Topological Euler numbers in a semi-stable degeneration of surfaces

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Abstract: The object of this paper is to study topological Euler numbers in a semi-stable degeneration of surfaces by using the semi-stable minimal model program. As its application, we find some restrictions of singularities in a semi-stable degeneration of surfaces with general fiber a minimal $\kappa = 0$ surface.

Key words: Algebraic surface; semi-stable degeneration; topological Euler number.

Introduction. Let $\mathcal{X} \to \Delta$ be a one parameter flat family of projective surfaces over a small disk in $\mathbf{C}$. We assume that a general fiber X_t for $t \in \Delta - \{0\}$ has nef canonical bundle. Then via log resolution, base change, normalization and special resolution of toric singularities one can obtain a new family $\mathcal{X} \to \Delta$ with smooth $\mathcal{X}$ and simple normal crossing X_0, called a semi-stable reduction [3]. Given a semi-stable reduction family of projective surfaces over Δ whose canonical bundle of a general fiber is nef, the following holds by semi-stable minimal model program of threefolds (cf. [5]).

Theorem A. *Semi-stable minimal model program (it may need base change) leads a degeneration $\pi : \mathcal{X} \to \Delta$ with the following properties:*

1. *$\mathcal{X}$ has $\mathbf{Q}$-factorial terminal singularities,*
2. *X_0 is a reduced Cartier divisor and is numerically zero relative to π,*
3. *$\pi : \mathcal{X} \to \Delta$ is dlt ($(\mathcal{X}, \pi^{-1}(t))$ is dlt for all $t \in \Delta$),*
4. *$K_{\mathcal{X}/\Delta}$ is π-nef.*

Let $\pi : \mathcal{X} \to \Delta$ be a semi-stable degeneration (in the sense of the minimal model program) of surfaces whose canonical bundle is relatively nef. Let (V, D_V) be a pair of a component and its double curve in the central fiber. Then (V, D_V) is a dlt pair with D_V a reduced Weil divisor (cf. [5]). The second Chern class of a dlt pair can be defined as an orbifold Euler number (cf. [12, 19]). Define $\mathrm{Sing}(V, D_V)$ to be the set of singular points of V outside D_V. Then

$$c_2(V, D_V) = e_{\mathrm{top}}(V) - e_{\mathrm{top}}(D_V) - \sum_{p \in \mathrm{Sing}(V, D_V)} (1 - 1/r(p))$$

where $r(p)$ is the local orbifold fundamental group. Bogomolov-Miyaoka-Yau inequality can be generalized to a dlt pair (cf. [11, 12, 19]), and therefore the following holds.

Theorem B. *Let $\mathcal{X} \to \Delta$ be a semi-stable degeneration (in the sense of the minimal model program) of surfaces whose canonical bundle is relatively nef. Let (V, D_V) be a pair of a component and its double curve in the central fiber. Then the following holds:*

1. $c_2(V, D_V) \geq 1/3(K_V + D_V)^2$,
2. $e_{\mathrm{top}}(V) - e_{\mathrm{top}}(D_V) \geq 0$, *and it is strictly positive if it has a singular point outside double curves.*

In the paper, our concern is to study the relation between $\sum_V e_{\mathrm{top}}(V) - e_{\mathrm{top}}(D_V)$ and $c_2(X_t)$ in a semi-stable degeneration of surfaces. Precisely, we prove the following by using the semi-stable minimal model program:

Theorem. *Let $\pi : \mathcal{X} \to \Delta$ be a semi-stable degeneration (in the sense of the minimal model program) of surfaces whose canonical bundle is relatively nef. Let (V, D_V) be a pair of a component and its double curve in the central fiber.*

Then $c_2(X_t) \geq \sum_V e_{\mathrm{top}}(V) - e_{\mathrm{top}}(D_V)$.

For a semi-stable reduction family of surfaces $\mathcal{X} \to \Delta$, we have the equality

$$c_2(X_t) = \sum_V e_{\mathrm{top}}(V) - e_{\mathrm{top}}(D_V)$$

by topological argument [15]. Theorem can be applied to the bounds of the number of components and to the restriction of singularities on the central fiber of semi-stable degeneration of surfaces. It is proved in [9] under the suitable condition (semi-stable degeneration with permissible singularities), and it can be generalized to stable log surfaces [10].

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1. Preliminaries. The notion of discrepancy is the fundamental measure of the singularities of (X, D) (cf. [4] or [5]).

Definition. Let X be a normal variety and $D = \sum d_i D_i$ an effective $\mathbf{Q}$-divisor such that $K_X + D$ is $\mathbf{Q}$-Cartier. Let $f\colon Y \to X$ be a proper birational morphism from a normal variety Y. Then we can write

$$K_Y + f_*^{-1}(D) \equiv f^*(K_X + D) + \sum a(E, D)E$$

where $f_*^{-1}(D)$ is the proper transform of D, the sum runs over distinct prime divisors $E \subset Y$, and $a(E, D) \in \mathbf{Q}$. This $a(E, D)$ is called the *discrepancy* of E with respect to (X, D); it only depends on the divisor E, and not on the partial resolution Y.

We define $\operatorname{discrep}(X, D)$ $= \inf_E\{a(E, D) \mid E \text{ is exceptional, } \operatorname{Center}_X(E) \neq \emptyset\}$. And we say that (X, D), or $K_X + D$ is

$$\left.\begin{array}{l}\text{terminal}\\ \text{canonical}\\ \text{purely log terminal}\\ \text{log canonical}\end{array}\right\} \text{ if } \operatorname{discrep}(X, D) \left\{\begin{array}{l} > 0,\\ \geq 0,\\ > -1,\\ \geq -1.\end{array}\right.$$

Moreover, (X, D) is *Kawamata log terminal* (klt) if (X, D) is purely log terminal and $d_i < 1$ for every i; and (X, D) is *divisorial log terminal* (dlt) if there exists a log resolution such that the exceptional locus consists of divisors with all $a(E, D) > -1$.

We work throughout over the complex number field $\mathbf{C}$. The notation here follows Hartshorne's Algebraic Geometry.

2. Proof of Theorem.

Theorem. *Let $\pi : \mathcal{X} \to \Delta$ be a semi-stable degeneration* (*in the sense of the minimal model program*) *of surfaces whose canonical bundle is relatively nef. Let (V, D_V) be a pair of a component and its double curve in the central fiber. Then $c_2(X_t) \geq \sum_V e_{\text{top}}(V) - e_{\text{top}}(D_V)$.*

Proof. We change a semi-stable degeneration $\mathcal{X} \to \Delta$ to another semi-stable degeneration $\mathcal{Y} \to \Delta'$ (relatively minimal permissible model, cf. [2, 9]) which admits a semi-stable model (in the sense of the semi-stable reduction theorem). Let the central fiber $Y_0 = \sum(W, D_W)$ of $\mathcal{Y}$. By this process, we can compare the second Chern class of the central fiber with that of a general fiber, the proof is given in [9]:

$$c_2(Y_t) = \sum_W e_{\text{top}}(W) - e_{\text{top}}(D_W).$$

When we change $\mathcal{X} \to \Delta$ to $\mathcal{Y} \to \Delta'$ there is no change of type of a singularity on the double curves of the central fiber, i.e., $\sum_V e_{\text{top}}(V) - e_{\text{top}}(D_V) = \sum_W e_{\text{top}}(W) - e_{\text{top}}(D_W)$ if there is no singular point outside double curves. The possible type of a singularity on the central fiber of $\mathcal{X}$ outside double curves is a rational double point or a quotient singularity of the form $1/(r^2 s)(1, dsr - 1)$ where d is prime to r (cf. [6]). The possible type of a singularity on the central fiber of $\mathcal{Y}$ is a quotient singularity of the form $1/(r^2)(1, dr - 1)$ where d is prime to r (cf. [2]). For the Milnor fiber F of a $\mathbf{Q}$-Gorenstein smoothing of a singularity of the form $1/(r^2 s)(1, dsr - 1)$ where d is prime to r, it holds $b_2(F) = s - 1$ (cf. [2, 6]). Since the change of $\mathcal{X} \to \Delta$ to $\mathcal{Y} \to \Delta'$ is obtained by some base change of Δ and simultaneous resolution of rational double points, the following inequality holds by decreasing the second Betti number of the central fiber via Milnor fiber:

$$\begin{aligned} c_2(X_t) &= c_2(Y_t)\\ &= \sum_W e_{\text{top}}(W) - e_{\text{top}}(D_W)\\ &\geq \sum_V e_{\text{top}}(V) - e_{\text{top}}(D_V). \end{aligned}$$

□

By Theorem B and Theorem, we have the following:

Corollary 1. *Let $\pi : \mathcal{X} \to \Delta$ be a semi-stable degeneration* (*in the sense of the minimal model program*) *of surfaces whose canonical bundle is relatively nef. Let (V, D_V) be a pair of a component and its double curve in the central fiber. Then the number of components on the central fiber, with $(K_V + D_V)^2 > 0$ or with singular points outside double curves, is bounded by $c_2(X_t)$.*

3. Application to a semi-stable degeneration of surfaces with $\boldsymbol{\kappa = 0}$. Let $\pi : \mathcal{X} \to \Delta$ be a semi-stable degeneration (in the sense of the minimal model program) of surfaces with general fiber a minimal $\kappa = 0$ surface. Assume that $mK_{X_t} \sim 0$ for $t \in \Delta - \{0\}$. Then $mK_{X_0} \sim 0$ by semi-stable minimal model program (cf. [5]). Before the minimal model program, the similar results were obtained by Kulikov, Morrison, Persson, Pinkham and others via elementary modifications [7, 8, 13, 16].

Therefore the index of $\mathcal{X}$ is bounded by the number m which is the smallest number such that $mK_{X_t} \sim 0$ for $t \in \Delta - \{0\}$. So on a semi-stable degeneration of K3 surfaces or abelian surfaces, $K_{\mathcal{X}/\Delta}$

is Cartier divisor. And on a semi-stable degeneration of Enriques surfaces, the example with the singular points of the index 2 outside double curves was given by Persson [15]. The examples with the singular points of the index 2 on the double curves can be constructed easily by using the involution action on the special degenerations of K3 surfaces (cf. [13, 17]). Also on semi-stable degenerations of hyperelliptic surfaces, the examples with the singular points of the index 2, 3, 4, 6 on the double curves can be constructed easily by using the action on the special degenerations of abelian surfaces (cf. [18]).

Let $\mathcal{X} \to \Delta$ be a semi-stable degeneration of hyperelliptic surfaces. Then the central fiber X_0 has no singular point outside double curves by Theorem B, and Theorem. So our concern is to study a semi-stable degeneration of Enrique surfaces.

Corollary 2. *Let $\mathcal{X} \to \Delta$ be a semi-stable degeneration (in the sense of the minimal model program) of surfaces. Assume that a general fiber is a minimal Enriques surface. Then the number of singular points outside double curves on the central fiber X_0 is bounded by 16. If X_0 is normal then this number is bounded by 10.*

Proof. Let (V, D_V) be a pair of a component and its double curve in the central fiber and let $\mathrm{Sing}(V, D_V)$ be the set of singular points of V outside D_V. Then (V, D_V) is a dlt pair with D_V a reduced Weil divisor (cf. [5]). The second Chern class of a dlt pair can be defined as an orbifold Euler number (cf. [12, 19]). Let $r(p)$ be the local orbifold fundamental group of a singular point $p \in \mathrm{Sing}(V, D_V)$.

The first statement holds directly by Theorem B and Theorem:

$$\begin{aligned} 12 &= c_2(X_t) \\ &\geq \sum_V e_{\mathrm{top}}(V) - e_{\mathrm{top}}(D_V) \\ &\geq \sum_V \sum_{p \in S_V} (1 - 1/r(p)) + \sum_V \sharp R_V \end{aligned}$$

where the set of singular points

$$R_V = \{\text{rational double points in} \, \mathrm{Sing}(V, D_V)\}$$

and the set of singular points $S_V = \mathrm{Sing}(V, D_V) - R_V$. Note that $r(p) \geq 4$ if $p \in S_V$.

Assume that X_0 is normal. We consider the global index one cover $\mathcal{Z}$ of $\mathcal{X}$ (cf. [5]). Then $\mathcal{Z} \to \Delta$ gives a semi-stable degeneration of K3 surfaces (in the sense of the minimal model program) and the central fiber Z_0 of $\mathcal{Z}$ is normal with at most rational double points. For the Milnor fiber F of a rational double point or a quotient singularity of the form $1/(r^2 s)(1, dsr-1)$ for $s > 1$ where d is prime to r, it holds $b_2(F) \geq 1$ (cf. [2, 6]). Note that $b_2(X_t) = 10$.

If there is a rational double point or a quotient singularity of the form $1/(r^2 s)(1, dsr-1)$ for $s > 1$ where d is prime to r, each point decreases topological Euler number by more than or equal to 1. Therefore we may assume that singularities are of the form $1/(r^2)(1, dr-1)$ where d is prime to r. Since the index of singularity is only 2, the form of a singularity is $1/4(1,1)$. And the corresponding singular point on Z_0 is an ordinary double point.

The involution σ induces a quotient $Z_0 \to X_0$. Let Z be the minimal resolution of Z_0. Consider the topological Lefschetz formula and the holomorphic Lefschetz formula [1]:

$$e_{\mathrm{top}}(Z^\sigma) = \sum (-1)^i \, \mathrm{Tr}(\sigma^* : H^i(Z, \mathbf{Z}))$$
$$\sum (-1)^i \, \mathrm{Tr}(\sigma^* : H^i(Z, \mathcal{O}_Z)) = 0.$$

Therefore σ^* acts on $H^2(Z, \mathcal{O}_Z)$ as -1 by the holomorphic Lefshetz formula, and it holds that 2 (the number of (-2) curves) $= e_{\mathrm{top}}(Z^\sigma) \leq 20$. □

Oguiso and Zhang [14] constructed an Enriques surface with a singularity of the form $1/(2^2 10)(1, 19)$. This example is the extremal case of a singularity of the form $1/(r^2 s)(1, sdr-1)$ where d is prime to r. The index one cover of this singularity is the form $xy = z^{20}$ (A_{19}-singularity). By some base change of Δ it can be changed to 10 ordinary double points, therefore it produces 10 singularities of the form $1/4(1,1)$ in an Enriques surface.

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