpha) = -\sum_{A\in\alpha} \mu(A)\log \mu(A),$$

and

$$h_\mu(\alpha, T) = \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right).$$

The *measure-theoretic entropy of* T is defined as

$$h_\mu(T) = \sup h_\mu(\alpha, T),$$

where the supremum is taken over all finite measurable partitions of X.

For every cover α of X, we write

$$w(\alpha) = \bigcup \{U \cap V:\ U, V \in \alpha;\ U \neq V\}.$$

If μ is a measure on the Borel sets of X, we say that *a cover α is μ-disjoint* whenever $\mu(w(\alpha))=0$. For $\mu\in M(X, T)$, it is easy to show that if α is μ-disjoint then so is $\bigvee_{i=0}^{n-1} T^{-i}\alpha$. It follows that $h_\mu(\alpha, T)$ is defined for any finite measurable μ-disjoint cover α of X.

By an essential member of a cover α we mean a member U such that $\alpha-\{U\}$ is not a cover.

PROPOSITION 1. *Let $\mu\in M(X, T)$ and let α be a finite measurable μ-disjoint cover of X. Then*

$$h_\mu(\alpha, T) \leqq h(\alpha, T).$$

PROOF. Fix a positive integer n and let β be the set of members of $\bigvee_{i=0}^{n-1} T^{-i}\alpha$ which have positive measure. Let k be the number of members of β. It follows from the fact that $\bigvee_{i=0}^{n-1} T^{-i}\alpha$ is μ-disjoint that each set of β is essential to $\bigvee_{i=0}^{n-1} T^{-i}\alpha$, so that $k\leqq N(\bigvee_{i=0}^{n-1} T^{-i}\alpha)$. On the other hand, it follows from the convexity of the function $t \log t$ that $H_\mu(\beta)\leqq\log k$. (See [4, p. 4].) We can conclude that

$$H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\beta\right) = H_\mu(\beta) \leqq \log N\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right).$$

We now divide by n and let n tend to infinity, obtaining the result.

To make use of Proposition 1 we must be able to compare $h(\alpha, T)$ with $h(T)$ for a μ-disjoint cover α of X. We let $p(\alpha)$ denote the order of α, the largest number of distinct members of α with a nonempty intersection.

PROPOSITION 2. *If α is a finite closed cover of X, then*

$$h(\alpha, T) \leqq h(T) + \log p(\alpha).$$

PROOF. For $x \in X$, let $\mathrm{St}(\alpha, x)$ denote the union of the members of α which contain x, and let $\mathrm{St}(\alpha) = \{\mathrm{St}(\alpha, x) : x \in X\}$. Let n be a positive integer. We claim that

$$N\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right) \leqq N\left(\bigvee_{i=0}^{n-1} T^{-i}\mathrm{St}(\alpha)\right) \cdot p(\alpha)^n.$$

For let γ be a subcover of $\bigvee_{i=0}^{n-1} T^{-i}\mathrm{St}(\alpha)$ of minimal cardinality. Then each member F of γ is of the form

$$F = F_0 \cap T^{-1}F_1 \cap \cdots \cap T^{-n+1}F_{n-1},$$
$$\text{where } F_i \in \mathrm{St}(\alpha), \quad i = 0, \cdots, n-1.$$

Now each F_i is a union of at most $p(\alpha)$ members of α, so F is a union of at most $p(\alpha)^n$ members of $\bigvee_{i=0}^{n-1} T^{-i}\alpha$. Hence, from γ we obtain a subcover of $\bigvee_{i=0}^{n-1} T^{-i}\alpha$ with at most

$$N\left(\bigvee_{i=0}^{n-1} T^{-i}\,\mathrm{St}(\alpha)\right) \cdot p(\alpha)^n$$

members. This proves the above inequality. If we now take the logarithm of both sides of the inequality, divide by n and let n tend to infinity, we obtain

$$h(\alpha, T) \leqq h(\mathrm{St}(\alpha), T) + \log p(\alpha).$$

We next claim that for each $x \in X$, x is an interior point of $\mathrm{St}(\alpha, x)$. This follows from the fact that the intersection of the complements of the sets of α which do not contain x is an open subset of $\mathrm{St}(\alpha, x)$. It now follows that $\mathrm{St}(\alpha)$ has an open refinement, so that $h(\mathrm{St}(\alpha), T) \leqq h(T)$, and the proposition is proved.

It should be remarked that a finite closed cover can yield entropy strictly greater than the topological entropy. (See [**3**, p. 45].)

The following theorem will be used to obtain $h_\mu(T)$ in terms of the entropy of finite, μ-disjoint closed covers of X. The theorem is due to Rohlin, [**4**]; it appears in this form in [**2**, p. 87].

THEOREM. *Let $\mu \in M(X, T)$, and let $\alpha_0 < \alpha_1 < \cdots$ be a sequence of finite μ-disjoint measurable covers of X such that the smallest sigma-*

algebra containing $\bigcup_{i=1}^{\infty} \bigcup_{k=1}^{\infty} T^{-i}\alpha_k$ *is the algebra of all Borel sets of* X. *Then*

$$h_\mu(T) = \lim_{k\to\infty} h_\mu(\alpha_k, T).$$

Our next step will be to show that for certain subsets of the Hilbert cube we can obtain a sequence $\alpha_0 < \alpha_1 < \cdots$ of covers to which we can apply Propositions 1 and 2, and Rohlin's Theorem.

Throughout the paper, Z^+ will denote the set of nonnegative integers.

Let I^n denote the n-cube,

$$I^n = \{u = (u_0, u_1, \cdots, u_{n-1}): 0 \leqq u_i \leqq 1 \text{ for } i = 0, \cdots, n-1\}.$$

We shall find it useful to let I^n have the metric d defined as follows:

$$d\,(u, v) = \max_{i=0,\cdots,n-1} |\mu_i - v_i| \qquad \text{for } \mu, v \in I^n.$$

For each positive integer n we define B_n to be the set of all sequences $x = (x(0), x(1) \cdots)$ of points in I^n; that is, $B_n = (I^n)Z^+$. We let B_n have the metric ρ defined as follows:

$$\rho(x, y) = \sum_{m=0}^{\infty} 2^{-m} d(x(m), y(m)).$$

We let $\sigma_n: B_n \to B_n$ be the shift transformation on B_n defined by: $\sigma_n(x)(m) = x(m+1)$ for $m \in N$, $x \in B_n$. Finally, we let $\pi_n: B_n \to I_n$ be the projection:

$$\pi_n(x) = x(0) \qquad \text{for } x \in B_n.$$

PROPOSITION 3. (B_n, σ_n) *is isomorphic to* (B_1, σ_1^n) *for any positive integer* n.

PROOF. Define $f: B_1 \to B_n$ as follows:

$$(f(x)(m))_i = x(nm + i), \qquad \text{for } 0 \leqq i < n, m \in Z^+.$$

It is straightforward to show that f is an isomorphism.

By mesh(α) we mean the supremum of the diameters of the sets in α.

PROPOSITION 4. *Let* α *be a cover of* I^n. *If* m *is a positive integer, then*

$$\operatorname{mesh}\left(\bigvee_{i=0}^{m-1} \sigma_n^{-i}(\pi_n^{-1}(\alpha))\right) \leqq m \cdot \operatorname{mesh}(\alpha) + 2^{-m+1}.$$

The proof is straightforward.

THEOREM 1. *Let ν be a finite measure on the Borel sets of I^n. Then there is a sequence $\beta_0 < \beta_1 < \cdots$ of closed ν-disjoint covers of I^n such that*

(1) $\text{mesh}(\beta_i) \to 0$.

(2) $p(\beta_i) \leqq n+1$ *for* $i \in Z^+$.

Before proving Theorem 1, we look at some corollaries. (Recall that $p(\beta_i)$ is the order of β_i.)

COROLLARY 1. *Let (X, T) be a subflow of (B_n, σ_n)*; i.e., *X is a closed σ_n-invariant subset of B_n and $T = \sigma_n | X$. Let $\mu \in M(X, T)$. Then $h_\mu(T) \leqq h(T) + \log(n+1)$.*

PROOF. Let $\beta_0 < \beta_1 < \cdots$ be a sequence obtained from Theorem 1, where ν is defined by the rule

$$\nu(A) = \mu(\pi_n^{-1}(A) \cap X) \qquad \text{for every Borel set } A \text{ of } I^n.$$

Let $\alpha_k = \{\pi_n^{-1}(F) \cap X : F \in \beta_k\}$, for $k \in N$. Then each α_k is a closed μ-disjoint cover of X. We claim that $\{\alpha_k\}$ satisfies the hypothesis of Rohlin's Theorem. For let $x \in X$ and let U be a neighborhood of x in X. Choose $\epsilon > 0$ so small that $\rho(x, y) < \epsilon$ implies $y \in U$, for $y \in X$. By Proposition 4, we can choose $k, m \in Z^+$ such that

$$\text{mesh}\left(\bigvee_{i=0}^{m-1} T^{-i}\alpha_k \right) < \epsilon.$$

Hence, there is a set $F \in \bigvee_{i=0}^{m-1} T^{-i}\alpha_k$ such that $x \in F \subset U$. This shows that every open set is a union of sets in the countable collection $\bigcup_{j=0}^{\infty} \bigcup_{r=1}^{\infty} \bigvee_{i=0}^{r-1} T^{-i}\alpha_j$. We now apply Rohlin's theorem and obtain $h_\mu(T) = \lim_{k\to\infty} h_\mu(\alpha_k, T)$. An application of Propositions 1 and 2 now completes the proof of the corollary.

COROLLARY 2. *Let (X, T) be a subflow of (B_n, σ_n) and let $\mu \in M(X, T)$. Then $h_\mu(T) \leqq h(T)$.*

PROOF. Let m be a positive integer. By Proposition 3, (X, T^m) is isomorphic to a subflow of (B_1, σ_1^{nm}) which is isomorphic to (B_{nm}, σ_{nm}). Hence, by Corollary 1, $h_\mu(T^m) - h(T^m) \leqq \log(nm+1)$. Now recalling that $h_\mu(T^m) = mh_\mu(T)$ and $h(T^m) = mh(T)$, we have $h_\mu(T) - h(T) = (1/m)(h_\mu(T^m) - h(T^m)) \leqq (1/m) \log(nm+1)$. We now observe that as m tends to infinity, $(1/m)\log(nm+1)$ tends to zero.

The proof of Theorem 1 proceeds by a series of lemmas. We let ν be a given finite measure on the Borel sets of I^n.

By an n-rectangle we mean a subset R of I^n of the form

$$R = \{x = (x_0, \cdots, x_{n-1}) \in I^n : a_i \leqq x_i \leqq b_i, i = 0, \cdots, n-1\},$$

where a_i, b_i are numbers such that $0 \leqq a_i < b_i \leqq 1$ for $i = 0, \cdots n-1$. The i-mesh of R is defined to be $b_i - a_i$.

By a rectangular cover of I^n we mean a finite cover of I^n consisting of n-rectangles. If α is a rectangular cover of I^n, we can write

$$(*)\qquad \begin{aligned} \alpha &= \{R_1, \cdots, R_q\}, \quad \text{where} \\ R_j &= \{x \in I^n \colon a_i^j \leqq x_i \leqq b_i^j,\ i = 0, \cdots, n-1\} \\ &\qquad\qquad \text{for } j = 1, \cdots, q. \end{aligned}$$

DEFINITION. Let α be a rectangular cover of I^n represented by (*). We define the i-mesh of α to be

$$L_i(\alpha) = \max_{j=1,\cdots,q} (b_i^j - a_i^j).$$

Note that $\max_{i=0,\cdots,n-1} L_i(\alpha)$ is the mesh of α with respect to the metric we defined on I^n.

DEFINITION. Let α be a rectangular cover of I^n represented by (*). For $x \in I^n$, we define

$$\xi(\alpha, x) = \text{the number of integers } i \in \{0, \cdots, n-1\}$$

such that there is a $j \in \{1, \cdots, q\}$ with $x_i = a_i^j$ or $x_i = b_i^j$. We say that α is *uneven* if

$$p(\alpha, x) \leqq \xi(\alpha, x) + 1 \qquad \text{for all } x \in I^n,$$

where $p(\alpha, x)$ is the number of distinct members of α containing x.

LEMMA 1. *Let α be an uneven ν-disjoint rectangular cover of I^n. Fix $R \in \alpha$ and fix $i \in \{0, \cdots n-1\}$. Then there is a decomposition of R,*

$$R = R' \cup R'' \text{ such that } \alpha' = \{R', R''\} \cup \{F \in \alpha \colon F \neq R\}$$

is an uneven ν-disjoint rectangular refinement of α, and such that the i-mesh of R' and R'' are both $\leqq \frac{2}{3}$ the i-mesh of R.

PROOF. Let α be represented by (*), where $R = R_1$. For $c \in (a_i^1, b_i^1)$ we define $H_c = \{x \in I^n : x_i = c\}$. We now choose

$$c_0 \in \left(\frac{2a_i^1 + b_i^1}{3}, \frac{a_i^1 + 2b_i^1}{3} \right)$$

such that $\nu(H_{c_0}) = 0$ and such that $c_0 \neq a_i^j$ and $c_0 \neq b_i^j$ for all $j = 2, \cdots, q$. This can be done because the interval mentioned above is uncountable, while $\nu(H_c) > 0$ can occur for at most a countable number of points c in the interval. We now define

$$R' = \{x \in R_1:\ a_i^1 \leqq x_i \leqq c_0\} \quad \text{and} \quad R'' = \{x \in R_2:\ c_0 \leqq x_i \leqq b_i^1\}.$$

Now the i-mesh of R' is $c_0 - a_i^1$ and the i_0-mesh of R'' is $b_i^1 - c_0$, and both of these numbers are $\leqq \frac{2}{3}(b_i^1 - a_i^1)$. It is straightforward to show that $w(\alpha') \subset w(\alpha) \cup H_{c_0}$, so that α' is ν-disjoint. To show that α' is uneven, let $x \in I^n$. We must show that $p(\alpha', x) \leqq \xi(\alpha', x) + 1$. If $x \in H_{c_0}$, then $\xi(\alpha', x) = \xi(\alpha, x) + 1$, and it follows that

$$p(\alpha', x) \leqq p(\alpha, x) + 1 \leqq \xi(\alpha, x) + 2 = \xi(\alpha', x) + 1.$$

If, on the other hand, $x \notin H_{c_0}$, then x is not in $R' \cap R''$, so

$$p(\alpha', x) = p(\alpha, x) \leqq \xi(\alpha, x) + 1 \leqq \xi(\alpha', x) + 1.$$

This completes the proof of Lemma 1.

Lemma 2. *Let α be an uneven ν-disjoint rectangular cover of I^n. Fix $i \in \{0, \cdots, n-1\}$. Then there is an uneven ν-disjoint rectangular refinement β of α such that*

$$L_i(\beta) \leqq \tfrac{2}{3} L_i(\alpha).$$

The proof is a successive application of Lemma 1 to the members R of α.

Lemma 3. *Let α be an uneven ν-disjoint rectangular cover of I_n. Then there is an uneven ν-disjoint rectangular refinement γ of α such that*

$$\text{mesh}(\gamma) \leqq \tfrac{2}{3}\text{mesh}(\alpha).$$

The proof of Lemma 3 is a successive application of Lemma 2, starting with $i = 0$, and continuing to $i = n - 1$.

Proof of Theorem 1. We let $\beta_0 = \{I^n\}$, and let β_1 be an uneven ν-disjoint rectangular refinement of β_0 obtained from Lemma 3. We continue applying Lemma 3 successively, obtaining $\beta_0 < \beta_1 < \cdots$, uneven ν-disjoint rectangular covers, such that mesh $\beta_k \leqq (\frac{2}{3})^k$. Now $\xi(\beta_k, x)$ is always $\leqq n$, so by unevenness, $p(\beta_k) \leqq n + 1$. This completes the proof of Theorem 1.

We now turn to the problem of showing that $h_\mu(T) \leqq h(T)$ in general.

By a *representation of* (X, T) we mean a homomorphism from (X, T) into some sequence flow (B_n, σ_n). If f is a continuous map from X into I^n, we can define a representation f^* of (X, T) in (B_n, σ_n) as follows:

$$f^*(x)(n) = f(T^n x) \qquad \text{for } n \in N,\ x \in X.$$

It is easily seen that all representations can be obtained this way. If ϕ is a representation of (X, T) in (B_n, σ_n), we write $T_\phi = \sigma_n | \phi(X)$, so that $(\phi(X), T_\phi)$ is a homomorphic image of (X, T). Let $R(X, T)$ be the set of all representations of (X, T). We include the following theorem for completeness, though it will not be used in the proof of our main result.

THEOREM 2. $h(T) = \sup\{h(T_\phi): \phi \in R(X, T)\}$.

PROOF. Let $\alpha = \{U_0, \cdots, U_{n-1}\}$ be an open cover of X. Define $f: X \to I^n$ as follows:

$$f(x)_i = d(x, X - U_i)/\delta \qquad \text{for } i = 0, \cdots, n-1,$$

where δ is the diameter of X. Let $\phi = f^*$. For $i = 0, \cdots, n-1$, let

$$V_i = \{y \in B_n: y(0)_i > 0\},$$

and let $\beta = \{V_i \cap \phi(X): i = 0, \cdots, n-1\}$. It is clear that β is an open cover of $\phi(X)$ and that $\phi^{-1}(\beta) = \alpha$. Hence $h(\alpha, T) = h(\beta, T_\phi) \leqq h(T_\phi)$. Now since α was an arbitrary finite open cover of X, $h(T) \leqq \sup\{h(T\phi): \phi \in R(X, T)\}$. The reverse inequality follows from general properties of topological entropy.

To prove an analogous result for measure theoretic entropy, we need the following

PROPOSITION 5. *Let α be a finite open cover of X and let $\mu \in M(X, T)$. Then α has a finite closed μ-disjoint refinement.*

PROOF. We can assume that each member of α is essential. Write $\alpha = \{U_0, \cdots, U_{n-1}\}$. Choose a closed cover $\{F_0, \cdots, F_{n-1}\}$ such that $F_i \subset U_i$ for $i = 0, \cdots, n-1$, and let f_i be a continuous real-valued function on X which is zero on F_i and one on $X - U_i$. For fixed $i \in \{0, \cdots, n-1\}$, notice that $\{f_i^{-1}(r): 0 < r < 1\}$ is an uncountable pairwise disjoint collection of closed sets of X, and hence the sets $f_i^{-1}(r)$ cannot all have positive measure. Choose $r_i \in (0, 1)$ such that $\mu(f_i^{-1}(r_i)) = 0$. Let

$$V_i = \{x \in X: f_i(x) < r_i\},$$

and let $\beta = \{V_0, \cdots, V_{n-1}\}$. It is clear that β is an open cover and that $\bar{\beta} = \{\overline{V}_0, \cdots, \overline{V}_{n-1}\}$ is a refinement of α. Furthermore, $\mu(\overline{V}_i - V_i) = 0$ for $i = 0, \cdots, n-1$. We now define $G_0, \cdots, G_{n-1}$ as follows:

$$G_0 = \overline{V}_0; \qquad G_1 = \overline{V}_1 - V_0; \text{ in general,}$$

$G_i = \overline{V}_i - \bigcup_{j<i} V_j$ for $i = 0, \cdots n-1$. It is clear that $\beta = \{G_0, \cdots, G_{n-1}\}$

is a closed refinement of α. The fact that β is μ-disjoint follows from the inclusion $G_i \cap G_j \subset \overline{V}_i - V_i$, for $j < i$. This completes the proof.

If $\phi:(X, T) \to (Y, S)$ is a homomorphism and if $\mu \in M(X, T)$, we can define a measure $\phi(\mu) \in M(Y, S)$ as follows:

$$\phi(\mu)(A) = \mu(\phi^{-1}(A))$$

for every Borel set $A \subset Y$.

THEOREM 3. *If $\mu \in M(X, T)$, then*

$$h_\mu(T) = \sup\{h_{\phi(\mu)}(T_\phi): \phi \in R(X, T)\}.$$

PROOF. For each positive integer m, choose an open cover α_m of X with mesh $\leqq 1/m$. Now by Proposition 5 there is a finite closed μ-disjoint refinement β_m of α_m. Let $\gamma_m = \beta_1 \vee \beta_2 \vee \cdots \vee \beta_m$. It is clear that γ_m is also a finite closed μ-disjoint refinement of α_m. As in the proof of Corollary 1, we can show that the sigma-algebra generated by $\bigcup_{m=1}^{\infty} \gamma_m$ is the algebra of Borel sets of X. We can now apply Rohlin's theorem and obtain

$$h_\mu(T) = \lim_{m \to \infty} h_\mu(\gamma_m, T).$$

We now fix m and write $\gamma_m = \{U_0, \cdots, U_{n-1}\}$. We define $f: X \to I^n$ as follows:

$$f(x)_i = d(x, U_i)/\delta \qquad \text{for } i = 0, \cdots, n-1,$$

where δ is the diameter of X. We let $\phi = f^*$. Next, we let

$$V_i = \{y \in B_n: y(0)_i = 0\} \qquad \text{for } i = 0, \cdots, n-1,$$

and $\gamma'_m = \{V_i \cap \phi(X): i = 0, \cdots, n-1\}$. It is clear that γ'_m is a closed cover of $\phi(X)$ and that $\phi^{-1}(\gamma'_m) = \gamma_m$. It follows that γ'_m is $\phi(\mu)$-disjoint and hence we have

$$h_\mu(\gamma_m, T) = h_{\phi(\mu)}(\gamma'_m, T_\phi) \leqq h_{\phi(\mu)}(T_\phi).$$

We now let m tend to infinity and obtain

$$h_\mu(T) \leqq \sup\{h_{\phi(\mu)}(T_\phi): \phi \in R(X, T)\}.$$

The reverse inequality follows from general properties of entropy.

THEOREM 4. *If $\mu \in M(X, T)$, then $h_\mu(T) \leqq h(T)$.*

The proof is an application of Theorem 3 and Corollary 2.

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