Topological identification in networks of dynamical systems

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***Abstract*— The paper deals with the problem of identifying the topological structure of a network of dynamical systems. The dependencies among the measured signals are assumed linear and the approach is non causal, that is data are assumed to be analized off-line. A distance function is defined in order to evaluate the "closeness" of two processes and a few useful mathematical properties are derived. Theoretical results to guarantee the correctness of the identification procedure are provided as well.**

I. INTRODUCTION

In the recent years, under the influence of improved numerical tools, a significant interest for complex systems has been shown in many scientific fields. In particular, attention has been focused on networks, highlighting the emergence of complicated phenomena resulting from the connection of simple models. To this regard, a relevant impulse has been provided by the advances in neural networks theory, that has contributed to underline the importance of connectivity and link topology in the realization of complex dynamics [1]. As a consequence, graph theory [2] has been succesfully exploited to perform novel modeling approaches in several fields, such as Economy (see e.g. [3], [4], [5]), Biology (see e.g. [6], [7]) and Ecology (see e.g. [8], [9], [10]), especially when the investigated phenomena were characterized by spatial distribution.

In this paper, we will focus our attention on tree topology networks. Though its reduced complexity with respect to cyclic link structures, the tree connection model turns out to be particularly suitable to represent a large variety of processes. In particular, the tree network scheme results effective in the description of systems with transportation, such as water and power supply, air and rail trafic, vascular systems of living organisms and channel and drainage networks (see e.g. [11], [9], [12], [13], [14]). It is worth to highlight that this kind of models is deeply related to the idea of delay, that characterizes the connections as transportation media. Moreover, it is important to recall that in linear dynamical system theory the transfer function is a powerful representation tool for delayed processes [15], [16].

In this manuscript we will develope a rigorous mathematical method to exactly identify the connections scheme of a tree topology network of linear dynamical systems, providing a theoretical background for linear network modeling. In particular, in Section II we will introduce definitions and preliminary results, which are useful to characterize the mathematical framework. In Section III the main results about topology reconstruction will be presented and a method for the exact connection scheme identification will be reported as well. In Section IV a practical implementation of the proposed techinque will be illustrated by means of some numerical examples. Some final conclusions in Section V will end the manuscript.

Notation:
$E[\cdot]$: mean operator;
$R_{XY}(\tau) \doteq E[X(t)Y(t+\tau)]$: cross-covariance function of stationary processes;
$R_X(\tau) \doteq R_{XX}(\tau)$: autocovariance;
$\rho_{XY} \doteq \frac{R_{XY}}{\sqrt{R_X R_Y}}$: correlation index;
$\mathcal{Z}(\cdot)$: Zeta-transform of a signal;
$\Phi_{XY}(z) \doteq \mathcal{Z}(R_{XY}(\tau))$: cross-power spectral density;
$\Phi_X(z) \doteq \Phi_{XX}(z)$: power spectral density;
with abuse of notation, $\Phi_X(\omega) = \Phi_X(e^{i\omega})$;
$\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$: ceiling and floor function respectively;
$(\cdot)^*$: complex conjugate.

II. PROBLEM SET UP

Let us consider a network of n time-discrete SISO linear dynamical systems affected by additive noises. Then, let $H_j(z)$ be the transfer function of the j-th system, $\{X_j(k)\}_{k\in\mathbb{Z}}$ and $\{U_j(k)\}_{k\in\mathbb{Z}}$ respectively its output and input signals and $\{\varrho_j(k)\}_{k\in\mathbb{Z}}$ a zero-mean wide-sense stationary noise. Hence, each system can be represented according to the model:

$$X_j(k) = H_j(z)U_j(k) + \varrho_j(k) \quad \forall j = 1,\dots,n\,. \qquad (1)$$

We stress that no assumptions on the causality of $H_j(z)$ have been done. Moreover, it holds that:

$$E[\varrho_j \varrho_i] = 0 \quad \forall j \neq i\,. \qquad (2)$$

Then, let us suppose that the systems of the network are connected to form a tree topology, so that the input signal U_i of each node results the output of another process and the presence of cycles is prevented.

In this paper we will formally address this kind of network according to the following definition.

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Definition 1: Consider the ensemble of a rooted tree topology of n nodes N_j and a corresponding set of n linear time-discrete SISO systems affected by noise, described according to the model (1). Namely, assume N_i as the root node. Moreover, let $\{\varrho_j\}_{j=1,\dots,n}$ be zero-mean wide-sense stationary random processes satisfying (2), i.e. mutually not correlated zero-mean noises. Then, we define *Linear Cascade Model Tree (LCMT)* a dynamical network defined by the equation system

$$\begin{cases} X_1 = H_1(z)X_{\pi(1)} + \varrho_1 \\ \dots \\ X_n = H_n(z)X_{\pi(n)} + \varrho_n , \end{cases} \tag{3}$$

where $H_i(z) \equiv 0$ and $\pi : I \doteq \{1,\dots,n\} \to I$ is such that the map π^k has the unique fixed point i $\forall k \in \mathbb{N}$.

Definition 2: A LCMT is *well-posed* if $\Phi_{\varrho_j}(\omega) > 0$ for all ϱ_j, and for all ω

Assuming to have a complete knowledge of each stochastic process $\{X_i\}_{i=1,\dots,n}$, we are interested in the identification of the links, which describe the tree characterizing the network topology.

To this aim, let us recall some preliminary mathematical results, which will turn out to be useful in the following developments.

Let us consider two stochastic processes X_i, X_j and let $W_{ji}(z)$ be a time-discrete SISO transfer function. Hence, consider the quadratic cost

$$E\left[(\varepsilon_Q)^2\right] \tag{4}$$

where

$$\varepsilon_Q \doteq Q(z)(X_j - W_{ji}(z)X_i)$$

and $Q(z)$ is an arbitrary stable and causally invertible time-discrete transfer function weighting the error

$$e_{ji} \doteq X_j - W_{ji}(z)X_i \,.$$

Then, the computation of the transfer function $\hat{W}(z)$ that minimizes the quadratic cost (4) is a well-known problem in scientific literature and its solution is referred to as the Wiener filter [16].

Proposition 3 (Wiener filter): The Wiener filter modeling X_j by X_i is the linear stable filter $\hat{W}_{ji}$ minimizing the filtered quantity (4). Its expression is given by

$$\hat{W}_{ji}(z) = \frac{\Phi_{X_i X_j}(z)}{\Phi_{X_i}(z)} \tag{5}$$

and it does not depend upon $Q(z)$. Moreover, the minimized cost is equal to

$$\min E\left[\varepsilon_Q^2\right] = \\ = \frac{1}{2\pi}\int_{-\pi}^{\pi} |Q(\omega)|^2 \left(\Phi_{X_j}(\omega) - |\Phi_{X_j X_i}(\omega)|^2 \Phi_{X_i}^{-1}(\omega)\right) d\omega \,.$$

Moreover, the corresponding error

$$\hat{e}_{ji} \doteq X_j - \hat{W}_{ji}(z)X_i$$

is uncorrelated to X_i, i.e.

$$E[\hat{e}_{ji}X_i] = 0 \,. \tag{6}$$

Proof: See, for example, [16]. ■

Since the weighting function $Q(z)$ does not affect the Wiener filter, but only the energy of the filtered error, we can choose $Q(z)$ equal to $F_j(z)$, the inverse of the spectral factor of $\Phi_{X_j}(z)$, that is

$$\Phi_{X_j}(z) = F_j^{-1}(z)(F_j^{-1}(z))^* \,. \tag{7}$$

In particular, it is worth to recall that $F_j(z)$ is stable and causally invertible [17]. Therefore, the minimum of cost (4) assumes the value

$$\min E[\varepsilon_{F_j}^2] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(1 - \frac{|\Phi_{X_j X_i}(\omega)|^2}{\Phi_{X_i}(\omega)\Phi_{X_j}(\omega)}\right) d\omega \,. \tag{8}$$

Observe that, due to such choice of $Q(z)$, the cost turns out to explicitly depend on the *coherence function* of the two processes:

$$C_{X_i X_j}(\omega) \doteq \frac{|\Phi_{X_j X_i}(\omega)|^2}{\Phi_{X_i}(\omega)\Phi_{X_j}(\omega)} \,. \tag{9}$$

Let us underline that the coherence function is not negative and symmetric with respect to ω. Moreover, it is also well-known that the cross-spectral density satisfies the Schwartz inequality and, thus, the coherence function results limited between 0 and 1. Therefore, according to the previous results, the cost (8) turns out to be adimensional and not depending on the energy of the stochastic processes X_i and X_j.

The following result holds.

Proposition 4: The binary function

$$d(X_i, X_j) \doteq \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \left(1 - C_{X_i X_j}(\omega)\right) d\omega\right]^{1/2} \tag{10}$$

is a metric.

Proof: The only non trivial property to prove is the triangle inequality. Let $\hat{W}_{ji}(z)$ be the Wiener filter between X_i, X_j computed according to (5) and e_{ji} the relative error. The following relations hold:

$$X_3 = \hat{W}_{31}(z)X_1 + e_{31}$$
$$X_3 = \hat{W}_{32}(z)X_2 + e_{32}$$
$$X_2 = \hat{W}_{21}(z)X_1 + e_{21}.$$

Since $\hat{W}_{31}(z)$ is the Wiener filter between the two processes X_1 and X_3, it performs better at any frequency than any other linear filter, such as $\hat{W}_{32}(z)\hat{W}_{21}(z)$. So we have

$$\Phi_{e_{31}}(\omega) \le \Phi_{e_{32}}(\omega) + |\hat{W}_{32}(\omega)|^2\Phi_{e_{21}}(\omega) + \\ + \Phi_{e_{32}e_{21}}(\omega)\hat{W}_{32}^*(\omega) + \hat{W}_{32}(\omega)\Phi_{e_{21}e_{32}}(\omega) \le \\ \le (\sqrt{\Phi_{e_{32}}(\omega)} + |\hat{W}_{32}(\omega)|\sqrt{\Phi_{e_{21}}(\omega)})^2 \quad \forall\, \omega \in \mathbb{R}.$$

For the sake of simplicity we neglect to explicitly write the argument ω in the following passages. Normalizing with respect to Φ_{X_3}, we find

$$\frac{\Phi_{e_{31}}}{\Phi_{X_3}} \le \frac{1}{\Phi_{X_3}}(\sqrt{\Phi_{e_{32}}} + |\hat{W}_{32}|\sqrt{\Phi_{e_{21}}})^2$$

and considering the 2-norm properties

$$\left(\int_{-\pi}^{\pi} \frac{\Phi_{e_{31}}}{\Phi_{X_3}} d\omega\right)^{\frac{1}{2}} \le \\ \le \left(\int_{-\pi}^{\pi} \frac{\Phi_{e_{32}}}{\Phi_{X_3}} d\omega\right)^{\frac{1}{2}} + \left(\int_{-\pi}^{\pi} \frac{|\Phi_{X_3 X_2}|^2}{\Phi_{X_3}\Phi_{X_2}} \frac{\Phi_{e_{21}}}{\Phi_{X_2}} d\omega\right)^{\frac{1}{2}}$$

where we have substituted the expression of $\hat{W}_{32}$. Finally, considering that

$$0 \leq \frac{|\Phi_{X_3X_2}|^2}{\Phi_{X_3}\Phi_{X_2}} \leq 1,$$

we find

$$d(X_1, X_3) \leq d(X_1, X_2) + d(X_2, X_3).$$

■

III. Main result

In this section we show the main theoretical contributions of the paper. In particular the aim is to provide sufficient conditions to guarantee the exact reconstruction of the network topology. We first need to introduce a few definitions and technical lemmas.

Definition 5: We define "path" from N_i to N_j a finite sequence of $l > 0$ nodes $N_{\pi_1}, ..., N_{\pi_l}$ such that

- $N_{\pi_1} = N_i$
- $N_{\pi_l} = N_j$
- N_{π_i} and $N_{\pi_{i+1}}$ are linked by an arc of the tree for $i = 1, ..., l-1$
- $N_{\pi_i} \neq N_{\pi_j}$ for $i \neq j$.

The topology we are considering is given by a rooted tree (that is the pair made of a tree and one of its nodes N_r, named as "root"). Since a tree is a connected graph there is always a path between two nodes and, since the are no cycles, such a path is also unique.

The presence of a special node labeled as "root" induces a natural relation of "order" among the nodes in the following way

Definition 6: Given a rooted tree, consider the path from N_r to another node N_j. A node N_i is said to be an ancestor of N_j if $N_i \neq N_j$ and if it belongs to the path from N_r to N_j. Alternatively, we say that N_j is a descendant of N_i.

We also say that N_i is a parent of N_j (or that N_j is a child of N_i) if, in addition, N_j and N_i are connected by an arc.

It is straightforward to prove that the root is an ancestor to all the other nodes and that every node but the root has exactly one parent. A useful result is the following, showing that, in an LMCT, if N_j is a descendant of N_i the signal X_i is uncorrelated with the noise ϱ_j

Lemma 7: Given a LMCT $\mathcal{T}$, consider a node N_j and a node $N_i \neq N_j$ which is not a descendant of N_j. Then it holds that $E[\varrho_j X_i] = 0$.

Proof: Let N_r be the root of $\mathcal{T}$ and $N_{\pi_1}, ..., N_{\pi_l}$ the path from N_r to N_i. Exploiting the linear dependencies among the signals of the LMCT, X_i can be espressed in terms of the noises $\varrho_{\pi_1}, ..., \varrho_{\pi_l}$

$$X_i = \sum_{q=1}^{l} W_{i\pi_q} \varrho_{\pi_q} \tag{11}$$

where

$$W_{i\pi_q} = \prod_{h=q}^{l-1} H_{\pi_h}. \tag{12}$$

Since N_i is not a descendant of N_j and $N_i \neq N_j$, we have that $\varrho_{\pi_q} \neq \varrho_j$ for $q = 1, ..., l$, thus

$$E[\varrho_j X_i] = E\left[\varrho_j \sum_{q=1}^{l} W_{i\pi_q} \varrho_{\pi_q}\right] = 0 \tag{13}$$

■

The two following lemmas provide two important inequalities about the coherence functions related to the network signals.

Lemma 8: Consider a LCMT $\mathcal{T}$ and three nodes N_i, N_j and N_k such that

- N_k is a descendant of N_j
- N_i is not a descendant of N_j and $N_i \neq N_j$

Then we have that $C_{X_iX_j} \geq C_{X_iX_k}$. Moreover, if $\mathcal{T}$ is well-posed then the inequality is strict.

Proof: Consider the path from N_j to N_k described by the sequence $N_{\pi_1}, ..., N_{\pi_l}$. Exploiting the linear relations (1), the process X_k can be expressed in terms of X_j and of the noises acting on the nodes $N_{\pi_2}, ..., N_{\pi_l}$ which are all descendants of N_j.

$$X_k = W_{k\pi_1} X_j + \sum_{q=2}^{l} W_{k\pi_q} \varrho_{\pi_q} \tag{14}$$

where $W_{i\pi_q}$ is defined as in (12). Now, we intend to evaluate the coherence between X_i and X_j. From the assumption on N_i, it follows that N_i is not on the path from N_j to N_k. In other words, N_i is not a descendant of N_{π_q} and $N_i \neq N_{\pi_q}$ for $q = 1, ..., l$. We can write

$$C_{X_iX_k} = \frac{|\Phi_{X_iX_k}|^2}{\Phi_{X_i}\Phi_{X_k}} = \tag{15}$$

$$= \frac{|W_{k\pi_1}|^2 |\Phi_{X_iX_j}|^2}{\Phi_{X_i}[\Phi_{X_j}|W_{k\pi_1}|^2 + \sum_{q=2}^{l} |W_{k\pi_q}|^2 \Phi_{\varrho_{\pi_q}}]} \tag{16}$$

where the last equality holds because of lemma 7 . Collecting the factor $\Phi_{X_j}|W_{k\pi_1}|^2$, we obtain

$$C_{X_iX_k} = \frac{|\Phi_{X_iX_j}|^2}{\Phi_{X_i}\Phi_{X_j}\left[1 + \frac{\sum_{q=2}^{l} |W_{k\pi_q}|^2 \Phi_{\varrho_{\pi_q}}}{\Phi_{X_j}|W_{k\pi_1}|^2}\right]} \leq C_{X_iX_j} \tag{17}$$

where the inequality is strict if $\sum_{q=2}^{l} |W_{k\pi_q}|^2 \Phi_{\varrho_{\pi_q}} > 0$. ■

Lemma 9: Consider a LCMT $\mathcal{T}$ and three different nodes N_i, N_j and N_k such that

- N_k is a child of N_j
- $N_i \neq N_j, N_k$ and it is not a descendant of N_k

Then $C_{X_jX_k} \geq C_{X_iX_k}$. Moreover, if $\mathcal{T}$ is well-posed the inequality is strict.

Proof: Assume that $X_k = H_{kj} X_j + \varrho_k$ and let us distinguish two possible scenarios.

Case A. First, consider the case where N_j is a descendant of N_i. Consider the path from N_i to N_j described by the sequence of l nodes $N_{\pi_1}, ..., N_{\pi_l}$ where $N_{\pi_1} = N_i$ and $N_{\pi_l} = N_j$. The process X_j can be expressed in terms of

X_i and of the noises acting on the nodes $N_{\pi_2}, ..., N_{\pi_l}$ which are all descendants of N_i.

$$X_j = W_{j\pi_1} X_i + \sum_{q=2}^{l} W_{j\pi_q} \varrho_{\pi_q} . \tag{18}$$

Exploiting Lemma 7 we can evaluate the following quantities

$$C_{X_i X_k} = \frac{|\Phi_{X_i X_k}|^2}{\Phi_{X_i}\Phi_{X_k}} = \frac{|W_{j\pi_1}|^2 |H_{kj}|^2 |\Phi_{X_i}|^2}{\Phi_{X_i}\Phi_{X_k}} = \\ = \frac{|W_{j\pi_1}|^2 |H_{kj}|^2 \Phi_{X_i}}{\Phi_{X_k}} \tag{19}$$

and

$$C_{X_j X_k} = \frac{|\Phi_{X_j X_k}|^2}{\Phi_{X_j}\Phi_{X_k}} = \frac{|H_{kj}|^2 |\Phi_{X_j}|^2}{\Phi_{X_j}\Phi_{X_k}} = \\ = \frac{|H_{kj}|^2}{\Phi_{X_k}} \left[\Phi_{X_i} |W_{j\pi_1}|^2 + \sum_{q=2}^{l} |W_{j\pi_q}|^2 \Phi_{\varrho_{\pi_q}} \right] . \tag{20}$$

By inspection we have the assertion.

Case B. Now we are left to consider the case where N_j is not a descendant of N_i. Then, also N_k is not a descendant of N_i. By hypothesis, N_i is not a descendant of N_k, either. Thus, they must have a common ancestor N_d, such that the two paths from N_d to N_k and from N_d to N_i have only N_d in common. Consider the path from N_d to N_i, such that it is possible to write

$$X_i = W_{i\pi_1} X_d + \sum_{q=2}^{l} W_{i\pi_q} \varrho_{\pi_q} . \tag{21}$$

Exploiting Lemma 7, we have

$$C_{X_i X_k} = \frac{|\Phi_{X_i X_k}|^2}{\Phi_{X_i}\Phi_{X_k}} = \tag{22}$$

$$= \frac{|\Phi_{X_k X_d}|^2}{\Phi_{X_k}\left[\Phi_{X_d} + \sum_{q=2}^{l} |W_{j\pi_q}|^2 \Phi_{\varrho_{\pi_q}}\right]} \leq \tag{23}$$

$$\leq C_{X_k X_d} \tag{24}$$

If $N_d = N_j$, we have the assertion. If $N_d \neq N_j$, then N_j must be a descendant of N_d. We are in a situation equivalent to case A: there is a node N_d such that N_j is one of its descendants. As a consequence, we can state that

$$C_{X_k X_d} \leq C_{X_k X_j} . \tag{25}$$

Combining the last two inequalities, we conclude that the lemma holds also in this case. ∎

All the previous lemmas are functional to the show that the coherence distance (10) is minimal between two contiguous nodes, as summarized in this theorem.

Theorem 10: Given a LCMT $\mathcal{T}$, consider a node N_a and a node $N_b \neq N_a$ which is not directly linked to it. Then there exists a node N_c directly linked to N_a such that

$$d(N_a, N_c) \leq d(N_a, N_b) \tag{26}$$

where the inequality is strict if $\mathcal{T}$ is well-posed.

Proof: First, consider the case where N_b is a descendant of N_a. Name N_c the child of N_a on the path linking it to N_b. Since N_c is directly linked to N_a, we have $N_b \neq N_c$. Moreover N_b is a descendant of N_c. We are allowed to apply Lemma 8 with $N_i = N_a$, $N_j = N_c$ and $N_k = N_b$ to have the assertion.

Now, consider the case where N_b is not a descendant of N_a. N_a can not be the root, otherwise N_b would be one of its descendants. Thus N_a has a parent and let us name it N_c. N_b can not be N_c because it is not directly linked to N_a. Applying Lemma 9 with $N_i = N_b$, $N_j = N_c$ and $N_k = N_a$ and by the definition of the coherence distance (10), we have the assertion. ∎

When we are performing our observations during a time horizon t which approaches infinity the estimates of the spectral and cross-spectral densities converge to the actual values. Hence, for t "sufficiently large" such quantities can be assumed "exact". Since we are dealing with stochastic processes such a working hypothesis is necessary and quite reasonable.

We are ready to show the main contribution of the paper

Theorem 11: Consider a well-posed LCMT $\mathcal{T}$ and assume to observe the signals X_j during a time horizon t. Compute an estimate of the coherence based distances $d_{ij} = d(X_i, X_j)$ among the nodes N_j and evaluate the relative Minimum Spanning Tree (MST). When t approaches infinity, the corresponding topology is equivalent to the unique MST T associated to the coherence metric.

Proof: The proof consists in showing that the MST T associated to the distance (10) is unique and corresponds to the LCMT topology. We will prove this result by induction on the number n of nodes of the LCMT.

The basic induction step consists in observing that theorem is true for $n = 2$.

Now assume the theorem true for a LCMT with n nodes. Given a LCMT $\mathcal{T}$ with $n + 1$ nodes, remove one of its "leaves". By leaf we mean a non-root node with no descendants. This operation is always possible since any rooted tree with at least two nodes has at least one leaf. Without loss of generality, let the removed leaf be N_{n+1} and let N_i be its parent. Now we have a LCMT $\mathcal{T}'$ with n nodes and with the same topology of $\mathcal{T}$ apart from the removed arc $(i, n + 1)$. Using the induction hypothesis, we know that the topology of $\mathcal{T}'$ is given by the unique MST T' obtained considering the distances among the nodes $N_1, ..., N_n$. Now compute

$$i^* = \arg\min_{j<N+1} d(X_i, X_{n+1}). \tag{27}$$

The solution of such a minimization problem is unique since the LCMT $\mathcal{T}$ is well posed. Because of Theorem 10, the arc $(i^*, N + 1)$ belongs to the topology of $\mathcal{T}$, so we conclude $i^* = i$. Let T be the spanning tree obtained by adding the arc $(i, N+1)$ to T'. So far, we have shown that T represents the topology of $\mathcal{T}$. We have to prove that T is the unique MST related to the distance (10) among the nodes $N_1, ..., N_{n+1}$. Suppose, by contradiction, that there is a minimum spanning tree $\bar{T} \neq T$ with weight lesser or equal than the weight of T. The only arc of $\bar{T}$ incident to the node N_{n+1} is $(i, n+1)$. If there were another arc $(k, n + 1)$ in $\bar{T}$ we could replace it with the arc (k, i) obtaining a spanning tree with inferior cost. Indeed, by Lemma 9, we would have

$$d(X_k, X_i) < d(X_{n+1}, X_i). \quad (28)$$

So, if $\bar{T}$ is a minimum spanning tree, then X_{n+1} can be connected only to X_i. Let $\bar{T}'$ be the tree obtained by $\bar{T}$ removing the arc $(i, n+1)$. $\bar{T}'$ is the minimum spanning tree for the nodes $N_1, ..., N_n$ since it has been obtained from $\bar{T}$ removing the node N_{n+1} which has a single connection. However, by the induction hypothesis, there is a unique MST T' among the nodes $N_1, ..., N_n$. Thus we have that $\bar{T}' = T'$. It immediately follows the contradiction that $\bar{T} = T$. ■

So far, we have assumed that the dynamics of the network is described by a rooted tree. Moreover, the previous theorem proves that the topology structure can be correctly identified evaluating the MST according to the distance (10). However, no information is recovered about the root node. The following result shows that such an information is not necessary. Indeed, from a modeling point of view, the choice of the root can be arbitrary (as far as we are considering non-causal transfer functions linking the processes X_j).

Theorem 12: Given a LCMT $\mathcal{T}$ whose root is the node N_j and given one of its children N_i, it is possible to define another LCMT $\mathcal{T}^*$ with the same tree structure and described by the same processes X_k, $k = 1, ..., n$, such that its root is N_i.

Proof: Consider the Wiener Filter W_{ji} modeling the signal X_j, seen as the output, when X_i is the input

$$X_j = W_{ji} X_i + e_{ji}. \quad (29)$$

Now, consider a rooted tree with the same topology of $\mathcal{T}$ but with N_i as the root. Define $H^*_k = H_k$ and $\varrho^*_k = \varrho_k$ for all $k \neq i, j$. Conversely, define

$$H^*_j = W_{ji} \qquad \varrho^*_j = e_{ji} \quad (30)$$

$$\varrho^*_i = X_i. \quad (31)$$

To show that the new dynamical network with N_i as root and described by the filters H^*_k is an LCMT, we need to prove that, for $h \neq k$,

$$E[\varrho^*_h \varrho^*_k] = 0. \quad (32)$$

There are three possible scenarios.
If $h = i$ and $k = j$ or $h = j$ and $k = i$, then

$$E[\varrho^*_h \varrho^*_k] = 0. \quad (33)$$

because of the Wiener Filter properties.
If $h = i, j$ and $k \neq i, j$ (or equivalently $h \neq i, j$ and $k = i, j$), then lemma 7 can be applied.
If $h \neq i, j$ and $k \neq i, j$, then

$$E[\varrho^*_h \varrho^*_k] = E[\varrho_h \varrho_k] \quad (34)$$

and we have the assert because ϱ_h and ϱ_k are two noise signals of the original LCMT $\mathcal{T}$. ■

It is straightforward to show that, starting from an LCMT $\mathcal{T}$, we can arbitrary define a LCMT $\mathcal{T}^*$ having an arbitrary node as root. Indeed it is sufficient to iteratively apply theorem 12 along the path starting from the original root to the new one.

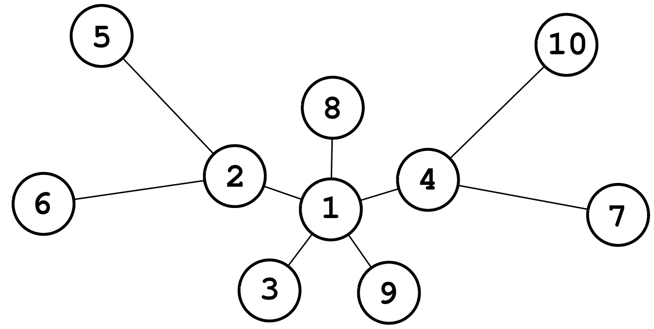

Fig. 1. The figure illustrates the topology of the 10 nodes network analyzed in the numerical examples paragraph. Each node is responsable for a process X_j, while the arcs describe the connections among them, according to the linear SISO model (**??**). For the data generation we have considered only transfer functions of at most the second order. The noises ϱ_j have been assumed to provide half the power of the affected processes. The samples have been collected over 1000 time steps.

IV. NUMERICAL EXAMPLES

In this section we introduce a suitable framework to illustrate the application of the previous theoretical results to numerical analysis. It is worth to observe that the previous results have been developed for the most general class of linear models. Indeed, no assumptions have been done on the order and causalty property of the considered transfer functons.

Moreover, let us hiligth that the coherence based analysis must be realized "off-line", since the processes have to be evaluated over their entire time span. Thus, because the coherence function can be numerically computed only over limited intervals, in the following examples we will consider sufficently long time spans to reduce the numerical error.

Hence, let us build the original dynamical networks according the following rules:

- each system is described according to the model (1);
- each transfer function H_j is randomly generated and such that it is causal and at most of the second order;
- the tree topology is randomly chosen;
- the noises ϱ_j are numerically generated with a pseudo-random algorithm;
- the noise-to-signal ratio of each system is equal to one.

Then, such networks are simulated over 1000 time steps and the related data X_j are collected. The corresponding coherence based distances are evaluated and used for the extraction of the MST, that defines the link topology.

The above procedure will be first applyed to a ten node network. In particular, to test the numerical reliability of the topological identification technique, we repeat such analysis several times, so that a significant number of network configurations is considered. The corresponding results fit the expectations and the real topology is correctly identified each time. In Fig. 1 one of the considered network configurations is depicted, while the related coherence based distance matrix is reported in Table I.

To provide a further test, a new set of similar simulations is performed with a network of fifty dynamical systems, under the same assumptions used in the previous case. Fig. 2 presents one of the considered network configurations. For

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_1	0	0,7299	0,6675	0,7351	0,8316	0,8542	0,8297	0,7055	0,6549	0,8298
X_2	0,7299	0	0,8065	0,8353	0,6934	0,7358	0,8786	0,8483	0,8299	0,8717
X_3	0,6675	0,8065	0	0,8216	0,8744	0,8807	0,8750	0,8262	0,7841	0,8821
X_4	0,7351	0,8353	0,8216	0	0,8662	0,8722	0,7404	0,8502	0,8198	0,7039
X_5	0,8316	0,6934	0,8744	0,8662	0	0,8540	0,8919	0,8995	0,8730	0,8846
X_6	0,8542	0,7358	0,8807	0,8722	0,8540	0	0,8934	0,8984	0,8796	0,8944
X_7	0,8297	0,8786	0,8750	0,7404	0,8919	0,8934	0	0,8838	0,8694	0,8346
X_8	0,7055	0,8483	0,8262	0,8502	0,8995	0,8984	0,8838	0	0,8167	0,8908
X_9	0,6549	0,8299	0,7841	0,8198	0,8730	0,8796	0,8694	0,8167	0	0,8715
X_{10}	0,8298	0,8717	0,8821	0,7039	0,8846	0,8944	0,8346	0,8908	0,8715	0

TABLE I

THE COHERENCE BASED DISTANCE MATRIX ASSOCIATED TO THE NETWORK TOPOLOGY DEPICTED IN FIG. 1

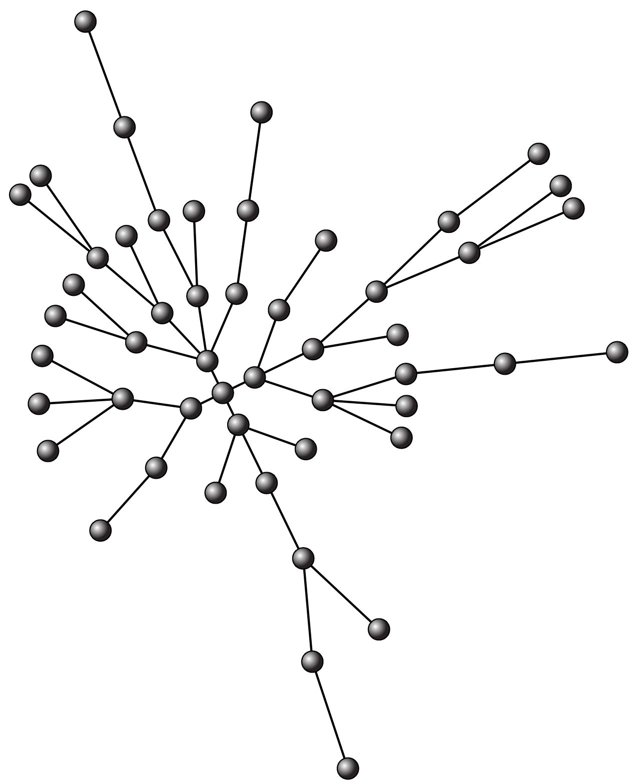

Fig. 2. A representative topological configuration of the 50 nodes network case considered of the numerical examples paragraph. The example has been designed according to the same assumptions of the ten node network of Figure 1.

a space limitation issue, we do not report in this manuscript the corresponding coherence based distance matrix. Nontheless, the computation of the related MST has succesfully identified the real network topology in any of the performed simulations.

V. CONCLUSIONS

This work has illustrated a simple but effective procedure to identify the structure of a network of linear dynamical systems when the topology is described by a tree. To the best knowledge of the authors, the problem of identifying a network has not yet been tackled in scientific literature. The approach followed in this paper is based on the definition of a distance function in order to evaluate if there exists a direct link between two nodes. A few theoretical results are provided, in particular to guarantee the correctness of the identification procedure.

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