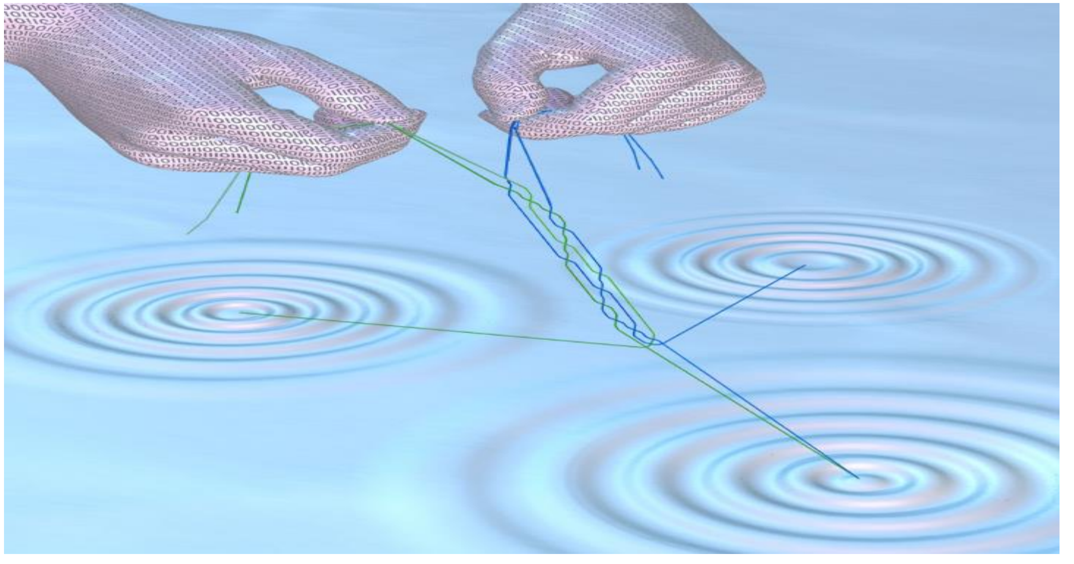
# Topological Quantum Computation

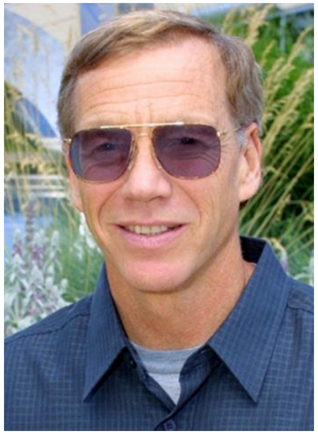



**Zhenghan Wang**
**Microsoft Station Q & UC Sana Barbara**


**Texas, March 26, 2015**

*P/NP*, and the quantum field computer

MICHAEL H. FREEDMAN



Classical Physics — Turing Model

Quantum Mechanics — Quantum Computing

Quantum Field Theory — ???

String Theory — ??????

Fault-tolerant quantum computation by anyons

A.Yu. Kitaev

**Quantum field computing is the same as quantum computing.**

True for TQFTs
(Freedman, Kitaev, Larsen, W.)

# Quantum Computation

- There is a serious prospect for quantum physics to change the face of information science.
- Theoretically, the story is quite compelling:
  - Shor's factoring algorithm (1994)
  - Fault tolerance ~1996-1997 independently
    - **P. Shor**
    - **A. M. Steane**
    - **A. Kitaev**
- But for the last twenty years the most interesting progress has been to build a quantum computer.

# Why Quantum More Powerful?

- Superposition

A (classical) **bit** is given by a physical system that can exist in one of two distinct states:
0 or 1

A **qubit** is given by a physical system that can exist in a linear combination of two distinct quantum states: $|0\rangle$ or $|1\rangle$

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

$\alpha, \beta \in \mathbb{C}$
$|\alpha|^2 + |\beta|^2 = 1$ $\qquad |\psi> \in CP^1$

- Entanglement

Quantum states need not be products. For example:

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right)$$
$$\neq |\psi_A\rangle \otimes |\phi_B\rangle$$

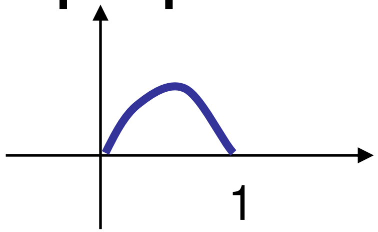
This is the property that enables quantum state teleportation and Einstein's "spooky action at a distance."

- Classical information source is modeled by a random variable X

The bit---a random variable $X \in \{0,1\}$ with equal probability.

Physically, it is a switch

$$I_X(p) = -\sum_{i=1}^{n} p_i \log_2 p_i \,,$$




- A state of a quantum system is an information source

The qubit---a quantum system whose states given by non-zero vectors in $\mathbb{C}^2$ up to non-zero scalars.

Physically, it is a 2-level quantum system.

Paradox: A qubit contains both more and less than 1 bit of information.

The average amount information of a qubit is $\frac{1}{2ln2}$.

A computing problem is given by a family of Boolean maps $\{0,1\}^n \longrightarrow \{0,1\}^{m(n)}$

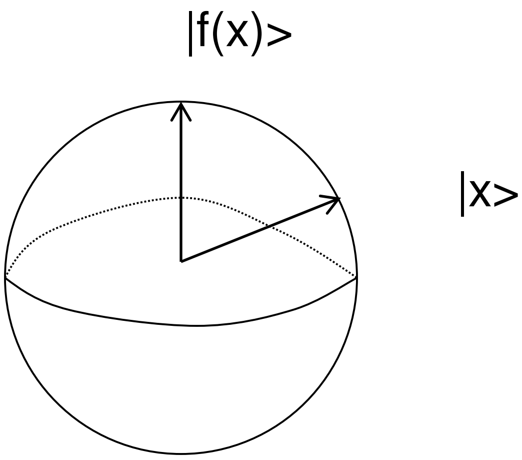
Name: Factoring
Instance: an integer N>0
Question: Find the largest prime factor of N

Encode N as a bit string of length=n~ $\log_2$ N,
the factoring problem is a family of Boolean functions
$f_n: \{0,1\}^n \longrightarrow \{0,1\}^{m(n)}$

e.g. n=4, $f_4(1111) = 101$, i.e., $f_4(15) = 5$

# How Quantum Computers Work

**Given a Boolean map f: $\{0,1\}^n \longrightarrow \{0,1\}^n$, for any $x \in \{0,1\}^n$, represent x as a basis $|x\rangle \in (\mathbb{C}^2)^{\otimes n}$, then find a unitary matrix U so that $U(|x\rangle) = |f(x)\rangle$.**




**Basis of $(\mathbb{C}^2)^{\otimes n}$ is in1-1 correspondence with n-bit strings or $0,1,\ldots,2^n-1$**

# Problems

- **x, f(x) does not have same # of bits**
- **f(x) is not reversible**
- **The final state is a linear combination**
- …
- **Not every $U_x$ is physically possible**

# Gate Set

Fix a collection of unitary matrices (called gates) and use only compositions of local unitaries from this gate set

e.g. standard gate set

H=$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$---Hadmard,   T=$\begin{pmatrix}1 & 0\\ 0 & e^{\pi i/4}\end{pmatrix}$---$\frac{\pi}{8}$-gate

CNOT=$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}$, $|i, j >\rightarrow |i, (\mathrm{i}+\mathrm{j}) \bmod 2 >, \mathrm{i}, \mathrm{j} = 0 \text{ or } 1$

# Universality

- **Fix a gate set S, a quantum circuit on n-qubits $(\mathbb{C}^2)^{\otimes n}$ is a composition of finitely many matrices $g_i$, where each $g_i$ is of the form $id \otimes g \otimes id$, where each $g \in S$ is a gate.**
- **Universality: A gate set S is universal if the collection of all quantum circuits form a dense subset of the union $\bigcup_{n=1}^{\infty} PSU(2^n)$.**

**The class BQP (bounded error quantum polynomial-time)**
**Fix a physical universal gate set**

**A computing problem $f_n: \{0,1\}^n \rightarrow \{0,1\}^{m(n)}$ is in BQP if**

**1) there exists a classical algorithm of time poly (n) (i.e. a Turing machine) that computes a function $x \rightarrow D_x$, where $x \in \{0,1\}^n$, and $D_x$ encodes a poly(n)-qubit circuit $U_x$.**

**2) when the state $U_x|0\cdots 0\rangle$ is measured in the standard basis $\{|i_1\cdots i_{p(n)}\rangle\}$, the probability to observe the value $f_n(x)$ for any $x \in\{0,1\}^n$ is at least ¾.**

**Remarks:**

1) Any function that can be computed by a QC can be computed by a TM.

2) Any function can be efficiently computed by a TM can be computed efficiently by a QC, i.e. BPP $\subseteq$BQP

Factoring is in **BQP** (Shor's algorithm), but not known in **FP** (although **Primality** is in P).

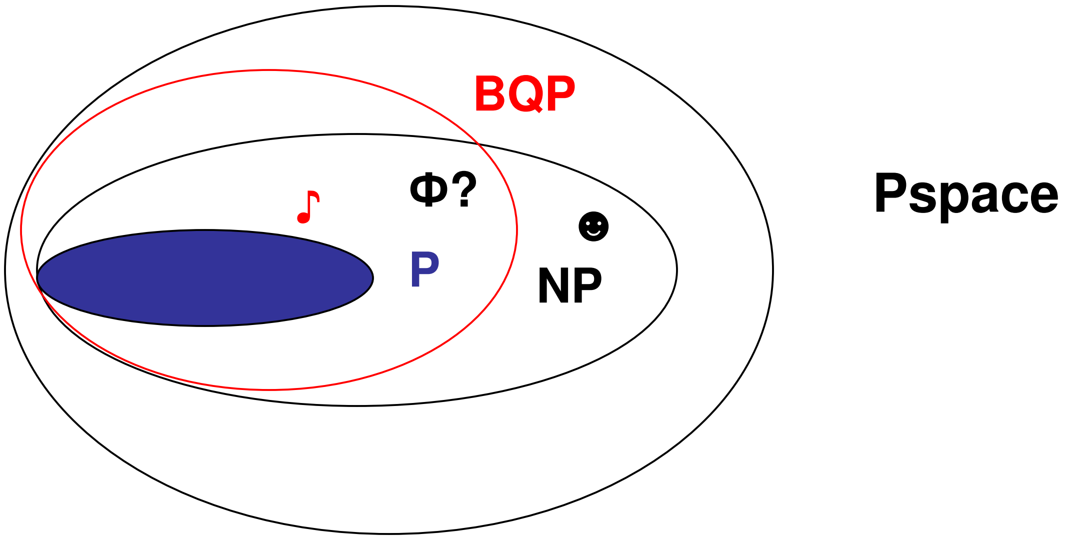
Given an n bit integer $N \sim 2^n$

Classically $\sim e^{c\, n^{1/3}\, \mathrm{poly}(\log n)}$

Quantum mechanically $\sim n^2$ poly (log n)

For $N=2^{500}$, classically ~ billion years

Quantum computer ~ few days



## Can We Build a Large Scale Universal QC?

Yes theoretically. Fault-tolerant quantum computation theory shows if hardware can be built up to the accuracy threshold $\sim 10^{-4}$, then a scalable QC can be built.

But in reality, the obstacle is mistakes and errors (decoherence)

Classical error correction by redundancy

$$0 \to 000,\ 1 \to 111$$

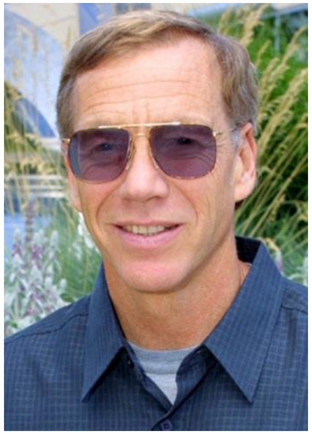
Not available due to the No-cloning theorem:

The cloning map $|\psi\rangle \otimes |0\rangle \to |\psi\rangle \otimes |\psi\rangle$ is not linear.

# Key “Post-Shor” Idea

Peter Shor
Shor’s Factoring Algorithm

To use topology to protect quantum information

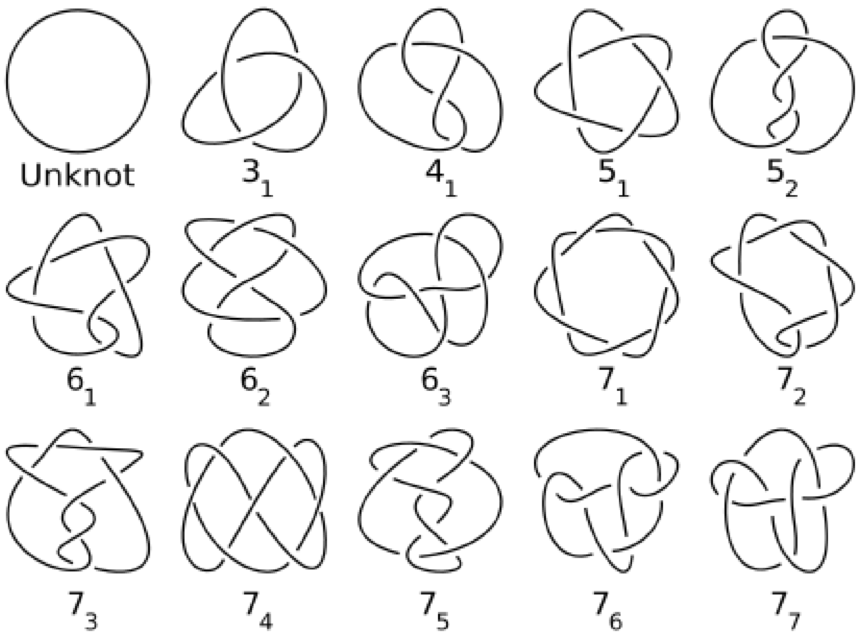


Michael Freedman

Alexei Kitaev

# Why Topology?

- **Topology** is usually conceived of as that part of geometry which survives deformation.




- But, equally, **topology** is that part of quantum physics which is robust to deformation (error).

# A Revolutionary New Idea

If a physical system were to have quantum *topological (necessarily nonlocal) degrees of freedom,* which were *insensitive to local probes*, then information contained in them would be *automatically protected against errors* caused by *local interactions with the environment*.

This would be fault tolerance guaranteed by physics at the hardware level, with no further need for quantum error correction, i.e. topological protection.

**Alexei Kitaev**

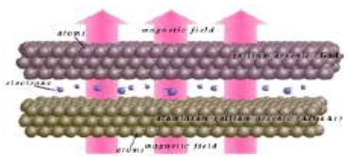
# Topological Phases of Matter

A topological quantum phase is represented by a quantum theory whose low energy physics in the thermodynamic limit is modeled by a stable unitary topological quantum field theory (TQFT).

## 2D Topological Phases in Nature

- Quantum Hall States

  1980 Integral Quantum Hall Effect --- von Klitzing (1985 Nobel)

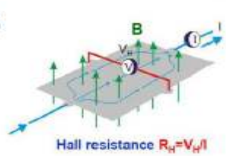


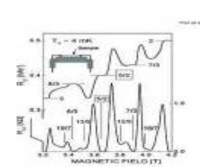
  1982 Fractional QHE---Stormer, Tsui, Gossard at $\nu = \frac{1}{3}$

  (1998 Nobel for Stormer, Tsui, and Laughlin)

  1987 Non-abelian FQHE???---R. Willett et al at $\nu = \frac{5}{2}$

- Topological superconductors
- Topological insulators
- …



$R_H = \nu^{-1} \frac{h}{e^2}$

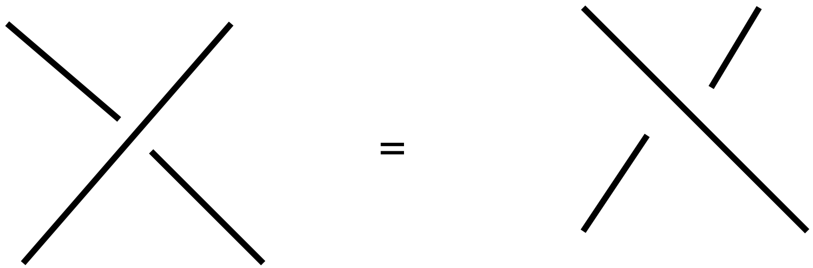
# Anyons

- **Anyons:** quasi-particles or topological excitations in topological phases of matter. Their statistics are more general than bosons/fermions, can be even non-abelian---$k \times k$ matrices.
- **Models:** Simple objects in unitary modular tensor categories.

# Statistics of Particles

In $R^3$, particles are either bosons or fermions

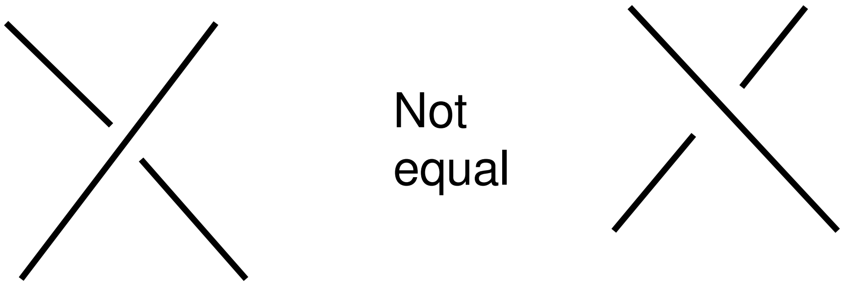
Worldlines (curves in $R^3 x R$) exchanging two identical particles depend only on permutations



**Statistics is $\lambda: S_n \rightarrow Z_2$**

# Braid Statistics

In $R^2$, an exchange is of infinite order




**Braids form groups $B_n$**

**Statistics is $\lambda: B_n \rightarrow U(k)$**

**If k>1, non-abelian anyons**

# Non-abelian Statistics

**If the ground state is not unique, and has a basis $\psi_1, \psi_2, \ldots, \psi_k$**

**Then after braiding some particles:**

$$\psi_1 \longrightarrow a_{11}\psi_1 + a_{12}\psi_2 + \ldots + a_{k1}\psi_k$$

$$\psi_2 \longrightarrow a_{12}\psi_1 + a_{22}\psi_2 + \ldots + a_{k2}\psi_k$$

.......

$$\lambda : B_n \longrightarrow U(k)$$

**Do non-abelian anyons exist?**

# The search for Majoranas or the Jones rep of the braid group at 4$^{th}$ root of unity in the real world

Of all exotic excitations believed to exist in topological quantum physics, we are the closest to detecting and harnessing Majoranas.

**perspective**

## Majorana returns

F. Wilczek, Nature Physics'09

Physics Today / Volume 64 / Issue 3 / SEARCH AND DISCOVERY

**Physics Today - March 2011**

**The expanding search for Majorana particles**

Barbara Goss Levi

Science, April (2011)

**NEWS**

## Search for Majorana Fermions Nearing Success at Last?

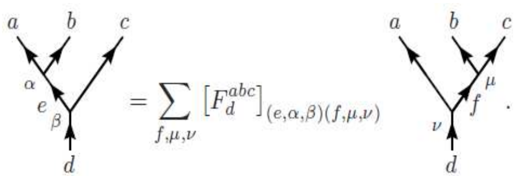
Researchers think they are on the verge of discovering weird new particles that borrow a trick from superconductors and could give a big boost to quantum computers

# Anyon Models

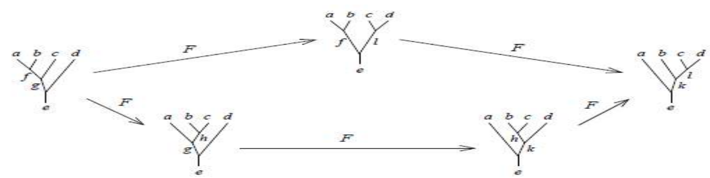
- **Label set:** a finite set L={a,b,c,…} of anyon types or labels with an involution and a trivial type. E.g. any finite group G.
- **Fusion rules:** $\{{N_{ab}}^{c}$, a,b,c∈L}. The fusion rules determine when two anyons of types a,b are fused, whether or not anyons of type c appear, i.e. if ${N_{ab}}^{c}$ is ≥ 1 or =0.
- The Frobenius-Perron eigenvalue of the matrix $N_a$ is the quantum dimension of a.
- Others

# Anyon Model $\mathcal{C}$ = UMTC

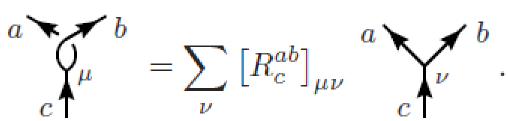
**A modular tensor category = a non-degenerate braided spherical fusion category: a collection of numbers {L, $N_{ab}^{c}$, $F_{d;nm}^{abc}$, $R_{c}^{ab}$} that satisfy some polynomial constraint equations.**



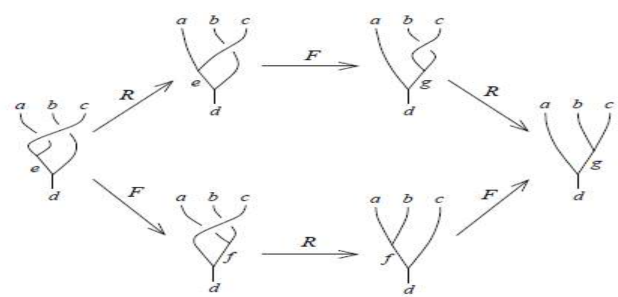

6j symbols for recoupling



Pentagons for 6j symbols



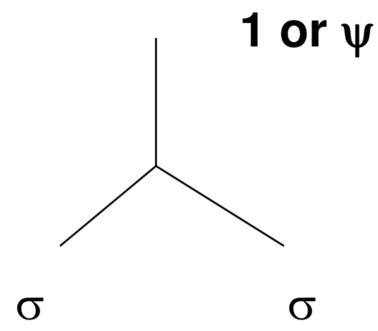

R-symbol for braiding



Hexagons for R-symbols

# Ising Theory

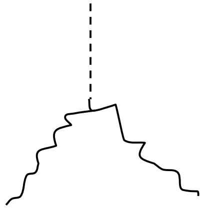
**Ising=M(3,4) minimal model**
**=TL at 4th root**



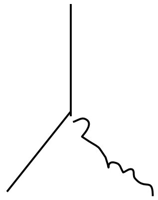

**Particle types: {1, σ, ψ}**

**q-dimensions: $\{1, \sqrt{2}, 1\}$**

**Fusion rules:**



$$\sigma^2 \cong 1+\psi,$$
$$\psi^2 \cong 1,$$
$$\sigma\ \psi \cong \psi\ \sigma \cong \sigma$$

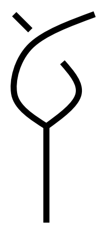


1---ground state
ψ---Majarona fermion
σ---Ising anyon

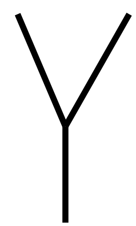
# Fibonacci Theory

## $G_2$ level=1 CFT, c=14/5 mod 8

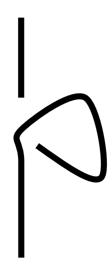
- Particle types: $\{1,\tau\}$, $\tau$---Fib anyon
- Quantum dimensions: $\{1,\phi\}$, $\phi$=golden ratio
- Fusion rules: $\tau^2=1\oplus\tau$
- Braiding:



$=e^{3\pi i/5}$



- Twist:

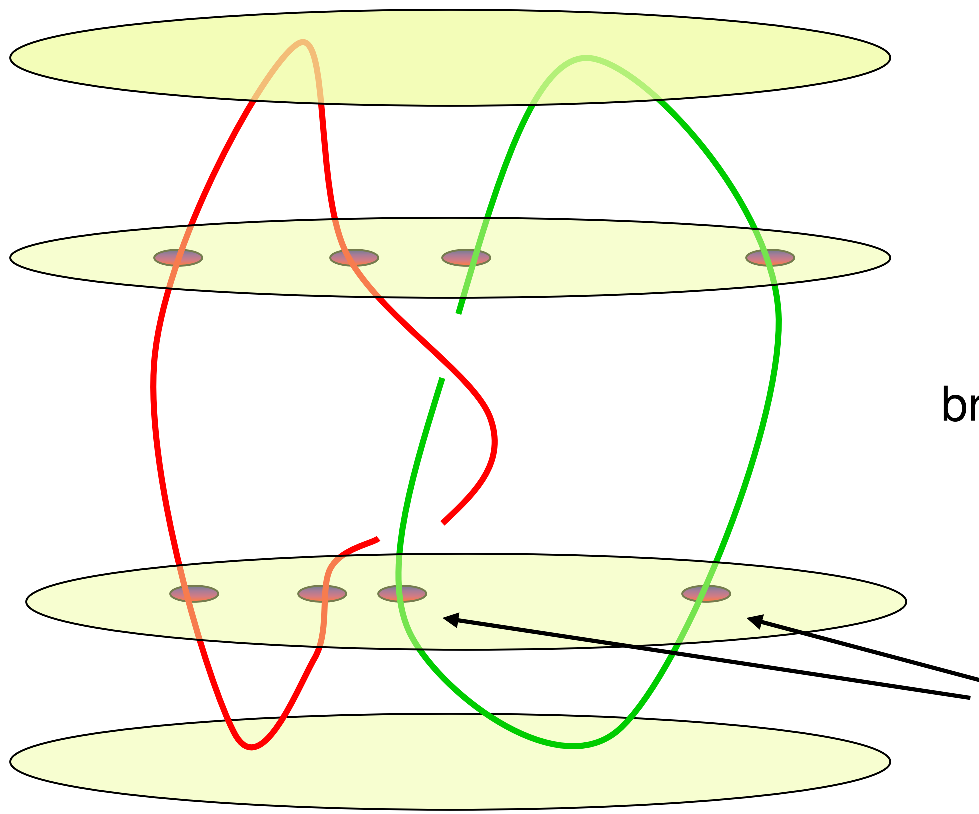


$=e^{4\pi i/5}$

# Topological Quantum Computation

**Freedman 97, Kitaev 97, FKW 00, FLW 00**

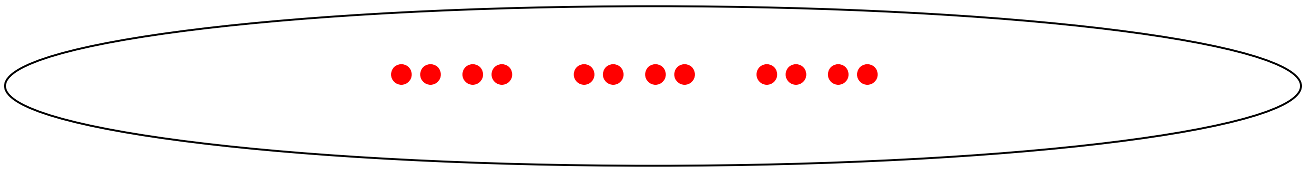
| <u>**Computation**</u> | <u>**Physics**</u> |
|---|---|
| readout | fusion |
| applying gates | braiding particles |
| initialize | create anyons |



# Anyonic Quantum Computer

**For n qubits, consider the 4n anyons**

**$\rho$: $B_{4n} \longrightarrow U(N_{4n})$, e.g. $N_{4n}$---4n-2 Fib number**



**Given a quantum circuit on n qubits**

$$U_L: (C^2)^{\otimes n} \longrightarrow (C^2)^{\otimes n}$$

**Topological compiling: find a braid $b \in B_{4n}$ so that the following commutes for any $U_L$:**

$$\begin{tikzcd}
(C^2)^{\otimes n} \arrow[r] \arrow[d, "U_L"'] & V_{4n} \arrow[d, "\rho(b)"] \\
(C^2)^{\otimes n} \arrow[r] & V_{4n}
\end{tikzcd}$$

**$V_{4n}$-gs of 4n anyons**

# Mathematical Theorems

Theorem 1 (Freedman-Kitaev-W.): Any unitary (2+1)-TQFT can be efficiently simulated by the quantum circuit model.

There are efficient additive approximation algorithms of quantum invariants by the quantum circuit model.

Theorem 2 (Freedman-Larsen-W.): Anyonic quantum computers based on RT/WCS $SU(2)$-TQFTs at level $k$ are braiding universal except $k = 1, 2, 4$.

The approximation of Jones poly of links at the $(k+2)^{\text{th}}$ root of unity ($k \neq 1, 2, 4$) is a BQP-complete problem.

Theorem 3 (Cui-W., Levaillant-Bauer-Bonderson-Freedman-W.): Anyonic model based on $SU(2)$ at level $k = 4$ is universal for quantum computation if braidings are supplemented with (projective or interferometric) measurements in the middle of computation.

# Universality of Braiding Gates

In 1981, Jones proved that the images of his unitary representation $\rho_{r,a}(B_n)$ of the braid groups are infinite

(same as anyon statistics in Reshetikhin-Turaev/Witten-Chern-Simons $SU(2)_k$-TQFTs for k=r-2)

**if r ≠ 1,2,3,4,6, n≥3 or r = 10, n≠4**,
and asked:

What are the closed images of $\rho_{r,a}(B_n)$ in the unitary groups?

Density Theorem (FLW):

Always contain SU($N_{r,a}$) if r ≠ 1, 2, 3, 4, 6 and n≥3 or r=10, also n≠4.

Others are finite groups which can be identified.

Proof is a general solution of 2-eigenvalue problem and generalized to 3-eigenvalue by Larsen-Rowell-W.

# Computational Power of Braiding Gates

- Ising anyon $\sigma$ does not lead to universal braiding gates, but Fib anyon $\tau$ does

- Quantum dimension of Ising anyon $\sigma$ has quantum dimension=$\sqrt{2}$, while Fib anyon $\tau$ has quantum dimension $\phi=(\sqrt{5}+1)/2$---golden ratio

- Given an anyon type x, when does it lead to universal braiding gate sets?

**Rowell conjecture:** **Braid universal iff $d_x^2 \neq integer$**

**Quantum Mathematics:**

**Quantization and Categorification**

**TQFT and Higher Category**