Topological Persistence for Circle-Valued Maps

Dan Burghelea · Tamal K. Dey

Abstract We study *circle-valued maps* and consider the *persistence of the homology of their fibers*. The outcome is a finite collection of computable invariants which answer the basic questions on persistence and in addition encode the topology of the source space and its relevant subspaces. Unlike persistence of real-valued maps, circle-valued maps enjoy a different class of invariants called *Jordan cells* in addition to bar codes. We establish a relation between the homology of the source space and of its relevant subspaces with these invariants and provide a new algorithm to compute these invariants from an input matrix that encodes a circle-valued map on an input simplicial complex.

Keywords Topological persistence · Persistence for circle-valued maps · Bar codes · Jordan cells

1 Introduction

Data analysis provides many scenarios where one ends up with a nice space, most often a simplicial complex, a smooth manifold, or a stratified space equipped with a real-valued or a circle-valued map. The persistence theory, introduced in [14], provides a useful tool for analyzing real-valued maps with the help of homology. A similar theory for circle-valued maps has not yet been developed in the literature. The work

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in [9] introduces the concept of circle-valued maps in the context of persistence by deriving a circle-valued map for given data by using the existing persistence theory. In contrast, we develop a persistence theory for circle-valued maps.

One place where circle-valued maps appear naturally is the area of dynamics of vector fields. Many dynamics are described by vector fields which admit a minimizing action (in mathematical terms a Lyapunov closed one form). Such actions can be interpreted as 1-cocycles which are intimately connected to circle-valued maps as shown in [3]. Consequently, a notion of persistence for circle-valued maps also provides a notion of persistence for 1-cocycles which appear in some data analysis problems [20,21]. In summary, persistence theory for circle-valued maps promises to play the role for some vector fields as does the standard persistence theory for the scalar fields [6,7,14,22].

One of the main concepts of the persistence theory is the notion of *bar codes* [22]— invariants that characterize a real-valued map at the homology level. The angle (circle) valued maps, when characterized at homology level, require a new invariant called *Jordan cells* in addition to the refinement of the bar codes into four types.

The standard persistence [14,22] which we refer as *sublevel persistence* deals with the change in the homology of the sublevel sets which cannot make sense for a circle-valued map. However, the change in the homology of the level sets can be considered for both real- and circle-valued maps. The notion of persistence, when considered for the level sets of a real-valued map [10] is referred here as *level persistence*. It refines the sublevel persistence. The zigzag persistence introduced in [4] provides complete invariants (bar codes) for level persistence of (tame) real-valued maps. They are defined by using representation theory for linear quivers.

The change in homology of the level sets of a (tame) circle-valued map is more complicated because of the *return* of the level to itself when one goes along the circle. It turns out that representation theory of cyclic quivers provides the complete invariants for persistence in the homology of the level sets of the circle-valued maps. This notion of persistence is called here the *persistence for circle-valued maps* and its invariants, *bar codes* and *Jordan cells* are shown to be effectively computable.

Our results include a derivation of the homology for the source space and its relevant subspaces in terms of the invariants (Theorems 3.1 and 3.2). The result also applies to real-valued maps as they are special cases of the circle-valued maps. This leads to a result (Corollary 3.4) which to our knowledge has not yet appeared in the literature.[1] A number of other topological results which cannot be derived from any of the previously defined persistence theories are described in [2] providing additional motivation for this work.

After developing the results on invariants, we propose a new algorithm to compute the bar codes and Jordan cells. For a simplicial complex, the entire computation can be done by manipulating the original matrix that encodes the input complex and the map. The algorithm first builds a block matrix from the original incidence matrix which encodes linear maps induced in homology among regular and critical level sets, more precisely the quiver representations ρ_r described in Sect. 4. Next, it iteratively reduces

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[1] It was brought to our attention by David Cohen-Steiner that the extended persistence proposed in [7] allows similar connections between homology of source spaces and persistence.

this new matrix eliminating and hence computing the bar codes. The resulting matrix which is invertible can be further processed to Jordan canonical form [12] providing Jordan cells. The algorithm for zigzag persistence [4] when applied to what we refer in Sect. 3 as the *infinite cyclic covering map* $\tilde{f}$ can compute bar codes but not Jordan cells. In contrast, our method can compute the bar codes and Jordan cells simultaneously by manipulating matrices and can also be used as an alternative to compute the bar codes in zig-zag persistence.

Notations. We list here some of the notations that are used throughout.

- For rth homology group of a topological space X under an a priori fixed field κ, we write $H_r(X)$ instead of $H_r(X; \kappa)$.
- For a map $f : X \to Y$ and $K \subseteq Y$ we write $X_K := f^{-1}(K)$.
- We use $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{>0}$ for non-negative and positive integers respectively.
- In our exposition, we need to use open, semi-open, and closed intervals denoted as (a, b), $(a, b]$ or $[a, b)$, and $[a, b]$ respectively. To denote an interval, in general, we use the notation $\{a, b\}$ where "{" stands for either "[" or "(".
- For a linear map $\alpha : V \to W$ between two vector spaces we write:

$$
\begin{aligned}
\ker \alpha &:= \big\{ v \in V \mid \alpha(v) = 0 \big\}, \\
\operatorname{img} \alpha &:= \big\{ w \in \alpha(V) \subseteq W \big\}, \\
\operatorname{coker} \alpha &:= W/\alpha(V).
\end{aligned}
$$

- A matrix A is said to be in *column echelon* form if all zero columns, if any, are on the right to nonzero ones and the leading entry (the first nonzero number from below) of a nonzero column is always strictly below of the leading entry of the next column. Similarly, A is said to be in *row echelon* form if all zero rows, if any, are below nonzero ones and the leading entry (the first nonzero number from the right) of a nonzero row is always strictly to the right of the leading entry of the row below it.
  If A is an $m \times n$ matrix (m rows and n columns), there exist an invertible $n \times n$ matrix $R(A)$ and an invertible $m \times m$ matrix $L(A)$ so that $A \cdot R(A)$ is in column echelon form and $L(A) \cdot A$ is in row echelon form. Algorithms for deriving the column and row echelon form can be found in standard books on linear algebra.

2 Definitions and Background

We begin with the technical definition of tameness of a map.

For a continuous map $f : X \to Y$ between two topological spaces X and Y, let $X_U = f^{-1}(U)$ for $U \subseteq Y$. When $U = y$ is a single point, the set X_y is called a *fiber* over y and is also commonly known as the level set of y. We call the continuous map $f : X \to Y$ *good* if every $y \in Y$ has a contractible neighborhood U so that the inclusion $X_y \to X_U$ is a homotopy equivalence. The continuous map $f : X \to Y$ is a *fibration* if each $y \in Y$ has a neighborhood U so that the maps $f : X_U \to U$ and $pr : X_y \times U \to U$ are fiber wise homotopy equivalent. This means that there exist continuous maps $l : X_U \to X_y \times U$ with $pr|_U \cdot l|_U = f|_U$ which, when restricted to the fiber for any $z \in U$, are homotopy equivalences. In particular, f is good.

Definition 2.1 A proper continuous map $f : X \to Y$ is *tame* if it is good, and for some discrete closed subset $S \subset Y$, the restriction $f : X \setminus f^{-1}(S) \to Y \setminus S$ is a fibration. The points in $S \subset Y$ which prevent f to be a fibration are called *critical values*.

If $Y = \mathbb{R}$ and X is compact or $Y = \mathbb{S}^1$,[2] then the set of critical values is finite, say $s_1 < s_2 < \cdots s_k$. The fibers above them, X_{s_i}, are referred to as *singular fibers*. All other fibers are called *regular*. In the case of $\mathbb{S}^1$, s_i can be taken as angles and we can assume that $0 < s_i \leq 2\pi$. Clearly, for the open interval (s_{i-1}, s_i) the map $f : f^{-1}(s_{i-1}, s_i) \to (s_{i-1}, s_i)$ is a fibration which implies that all fibers over angles in (s_{i-1}, s_i) are homotopy equivalent with a fixed regular fiber, say X_{t_i}, with $t_i \in (s_{i-1}, s_i)$.
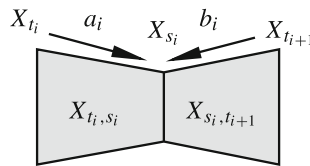
In particular, there exist maps $a_i : X_{t_i} \to X_{s_i}$ and $b_i : X_{t_{i+1}} \to X_{s_i}$, unique up to homotopy defined as follows: If t_i and t_{i+1} are contained in $U_i \subset Y$ where the inclusion $X_{s_i} \subset X_{U_i}$ is a homotopy equivalence with a homotopy inverse $r_i : X_{U_i} \to X_{s_i}$, then a_i and b_i are the restrictions of r_i to X_{t_i} and $X_{t_{i+1}}$ respectively. If not, in view of the tameness of f, one can find t_i' and t_{i+1}' in U_i so that X_{t_i} and $X_{t_{i+1}}$ are homotopy equivalent to $X_{t_i'}$ and $X_{t_{i+1}'}$ respectively and compose the restrictions of r_i with these homotopy equivalences.

These maps determine homotopically $f : X \to Y$, when $Y = \mathbb{R}$ or $\mathbb{S}^1$. For simplicity in writing, when $Y = \mathbb{R}$ we put $t_{k+1} \in (s_k, \infty)$ and $t_1 \in (-\infty, s_1)$ and when $Y = \mathbb{S}^1$ we put $t_{k+1} = t_1 \in (s_k, s_1 + 2\pi)$. All scalar or circle-valued simplicial maps on a simplicial complex, and all smooth maps with generic isolated critical points on a smooth manifold or stratified space are tame. In particular, Morse maps are tame.

2.1 Persistence and Invariants for Real-Valued Maps

Since our goal is to extend the notion of persistence from real-valued maps to circle-valued maps, we first summarize the questions that the persistence answers when applied to real-valued maps, and then develop a notion of persistence for circle-valued maps which can answer similar questions and more. We fix a field κ and write $H_r(X)$ to denote the homology vector space of X in dimension r with coefficients in a field κ.

Sublevel persistence. The persistent homology introduced in [14] and further developed in [22] is concerned with the following questions:

Q1. Does the class $x \in H_r(X_{(-\infty,t]})$ originate in $H_r(X_{(-\infty,t'']})$ for $t'' < t$? Does the class $x \in H_r(X_{(-\infty,t]})$ vanish in $H_r(X_{(-\infty,t']})$ for $t < t'$?

Q2. What are the smallest t' and largest t'' such that this happens?

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[2] Since the map f is proper and $\mathbb{S}^1$ compact, so is X.

This information is contained in the inclusion induced linear maps $H_r(X_{(-\infty,t]}) \rightarrow H_r(X_{(-\infty,t']})$ where $t' \geq t$ and is known as persistence. Since the involved subspaces are sublevel sets, we refer to this persistence as *sublevel persistence*. When f is tame, the persistence for each $r = 0, 1, \ldots, \dim X$, is determined by a finite collection of invariants referred to as *bar codes* [22]. For sublevel persistence the bar codes are a collection of *closed intervals* of the form $[s, s']$ or $[s, \infty)$ with s, s' being the critical values of f. From these bar codes one can derive the Betti numbers of $X_{(-\infty,a]}$, the dimension of $\mathrm{img}(H_r(X_{(-\infty,t]}) \rightarrow H_r(X_{(-\infty,t']}))$ and get the answers to questions Q1 and Q2. For example, the number of r-bar codes which contain the interval $[a, b]$ is the dimension of $\mathrm{img}(H_r(X_{(-\infty,a]}) \rightarrow H_r(X_{(-\infty,b]}))$. The number of r-bar codes which identify to the interval $[a, b]$ is the maximal number of linearly independent homology classes born exactly in $X_{(-\infty,a]}$ but not before and die exactly in $H_r(X_{-\infty,b]})$ but not before.

Level persistence. Instead of sublevels, if we use levels, we obtain what we call level persistence. The level persistence was first considered in [10] but was better understood computationally when the zig-zag persistence was introduced in [4]. Level persistence is concerned with the homology of the fibers $H_r(X_t)$ and addresses questions of the following type.

Q1. Does the image of $x \in H_r(X_t)$ vanish in $H_r(X_{[t,t']})$, where $t' > t$ or in $H_r(X_{[t'',t]})$, where $t'' < t$?

Q2. Can x be detected in $H_r(X_{t'})$ where $t' > t$ or in $H_r(X_{t''})$ where $t'' < t$? The precise meaning of *detection* is explained below.

Q3. What are the smallest t' and the largest t'' for the answers to Q1 and Q2 to be affirmative?

To answer such questions one needs information about the following inclusion induced linear maps:

$$H_r(X_t) \rightarrow H_r\big(X_{[t,t']}\big) \leftarrow H_r(X_{t'}).$$

The *level persistence* is the information provided by this collection of vector spaces and linear maps for all t, t'.

We say that $x \in H_r(X_t)$ is dead in $H_r(X_{[t,t']})$, $t' > t$, if its image by $H_r(X_t) \rightarrow H_r(X_{[t,t']})$ vanishes. Similarly, x is dead in $H_r(X_{[t'',t]})$, $t'' < t$, if its image by $H_r(X_t) \rightarrow H_r(X_{[t'',t]})$ vanishes.

We say that $x \in H_r(X_t)$ is detected in $H_r(X_{t'})$, $t' > t$ (resp. $t'' < t$), if its image in $H_r(X_{[t,t']})$ (resp. in $H_r(X_{[t'',t]})$) is nonzero and is contained in the image of $H_r(X_{t'}) \rightarrow H_r(X_{[t,t']})$ (resp. $H_r(X_{t''}) \rightarrow H_r(X_{[t'',t]})$). In Fig. 1, the class consisting of the sum of two circles at level t is not detected on the right, but is detected at all levels on the left up to (but not including) the level t'. In case of a tame map the collection of the vector spaces and linear maps is determined up to coherent isomorphisms by a collection of invariants called *bar codes for level persistence* which are intervals of the form $[s, s'], (s, s'), (s, s'], [s, s')$ with s, s' critical values as opposed to the *bar codes for sublevel persistence* which are intervals of the form $[s, s'], [s, \infty)$ with s, s' critical values. These bar codes are called *invariants* because two tame maps $f : X \rightarrow \mathbb{R}$ and $g : Y \rightarrow \mathbb{R}$ which are fiber wise homotopy equivalent have the same associated bar codes. In the case of level persistence the open end of an interval signifies the death
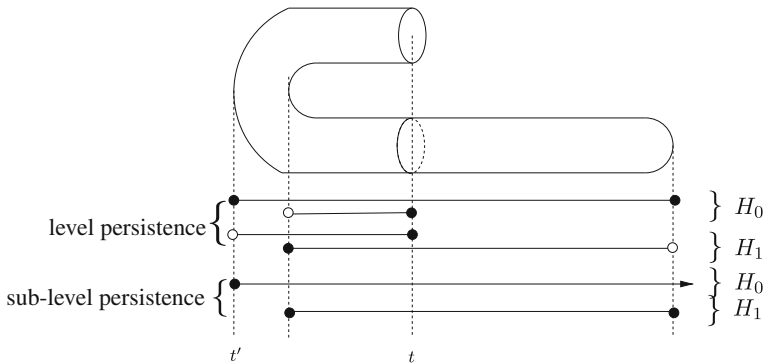
Fig. 1 Bar codes for level and sublevel persistence

of a homology class at that end (left or right) whereas a closed end signifies that a homology class cannot be detected beyond this level (left or right). In the case of the sublevel persistence the left end signifies *birth* while the right *death*. Level persistence provides considerably more information than the sublevel persistence. The bar codes of the sublevel persistence can be recovered from the ones of level persistence. Precisely a level bar code $[s, s']$ gives a sublevel bar code $[s, \infty)$ and a level bar code $[s, s')$ gives a sublevel bar code $[s, s']$; the sublevel persistence does not see any of the level bar codes (s, s') or $(s, s']$. It turns out that the bar codes of the level persistence can also be recovered from the bar codes of the sublevel persistence of f and additional maps canonically associated to f.

In Fig. 1, we indicate the bar codes both for sublevel and level persistence[3] for some simple map $f : X \to \mathbb{R}$ in order to illustrate their differences. The space X is a tube open on one end and f is the height function laid horizontally.

3 Persistence for Circle-Valued Maps

Let $f : X \to \mathbb{S}^1$ be a circle-valued map. The sublevel persistence for such a map cannot be defined since circularity in values prevents defining sublevels. Even level persistence cannot be defined as per se since the intervals may repeat over values. To overcome this difficulty we associate the infinite cyclic covering map $\tilde{f} : \tilde{X} \to \mathbb{R}$ for f. It is defined by the commutative diagram:

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\ \tilde{f}\ } & \mathbb{R} \\
{\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle p} \\
X & \xrightarrow{\ f\ } & \mathbb{S}^1
\end{array}
$$

The map $p : \mathbb{R} \to \mathbb{S}^1$ is the universal covering of the circle (the map which assigns to the number $t \in \mathbb{R}$ the angle $\theta = t \,(\mathrm{mod}\ 2\pi)$ and ψ is the pull back of p by the

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[3] The white circles indicate open ends and the dark circles indicate closed ends.

map f which is an infinite cyclic covering. Notice that if $p(t) = \theta$ then $\tilde{X}_t$ and X_θ are identified by ψ. If $x \in H_r(X_\theta) = H_r(\tilde{X}_t)$, $p(t) = \theta$, the questions Q1, Q2, Q3 for f and X can be formulated in terms of the level persistence for $\tilde{f}$ and $\tilde{X}$.

Suppose that $x \in H_r(\tilde{X}_t) = H_r(X_\theta)$ is detected in $H_r(\tilde{X}_{t'})$ for some $t' \geq t + 2\pi$. Then, in some sense, x returns to $H_r(X_\theta)$ going along the circle $\mathbb{S}^1$ one or more times. When this happens, the class x may change in some respect. This gives rise to new questions that were not encountered in sublevel or level persistence.

Q4. When $x \in H_r(X_\theta)$ returns, how does the "returned class" compare with the original class x? It may disappear after going along the circle a number of times, or it might never disappear and if so how does this class change after its return.

To answer Q1–Q4 one has to record information about $H_r(X_\theta) \to H_r(X_{[\theta,\theta']}) \leftarrow H_r(X_{\theta'})$ for any pair of angles θ and θ' which differ by at most 2π. This information is referred to as the *persistence for the circle-valued map f*.

When f is tame, this is again completely determined up to coherent isomorphisms by a finite collection of invariants. However, unlike sublevel and level persistence for real-valued maps, the invariants include structures other than bar codes called *Jordan cells*. Specifically, for any $r = 0, 1, \ldots, \dim(X)$ we have two types of invariants:

- *Bar codes*: Intervals with ends s, s' $0 < s \leq 2\pi$, $s \leq s' < \infty$, that are closed or open at s or s', precisely of one of the forms $[s, s']$, $(s, s']$, $[s, s')$, and (s, s'). These intervals can be geometerized as "spirals" with equations in (1). For any interval $\{s, s'\}$ the spiral is the plane curve (see Fig. 3 in Sect. 4)

$$\begin{aligned} x(\theta) &= (\theta + 1 - s)\cos\theta \\ y(\theta) &= (\theta + 1 - s)\sin\theta \end{aligned} \quad \text{with } \theta \in \{s, s'\}. \tag{1}$$

- *Jordan cells*. A Jordan cell is a pair (λ, k), $\lambda \in \overline{\kappa} \setminus 0$, $k \in \mathbb{Z}_{>0}$, where $\overline{\kappa}$ denotes the algebraic closure of the field κ. It corresponds to a $k \times k$ matrix of the form

$$\begin{pmatrix} \lambda & 1 & 0 \cdots & 0 \\ 0 & \lambda & 1 \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & \lambda & 1 \\ 0 & \cdots & 0 & \lambda \end{pmatrix}. \tag{2}$$

- *r-Invariants*. Given a tame map $f : X \to \mathbb{S}^1$, the collection of bar codes and Jordan cells for each dimension $r \in \{0, 1, 2, \ldots, \dim X\}$ constitute the *r-invariants* of the map f.

We will define all of the above items in the next section using quiver representations.

The bar codes for f can be inferred from $\tilde{f} : \tilde{X}_{[a,b]} \to \mathbb{R}$ with $[a, b]$ being any large enough interval. Specifically, the bar codes of $f : X \to \mathbb{S}^1$ are among the ones of $\tilde{f} : \tilde{X}_{[a,b]} \to \mathbb{R}$ for $(b - a)$ being at most $\sup_\theta \dim H_r(X_\theta)$.

The Jordan cells cannot be derived from $\tilde{f} : \tilde{X} \to \mathbb{R}$ or any of its truncations $\tilde{f} : \tilde{X}_{[a,b]} \to \mathbb{R}$ unless additional information, like the deck transformation of $\tilde{X}$, is

provided. The end points of any bar code for f correspond to critical angles, that is, s and s' (mod 2π) of a bar code interval $\{s, s'\}$ are critical angles for f. One can recover the following information from the bar codes and Jordan cells:

1. The Betti numbers of each fiber.
2. The Betti numbers of the source space X.
3. The dimension of the kernel and the image of the linear map induced in homology by the inclusion $X_\theta \subset X$ as well as other additional topological invariants not discussed here [2].

Theorems 3.1 and 3.2 make the above statement precise. Let B be a bar code described by a spiral in (1) and θ be any angle. Let $n_\theta(B)$ denote the cardinality of the intersection of the spiral with the ray originating at the origin and making an angle θ with the x-axis. For the Jordan cell $J = (\lambda, k)$, let $n(J) = k$ and $\lambda(J) = \lambda$. Furthermore, let $\mathcal{B}_r$ and $\mathcal{J}_r$ denote the set of bar codes and Jordan cells for r-dimensional homology. We have the following results.

Theorem 3.1 $\dim H_r(X_\theta) = \sum_{B \in \mathcal{B}_r} n_\theta(B) + \sum_{J \in \mathcal{J}_r} n(J)$.

Theorem 3.2 $\dim H_r(X) = \#\{B \in \mathcal{B}_r | \text{both ends closed}\} + \#\{B \in \mathcal{B}_{r-1} | \text{both ends open}\} + \#\{J \in \mathcal{J}_r | \lambda(J) = 1\} + \#\{J \in \mathcal{J}_{r-1} | \lambda(J) = 1\}$.

Using the same arguments as in the proof of the above Theorems one can derive:

Proposition 3.3 $\dim \text{img}(H_r(X_\theta) \to H_r(X)) = \#\{B \in \mathcal{B}_r \,|\, n_\theta(B) \neq 0 \text{ and both ends closed}\} + \#\{J \in \mathcal{J}_r | \lambda = 1\}$

A real-valued tame map $f : X \to \mathbb{R}$ can be regarded as a circle-valued tame map $f' : X \to \mathbb{S}^1$ by identifying $\mathbb{R}$ to $(0, 2\pi)$ with critical values $t_1, \ldots, t_m$ becoming the critical angles $\theta_1, \ldots, \theta_m$ where $\theta_i = 2 \arctan t_i + \pi$. The map f' in this case will not have any Jordan cells and the bar codes will be the same as level persistence bar codes. We have the following corollary:

Corollary 3.4 $\dim H_r(X_\theta) = \sum_{B \in \mathcal{B}_r} n_\theta(B)$ and $\dim H_r(X) = \#\{B \in \mathcal{B}_r | \text{both ends closed}\} + \#\{B \in \mathcal{B}_{r-1} | \text{both ends open}\}$.

Theorem 3.1 is quite intuitive and is in analogy with the derived results for sublevel and level persistence [4,22]. Theorem 3.2 is more subtle. Its counterpart for real-valued function (Corollary 3.4) has not yet appeared in the literature though a related result for homology of source space can be derived from extended persistence [7]. The proofs of these results require the definition of the bar codes and Jordan cells which appear in the next section. The proofs are sketched in Sect. 5.

Questions Q1–Q3 can be answered using the bar codes. Question Q4 about returned homology can be answered using the bar codes and Jordan cells.

Figure 2 indicates a tame map $f : X \to \mathbb{S}^1$ and the corresponding invariants, bar codes, and Jordan cells. The space X is obtained from Y in the figure by identifying its right end Y_1 (a union of three circles) to the left end Y_0 (again a union of three circles) following the map $\phi : Y_1 \to Y_0$. The map $f : X \to \mathbb{S}^1$ is induced by the projection

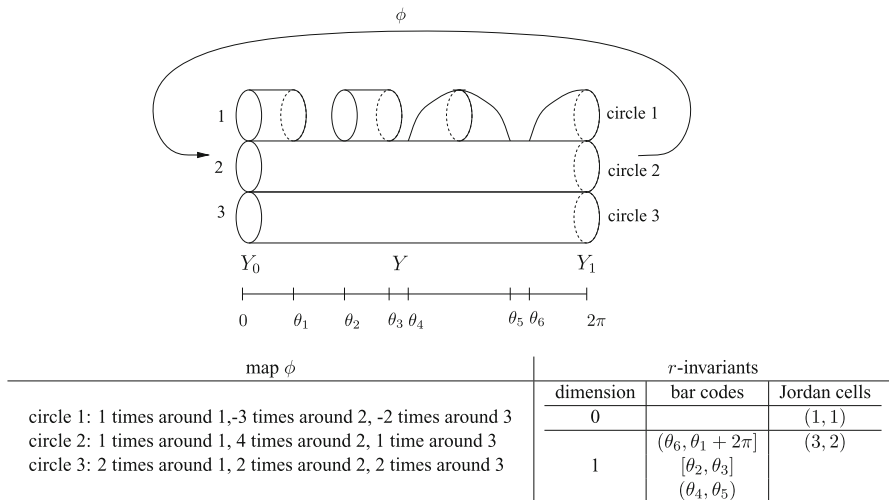| map ϕ	r-invariants		
	dimension	bar codes	Jordan cells
circle 1: 1 times around 1,-3 times around 2, -2 times around 3	0		$(1,1)$
circle 2: 1 times around 1, 4 times around 2, 1 time around 3		$(\theta_6, \theta_1 + 2\pi]$	$(3,2)$
circle 3: 2 times around 1, 2 times around 2, 2 times around 3	1	$[\theta_2, \theta_3]$	
		(θ_4, θ_5)	

Fig. 2 Example of r-invariants for a circle-valued map

of Y on the interval $[0, 2\pi]$. We have $H_1(Y_1) = H_1(Y_0) = \kappa \oplus \kappa \oplus \kappa$ and ϕ induces a linear map in 1-homology represented by the matrix [4]

$$\begin{pmatrix} 1 & 1 & 2 \\ -3 & 4 & 2 \\ -2 & 1 & 2 \end{pmatrix}.$$

The first generator (circle 1) of $H_1(\tilde{X}_{2\pi})$ is dead in $H_1(\tilde{X}_{[\theta, 2\pi]})$ for $\theta \leq \theta_6$ but not for $\theta \in (\theta_6, 2\pi]$ and is detected in $H_1(\tilde{X}_{2\pi+\theta})$ for $0 \leq \theta \leq \theta_1$ but not for $\theta > \theta_1$. It generates a bar code $(\theta_6, 2\pi + \theta_1]$. The other two (circle 2 and 3) never die and provide a Jordan cell $(3, 2)$. In Appendix we show how our algorithm can be used to compute the bar codes and Jordan cells for the above example.

4 Representation Theory and its Invariants

The invariants for the circle-valued map are derived from the representation theory of quivers. The quivers are directed graphs. The representation theory of simple quivers such as paths with directed edges was described by Gabriel [15] and is at the heart of the derivation of the invariants for zig-zag and then level persistence in [4]. For circle-valued maps, one needs representation theory for circle graphs with directed edges. This theory appears in the work of Nazarova [18], and Donovan and Freislich [11].

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[4] Each circle is oriented counterclockwise and represents a 1-dimensional homology class; "k times ($-k$ times) around the circle" means "going around k times counter clockwise (clockwise respectively)".
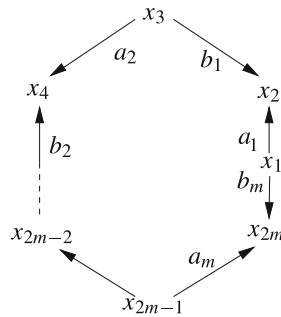
Let G_{2m} be a directed graph with $2m$ vertices, $x_1, x_1, \ldots, x_{2m}$. Its underlying undirected graph is a simple cycle. The directed edges in G_{2m} are of two types: *forward* $a_i : x_{2i-1} \to x_{2i}$, $1 \le i \le m$, and *backward* $b_i : x_{2i+1} \to x_{2i}$, $1 \le i \le m-1$, $b_m : x_1 \to x_{2m}$.

We think of this graph as being located on the unit circle centered at the origin o in the plane.

A representation ρ on G_{2m} is an assignment of a vector space V_x to each vertex x and a linear map $\ell_e : V_x \to V_y$ for each oriented edge $e = \{x, y\}$. Two representations ρ and ρ' are isomorphic if for each vertex x there exists an isomorphism from the vector space V_x of ρ to the vector space V_x' of ρ', and these isomorphisms commute with the linear maps $V_x \to V_y$ and $V_x' \to V_y'$. A *non-trivial* representation assigns at least one vector space which is not zero-dimensional. A representation is *indecomposable* if it is not isomorphic to the sum of two non-trivial representations.

Given two representations ρ and ρ', their sum $\rho \oplus \rho'$ is a representation whose vector spaces are the direct sums $V_x \oplus V_x'$ related by linear maps that are the direct sums $\ell_e \oplus \ell_e'$. It is not hard to observe that each representation has a decomposition as a sum of indecomposable representations unique up to isomorphisms.

We provide a description of indecomposable representations of the quiver G_{2m}. For any triple of integers $\{i, j, k\}$, $1 \le i, j \le m$, $k \ge 0$, one may have any of the four representations, $\rho^I([i, j]; k)$, $\rho^I((i, j]; k)$, $\rho^I([i, j); k)$, and $\rho^I((i, j); k)$ defined below. For any Jordan cell (λ, k) one has the representation $\rho^J(\lambda, k)$ defined below. The exponents I and J indicate that these representations are associated with a bar code (interval) or a Jordan cell respectively and hence we call them bar code and Jordan cell representations.

- Bar code representation $\rho^I(\{i, j\}; k)$: Suppose that the evenly indexed vertices $\{x_2, x_4, \ldots, x_{2m}\}$ of G_{2m} which are the targets of the directed arrows correspond to the angles $0 < s_1 < s_2 < \cdots < s_m \le 2\pi$. Draw the spiral curve given by (1) for the interval $\{s_i, s_j + 2k\pi\}$; refer to Fig. 3.
  For each x_i, let $\{e_i^1, e_i^2, \ldots\}$ denote the ordered set (possibly empty) of intersection points of the ray ox_i with the spiral. While considering these intersections, it is important to realize that the point $(x(s_i), y(s_i))$ (resp. $(x(s_j + 2k\pi), y(s_j + 2k\pi))$) does not belong to the spiral (1) if $\{i, j\}$ is open at i (resp. j). For example, in Fig. 3, the last circle on the ray ox_{2j} is not on the spiral since $[i, j)$ in $\rho^I([i, j); 2)$ is open at right.
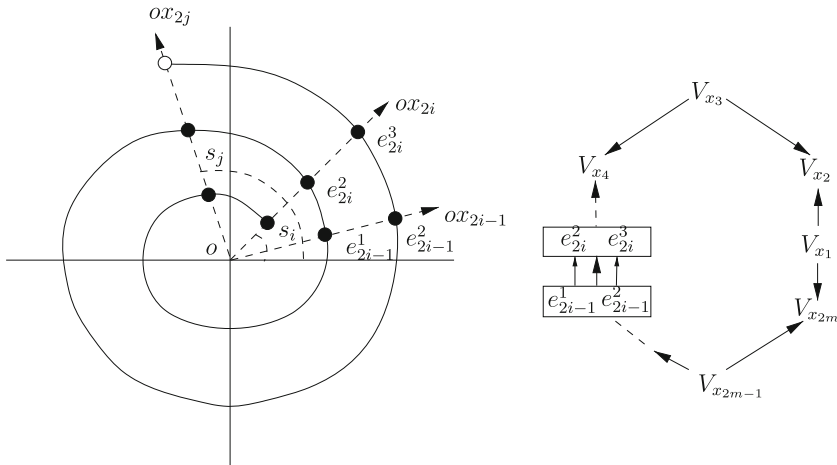
Fig. 3 The spiral for $[s_i, s_j + 4\pi)$

Let V_{x_i} denote the vector space generated by the base $\{e_i^1, e_i^2, \ldots\}$. Furthermore, let $\alpha_i : V_{x_{2i-1}} \to V_{x_{2i}}$ and $\beta_i : V_{x_{2i+1}} \to V_{x_{2i}}$ be the linear maps defined on bases and extended by linearity as follows: assign the vector $e_{2i}^h \in V_{x_i}$ to $e_{2i\pm1}^\ell$ if e_{2i}^h is an adjacent intersection point to the points $e_{2i\pm1}^\ell$ on the spiral. If e_{2i}^ℓ does not exist, assign zero to $e_{2i\pm1}^\ell$. If $e_{2i\pm1}^\ell$ does not go to zero, h has to be l, $l-1$, or $l+1$. The construction above provides a representation on G_{2m} which is indecomposable. Once the angles s_i are associated to the vertices x_{2i} one can also think of these representations $\rho^I(\{i, j\}; k)$ as the bar codes $[s_i, s_j + 2k\pi]$, $(s_i, s_j + 2k\pi]$, $[s_i, s_j + 2k\pi)$, and $(s_i, s_j + 2k\pi)$.

- Jordan cell representation $\rho^J(\lambda, k)$: Assign the vector space with the base $\{e_1, e_2, \ldots, e_k\}$ to each x_i and take all linear maps α_i but one (say α_1) and β_i the identity. The linear map α_1 is given by the Jordan matrix defined by (λ, k) in (2). Again this representation is indecomposable.

It follows from the work of [11,18] that when κ is algebraically closed,[5] the bar code and Jordan cell representations are all and only indecomposable representations of the quiver G_{2m}. The collection of all bar code and Jordan cell representations of a representation ρ constitutes its *invariants*.

Now, consider the representation ρ on the graph G_{2m} given by the vector spaces $V_{2i-1} := V_{x_{2i-1}}$, $V_{2i} := V_{x_{2i}}$ and the linear maps α_i and β_i. To such a representation ρ, we associate a map $M_\rho : \bigoplus_{1 \leq i \leq m} V_{2i-1} \to \bigoplus_{1 \leq i \leq m} V_{2i}$ which is represented by a block matrix also denoted as M_ρ:

---

[5] When κ is not algebraically closed Jordan cells have to be replaced by conjugate classes of indecomposable (not conjugated to a direct sum of matrices) matrices with entries in κ.

$$\begin{pmatrix}
\alpha_1 & -\beta_1 & 0 & \cdots & \cdots & & 0 \\
0 & \alpha_2 & -\beta_2 & \cdots & \cdots & & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \\
0 & v\cdots & \cdots & \cdots & \cdots\alpha_{m-1} & -\beta_{m-1} \\
-\beta_m & \cdots & \cdots & \cdots & & \cdots & \alpha_m
\end{pmatrix}$$

For this representation we define its dimension characteristic as the $2m$-tuple of positive integers

$$\dim \rho = (n_1, r_1 \cdots n_m, r_m)$$

with $n_i = \dim V_{x_{2i-1}}$ and $r_i = \dim V_{x_{2i}}$ and denote by $\ker \rho := \ker M_\rho$ and $\operatorname{coker} \rho = \operatorname{coker} M_\rho$. For the sum of two such representations $\rho = \rho_1 \oplus \rho_2$ we have:

Proposition 4.1 1. $\dim(\rho_1 \oplus \rho_2) = \dim \rho_1 + \dim \rho_2$,
2. $\dim \ker(\rho_1 \oplus \rho_2) = \dim \ker \rho_1 + \dim \ker \rho_2$,
3. $\dim \operatorname{coker}(\rho_1 \oplus \rho_2) = \dim \operatorname{coker} \rho_1 + \dim \operatorname{coker} \rho_2$.

The description of a bar code representation permits explicit calculations.

Proposition 4.2 1. *If $i \leq j$ then*
  (a) $\dim \rho^I([i, j]; k)$ *is given by:*
      $n_l = k + 1$ *if $(i + 1) \leq l \leq j$ and k otherwise,*
      $r_l = k + 1$ *if $i \leq l \leq j$ and k otherwise*
  (b) $\dim \rho^I((i, j]; k)$ *is given by:*
      $n_l = k + 1$ *if $(i + 1) \leq l \leq j$ and k otherwise,*
      $r_l = k + 1$ *if $(i + 1) \leq l \leq j$ and k otherwise,*
  (c) $\dim \rho^I([i, j); k)$ *is given by:*
      $n_l = k + 1$ *if $(i + 1) \leq l \leq j$ and k otherwise,*
      $r_l = k + 1$ *if $i \leq l \leq (j - 1)$ and k otherwise,*
  (d) $\dim \rho^I((i, j); k)$ *is given by:*
      $n_l = k + 1$ *if $(i + 1) \leq l \leq j$ and k otherwise,*
      $r_l = k + 1$ *if $(i + 1) \leq l \leq (j - 1)$ and k otherwise*
2. *If $i > j$ then similar statements hold.*
  (a) $\dim \rho^I([i, j]; k)$ *is given by:*
      $n_l = k$ *if $(j + 1) \leq l \leq i$ and $k + 1$ otherwise;*
      $r_l = k$ *if $(j + 1) \leq l \leq (i - 1)j$ and $k + 1$ otherwise*
  (b) $\dim \rho^I((i, j]; k)$ *is given by:*
      $n_l = k$ *if $(j + 1) \leq l \leq i$ and $k + 1$ otherwise.*
      $r_l = k$ *if $(j + 1) \leq l \leq i$ and $k + 1$ otherwise,*
  (c) $\dim \rho^I([i, j); k)$ *is given by:*
      $n_l = k$ *if $(j + 1) \leq l \leq i$ and $k + 1$ otherwise;*
      $r_l = k$ *if $j \leq l \leq (i - 1)$ and $k + 1$ otherwise,*
  (d) $\dim \rho^I((i, j); k)$ *is given by:*
      $n_l = k$ *if $(j + 1) \leq l \leq i$ and $k + 1$ otherwise;*
      $r_l = k$ *if $j \leq l \leq i$ and $k + 1$ otherwise.*
3. $\dim \rho^J(\lambda, k)$ *is given by* $n_i = r_i = k$

Proposition 4.3   1. $\dim \ker \rho^I([i, j]; k) = 0$, $\dim \mathrm{coker}\, \rho^I([i, j]; k) = 1$,

2. $\dim \ker \rho^I([i, j); k) = 0$, $\dim \mathrm{coker}\, \rho^I([i, j); k) = 0$,
3. $\dim \ker \rho^I((i, j]; k) = 0$, $\dim \mathrm{coker}\, \rho^I((i, j]; k) = 0$,
4. $\dim \ker \rho^I((i, j); k) = 1$, $\dim \mathrm{coker}\, \rho^I((i, j); k) = 0$,
5. $\dim \ker \rho^J(\lambda, k) = 0$ *(resp. 1) if* $\lambda \neq 1$ *(resp. 1),*
6. $\dim \mathrm{coker}\, \rho^J(\lambda, k) = 0$ *(resp. 1) if* $\lambda \neq 1$ *(resp. 1).*

Observation 4.4 *A representation ρ has no indecomposable components of type ρ^I in its decomposition iff all linear maps $\alpha_i's$ and $\beta_i's$ are isomorphisms. For such a representation, starting with an index i, consider the linear isomorphism*

$$T_i = \beta_i^{-1} \cdot \alpha_i \cdot \beta_{i-1}^{-1} \cdot \alpha_{i-1} \cdots \beta_2^{-1} \cdot \alpha_2 \cdot \beta_1^{-1}$$
$$\cdot \, \alpha_1 \cdot \beta_m^{-1} \cdot \alpha_m \cdot \beta_{m-1}^{-1} \cdot \alpha_{m-1} \cdots \beta_{i+1}^{-1} \cdot \alpha_{i+1}.$$

The Jordan canonical form [12] *of the isomorphism T_i is independent of i and is a block diagonal matrix with the diagonal consisting of Jordan cells $(\lambda, k)s$. Clearly, ρ is the direct sum of $\rho^J(\lambda, k)s$, the Jordan cell representations of ρ.*

Definition 4.5 (*r-Invariants.*) Let f be a circle-valued tame map defined on a topological space X. For f with m critical angles $0 < s_1 < s_2, \ldots, s_m \leq 2\pi$, consider the quiver G_{2m} with the vertices x_{2i} identified with the angles s_i and the vertices x_{2i-1} identified with the angles t_i that satisfy $0 < t_1 < s_1 < t_2 < s_2, \ldots, t_m < s_m$.

    For any r, consider the representation ρ_r of G_{2m} with $V_{x_i} = H_r(X_{x_i})$ and the linear maps α_is and β_is induced in the r-homology by maps $a_i : X_{x_{2i-1}} \to X_{x_{2i}}$ and $b_i : X_{x_{2i+1}} \to X_{x_{2i}}$ described in Sect. 2. The bar code and Jordan cell representations of ρ_r are independent of the choice of t_is and are collectively referred as the *r-invariants* of f.

5 Proof of the Main Results

Figure 2 and the bar codes listed below suggest why a semi-closed (one end open and the other closed) bar code does not contribute to the homology of the total space X and why a closed bar code (both ends closed) in $\mathcal{B}_r$ contributes one unit while an open (both ends open) bar code in $\mathcal{B}_{r-1}$ contributes one unit to the $H_r(X)$. Indeed, in our example, a semi-closed bar code in $\mathcal{B}_1$ adds to the total space a cone over $\mathbb{S}^1$, which is a contractible space. It gets glued to the total space along a generator of the cone (a segment connecting the apex to $\mathbb{S}^1$), again a contractible space. A closed bar code in $\mathcal{B}_1$ adds a cylinder of $\mathbb{S}^1$ whose H_1 has dimension 1. It gets glued to the total space along a generator of the cylinder (a segment connecting the same point on the two copies of $\mathbb{S}^1$), again a contractible space. An open bar code in $\mathcal{B}_1$ adds the suspension over $\mathbb{S}^1$, topologically a 2-sphere which gets glued along a meridian, a contractible space. This contributes a dimension to H_2.

    The lack of contribution of a Jordan cell with $\lambda \neq 1$ as well as the contribution of one unit of a Jordan cell in $\mathcal{J}_r$ with $\lambda = 1$ to both r and $r + 1$ dimensional homology of the total space should not be a surprise for the reader familiar with the calculation of the homology of mapping torus.

Below we explain rigorously but schematically the arguments for the proof of Theorems 3.1, 3.2, and Corollary 3.4.

The proof of Theorem 3.1 is a consequence of Propositions 4.1 and 4.2. The proof of Theorem 3.2 proceeds along the following lines.

First observe that, up to homotopy, the space X can be regarded as the iterated mapping torus $\mathcal{T}$ described below. Consider the collection of spaces and continuous maps:

$$X_m = X_0 \overset{b_0=b_m}{\longleftarrow} R_1 \overset{a_1}{\longrightarrow} X_1 \overset{b_1}{\longleftarrow} R_2 \overset{a_2}{\longrightarrow} X_2 \cdots X_{m-1} \overset{b_{m-1}}{\longleftarrow} R_m \overset{a_m}{\longrightarrow} X_m$$

with $R_i := X_{t_i}$ and $X_i := X_{s_i}$ and denote by $\mathcal{T} = T(\alpha_1 \cdots \alpha_m; \beta_1 \cdots \beta_m)$ the space obtained from the disjoint union

$$\left(\bigsqcup_{1 \le i \le m} R_i \times [0, 1] \right) \sqcup \left(\bigsqcup_{1 \le i \le m} X_i \right)$$

by identifying $R_i \times \{1\}$ to X_i by α_i and $R_i \times \{0\}$ to X_{i-1} by β_{i-1}. Denote by $f^{\mathcal{T}} : \mathcal{T} \to [0, m]$ where $f^{\mathcal{T}} : R_i \times [0, 1] \to [i - 1, i]$ is the projection on $[0, 1]$ followed by the translation of $[0, 1]$ to $[i - 1, i]$. This map is a homotopical reconstruction of $f : X \to \mathbb{S}^1$ provided that, with the choice of angles t_i, s_i and maps a_i b_i described in Sect. 2, $X_i := f^{-1}(s_i)$, $R_i := f^{-1}(t_i)$.

Let $\mathcal{P}'$ denote the space obtained from the disjoint union

$$\left(\bigsqcup_{1 \le i \le m} R_i \times (\varepsilon, 1] \right) \sqcup \left(\bigsqcup_{1 \le i \le m} X_i \right)$$

by identifying $R_i \times \{1\}$ to X_i by α_i, and $\mathcal{P}''$ denote the space obtained from the disjoint union

$$\left(\bigsqcup_{1 \le i \le m} R_i \times [0, 1 - \varepsilon) \right) \sqcup \left(\bigsqcup_{1 \le i \le m} X_i \right)$$

by identifying $R_i \times \{0\}$ to X_{i-1} by β_{i-1}.

Let $\mathcal{R} = \bigsqcup_{1 \le i \le m} R_i$ and $\mathcal{X} = \bigsqcup_{1 \le i \le m} X_i$. Then, one has:

1. $\mathcal{T} = \mathcal{P}' \cup \mathcal{P}''$,
2. $\mathcal{P}' \cap \mathcal{P}'' = (\bigsqcup_{1 \le i \le m} R_i \times (\varepsilon, 1 - \varepsilon)) \sqcup \mathcal{X}$, and
3. the inclusions $(\bigsqcup_{1 \le i \le m} R_i \times \{1/2\}) \sqcup \mathcal{X} \subset \mathcal{P}' \cap \mathcal{P}''$ as well as the obvious inclusions $\mathcal{X} \subset \mathcal{P}'$ and $\mathcal{X} \subset \mathcal{P}''$ are homotopy equivalences.

The Mayer–Vietoris long exact sequence leads to the diagram

$$
\begin{array}{ccc}
H_r(\mathcal{R}) & \xrightarrow{\;\;M_{\rho r}\;\;} & H_r(\mathcal{X}) \\
\Big\uparrow{\scriptstyle pr_1} & & \Big\uparrow{\scriptstyle (Id,-Id)} \\
\longrightarrow H_{r+1}(\mathcal{T}) \xrightarrow{\;\partial_{r+1}\;} H_r(\mathcal{R}) \oplus H_r(\mathcal{X}) \xrightarrow{\;N\;} H_r(\mathcal{X}) \oplus H_r(\mathcal{X}) \xrightarrow{\;(i^r,-i^r)\;} H_r(\mathcal{T}) \longrightarrow \\
\Big\uparrow{\scriptstyle in_2} & & \Big\uparrow{\scriptstyle \Delta} \\
H_r(\mathcal{X}) & \xrightarrow{\;\;Id\;\;} & H_r(\mathcal{X})
\end{array}
$$

Here Δ denotes the diagonal, in_2 the inclusion on the second component, pr_1 the projection on the first component, i^r the linear map induced in homology by the inclusion $\mathcal{X} \subset \mathcal{T}$, and M_{ρ_r} the map given by the matrix

$$
\begin{pmatrix}
\alpha_1^r & -\beta_1^r & 0 & \cdots & & \cdots & 0 \\
0 & \alpha_2^r & -\beta_2^r & \cdots & & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \\
0 & \cdots & \cdots & \cdots & \cdots & \alpha_{m-1}^r & -\beta_{m-1}^r \\
-\beta_m^r & \cdots & \cdots & \cdots & & \cdots & \alpha_m^r
\end{pmatrix}
\tag{3}
$$

with $\alpha_i^r : H_r(R_i) \to H_r(X_i)$ and $\beta_i^r : H_r(R_{i+1}) \to H_r(X_i)$ induced by the maps α_i and β_i, and N defined by

$$
\begin{pmatrix}
\alpha^r & Id \\
-\beta^r & Id
\end{pmatrix}
$$

where α^r and β^r are the matrices

$$
\begin{pmatrix}
\alpha_1^r & 0 & \cdots & \cdots & 0 \\
0 & \alpha_2^r & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \alpha_{m-1}^r
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & \beta_1^r & 0 & \cdots & 0 \\
0 & 0 & \beta_2^r & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & 0 & \beta_{m-1}^r \\
\beta_m^r & 0 & \cdots & 0 & 0
\end{pmatrix}.
$$

From the diagram above we retain only the long exact sequence

$$
\cdots \to H_r(\mathcal{R}) \xrightarrow{\;M_{\rho r}\;} H_r(\mathcal{X}) \to H_r(\mathcal{T}) \to H_{r-1}(\mathcal{R}) \xrightarrow{\;M_{\rho_{r-1}}\;} H_{r-1}(\mathcal{X}) \to \cdots
\tag{4}
$$

from which we derive the short exact sequence

$$
0 \to \operatorname{coker} \rho_r \to H_r(\mathcal{T}) \to \ker \rho_{r-1} \to 0
\tag{5}
$$

and then

$$H_r(\mathcal{T}) = \operatorname{coker} \rho_r \oplus \ker \rho_{r-1} \tag{6}$$

Theorem 3.2 follows from Propositions 4.1, 4.3 and Eqs. (6) above. A specified decomposition of ρ_r and ρ_{r-1} into indecomposable representations and a splitting in the sequence (5) provide specified elements in $H_r(X_\theta)$ and $H_r(\mathcal{T})$ which can be compared. This leads to the verification of Proposition 3.3.

6 Algorithm

Given a circle-valued tame map $f : X \to \mathbb{S}^1$, we now present an algorithm to compute the bar codes and the Jordan cells when X is a finite simplicial complex, and f is generic and linear. This makes the map tame. Genericity means that f is injective on vertices. To explain linearity we recall that, for any simplex $\sigma \in X$, the restriction $f|_\sigma$ admits liftings $\hat{f} : \sigma \to \mathbb{R}$, i.e. $\hat{f}$ is a continuous map which satisfies $p \cdot \hat{f} = f|_\sigma$. The map $f : X \to \mathbb{S}^1$ is called *linear* if for any simplex σ, at least one of the liftings (and then any other) is linear.

Our algorithm takes the simplicial complex X equipped with the map f as input and, for any r, computes the matrix M_{ρ_r} of the representation ρ_r for f. This requires recognizing the critical values $s_1, s_2, \ldots, s_m \in \mathbb{S}^1$ of f, and for conveniently chosen regular values $t_1, t_2, \ldots, t_m \in \mathbb{S}^1$, determining the vector spaces $V_{2i-1} = H_r(X_{t_i})$, $V_{2i} = H_r(X_{s_i})$ with the linear maps α_i and β_i as matrices. We consider the block matrix $M_{\rho_r} : \bigoplus_{1 \le i \le m} V_{2i-1} \to \bigoplus_{1 \le i \le m} V_{2i}$ described in the previous section.

We compute the bar codes from the block matrix M_{ρ_r} first, and then the Jordan cells. The algorithm consists of three steps. We describe the first and second steps in sufficient details. The third step is a routine application of Observation 4.1 and is accomplished by standard algorithms in linear algebra (reduction of the matrix to the canonical Jordan form).

- **Step 1.** Compute the matrices α_i, β_i that constitute the matrix M_{ρ_r} of the representation ρ_r.
- **Step 2.** Process the matrix of M_{ρ_r} to derive the bar codes ending up with a representation ρ'_r whose all α'_is and β'_is are invertible matrices.
- **Step 3.** Compute the Jordan cells of ρ_r from the representation ρ'_r.

Step 1. In Step 1 we begin with the incidence matrix of the input simplicial complex X equipped with the map $f : X \to \mathbb{S}^1$. Let the angles $0 \le s_1 < s_2 \cdots s_m \le 2\pi$ be the critical values of f. Choose a collection of regular angles $0 < t_1 < t_2 \cdots t_m < 2\pi$ with $t_i < s_i < t_{i+1} < s_{i+1}$. Consider a canonical subdivision of X into a cell complex so that $X_{[t_i, t_{i+1}]}$, and X_{t_i} are subdivided into subcomplexes R_i and X_i as follows. For any open simplex σ we associate the open cells:

1. $\sigma(i) := \sigma \cap X_{t_i}$ with $\dim(\sigma(i)) = \dim \sigma - 1$ if the intersection is nonempty
2. $\sigma\langle i \rangle := \sigma \cap X_{(t_i, t_{i+1})}$ with $\dim \sigma\langle i \rangle = \dim \sigma$ if the intersection is nonempty.

The cells of X_i are exactly of the form $\sigma(i)$ and their incidences are given as $I(\sigma(i), \tau(i)) = I(\sigma, \tau)$ where $I(\sigma, \tau) = 0, +1,$ or -1 depending on whether τ is

a coface of σ and whether their orientations match or not. The cells of R_i consist of cells of X_i, X_{i+1}, and all cells of the form $\sigma\langle i\rangle$. The incidences are given as $I(\sigma\langle i\rangle, \tau\langle i\rangle) = I(\sigma, \tau)$, $I(\sigma(i), \sigma\langle i\rangle) = 1$, and $I(\sigma(i+1), \sigma\langle i\rangle) = -1$. All other incidences are zero. Assume that we are given a total order for the simplices of X that is compatible with f and also the incidence relations. This induces a total order for the cells in X_i and X_{i+1} and also the cells in $R_i' = R_i \setminus X_i \sqcup X_{i+1}$ for any $1 \leq i \leq m$ with $X_{m+1} := X_1$. Impose a total order on R_i by juxtaposing the total orders of X_i, X_{i+1}, and R_i' in this sequence. Clearly, the incidence matrix for R_i can be derived from the incidence matrix of X.

The incidence matrix of $A = X_i \sqcup X_{i+1}$ appears in the upper left corner of the matrix for $R := R_i$. We obtain the matrices for the linear maps $\alpha_i : H_r(X_{t_i}) \rightarrow H_r(X_{s_i})$ and $\beta_i : H_r(X_{t_{i+1}}) \rightarrow H_r(X_{s_i})$ by using the persistence algorithm [5,22] on R and A as follows. First, we run the persistence algorithm on the incidence matrix for A to compute a base of the homology group $H_r(A)$. We continue the procedure by adding the columns and rows of the matrix for R to obtain a base of $H_r(R)$. It is straightforward to compute a matrix representation of the inclusion induced linear map $H_r(A) \rightarrow H_r(R)$ with respect to the bases computed by the persistence algorithm.

Step 2. Step 2 takes the matrix representation M_{ρ_r} constructed out of matrices α_i, β_i computed in step 1, and uses four elementary transformations $T_1(i)$, $T_2(i)$, $T_3(i)$, and $T_4(i)$ defined below to transform M_{ρ_r} to $M_{\rho_r'} = T_{...}(\cdots)M_{\rho_r}$, whose total number of rows and columns is strictly smaller than that of M_{ρ_r}. For convenience, let us write $\rho = \rho_r$ and $\rho' = \rho_r'$. Each elementary transformation T modifies the representation ρ to the representation ρ' while keeping indecomposable Jordan cell representations unaffected but possibly changing the bar code representations. Some of these bar code representations remain the same, some are eliminated, and some are shortened by one unit as described below. For each elementary transformation we record the changes to reconstruct the original bar codes. The elementary transformations are applied as long as the linear maps α_i or β_i satisfy some injectivity and surjectivity property. When no such transformation is applicable, the algorithm terminates with all α_i and β_i being necessarily invertible matrices. At this point the bar codes can be reconstructed reading backwards the eliminations/modifications performed. The Jordan cells then can be obtained as detailed in Step 3.

The elementary transformations modify the bar codes as follows:

- $T_1(i)$ shortens the bar codes$(i-1, k\}$ to $(i, k\}$ if $i \geq 2$ and shortens the bar codes $(m, k\}, m < k$, to $(1, k-m\}$ if $i = 1$.
- $T_2(i)$ shortens the bar codes $\{l, i+km\}$ to $\{l, i-1+km\}$ for $k \geq 0$.
- $T_3(i)$ shortens the bar codes $[i, k\}$ to $[i+1, k\}$ for $i < m$ and to $[1, k-m\}$ if $i = m$.
- $T_4(i)$ shortens the bar codes $\{l, (i+1)+km)$ to $\{l, i+km)$ for $k \geq 0$.

It is understood that if an elementary transformation applied to a bar code provides an interval which is not a bar code, then the bar code is eliminated. Consequently $T_1(i)$ eliminates the bar codes $(i-1, i)$, $(i-1, i]$,[6] $T_2(i)$ eliminates the bar codes

---

[6] If $i = 1$ eliminates the bar codes $(m, m+1)$ and $(m, m+1]$.

$[i, i]$, $(i - 1, i]$, $T_3(i)$ eliminates the bar codes $[i, i + 1)$, $[i, i]$, and $T_4(i)$ eliminates the bar codes $(i, i + 1)$, $[i, i + 1)$.

To decide how many bar codes are eliminated one uses Proposition 6.1 below. Let $\#\{i, j\}_\rho$ denote the number of bar codes of type $\{i, j\}$.

Proposition 6.1 1. $\#(i, i + 1)_\rho = \dim(\ker \beta_i \cap \ker \alpha_{i+1})$
2. $\#[i, i]_\rho = \dim(V_{2i}/((\beta_i(V_{2i+1}) + \alpha_i(V_{2i-1})))$
3. $\#(i, i + 1]_\rho = \dim(\beta_i(V_{2i+1}) + \alpha_i(\ker \beta_{i-1})) - \dim(\beta_i(V_{2i+1}))$
4. $\#[i, i + 1)_\rho = \dim(\alpha_i(V_{2i-1}) + \beta_i(\ker \alpha_{i+1})) - \dim(\alpha_i(V_{2i-1}))$

The following diagrams define the elementary transformations and indicate the relation between the representation $\rho = \{V_i, \alpha_i, \beta_i\}$ and the representation $\rho' = \{V'_i, \alpha'_i, \beta'_i\}$ obtained after applying an elementary transformation.

- Transformation $T_1(i)$:
$$V'_{2i-1} = V_{2i-1}/\ker \beta_{i-1}, \quad V'_{2i} = V_{2i}/\alpha_i(\ker \beta_{i-1}), \quad V'_k = V_k, k = 2i, 2i - 1$$

- Transformation $T_2(i)$:
$$V'_{2i} = \beta_i(V_{2i+1}), \quad V'_{2i-1} = \alpha_i^{-1}(\beta_i(V_{2i+1})), \quad V'_k = V_k, k \neq 2i - 1, 2i$$

- Transformation $T_3(i)$:
$$V'_{2i} = \alpha_i(V_{2i-1}), \quad V'_{2i+1} = \beta_i^{-1}(\alpha_i(V_{2i-1})), \quad V'_k = V_k, k \neq 2i, 2i + 1$$

- Transformation $T_4(i)$:
$$V'_{2i+1} = V_{2i+1}/\ker \alpha_{i+1}, \quad V'_{2i} = V_{2i}/\beta_i(\ker \alpha_{i+1}), \quad V'_k = V_k, k \neq 2i, 2i + 1.$$

The verification of the properties stated above and the proof of Proposition 6.1 are straightforward for indecomposable representations described in Sect. 4 and therefore for arbitrary representations.

As one can see from the diagrams above, when β_{i-1} is injective, the representations ρ and ρ' are the same and we say that $T_1(i)$ is not applicable. Similarly, when β_i is surjective, $T_2(i)$ is not applicable, when α_i is surjective, $T_3(i)$ is not applicable, and when α_{i+1} is injective, $T_4(i)$ is not applicable. When all α_i, β_i are invertible, no elementary transformation is applicable and at this stage the algorithm (Step 2) terminates.

To explain how the algorithm works, it is convenient to consider the following block matrices B_{2i-1} and B_{2i}, $i = 1, \ldots, m$, which become the sub-matrices of M_{ρ_r} in (3) when the entries β_i are replaced with $-\beta_i$. Let

$$B_{2i-1} = \begin{pmatrix} \alpha_i & \beta_i \\ 0 & \alpha_{i+1} \end{pmatrix}, \quad B_{2i} = \begin{pmatrix} \beta_i & 0 \\ \alpha_{i+1} & \beta_{i+1} \end{pmatrix} \tag{7}$$

for $i = 1, 2, \ldots, (m-1)$ and

$$B_{2m-1} = \begin{pmatrix} \alpha_m & \beta_m \\ 0 & \alpha_1 \end{pmatrix}, \quad B_{2m} = \begin{pmatrix} \beta_m & 0 \\ \alpha_1 & \beta_1 \end{pmatrix}. \tag{8}$$

We modify M_ρ by modifying successively each block B_k. When $m > 1$ the algorithm iterates over the blocks in multiple passes. In a single pass, it processes the blocks $B_1, B_2, \ldots, B_{2m}$ in this order.

When

$$B_{2(i-1)} = \begin{pmatrix} \beta_{i-1} & 0 \\ \alpha_i, & \beta_i \end{pmatrix}$$

is processed then:

1. If β_{i-1} is not injective, we apply $T_1(i)$. This boils down to changing the bases of V_{2i-1} and V_{2i} so that the matrix $B_{2(i-1)}$ becomes

$$\left(\begin{array}{c|c||c} \beta_{i-1,1} & 0 & 0 \\ \hline \alpha^1_{i,1} & \alpha^1_{i,2} & \beta^1_i \\ \alpha^2_{i,1} & 0 & \beta^2_i \end{array} \right)$$

with $(\beta_{i-1,1} \, 0)$ in column echelon form and

$$\begin{pmatrix} \alpha^1_{i,2} \\ 0 \end{pmatrix}$$

in row echelon form.

In this block matrix the first and third columns correspond to V'_{2i-1} and V_{2i+1} respectively, and the first and third rows to $V_{2(i-1)}$ and V'_{2i} respectively. The second column and row become "irrelevant" as a result of which the modified block matrix $B_{2(i-1)}$ becomes

$$\begin{pmatrix} \beta'_{i-1} & 0 \\ \alpha'_i & \beta'_i \end{pmatrix} = \begin{pmatrix} \beta_{i-1,1} & 0 \\ \alpha^2_{i,1} & \beta^2_i \end{pmatrix}.$$

2. If β_i is not surjective, we apply $T_2(i)$. This boils down to changing the bases of V_{2i-1} and V_{2i} so that the matrix $B_{2(i-1)}$ becomes

$$\left(\begin{array}{cc|c} \beta_{i-1,1} & \beta_{i-1,2} & 0 \\ \hline \alpha^1_{i,1} & \alpha^1_{i,2} & \beta^1_i \\ \hline \alpha^2_{i,1} & 0 & 0 \end{array} \right)$$

with

$$\begin{pmatrix} \beta^1_i \\ 0 \end{pmatrix}$$

in row echelon form and $(\alpha^2_{i,1}\ 0)$ in column echelon form.

In this block matrix the second and third columns correspond to V'_{2i-1} and V_{2i+1} respectively, and the first and second rows to $V_{2(i-1)}$ and V'_{2i} respectively. We make the first column and third row "irrelevant" as a result of which the modified block matrix $B_{2(i-1)}$ becomes

$$\begin{pmatrix} \beta'_{i-1} & 0 \\ \alpha'_i & \beta'_i \end{pmatrix} = \begin{pmatrix} \beta_{i-1,2} & 0 \\ \alpha^1_{i,2} & \beta^1_i \end{pmatrix}.$$

When B_{2i-1} is processed then:

3. If α_i is not surjective, we apply $T_3(i)$. This boils down to changing the bases of V_{2i+1} and V_{2i} so that the matrix B_{2i-1} becomes

$$\left(\begin{array}{c|c|c} \alpha^1_i & \beta^1_{i,1} & \beta^1_{i,2} \\ \hline 0 & \beta^2_{i,1} & 0 \\ \hline 0 & \alpha_{i+1,1} & \alpha_{i+1,2} \end{array} \right)$$

with

$$\begin{pmatrix} \alpha^1_i \\ 0 \end{pmatrix}$$

in row echelon form and $\beta^2_{i,1}\ 0$ in column echelon form.

In this block matrix the first and third columns correspond to V_{2i-1} and V'_{2i+1} respectively, and the first and third rows to V'_{2i} and V_{2i+2} respectively. We make the second column and second row "irrelevant" as a result of which the modified block matrix B_{2i-1} becomes

$$\begin{pmatrix} \alpha'_i & \beta'_i \\ 0 & \alpha'_{i+1} \end{pmatrix} = \begin{pmatrix} \alpha^1_i & \beta^1_{i,2} \\ 0 & \alpha_{i+1,2} \end{pmatrix}.$$

4. If α_{i+1} is not injective, we apply $T_4(i)$. This boils down to changing the bases of V_{2i+1} and V_{2i} so that the matrix B_{2i-1} becomes

$$\left(\begin{array}{c|c|c} \alpha^1_i & \beta^1_{i,1} & \beta^1_{i,2} \\ \hline \alpha^2_i & \beta^2_{i,1} & 0 \\ \hline 0 & \alpha_{i+1,1} & 0 \end{array} \right)$$

with $(\alpha_{i+1,1}\ 0)$ in column echelon form and

$$\begin{pmatrix} \beta^1_{i,2} \\ 0 \end{pmatrix}$$

in row echelon form.

In this block matrix first and second columns correspond to V_{2i-1} and V'_{2i+1} respectively, and second and third rows to V'_{2i} and $V_{2(i+1)}$ respectively. We make the third column and first row "irrelevant" as a result of which the modified block matrix B_{2i-1} becomes

$$\begin{pmatrix} \alpha'_i & \beta'_i \\ 0 & \alpha'_{i+1} \end{pmatrix} = \begin{pmatrix} \alpha^2_i & \beta^2_{i,1} \\ 0 & \alpha_{i+1,1} \end{pmatrix}.$$

Explicit formulae for α's and β's are given at the end of this section. At each pass the algorithm may eliminate or change bar codes, and if this happens, the matrix has less columns or rows. If this does not happen, the algorithm terminates, and indicates that there is no more bar code left. At termination, all α_i and β_i become isomorphisms. The bar codes can be recovered by keeping track of all eliminations of the bar codes after each elementary transformation. A bar code which is not eliminated in a pass gets shrunk by exactly two units, during that pass, that is, a bar code $\{i, j\}$ shrinks to $\{i + 1, j - 1\}$ by exactly two distinct elementary transformations. by elementary transformations. For example if $m = 5$ the bar code $(1, 5]$ during the pass became $(2, 4]$ as result of applying $T_1(1)$ when inspecting B_1 and $T_2(5)$ when inspecting B_9.

When a bar code $[i, i]$ is eliminated, say, in the kth pass, we know that it corresponds to a bar code $[i - k + 1, i + k - 1]$ in the original representation. Similarly, other bar codes of type $\{i, i + 1\}$ eliminated at the kth pass correspond to the bar code $\{i - k + 1, i + k\}$. In both cases, the multiplicity of the bar codes can be determined from the multiplicity of the eliminated bar codes thanks to Proposition 6.1.

When $m = 1$, the operations on above minors are not well defined. In this case we extend the quiver G_2 to G_4 ($m = 2$) by adding fake levels t_2, s_2 where $H_r(X_{t_2}) = H_r(X_{s_2}) = H_r(X_{s_1})$ and α_2, β_2 are identities.[7]

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[7] Other easier methods can also be used in this case.

A high level pseudocode for the Step 2 can be written as follows:

Algorithm BARCODE(M_ρ)

Consider the block sub-matrices $B_1, \ldots, B_m$ of M_ρ;

Repeat

for $j := 1$ to $2m$ do

1. if $j = 2i - 1$ is odd

A. if α_{i+1} is not injective, update $B_{2i-1} := T_4(i)(B_{2i-1})$.

B. if α_i is not surjective, update $B_{2i-1} := T_3(i)(B_{2i-1})$.

C. delete any rows and columns rendered irrelevant.

2. if $j = 2i$ is even

A. if β_{i+1} is not surjective, update $B_{2i} := T_2(i)(B_{2i})$.

B. if β_i is not injective, update $B_{2i} := T_1(i)(B_{2i})$.

C. delete any rows and columns rendered irrelevant.

endfor

until M_ρ is not empty or has not been updated.

Output M_ρ.

Example To illustrate how step 2 works, we consider a representation given by

$$\alpha_1 = \begin{pmatrix} 1 & 1 & 2 \\ -3 & 4 & 2 \\ -2 & 1 & 2 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \alpha_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad \alpha_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (9)$$

$$\beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \beta_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad \beta_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \beta_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

The reader can notice that this is the representation ρ_1 for a simplified version of the example provided in Fig. 2 with the cylinder between the critical values θ_2 and θ_3 removed.

- Inspect B_1 and B_2. No changes are necessary.
- Inspect B_3. Since α_3 is not injective, one modifies the block by applying $T_4(2)$ which makes both α_3 and β_2 equal to

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- Inspect the blocks B_4, B_5, B_6, B_7. No changes are necessary.
- Inspect B_8. Since β_4 is not injective, one modifies the block by applying $T_1(1)$ which leads to

$$\alpha_1 = \begin{pmatrix} -4 & 3 \\ -3 & 0 \end{pmatrix} \quad \text{and} \quad \beta_1 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Indeed the block B_8 is given by

$$\left(\begin{array}{c|c} \beta_4 & 0 \\ \hline \alpha_1 & \beta_1 \end{array}\right) = \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 2 & 1 & 0 \\ -3 & 4 & 2 & 0 & 1 \\ -2 & 1 & 2 & 0 & 0 \end{array}\right)$$

Since β_4 is already in column echelon form one only has to change the base of V_2 to bring the last column of α_1 in row echelon form which ends up with

$$\left(\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 2 & 1 & 0 \\ -4 & 3 & 0 & -1 & 1 \\ -3 & 0 & 0 & -1 & 0 \end{array}\right)$$

Therefore

$$\alpha_1' = \begin{pmatrix} -4 & 3 \\ -3 & 0 \end{pmatrix}, \beta_1' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \beta_4' = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

The algorithm stops as all α_i's and β_i's are at this time invertible. The last transformation $T_1(1)$ has eliminated only the bar code $(4, 5]$, and the previous, which was the first transformation, $T_4(2)$, has eliminated only the bar code $(2, 3]$. This can be concluded from Proposition 6.1. In view of the properties of these two transformations, one concludes that these were the only two bar codes.

Step 3. At termination, all α_i and β_i become isomorphisms because otherwise one of the transformations would be applicable. The Jordan cells can be recovered from the Jordan decomposition of the matrix

$$T = \beta_{i-1}^{-1} \cdot \alpha_{i-1} \cdot \beta_{i-2}^{-1} \cdots \beta_1^{-1} \cdot \alpha_1 \cdot \beta_m^{-1} \cdot \alpha_m \cdots \beta_{i+1}^{-1} \cdot \alpha_{i+1} \cdot \beta_i^{-1} \cdot \alpha_i \quad \text{for any } i.$$

Standard linear algebra routines permit the calculation of the Jordan cells for familiar algebraic closed fields. Note that if κ is not algebraically closed, Step 1 and Step 2 can still be performed and the matrix T can be obtained. In this case it may not be possible to decompose the matrix T in Jordan cells unless we consider the algebraic closure of κ. It is however possible to decompose the matrix T up to conjugacy as a sum of indecomposable invertible matrices while remaining in the class of matrices with coefficients in the field κ. This is the case for the field $\kappa = \mathbb{Z}_2$.

In the example above

$$T = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

provides the Jordan cell ($\lambda = 3, k = 2$).

6.1 Implementation of $T_1(i)$, $T_2(i)$, $T_3(i)$ and $T_4(i)$.

1. $T_1(i)$ acts on the block matrix

$$B_{2(i-1)} = \begin{pmatrix} \beta_{i-1} & 0 \\ \alpha_i & \beta_i \end{pmatrix}.$$

First we modify $B_{2(i-1)}$ to the block matrix

$$\begin{pmatrix} \beta_{i-1,1} & 0 & 0 \\ \alpha_{i,2} & \alpha_{i,2} & \beta_i \end{pmatrix}$$

where $\beta_{i-1,1} \quad 0 = \beta_{i-1} \cdot R(\beta_{i-1})$ and $\alpha_{i,1} \quad \alpha_{i,2} = \alpha_i \cdot R(\beta_{i-1})$. Recall the definition of $R(\cdot)$ and $L(\cdot)$ given under notations in the introduction. Then, one passes to the block matrix

$$\begin{pmatrix} \beta_{i-1,1} & 0 & 0 \\ \alpha_{i,2}^1 & \alpha_{i,2}^1 & \beta_i^1 \\ \alpha_{i,2}^2 & 0 & \beta_i^2 \end{pmatrix}$$

with

$$\begin{pmatrix} \alpha_{i,2}^1 \\ 0 \end{pmatrix} = L(\alpha_{i,2}) \cdot \alpha_{i,2}, \quad \begin{pmatrix} \alpha_{i,1}^1 \\ \alpha_{i,2}^2 \end{pmatrix} = L(\alpha_{i,2}) \cdot \alpha_{i,1} \quad \text{and} \quad \begin{pmatrix} \beta_i^1 \\ \beta_i^2 \end{pmatrix} = L(\alpha_{i,2})\beta_i.$$

The modified block matrix is

$$\begin{pmatrix} \beta_{i-1,1} & 0 \\ \alpha_{i,1}^2 & \beta_i^2 \end{pmatrix}.$$

2. $T_2(i)$ acts on the block matrix

$$B_{2(i-1)} = \begin{pmatrix} \beta_{i-1} & 0 \\ \alpha_i & \beta_i \end{pmatrix}.$$

First we modify $B_{2(i-1)}$ to the block matrix

$$\begin{pmatrix} \beta_{i-1} & 0 \\ \alpha_i^1 & \beta_i^1 \\ \alpha_i^2 & 0 \end{pmatrix}$$

where

$$\begin{pmatrix} \beta_i^1 \\ 0 \end{pmatrix} = L(\beta_i) \cdot \beta_i \quad \text{and} \quad \begin{pmatrix} \alpha_i^1 \\ \alpha_i^2 \end{pmatrix} = L(\beta_i) \cdot \alpha_i.$$

Then, one passes to the block matrix

$$\begin{pmatrix} \beta_{i-1,1} & \beta_{i-1,2} & 0 \\ \alpha_{i,1}^1 & \alpha_{i,2}^1 & \beta_i^1 \\ \alpha_{i,1}^2 & 0 & 0 \end{pmatrix}$$

with

$$\begin{pmatrix} \alpha_{i,1}^2 \\ 0 \end{pmatrix} = \alpha_{i,1}^2 \cdot R(\alpha_{i,1}^2), \begin{pmatrix} \alpha_{i,1}^1 \\ \alpha_{i,2}^1 \end{pmatrix} = \alpha_{i,1} R(\alpha_{i,1}^2), \quad \text{and} \quad \begin{pmatrix} \beta_{i-1,1} \\ \beta_{i-1,2} \end{pmatrix} = \beta_{i-1} R(\alpha_{i,1}).$$

The modified block matrix is

$$\begin{pmatrix} \beta_{i-1,2} & 0 \\ \alpha_{i,2}^1 & \beta_i^1 \end{pmatrix}.$$

3. $T_3(i)$ acts on the block matrix

$$B_{2i-1} = \begin{pmatrix} \alpha_i & \beta_i \\ 0 & \alpha_{i+1} \end{pmatrix}.$$

First we modify B_{2i-1} to the block matrix

$$\begin{pmatrix} \alpha_i^1 & \beta_i^1 \\ 0 & \beta_i^2 \\ 0 & \alpha_{i+1} \end{pmatrix}$$

where

$$\begin{pmatrix} \alpha_i^1 \\ 0 \end{pmatrix} = \alpha_i \cdot R(\alpha_i) \text{ and } \begin{pmatrix} \beta_i^1 \\ \beta_i^2 \end{pmatrix} = \beta_i \cdot R(\alpha_i).$$

Then, one passes to the block matrix

$$\begin{pmatrix} \alpha_i^1 & \beta_{i,1}^1 & \beta_{i,2}^1 \\ 0 & \beta_{i,1}^2 & 0 \\ 0 & \alpha_{i+1,1} & \alpha_{i+1,2} \end{pmatrix}$$

with

$$\begin{pmatrix} \beta_{i,1}^2 & 0 \end{pmatrix} = \beta_i^2 \cdot R(\beta_i^2), \begin{pmatrix} \beta_{i,1}^1 & \beta_{i,2}^1 \end{pmatrix} = \beta_i^1 \cdot R(\beta_i^2)$$

and $(\alpha_{i+1,1}\ \alpha_{i+1,2}) = \alpha_{i+1} \cdot R(\beta_i^2)$. The modified block matrix is

$$\begin{pmatrix} \alpha_i^1 & \beta_{i,2}^1 \\ 0 & \alpha_{i+1,2} \end{pmatrix}.$$

4. $T_4(i)$ acts on the block matrix

$$B_{2i-1} = \begin{pmatrix} \alpha_i & \beta_i \\ 0 & \alpha_{i_1} \end{pmatrix}.$$

First one modifies B_{2i-1} to the block matrix

$$\begin{pmatrix} \alpha_i & \beta_{i,1} & \beta_{i,2} \\ 0 & \alpha_{i+1,1} & 0 \end{pmatrix}$$

where

$$\begin{pmatrix} \alpha_{i+1,1} & 0 \end{pmatrix} = \alpha_{i+1} \cdot R(\alpha_{i+1}) \quad \text{and} \quad \begin{pmatrix} \beta_{i,1} & \beta_{i,2} \end{pmatrix} = \beta_i \cdot R(\alpha_{i+1}).$$

Then, one passes to the block matrix

$$\begin{pmatrix} \alpha_i^1 & \beta_{i,1}^1 & \beta_{i,2}^1 \\ \alpha_i^2 & \beta_{i,1}^2 & 0 \\ 0 & \alpha_{i+1,1}^2 & 0 \end{pmatrix}$$

with

$$\begin{pmatrix} \beta_{i,2}^1 \\ 0 \end{pmatrix} = L(\beta_{i,2}) \cdot \beta_{i,2}, \quad \begin{pmatrix} \beta_{i,1}^1 \\ \beta_{i,1}^2 \end{pmatrix} = L(\beta_{i,2}) \cdot \beta_{i,1} \quad \text{and} \quad \begin{pmatrix} \alpha_i^1 \\ \alpha_i^2 \end{pmatrix} = L(\beta_{i,2}) \cdot \alpha_i.$$

The modified block matrix is

$$\begin{pmatrix} \alpha_i^2 & \beta_{i,1}^2 \\ 0 & \alpha_{i+1,1}^2 \end{pmatrix}.$$

6.2 Time Complexity

Let the input complex X have n simplices in total on which the circle-valued map f is defined which has m critical values.

Then, step 1 takes $O(nd)$ time to detect all the critical values where $d \leq n$ is the maximum degree of any vertex. The critical values can be computed by looking at the simplices adjacent to each of the vertices. To compute the matrices α_i and β_i, we set up the matrices of size $O(n) \times O(n)$ and run persistence on them. Using the algorithm of [17], this can be achieved in $O(M(n))$ time where $M(n)$ is the time complexity

of multiplying two $n \times n$ matrices.[8] Since we perform this operations for each of the critical levels and the spaces between them, we have $O(m M(n))$ total time complexity for step 1.

In step 2, we process the matrix M_{ρ_r} iteratively until all bar code representations are removed. In each pass except the last one, we are guaranteed to shrink a bar code by at least one unit. Therefore, the total number of passes is bounded from above by the total length of all bar codes. Theorem 3.1 implies that a bar code cannot come back to the same level more than $\max_{s_i} \dim H_r(X_{s_i})$ times which can be at most $O(n)$. Therefore, any bar code has a length of at most $O(nm)$ giving a total length of $O(n^2 m)$ over all bar codes. Hence, the repeat loop in the algorithm BARCODE cannot have more that $O(n^2 m)$ iterations. In each iteration, we reduce the block matrices each of which can be done with $O(M(n))$ matrix multiplication time [16]. Since there are at most $O(m)$ block matrices to be considered, we have $O(m M(n))$ time per iteration giving a total of $O(n^2 m^2 M(n))$ time for step 2.

Step 3 is performed on the resulting matrix from step 2 which has $O(mn) \times O(mn)$ size. This can again be performed by matrix multiplication which takes $O(M(mn))$ time.

Therefore, the entire algorithm has time complexity of $O(m^2 n^2 M(n) + M(mn))$.

7 Conclusions

We have analyzed circle-valued maps from the perspective of topological persistence. We show that the notion of persistence for such maps incorporate an invariant that is not encountered in persistence studied erstwhile. Our results also shed lights on computing homology vector spaces and other topological invariants from bar codes and Jordan cells (Theorems 3.1 and 3.2). We have given an algorithm to compute the bar codes and the Jordan cells; the algorithms can also be adapted to compute zig-zag persistence. In a subsequent work, Burghelea and Haller have derived more subtle topological invariants like Novikov homology, monodromy [2], Reidemeister torsion, and others from bar codes and Jordan cells confirming their mathematical relevance. We have not treated in this paper the stability of the invariants; see [2] for partial answer.

The standard persistence is related to Morse theory. In a similar vein, the persistence for circle-valued map is related to Morse Novikov theory [19]. The work of Burghelea and Haller applies Morse Novikov theory to instantons and closed trajectories for vector field with Lyapunov closed one form [1]. The results in this paper will very likely provide additional insight on the dynamics of these vector fields and have implications in computational topology in particular and algebraic topology in general.

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[8] We have $M(n) = O(n^\omega)$ where $\omega < 2.376$ [8].

Appendix

In this appendix we explain the calculation of the r-invariants for the example depicted in Fig. 2. The representation ρ_0 has vector spaces that are all one dimensional and maps $\alpha_i = \beta_i$ that are all identity. Hence, there is no bar code, but one Jordan cell $\lambda = 1, k = 1$.

It is not hard to recognize from Fig. 2 that the maps for the representation ρ_1 are given by:

$$\alpha_1 = \begin{pmatrix} 1 & 1 & 2 \\ -3 & 4 & 2 \\ -2 & 1 & 2 \end{pmatrix}; \quad \alpha_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \alpha_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \alpha_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

$$\alpha_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad \alpha_6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

$$\beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \beta_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \beta_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \beta_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix};$$

$$\beta_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \beta_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We proceed with the step 2 of the algorithm.

- inspect B_1—no change for $\rho = \rho_1$; inspect B_2—no change.
- inspect B_3—since α_2 is not surjective apply $T_3(2)$. This changes $\alpha_2, \beta_2, \alpha_3$ into

$$\alpha_2' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \beta_2' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \alpha_3' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Update and continue.
- inspect B_4—no changes.
- inspect B_5—since α_3 is not surjective, apply $T_3(3)$. This changes α_3 and β_3 into

$$\alpha_3' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \beta_3' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Update and continue.
- inspect B_6—no changes.
- inspect B_7—since α_5 is not injective, apply $T_4(4)$. This changes β_4 and α_5 into

$$\alpha_5' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \beta_4' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Update and continue.
- inspect B_8—no change; inspect B_9—no change; inspect B_{10}—no change; inspect B_{11}—no change.

- inspect B_{12}—since β_6 is not injective, apply $T_1(1)$. This changes β_6, α_1, β_1 to

$$\beta_6' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \alpha_1' = \begin{pmatrix} -4 & 3 \\ -3 & 0 \end{pmatrix}, \text{ and } \beta_1' = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Update.

Since at this time all α_i's and β_i's are invertible, step 2 terminates.

Book keeping. The last transformation $T_1(1)$ has eliminated the bar code $(\theta_6, \theta_1 + 2\pi]$ (by Proposition 6.1) and nothing else. This bar code was not the modification of any other bar code by the previous elementary transformations. The previous transformation $T_4(4)$ has eliminated the bar code (θ_4, θ_5) and nothing else (by Proposition 6.1). This bar code was not the modification of any other bar code by the previous transformations. The transformation $T_3(3)$ has eliminated the bar code $[\theta_3, \theta_3]$ (by Proposition 6.1) which was the modification of $[\theta_2, \theta_3]$ by $T_3(2)$. These are all bar codes as listed in the table in Sect. 3. To calculate the Jordan cells we use step 3. We calculate the Jordan cells of

$$\begin{pmatrix} -4 & 3 \\ -3 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}^{-1}$$

which is $(\lambda = 3, k = 2)$ as listed in the table in Sect. 3.

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