

TOPOLOGICAL PROPERTIES IN *BCC*-ALGEBRAS

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ABSTRACT. In this paper, we show how to associate certain topologies with special ideals of *BCC*-algebras on these *BCC*-algebras. We show that it is natural for *BCC*-algebras to be topological *BCC*-algebras with respect to these topologies. Furthermore, we show how certain standard properties may arise. In addition we demonstrate that it is natural for these topologies to have many clopen sets and thus to be highly connected via the ideal theory of *BCC*-algebras.

1. Introduction

In 1966, Y. Imai and K. Iséki [5] defined a class of algebras of type (2,0) called *BCK*-algebras which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of implication algebra (see [6]). The class of all *BCK*-algebras is a quasivariety. K. Iséki posed an interesting problem (solved by A. Wroński [12]) whether the class of *BCK*-algebras is a variety. In connection with this problem, Y. Komori [9] introduced a notion of *BCC*-algebras, and W. A. Dudek [1, 2] redefined the notion of *BCC*-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [4], J. Hao introduced the concept of ideals in a *BCC*-algebra and studied some related properties. In this paper, we address the issue of attaching topologies to *BCC*-algebras in as natural a manner as possible. It turns out that we may use the class of *BCC*-ideals of a *BCC*-algebra as the underlying structure whence a certain uniformity and thence a topology is derived which provides a natural connection between the notion of a *BCC*-algebra and the notion of a topology in that we are able to conclude that in this setting a *BCC*-algebra becomes a topological *BCC*-algebra. Other properties are also identified both in the *BCC*-algebra and in the topology, such as $\{0\}$ is closed if and only if the topology is Hausdorff and $\{0\}$ is open if and only if the topology is discrete among others.

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2. Preliminaries

By a *BCC-algebra* ([9]) we mean a non-empty set X with a constant 0 and a binary operation “ $*$ ” satisfying the following axioms: for all $x, y, z \in X$,

- (I) $((x * y) * (z * y)) * (x * z) = 0$,
- (II) $0 * x = 0$,
- (III) $x * 0 = x$,
- (IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

For brevity, we also call X a *BCC-algebra*. In X we can define a binary operation “ \leq ” by $x \leq y$ if and only if $x * y = 0$, is called a *BCC-order* on X . Then it is easy to show that \leq is a partial order on X .

In a *BCC-algebra*, the following hold: for any $x, y, z \in X$,

- (2.1) $x * x = 0$,
- (2.2) $(x * y) * x = 0$,
- (2.3) $x \leq y \Rightarrow x * z \leq y * z$,
- (2.4) $x \leq y \Rightarrow z * y \leq z * x$.

Any *BCK-algebra* is a *BCC-algebra*, but there are *BCC-algebras* which are not *BCK-algebras* (see [2]). Note that a *BCC-algebra* is a *BCK-algebra* if and only if it satisfies: (2.5) $(x * y) * z = (x * z) * y$.

A non-empty subset S of a *BCC-algebra* X is called a *subalgebra* of X if it is closed under the *BCC-operation*.

Definition 2.1. Let X be a *BCC-algebra*. An equivalence relation \sim on X is called a *left congruence* if $x \sim y$ implies $u * x \sim u * y$, where $x, y, u \in X$. An equivalence relation \sim on X is called a *right congruence* if $x \sim y$ implies $x * u \sim y * u$, where $x, y, u \in X$.

Definition 2.2. Let X be a *BCC-algebra*. An equivalence relation \sim on X is called a *congruence* if $x \sim y$, $u \sim v$ imply $x * u \sim y * v$, where $x, y, u, v \in X$.

Proposition 2.3. Let X be a *BCC-algebra* and \sim be an equivalence relation on X . Then \sim is a congruence if and only if it is both a left congruence and a right congruence.

Definition 2.4 ([4]). Let X be a *BCC-algebra* and $\emptyset \neq I \subseteq X$. I is called an *ideal* of X if it satisfies the following conditions: (i) $0 \in I$; (ii) $x * y, y \in I$ imply $x \in I$.

Theorem 2.5 ([4]). In a *BCC-algebra* X , any ideal of X is a subalgebra of X .

Definition 2.6 ([3]). Let X be a *BCC-algebra* and $\emptyset \neq I \subseteq X$. I is called a *BCC-ideal* of X if it satisfies the following conditions:

- (i) $0 \in I$;
- (ii) $(x * y) * z \in I$ and $y \in I$ imply $x * z \in I$.

Lemma 2.7 ([3]). In a *BCC-algebra* any *BCC-ideal* is an ideal.

Corollary 2.8 ([3]). Any *BCC-ideal* of a *BCC-algebra* is a subalgebra.

Remark. In a BCC-algebra, a subalgebra need not be an ideal and an ideal need not be a BCC-ideal, in general (see [3, 4]).

Theorem 2.9 ([3]). *If I is a BCC-ideal of a BCC-algebra X , then the relation defined on X by*

$$x \sim y \text{ if and only if } x * y, y * x \in I$$

is a congruence relation on X .

Theorem 2.10. *Let I be a BCC-ideal of a BCC-algebra X . If we define a binary operation on the quotient set $X/I := \{I_x | x \in X\}$ by $I_x * I_y = I_{x*y}$, then $(X/I; *, I_0)$ is a BCC-algebra, called the quotient algebra of X relative to I .*

Proof. If $I_x = I_{x'}$ and $I_y = I_{y'}$, then $x \sim x'$ and $y \sim y'$. Hence $x * y \sim x' * y'$ since \sim is a congruence relation. Therefore $I_x * I_y = I_{x*y} = I_{x'*y'} = I_{x'} * I_{y'}$. Thus “ $*$ ” is well-defined on X/I . It is easy to show (I), (II), and (III). Assume that $I_x * I_y = I_y * I_x = I_0$. Then $I_{x*y} = I_{y*x} = I_0$. Hence $x * y \sim 0$ and $y * x \sim 0$. Therefore $x * y \in I$ and $y * x \in I$ and so $I_x = I_y$. Thus $(X/I; *, I_0)$ is a BCC-algebra. \square

3. Uniformity in BCC-algebras

From now on, X is a BCC-algebra, unless otherwise is stated.

Let X be a non-empty set, and U and V be any subsets of $X \times X$. Define

$$\begin{aligned} U \circ V &:= \{(x, y) \in X \times X \mid \text{for some } z \in X, (x, z) \in U \text{ and } (z, y) \in V\}, \\ U^{-1} &:= \{(x, y) \in X \times X \mid (y, x) \in U\}, \\ \Delta &:= \{(x, x) \mid x \in X\}. \end{aligned}$$

Definition 3.1. By a *uniformity* on X , we mean a non-empty collection \mathcal{K} of subsets of $X \times X$ which satisfies the following conditions:

- (U₁) $\Delta \subseteq U$ for any $U \in \mathcal{K}$,
- (U₂) if $U \in \mathcal{K}$, then $U^{-1} \in \mathcal{K}$,
- (U₃) if $U \in \mathcal{K}$, then there exists a $V \in \mathcal{K}$ such that $V \circ V \subseteq U$,
- (U₄) if $U, V \in \mathcal{K}$, then $U \cap V \in \mathcal{K}$,
- (U₅) if $U \in \mathcal{K}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{K}$.

The pair (X, \mathcal{K}) is called a *uniform structure*.

Theorem 3.2. *Let A be a BCC-ideal of a BCC-algebra X . If we define*

$$U_A := \{(x, y) \in X \times X \mid x * y \in A \text{ and } y * x \in A\}$$

and let

$$\mathcal{K}^* := \{U_A \mid A \text{ is a BCC-ideal of } X\}.$$

Then \mathcal{K}^ satisfies the conditions (U₁)~(U₄).*

Proof. (U₁): If $(x, x) \in \Delta$, then $(x, x) \in U_A$ since $x * x = 0 \in A$. Hence $\Delta \subseteq U_A$ for any $U_A \in \mathcal{K}^*$.

(U₂): For any $U_A \in \mathcal{K}^*$,

$$\begin{aligned} (x, y) \in U_A &\Leftrightarrow x * y \in A \text{ and } y * x \in A \\ &\Leftrightarrow y \sim_A x \\ &\Leftrightarrow (y, x) \in U_A \\ &\Leftrightarrow (x, y) \in U_A^{-1}. \end{aligned}$$

Hence $U_A^{-1} = U_A \in \mathcal{K}^*$.

(U₃): For any $U_A \in \mathcal{K}^*$, the transitivity of \sim_A implies that $U_A \circ U_A \subseteq U_A$.

(U₄): For any U_I and U_J in \mathcal{K}^* , we claim that $U_I \cap U_J \in \mathcal{K}^*$.

$$\begin{aligned} (x, y) \in U_I \cap U_J &\Leftrightarrow (x, y) \in U_I \text{ and } (x, y) \in U_J \\ &\Leftrightarrow x * y, y * x \in I \cap J \\ &\Leftrightarrow x \sim_{I \cap J} y \\ &\Leftrightarrow (x, y) \in U_{I \cap J}. \end{aligned}$$

Since $I \cap J$ is a BCC-ideal of X , $U_I \cap U_J = U_{I \cap J} \in \mathcal{K}^*$. This proves the theorem. \square

Theorem 3.3. *Let $\mathcal{K} := \{U \subseteq X \times X \mid U_A \subseteq U \text{ for some } U_A \in \mathcal{K}^*\}$. Then \mathcal{K} satisfies the conditions for a uniformity on X and hence the pair (X, \mathcal{K}) is a uniform structure.*

Proof. By Theorem 3.2, the collection \mathcal{K} satisfies the conditions (U₁)~(U₄). It suffices to show that \mathcal{K} satisfies (U₅). Let $U \in \mathcal{K}$ and $U \subseteq V \subseteq X \times X$. Then there exists a $U_A \subseteq U \subseteq V$, which means that $V \in \mathcal{K}$. This proves the theorem. \square

Let $x \in X$ and $U \in \mathcal{K}$. Define

$$U[x] := \{y \in X \mid (x, y) \in U\}.$$

Theorem 3.4. *Let X be a BCC-algebra. Then*

$$\mathcal{T} := \{G \subseteq X \mid \forall x \in G, \exists U \in \mathcal{K}, U[x] \subseteq G\}$$

is a topology on X .

Proof. It is clear that \emptyset and the set X belong to \mathcal{T} . Also from the definition, it is clear that \mathcal{T} is closed under arbitrary unions. Finally to show that \mathcal{T} is closed under finite intersection, let $G, H \in \mathcal{T}$ and suppose $x \in G \cap H$. Then there exist U and $V \in \mathcal{K}$ such that $U[x] \subseteq G$ and $V[x] \subseteq H$. Let $W := U \cap V$. Then $W \in \mathcal{K}$. Also $W[x] \subseteq U[x] \cap V[x]$ and so $W[x] \subseteq G \cap H$. Therefore $G \cap H \in \mathcal{T}$. Thus \mathcal{T} is a topology on X . \square

Note that for any $x \in X$, $U[x]$ is an open neighborhood of x .

Definition 3.5. Let (X, \mathcal{K}) be a uniform structure. Then the topology \mathcal{T} is called the *uniform topology* on X induced by \mathcal{K} .

Proposition 3.6. *Topological space (X, \mathcal{T}) is completely regular.*

Proof. See [11]. □

Example 3.7. Let $X := \{0, 1, 2, 3\}$ be a BCC-algebra which is not a BCK-algebra with the following table ([4]):

*	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	1
3	3	3	3	0

Then it is easy to show that $A := \{0, 1, 2\}$, $\{0\}$, and X are the only BCC-ideals of X . We can see that $U_{\{0\}} = \Delta$, $U_A = \Delta \cup \{(0, 1), (1, 0), (0, 2), (2, 0), (1, 2), (2, 1)\}$ and $U_X = X \times X$. Therefore $\mathcal{K}^* = \{U_{\{0\}}, U_A, U_X\}$ and $\mathcal{K} = \{U \subseteq X \times X \mid U_A \subseteq U \text{ for some } U_A \in \mathcal{K}^*\}$. If we take $U := U_A$, then $U[0] = U[1] = U[2] = \{0, 1, 2\}$ and $U[3] = \{3\}$. Therefore $\mathcal{T} = \{G \subseteq X \mid \forall x \in G, \exists U \in \mathcal{K}, U[x] \subseteq G\} \supseteq \{X, \emptyset, \{3\}, \{0, 1, 2\}\}$. Since $\{X, \emptyset, \{3\}, \{0, 1, 2\}\}$ is a topology on X , the topology \mathcal{T} on X induced by the BCC-ideal $A = \{0, 1, 2\}$ relative to U_A is a finer topology than $\{X, \emptyset, \{3\}, \{0, 1, 2\}\}$. Let $A := \{0\}$. Then $U_A = \Delta$. If we take $U := U_A$, then $U[x] = \{x\}, \forall x \in X$ and we obtain $\mathcal{T} = 2^X$, the discrete topology. Moreover, if we take X as a BCC-ideal of X , then $U[x] = X, \forall x \in X$ and obtain $\mathcal{T} = \{\emptyset, X\}$, the indiscrete topology.

4. Topological property of space (X, \mathcal{T})

Let X be a BCC-algebra and C, D be subsets of X . We define a set $C * D$ as follows:

$$C * D := \{x * y \mid x \in C, y \in D\}.$$

Let X be a BCC-algebra and \mathcal{T} be a topology on the set X . Then we say that the pair (X, \mathcal{T}) is a topological BCC-algebra if the operation “ $*$ ” is continuous with respect to \mathcal{T} . The continuity of the operation “ $*$ ” is equivalent to the following property:

(C): If O is an open set and $a, b \in X$ such that $a * b \in O$, then there exist open sets O_1 and O_2 such that $a \in O_1, b \in O_2$ and $O_1 * O_2 \subseteq O$.

Theorem 4.1. *The pair (X, \mathcal{T}) is a topological BCC-algebra.*

Proof. Let us first prove (C). Indeed, assume that $x * y \in G$, with $x, y \in X$ and G an open subset of X . Then there exist $U \in \mathcal{K}, U[x * y] \subseteq G$ and a BCC-ideal I of X such that $U_I \subseteq U$. We claim that the following relation holds:

$$U_I[x] * U_I[y] \subseteq U[x * y].$$

Indeed, for any $h \in U_I[x]$ and $k \in U_I[y]$ we have that $x \sim_I h$ and $y \sim_I k$. Since \sim_I is a congruence relation, it follows that $x * y \sim_I h * k$. From that fact we have $(x * y, h * k) \in U_I \subseteq U$. Hence $h * k \in U_I[x * y] \subseteq U[x * y]$. Then $h * k \in G$. Thus condition (C) is verified. □

Theorem 4.2 ([11]). *Let X be a set and $\mathcal{S} \subseteq \mathcal{P}(X \times X)$ be a family such that for every $U \in \mathcal{S}$ the following conditions hold:*

- (a) $\Delta \subseteq U$,
- (b) U^{-1} contains a member of \mathcal{S} ,
- (c) there exists an $V \in \mathcal{S}$ such that $V \circ V \subseteq U$.

Then there exists a unique uniformity \mathcal{U} , for which \mathcal{S} is a subbase.

Theorem 4.3. *If we set $\mathcal{B} := \{U_I | I \text{ is a BCC-ideal of a BCC-algebra } X\}$, then \mathcal{B} is a subbase for a uniformity of X . We denote its associated topology by \mathcal{S} .*

Proof. Since \sim_I is an equivalence relation, it is clear that \mathcal{B} satisfies the axioms of Theorem 4.2. \square

In Example 3.7, we can see that $\mathcal{B} = \{U_{\{0\}} = \Delta, U_A = \Delta \cup \{(0, 1), (1, 0), (0, 2), (2, 0), (1, 2), (2, 1)\}, U_X = X \times X\}$.

We say that the topology σ is *finer* than τ if $\tau \subseteq \sigma$ as subsets of the power set. Then we have the following:

Corollary 4.4. *The topology \mathcal{S} is finer than \mathcal{T} .*

Theorem 4.5. *Let Λ be an arbitrary family of BCC-ideals of a BCC-algebra X which is closed under intersection. Then any BCC-ideal is a clopen subset of X .*

Proof. Let I be a BCC-ideal of X in Λ and $y \in I^c$. Then $y \in U_I[y]$ and we obtain that $I^c \subseteq \cup\{U_I[y] | y \in I^c\}$. We claim that $U_I[y] \subseteq I^c$ for all $y \in I^c$. Let $z \in U_I[y]$, then $y \sim_I z$. Hence $y * z \in I$. If $z \in I$, then $y \in I$, since I is a BCC-ideal of X , which is a contradiction. So $z \in I^c$ and we obtain

$$\cup\{U_I[y] | y \in I^c\} \subseteq I^c.$$

Hence $I^c = \cup\{U_I[y] | y \in I^c\}$. Since $U_I[y]$ is open for any $y \in X$, I is a closed subset of X . We show that $I = \cup\{U_I[y] | y \in I\}$. If $y \in I$ then $y \in U_I[y]$ and hence $I \subseteq \cup\{U_I[y] | y \in I\}$. Given $y \in I$, if $z \in U_I[y]$, then $y \sim_I z$ and so $z * y \in I$. Since $y \in I$ and I is a BCC-ideal of X , we have $z \in I$. Hence we get that $\cup\{U_I[y] | y \in I\} \subseteq I$, i.e., I is also an open subset of X . \square

In Example 3.7, the BCC-ideals A , $\{0\}$, and X are clopen subsets of X .

Theorem 4.6. *For any $x \in X$ and $I \in \Lambda$, $U_I[x]$ is a clopen subset of a BCC-algebra X .*

Proof. We show that $(U_I[x])^c$ is open. If $y \in (U_I[x])^c$, then $x * y \in I^c$ or $y * x \in I^c$. We assume that $y * x \in I^c$. By applying Theorems 4.1 and 4.2, we obtain $(U_I[y] * U_I[x]) \subseteq U_I[y * x] \subseteq I^c$. We claim that $U_I[y] \subseteq (U_I[x])^c$. If $z \in U_I[y]$, then $z * x \in (U_I[z] * U_I[x])$. Hence $z * x \in I^c$ then we get $z \in (U_I[x])^c$, proving that $(U_I[x])^c$ is open. Hence $U_I[x]$ is closed. It is clear that $U_I[x]$ is open. So $U_I[x]$ is a clopen subset of X . \square

A topological space X is connected if and only if it has only X and \emptyset as clopen subsets of X . Therefore we have the following:

Corollary 4.7. *The space (X, \mathcal{T}) is not a connected space.*

We denote by \mathcal{T}_Λ the uniform topology by an arbitrary family Λ . Especially, if $\Lambda = \{I\}$, we denote it by \mathcal{T}_I .

Theorem 4.8. $\mathcal{T}_\Lambda = \mathcal{T}_J$, where $J = \cap\{I \mid I \in \Lambda\}$.

Proof. Let \mathcal{K} and \mathcal{K}^* be as Theorems 3.2 and 3.3, respectively. Now consider $\Lambda_0 = \{J\}$. Define $(\mathcal{K}_0)^* := \{U_J\}$ and $\mathcal{K}_0 := \{U \mid U_J \subseteq U\}$. Let $G \in \mathcal{T}_\Lambda$. Given an $x \in G$, there exists $U \in \mathcal{K}$ such that $U[x] \subseteq G$. From $J \subseteq I$, we obtain that $U_J \subseteq U_I$, for any BCC-ideal I of X . Since $U \in \mathcal{K}$, there exists $I \in \Lambda$ such that $U_I \subseteq U$. Hence $U_J[x] \subseteq U_I[x] \subseteq G$. Since $U_J \in \mathcal{K}_0$, $G \in \mathcal{T}_J$. Hence $\mathcal{T}_\Lambda \subseteq \mathcal{T}_J$.

Conversely, if $H \in \mathcal{T}_J$, then for any $x \in H$, there exists $U \in \mathcal{K}_0$ such that $U[x] \subseteq H$. So $U_J[x] \subseteq H$ and hence Λ is closed under intersection, $J \in \Lambda$. Then we get $U_J \in \mathcal{K}$ and so $H \in \mathcal{T}_\Lambda$. Thus $\mathcal{T}_J \subseteq \mathcal{T}_\Lambda$. \square

Theorem 4.9. *Let I and J be BCC-ideals of a BCC-algebra X and $I \subseteq J$. Then J is clopen in the topological space (X, \mathcal{T}_I) .*

Proof. Consider $\Lambda = \{I, J\}$. Then by Theorem 4.8, $\mathcal{T}_\Lambda = \mathcal{T}_I$ and therefore J is clopen in the topological space (X, \mathcal{T}_I) . \square

Theorem 4.10. *Let I and J be BCC-ideals of a BCC-algebra X . Then $\mathcal{T}_I \subseteq \mathcal{T}_J$ if $J \subseteq I$.*

Proof. Let $J \subseteq I$. Consider: $\Lambda_1 := \{I\}, \mathcal{K}_1^* := \{U_I\}, \mathcal{K}_1 := \{U \mid U_I \subseteq U\}$ and $\Lambda_2 := \{J\}, \mathcal{K}_2^* := \{U_J\}, \mathcal{K}_2 := \{U \mid U_J \subseteq U\}$. Let $G \in \mathcal{T}_I$. Then for any $x \in G$, there exists $U \in \mathcal{K}_1$ such that $U[x] \subseteq G$. Since $J \subseteq I$, we have $U_J \subseteq U_I$. $U_I[x] \subseteq G$ implies $U_J[x] \subseteq G$. This proves that $U_J \in \mathcal{K}_2$ and so $G \in \mathcal{T}_J$. Thus $\mathcal{T}_I \subseteq \mathcal{T}_J$. \square

Recall that a uniform space (X, \mathcal{K}) is said to be *totally bounded* if for each $U \in \mathcal{K}$, there exist $x_1, \dots, x_n \in X$ such that $X = \cup_{i=1}^n U[x_i]$, and (X, \mathcal{K}) is said to be *compact* if any open cover of X has its finite subcover.

Theorem 4.11. *Let I be a BCC-ideal of a BCC-algebra X . Then the following conditions are equivalent:*

- (1) *the topological space (X, \mathcal{T}_I) is compact,*
- (2) *the topological space (X, \mathcal{T}_I) is totally bounded,*
- (3) *there exists $P = \{x_1, \dots, x_n\} \subseteq X$ such that for all $a \in X$ there exist $x_i \in P$ ($i = 1, \dots, n$) with $a * x_i \in I$ and $x_i * a \in I$.*

Proof. (1) \Rightarrow (2): It is clear by [11].

(2) \Rightarrow (3): Let $U_I \in \mathcal{K}$. Since (X, \mathcal{T}_I) is totally bounded, there exist $x_1, \dots, x_n \in I$ such that $X = \cup_{i=1}^n U_I[x_i]$. If $a \in X$, then there exists x_i such that $a \in U_I[x_i]$, therefore $a * x_i \in I$ and $x_i * a \in I$.

(3) \Rightarrow (1): For any $a \in X$, by hypothesis, there exists $x_i \in P$ with $a_i * x_i \in I$ and $x_i * a \in I$. Hence $a \in U_I[x_i]$. Thus $X = \cup_{i=1}^n U_I[x_i]$. Now let $X = \cup_{\alpha \in \Omega} O_\alpha$, where each O_α is an open set of X . Then for any $x_i \in X$ there exists $\alpha_i \in \Omega$ such that $x_i \in O_{\alpha_i}$, since O_{α_i} is an open set. Hence $U_I[x_i] \subseteq O_{\alpha_i}$. Hence $X = \cup_{i=1}^n U_I[x_i] \subseteq \cup_{i=1}^n O_{\alpha_i}$, i.e., $X = \cup_{i=1}^n O_{\alpha_i}$, which means that (X, \mathcal{T}_I) is compact. \square

Theorem 4.12. *If I is a BCC-ideal of a BCC-algebra X , then $U_I[x]$ is a compact set in a topological space (X, \mathcal{T}_I) , for any $x \in X$.*

Proof. Let $U_I[x] \subseteq \cup_{\alpha \in \Omega} O_\alpha$, where each O_α is an open set of X . Since $x \in U_I[x]$, then there exists $\alpha \in \Omega$ such that $x \in O_\alpha$. Hence $U_I[x] \subseteq O_\alpha$, proving that $U_I[x]$ is compact. \square

Definition 4.13. A topological BCC-algebra X is said to be *discrete* if every element admits a neighborhood consisting of that element only.

Proposition 4.14. *If $\{0\}$ is an open set in a topological BCC-algebra X , then X is discrete.*

Proof. Since $x * x = 0 \in \{0\}$ for any $x \in X$ and $\{0\}$ is open, there exist neighborhoods U and V of x such that $U * V = \{0\}$. Let $W := U \cap V$. Then $W * W \subseteq U * V = \{0\}$ and so $W * W = \{0\}$. It follows from $x \in W$ that $W = \{x\}$, which means that X is discrete. \square

Proposition 4.15. *Let X be a topological BCC-algebra. Then $\{0\}$ is closed in X if and only if X is Hausdorff.*

Proof. Assume that $\{0\}$ is closed and let $x, y \in X$ with $x \neq y$. Then either $x * y \neq 0$ or $y * x \neq 0$. We may assume that $x * y \neq 0$ without loss of generality. Then there exist neighborhoods U and V of x and y respectively such that $U * V \subseteq X - \{0\}$. It follows that $U \cap V = \emptyset$ and hence X is Hausdorff.

Conversely, let X be Hausdorff. We claim that $X - \{0\}$ is open. If $x \in X - \{0\}$, then $x \neq 0$ and so there exist disjoint neighborhoods U and V of x and 0 respectively. Therefore $0 \notin U$ and hence $U \subseteq X - \{0\}$, which means that $X - \{0\}$ is open. This completes the proof. \square

Proposition 4.16. *Let A be a BCC-ideal of a topological BCC-algebra X . If 0 is an interior point of A , then A is open.*

Proof. Let $x \in A$. Since $x * x = 0 \in A$ and 0 is an interior point of A , there exists a neighborhood U of 0 which is contained in A . Then there exist neighborhoods G and H of x such that $G * H \subseteq U \subseteq A$. On the other hand for every $y \in G$, $y * x \in G * H \subseteq A$. Since A is a BCC-ideal and $x \in A$, it follows that $y \in A$ so that $x \in G \subseteq A$. Hence A is open, proving the proof. \square

Proposition 4.17. *Let X be a topological BCC-algebra. If A is an open set in X which is also a BCC-ideal of X , then it is a closed set in X .*

Proof. Let A be a BCC-ideal which is an open set in X and let $x \in X - A$. Then there exists a neighborhood U of x such that $U * U \subseteq A$, since $x * x = 0 \in A$ and A is open. We claim that $U \subseteq X - A$. If $U \not\subseteq X - A$, then there exists $y \in U \cap A$. Note that $z * y \in U * U \subseteq A$ for all $z \in U$. Since $y \in A$ and A is a BCC-ideal, it follows that $z \in A$ which shows that $U \subseteq A$, a contradiction. Hence A is closed. \square

Proposition 4.18. *Let X be a topological BCC-algebra and $\{0\}$ be closed. Then $\bigcap \mathfrak{N}_0 = \{0\}$ where \mathfrak{N}_0 is the neighborhood system of 0 .*

Proof. Since $\{0\}$ is closed, by Proposition 4.15 X is Hausdorff. Given an element $x \in X \setminus \{0\}$, 0 has a neighborhood U such that $x \notin U$ and so $x \notin \bigcap \mathfrak{N}_0$. Hence $\bigcap \mathfrak{N}_0 = \{0\}$. \square

Let X be a BCC-algebra. For an arbitrary element $a \in X$ and any non-empty subset V of X , denote

$$V(a) := \{x \in X \mid x * a \in V \text{ and } a * x \in V\}.$$

Note that $V(a) \subseteq U(a)$ whenever $V \subseteq U \subseteq X$.

Theorem 4.19. *Let Ω be a filter base on a BCC-algebra X such that for every $p, q \in V \in \Omega$*

- (1) $0 * p \in V$,
- (2) $(x * p) * q = 0$ implies $x \in V$.

Then $\mathcal{T} := \{O \subseteq X \mid \forall a \in O, \exists V \in \Omega : V(a) \subseteq O\}$ is a topology on X and Ω is a local base at 0 .

Proof. Let $\mathcal{T} := \{O \subseteq X \mid \forall a \in O, \exists V \in \Omega : V(a) \subseteq O\}$. Clearly, $\emptyset, X \in \mathcal{T}$. Let $\{O_\alpha\}$ be a family of members of \mathcal{T} and let $a \in \bigcup O_\alpha$. Then $a \in O_\alpha$ for some α . It follows that there exists $V \in \Omega$ such that $V(a) \subseteq O_\alpha \subseteq \bigcup O_\alpha$ so that $\bigcup O_\alpha \in \mathcal{T}$. Assume that O_α and O_β belong to \mathcal{T} and let $a \in O_\alpha \cap O_\beta$. Then there exist $V_\alpha \in \Omega$ and $V_\beta \in \Omega$ such that $V_\alpha(a) \subseteq O_\alpha$ and $V_\beta(a) \subseteq O_\beta$, respectively. Since Ω is a filter base, there exists $V \in \Omega$ such that $V \subseteq V_\alpha \cap V_\beta$. Thus we have

$$V(a) \subseteq (V_\alpha \cap V_\beta)(a) \subseteq V_\alpha(a) \cap V_\beta(a) \subseteq O_\alpha \cap O_\beta$$

and so $O_\alpha \cap O_\beta \in \mathcal{T}$. This proves that \mathcal{T} is a topology on X (In this case we call it a *topology induced by Ω* , and is denoted by \mathcal{T}_Ω).

Now we will show that Ω is the filter base of a neighborhood of 0 with respect to the topology \mathcal{T} . Let $p \in V \in \Omega$. Then $0 * p \in V$ by (i), and since $(0 * p) * (0 * p) = 0$ it follows from (ii) that $0 \in V$, i.e., every element $V \in \Omega$ contains 0 . If $x \in V(p)$ then $x * p, p * x \in V$ and so $x * p = v$ for some $v \in V$. Hence $(x * p) * v = 0$ which implies that $x \in V$. Therefore $V(p) \subseteq V$ and $v \in \mathcal{T}$. Thus V is a neighborhood of 0 . If we let V be a neighborhood of 0 , then there exists a $U \in \Omega$ such that $U(0) \subseteq V$. Note that $0 * a \in U$ and $a * 0 \in U$ for some $a \in U$. Hence $a \in U(0)$ and $0 \in U \subseteq U(0) \subseteq V$. Thus Ω is a local base at 0 with respect to the topology \mathcal{T} . \square

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