

## TOPOLOGICAL ENTROPY OF A SEQUENCE OF MONOTONE MAPS ON CIRCLES

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ABSTRACT. In this paper, we prove that the topological entropy of a sequence of equi-continuous monotone maps  $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$  on circles is  $h(f_{1,\infty}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \prod_{i=1}^n |\deg f_i|$ . As applications, we give the estimation of the entropies for some skew products on annular and torus. We also show that a diffeomorphism  $f$  on a smooth 2-dimensional closed manifold and its extension on the unit tangent bundle have the same entropy.

### 1. Introduction

The concept of topological entropy was originally introduced by Adler, Konheim, and Mcandrew [1] as an invariant of topological conjugacy and a numerical measure for the complexity of a dynamical system. Later, Bowen [2] and Dinaburg [3] gave an equivalent definition when the space under consideration is metrizable. We can see [12] for the definition and main properties of it. With the development of the study of nonautonomous dynamical systems, recently, Kolyada and Snoha [7] introduced and studied the notion of topological entropy for a sequence of endomorphisms of a compact topological space. For other recent results about entropy one can see [4], [9], [11], etc.

The systems on circle play an important role in the study of one dimensional dynamical systems. In [5] and [12] the authors studied the entropies of homeomorphism and monotone continuous map on circle respectively. Our purpose is to study the topological entropy of a sequence

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of monotone maps on circles. In section 2, by estimating the cardinal of the spanning set and the separated set, we prove that the topological entropy of a sequence of equi-continuous monotone maps  $f_{1,\infty} = \{f_i\}_{i=1}^\infty$  is  $h(f_{1,\infty}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \prod_{i=1}^n |\deg f_i|$ . In section 3, as applications, we give the estimation of the entropies for some skew products on annular and torus. We also show that a  $C^1$  diffeomorphism  $f$  on a smooth 2-dimensional closed manifold  $M$  and its extension  $D^\sharp f$  on the unit tangent bundle  $SM$  have the same entropy, i.e.,  $h(f) = h(D^\sharp f)$ .

Let  $(X, d)$  be a compact metric space and  $\{f_i\}_{i=1}^\infty$  a sequence of continuous maps on  $X$ . The identity map on  $X$  will be denoted by  $Id$ . Let  $\mathbf{N}$  be the set of all positive integers. For any  $i \in \mathbf{N}$ , let  $f_i^0 = Id$  and for any  $i, n \in \mathbf{N}$ , let

$$f_i^n = f_{i+(n-1)} \circ \dots \circ f_{i+1} \circ f_i, \quad f_i^{-n} = (f_i^n)^{-1} = f_i^{-1} \circ f_{i+1}^{-1} \circ \dots \circ f_{i+(n-1)}^{-1}.$$

( $f^{-1}$  will be applied to sets, we don't assume that the maps  $f_i$  are invertible). Denote by  $f_{1,\infty}$  the sequence  $\{f_i\}_{i=1}^\infty$  and the dynamical system  $(X, \{f_i\}_{i=1}^\infty)$ . Finally, denote by  $f_{1,\infty}^{[n]}$  the sequence of maps  $\left\{ f_i^{[n]} = f_{(i-1)n+1}^n \right\}_{i=1}^\infty$ .

Let  $\{f_i\}_{i=1}^\infty$  be a sequence of continuous maps of compact metric space  $(X, d)$ . For any  $n \in \mathbf{N}$ , define a new metric  $d_n$  on  $X$  by

$$d_n(x, y) := \max_{0 \leq i \leq n-1} d(f_1^i(x), f_1^i(y)).$$

For any  $\varepsilon > 0$ , a subset  $E \subset X$  is said to be an  $(n, f_{1,\infty}, \varepsilon)$  *spanning set* of  $X$ , if for any  $x \in X$ , there exists  $y \in E$  such that  $d_n(x, y) \leq \varepsilon$ . Let  $r(n, f_{1,\infty}, \varepsilon)$  denote the smallest cardinality of any  $(n, f_{1,\infty}, \varepsilon)$ -spanning set of  $X$ . A subset  $F \subset X$  is said to be an  $(n, f_{1,\infty}, \varepsilon)$ -*separated set* of  $X$ , if  $x, y \in F, x \neq y$ , implies  $d_n(x, y) > \varepsilon$ . Let  $s(n, f_{1,\infty}, \varepsilon)$  denote the largest cardinality of any  $(n, f_{1,\infty}, \varepsilon)$ -separated set of  $X$ . It's easy to prove that (similar to the proof for the autonomous system in [12])

$$r(n, f_{1,\infty}, \varepsilon) \leq s(n, f_{1,\infty}, \varepsilon) \leq r(n, f_{1,\infty}, \frac{\varepsilon}{2}).$$

DEFINITION 1.1. Let  $f_{1,\infty} = \{f_i\}_{i=1}^\infty$  be a sequence of continuous maps of compact metric space  $(X, d)$ , then the *topological entropy* of  $f_{1,\infty}$  is defined by

$$h(f_{1,\infty}) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log r(n, f_{1,\infty}, \varepsilon) = \lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s(n, f_{1,\infty}, \varepsilon).$$

Furthermore, we can see the equivalent definition using open covers in [7].

Let  $S^1$  be a circle with the “geodesic” metric, in which  $S^1$  has length 1 and the distance between two points is the length of the shortest path joining them. Let  $f : S^1 \rightarrow S^1$  be a continuous surjective map and  $F : \mathbf{R}^1 \rightarrow \mathbf{R}^1$  a lift of  $f$ , we say  $f$  is monotone if  $F$  is monotone. Denote by  $\deg f$  the degree of  $f$  (see [13]).

## 2. The main result

The main result of this paper is:

**THEOREM 2.1.** *Let  $f_{1,\infty} = \{f_i\}_{i=1}^\infty$  be a sequence of equi-continuous monotone maps of  $S^1$ . Then*

$$h(f_{1,\infty}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \prod_{i=1}^n |\deg f_i|.$$

We will prove this theorem using the idea in [6]. Let  $f : S^1 \rightarrow S^1$  be a continuous monotone map,  $|\deg f| = k$ . Then for any  $x \in S^1$ ,  $f^{-1}(x)$  is a set consist of  $k$  points, denote  $f^{-1}(x) = \{x_1, x_2, \dots, x_n\}$ . Let  $\alpha_{f,1} = (x_1, x_2), \dots, \alpha_{f,k-1} = (x_{k-1}, x_k), \alpha_{f,k} = (x_k, x_1)$ . Then we get a finite partition  $\xi_f = \{\alpha_{f,1}, \alpha_{f,2}, \dots, \alpha_{f,k}\}$  of  $S^1$ , where  $f(\overline{\alpha_{f,i}}) = S^1$  and  $\alpha_{f,i} \cap \alpha_{f,j} = \emptyset$  for  $1 \leq i \neq j \leq k$ .

**LEMMA 2.2.** *Let  $f_{1,\infty} = \{f_i\}_{i=1}^\infty$  be a sequence of equi-continuous monotone maps of  $S^1$ . Then there exists a constant  $a > 0$ , such that for every  $f_i (i \geq 1)$  and any partition  $\xi_{f_i} = \{\alpha_{f_i,1}, \alpha_{f_i,2}, \dots, \alpha_{f_i,k_i}\}$  of  $S^1$  defined as above, we have*

$$\text{diam } \alpha_{f_i,j} \geq a, \quad 1 \leq j \leq k_i,$$

where  $k_i = |\deg f_i|$ .

*Proof.* Since  $\{f_i\}_{i=1}^\infty$  is equi-continuous, then for  $\varepsilon = \frac{1}{2}$ , there exists a constant  $a > 0$  such that

$$d(x, y) < a \implies d(f_i(x), f_i(y)) < \varepsilon, \quad \forall i \in \mathbf{N}, x, y \in S^1.$$

Note that for every  $f_i (i \geq 1)$  and any partition  $\xi_{f_i} = \{\alpha_{f_i,1}, \alpha_{f_i,2}, \dots, \alpha_{f_i,k_i}\}$  of  $S^1$  defined as above,  $f_i(\overline{\alpha_{f_i,j}}) = S^1 (1 \leq j \leq k_i)$  and  $\text{diam } S^1 = 1$ , we have  $\text{diam } \alpha_{f_i,j} \geq a, 1 \leq j \leq k_i$ . □

LEMMA 2.3. Let  $f_{1,\infty} = \{f_i\}_{i=1}^\infty$  be a sequence of equi-continuous monotone maps of  $S^1$ ,  $\{\xi_{f_i} = \{\alpha_{f_i,1}, \alpha_{f_i,2}, \dots, \alpha_{f_i,k_i}\}\}_{i=1}^\infty$  be any sequence of partitions of  $S^1$  defined as above. Then for the new sequence of partitions of  $S^1$

$$\left\{ \xi_{f_1^n} = \{f_1^{-(n-1)}(\alpha_{f_n,j}) \mid \alpha_{f_n,j} \in \xi_{f_n}, 1 \leq j \leq k_n\} \right\}_{n=1}^\infty,$$

we have

$$h(f_{1,\infty}) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{card } \xi_{f_1^n} \leq h(f_{1,\infty}) + \log 2.$$

*Proof.* For any  $x \in S^1$ , let  $B_d(x, \varepsilon) = \{y \in S^1 \mid d(x, y) < \varepsilon\}$ . By the definition of  $\xi_{f_1^n}$ , for any given  $n \in \mathbf{N}$ , there are  $n - 1$  new partitions of  $S^1$ :  $\xi'_{f_i} = \{\alpha'_{f_i,1}, \alpha'_{f_i,2}, \dots, \alpha'_{f_i,k_i}\}$ ,  $1 \leq i \leq n - 1$ , such that

$$\xi_{f_1^n} = \left\{ \bigcap_{i=1}^n f_1^{-(i-1)}(\alpha'_{f_i,j}) \mid \alpha'_{f_i,j} \in \xi'_{f_i}, 1 \leq j \leq k_i \right\}.$$

by Lemma 2.2,  $\text{diam } \alpha'_{f_i,j} \geq a$  for any  $1 \leq i \leq n - 1$ ,  $1 \leq j \leq k_i$ .

Let  $0 < \varepsilon < \frac{a}{2}$  (the meaning of  $a$  is in Lemma 2.2), and  $E$  be an  $(n, f_{1,\infty}, \varepsilon)$ -spanning set of minimal cardinality of  $S^1$ . It can be seen that for any  $x \in E$  and  $0 \leq i \leq n - 1$ , the  $\varepsilon$ -neighborhood  $\overline{B_d(f_1^i(x), \varepsilon)}$  of  $f_1^i(x)$  intersects at most 2 elements of  $\xi'_{f_i}$ . So  $\overline{B_{d_n}(x, \varepsilon)}$  intersects at most  $2^n$  elements of  $\xi_{f_1^n}$ . By the definition of spanning set,  $\bigcup_{x \in E} \overline{B_{d_n}(x, \varepsilon)} = S^1$ , then  $\text{card } \xi_{f_1^n} \leq 2^n \text{card } E$ . Therefore,

$$(1) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{card } \xi_{f_1^n} \leq h(f_{1,\infty}) + \log 2.$$

Now we take an arbitrary  $0 < \varepsilon < \frac{1}{2}$  and choose an  $(n, f_{1,\infty}, \varepsilon)$ -separated set  $F$  of maximal cardinality of  $S^1$ . By the definition of separated set, for any  $\alpha \in \xi_{f_1^n}$  and any two adjacent points  $x, y$  in  $\alpha \cap F$ , there exists  $j$  with  $0 \leq j \leq n - 1$  such that  $d(f_1^j(x), f_1^j(y)) > \varepsilon$ . Since  $f_1^j$  is monotone on  $\alpha$ , then  $f_1^j(x)$  and  $f_1^j(y)$  are also two adjacent points. Hence, for each  $0 \leq j \leq n - 1$ , there are at most  $M = \lceil \frac{1}{\varepsilon} \rceil + 1$  pairs adjacent points which are more than  $\varepsilon$  apart in  $f_1^j(\alpha \cap F)$ . We claim that there are at most  $nM + 1$  points in  $\alpha \cap F$ . In fact, if there are  $nM + 2$  points in  $\alpha \cap F$ , then there are at least  $nM + 1$  pairs adjacent points. As mentioned above, for any two adjacent points  $x, y$  in  $\alpha \cap F$ , there exists  $j$  with  $0 \leq j \leq n - 1$  such that  $d(f_1^j(x), f_1^j(y)) > \varepsilon$ . This implies that

there exists at least one  $0 \leq s \leq n - 1$  such that  $d(f_1^s(x), f_1^s(y)) > \varepsilon$  for at least  $M + 1$  pairs adjacent points. This contradicts with the definition of  $M$ .

In such a way, we have  $\text{card}(\alpha \cap F) \leq nM + 1$ . Hence,  $\text{card}F \leq (nM + 1) \text{card} \xi_{f_1^n}$ . Furthermore, we have

$$\frac{1}{n} \log s(n, f_{1,\infty}, \varepsilon) \leq \frac{1}{n} \log \text{card} \xi_{f_1^n} + \frac{1}{n} \log(nM + 1).$$

Letting  $n \rightarrow \infty$ , we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log s(n, f_{1,\infty}, \varepsilon) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{card} \xi_{f_1^n}.$$

Taking limits as  $\varepsilon$  goes to 0 establish the following inequality:

$$(2) \quad h(f_{1,\infty}) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{card} \xi_{f_1^n}.$$

Then (1) and (2) yields

$$h(f_{1,\infty}) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{card} \xi_{f_1^n} \leq h(f_{1,\infty}) + \log 2.$$

□

Let  $f_{1,\infty} = \{f_i\}_{i=1}^\infty$  be a sequence of equi-continuous monotone maps of  $S^1$ . It is easy to see that for any  $m \in \mathbf{N}$ , the sequence  $f_{1,\infty}^{[m]}$  still be a sequence of equi-continuous monotone maps of  $S^1$ . For simplification, we denote  $g_{1,\infty} = f_{1,\infty}^{[m]}$ , i.e.,  $g_{1,\infty} = \{g_i\}_{i=1}^\infty$ , where  $g_i = f_i^{[m]}$ ,  $i \in \mathbf{N}$ . Accordingly, we can construct a new sequence of finite partitions of  $S^1$   $\{\xi_{g_1^n}\}_{n=1}^\infty$  from the sequence of finite partitions of  $S^1$   $\{\xi_{f_i}\}_{i=1}^\infty$ , where  $\xi_{g_1^n} = \xi_{f_1^{nm}}$ ,  $n \in \mathbf{N}$ .

LEMMA 2.4. *Let  $m$  be any given positive integer. Then for the sequence of maps  $g_{1,\infty}$  defined as above and the relevant sequence of partition  $\{\xi_{g_1^n}\}_{n=1}^\infty$ , we have*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{card} \xi_{g_1^n} = m \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{card} \xi_{f_1^n}.$$

*Proof.* By Lemma 2.2, for any  $i \in \mathbf{N}$ ,  $\text{card} \xi_{f_i} \leq N := \lfloor \frac{1}{a} \rfloor + 1$ . Then, for any positive integer  $n = lm + j$ ,  $0 \leq j \leq m - 1$ , we have

$$\text{card} \xi_{f_1^{lm}} \leq \text{card} \xi_{f_1^n} \leq N^m \text{card} \xi_{f_1^{lm}}.$$

So

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{card } \xi_{f_1^n} &= \limsup_{l \rightarrow \infty} \frac{1}{lm} \log \text{card } \xi_{f_1^{lm}} \\ &= \frac{1}{m} \limsup_{l \rightarrow \infty} \frac{1}{l} \log \text{card } \xi_{g_1^l}. \end{aligned}$$

Therefore,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{card } \xi_{g_1^n} = m \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{card } \xi_{f_1^n}.$$

□

LEMMA 2.5. ([7]). *If  $f_{1,\infty} = \{f_i\}_{i=1}^\infty$  is a sequence of equi-continuous maps on a compact metric space, then for any  $m \in \mathbf{N}$ , we have*

$$h(f_{1,\infty}^{[m]}) = m \cdot h(f_{1,\infty}).$$

*Proof of Theorem 2.1.* For any  $\varepsilon > 0$ , take  $m \in \mathbf{N}$  such that  $\frac{\log 2}{m} < \varepsilon$ . Since  $f_{1,\infty}$  is a sequence of monotone equi-continuous maps on  $S^1$ , as mentioned above, it is easy to see that  $g_{1,\infty} = f_{1,\infty}^{[m]}$  is also a sequence of equi-continuous monotone maps on  $S^1$ . By Lemma 2.3, we get

$$h(f_{1,\infty}^{[m]}) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{card } \xi_{g_1^n} \leq h(f_{1,\infty}^{[m]}) + \log 2.$$

Using Lemmas 2.4 and 2.5, and notice the way  $m$  is taken, we get

$$h(f_{1,\infty}) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \text{card } \xi_{f_1^n} \leq h(f_{1,\infty}) + \varepsilon.$$

Since  $\varepsilon$  is arbitrary, noting that  $\text{card } \xi_{f_1^n} = \prod_{i=1}^n |\deg f_i|$ , we get immediately

$$h(f_{1,\infty}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \prod_{i=1}^n |\deg f_i|.$$

□

COROLLARY 2.6. *If  $f_{1,\infty} = \{f_i\}_{i=1}^\infty$  is a sequence of equi-continuous monotone maps of  $S^1$ , and the absolute values of the degrees of the mappings are the same, denote it by  $k$ , then  $h(f_{1,\infty}) = \log k$ .*

In particular, (Theorem in [5]) If  $f : S^1 \rightarrow S^1$  is a continuous monotone map, then  $h(f) = \log |\deg f|$ .

**COROLLARY 2.7.** *If every element of the sequence  $\{f_i\}_{i=1}^\infty$  on  $S^1$  is chosen from a set consisted of finite continuous monotone maps, then*

$$h(f_{1,\infty}) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \prod_{i=1}^n |\deg f_i|.$$

*Proof.* It is only to note that the continuous map on compact space is uniformly continuous, and finite uniformly continuous maps are equi-continuous.  $\square$

**COROLLARY 2.8.** *Let  $f$  be an expansive map of  $S^1$ , i.e.,  $f$  be of  $C^1$ , and for every lift  $F : R^1 \rightarrow R^1$  of it,  $|F'(x)| > 1, \forall x \in R$ . If  $\{f_i\}_{i=1}^\infty$  are generated by sufficiently small  $C^1$ -perturbation of  $f$ , then  $h(f_{1,\infty}) = \log |\deg f|$ .*

*Proof.* Note that the expansive map of  $S^1$  is strictly monotone and structurally stable ([13]). Also note the degree of the mapping is an invariant of topological conjugacy. Therefore, if every element of  $\{f_i\}_{i=1}^\infty$  is chosen from the sufficiently small  $C^1$ -neighborhood of  $f$ , then  $\{f_i\}_{i=1}^\infty$  must be a sequence of equi-continuous monotone mappings, and  $\deg f_i = \deg f, \forall i \in N$ . From Lemma 2.6, we have  $h(f_{1,\infty}) = \log |\deg f|$ .  $\square$

### 3. Applications

**PROPOSITION 3.1.** ([2]). *Let  $X, Y$  be compact metric spaces,  $F : X \times Y \rightarrow X \times Y, f : Y \rightarrow Y$  be continuous maps,  $\pi : X \times Y \rightarrow Y$  be a surjective continuous map, and satisfy  $\pi \circ F = f \circ \pi$ , that is,  $f$  and  $F$  are topological semi-conjugate and  $f$  is the factor of  $F$ . Then*

$$h(f) \leq h(F) \leq h(f) + \sup_{y \in Y} h(F, \pi^{-1}(y)).$$

Let  $X, Y$  be compact metric spaces. A continuous map  $F : X \times Y \rightarrow X \times Y$  is called a *skew-product*, if there exist a continuous map  $f$  of  $Y$  and a set of continuous maps  $\{g_x \mid x \in X\}$  of  $Y$  which depend on  $x$  continuously, such that  $F(x, y) = (f(x), g_x(y)), \forall x \in X, y \in Y$ . By Proposition 3.1, we can get that: for the skew-product  $F : X \times Y \rightarrow X \times Y$ , we have

$$h(f) \leq h(F) \leq h(f) + \sup_{x \in X} h(F, \pi^{-1}(x)),$$

where  $\pi : X \times Y \rightarrow Y, (x, y) \mapsto y$  is the natural projection.

PROPOSITION 3.2. ([10]). *If  $f$  is a piecewise monotone continuous self-map of  $I$ , then*

$$h(f) = \lim_{n \rightarrow \infty} \frac{1}{n} \log C_n,$$

where  $C_n$  denotes the number of pieces of monotonicity of  $f^n$ .

COROLLARY 3.3. (1) *Let  $F(x, y) = (f(x), g_x(y))$  be a skew product of annular  $I \times S^1$ . If  $f$  is piecewise monotone,  $\{g_x \mid x \in I\}$  is a sequence of equi-continuous monotone maps, then*

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \log C_n &\leq h(F) \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{n} \log C_n + \sup_{x \in I} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \prod_{i=0}^{n-1} |\deg g_{f^i(x)}|. \end{aligned}$$

(2) *Let  $F(x, y) = (f(x), g_x(y))$  be a skew product of torus  $S^1 \times S^1$ . If  $\{f\} \cup \{g_x \mid x \in S^1\}$  is a sequence of equi-continuous monotone maps, then*

$$\log |\deg f| \leq h(F) \leq \log |\deg f| + \sup_{x \in S^1} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \prod_{i=0}^{n-1} |\deg g_{f^i(x)}|.$$

*Proof.* Firstly, note that for any skew product  $F : X \times Y \rightarrow X \times Y$ , and any  $x \in X$ , we have

$$h(F, \pi^{-1}(x)) = h(\{g_{f^{i-1}(x)}\}_{i=1}^{\infty}).$$

From Propositions 3.1, 3.2 and Theorem 2.1, we can get (1). From Proposition 3.1, Corollary 2.6 and Theorem 2.1, we can get (2).  $\square$

Let  $(M, \rho)$  be a smooth 2-dimensional closed manifold (i.e.,  $M$  is compact and without boundary),  $TM$  be the tangent bundle of  $M$ . We denote  $|\cdot|$ ,  $\|\cdot\|$  and  $d(\cdot, \cdot)$ , respectively, the norm on  $TM$ , the operator norm and the metric on  $M$  induced by the Riemannian metric. Denote by  $SM = \bigcup_{x \in M} S_x M$  the unit tangent bundle of  $M$ , where  $S_x M = \{u \in T_x M \mid |u| = 1\}$ . Note that  $SM$  is a compact metric space and its metric  $d$  can be derived from  $\rho$ . That is, the restriction of  $d$  on  $S_x M$  is consistent with the restriction of the metric of  $T_x M$ , which derived from the inner product  $\rho_x$ , on  $S_x M$ .

Let  $f : M \rightarrow M$  be a  $C^1$  diffeomorphism,  $Df : TM \rightarrow TM$  be the tangent map of  $f$ . Let  $D^\sharp f : SM \rightarrow SM$ ,  $u \mapsto \frac{Df(x)u}{|Df(x)u|}$ ,  $u \in T_x M$ . Then  $(SM, D^\sharp f)$  is a compact topological system, we also call it the *extension*



of  $f$  on the unit tangent bundle. One can see [8] for some connections of the dynamics between  $f$  and its extension  $D^\sharp f$ .

PROPOSITION 3.4. *Let  $f : M \rightarrow M$  be a  $C^1$  diffeomorphism on a smooth two-dimensional closed Riemannian manifold  $M$ , and  $D^\sharp f$  be its extension on the unit tangent bundle  $SM$ . Then*

$$h(f) = h(D^\sharp f).$$

*Proof.* Let  $\pi : SM \rightarrow M, u \mapsto x, u \in S_x M$  be the natural projection. It is easy to verify that  $\pi \circ D^\sharp f = f \circ \pi$ . By Proposition 3.1, we have

$$(3) \quad h(f) \leq h(D^\sharp f) \leq h(f) + \sup_{x \in M} h(D^\sharp f, \pi^{-1}(x)).$$

Since  $M$  is compact,  $f$  is a  $C^1$  diffeomorphism, then we can take

$$M = \max_{x \in M} \|Df(x)\|, \quad m = \min_{x \in M} \|Df(x)\|.$$

For any  $x \in M, u, v \in S_x M$ , we have

$$\begin{aligned} & d(D^\sharp f(x)u, D^\sharp f(x)v) \\ &= \left| D^\sharp f(x)u - D^\sharp f(x)v \right| \\ &= \left| \frac{Df(x)u}{|Df(x)u|} - \frac{Df(x)v}{|Df(x)v|} \right| \\ &= \frac{1}{|Df(x)u| \cdot |Df(x)v|} \left| |Df(x)v| \cdot Df(x)u - |Df(x)u| \cdot Df(x)v \right| \\ &\leq \frac{1}{m^2} \left| |Df(x)v| \cdot [Df(x)(u - v)] - [ |Df(x)u| - |Df(x)v| ] \cdot Df(x)v \right| \\ &\leq \frac{1}{m^2} [M^2(u - v) + M|Df(x)(u - v)|] \\ &\leq \frac{2M^2}{m^2} |u - v|. \end{aligned}$$

This shows that  $\{D^\sharp f(x) \mid x \in M\}$  are equi-continuous with respect to  $d$ .

Since  $D^\sharp f(x) : S_x M \rightarrow S_{f(x)} M$  is a homeomorphism, then it is monotone and  $|\deg D^\sharp f(x)| = 1$ . Hence, from Theorem 2.1 and Corollary 2.6, we have  $h(D^\sharp f, \pi^{-1}(x)) = 0$  for any  $x \in M$ . Therefore, from (3) we have

$$h(f) = h(D^\sharp f).$$

□

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