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Topological Quantum Field Theories, Moduli Spaces and Flat Gauge Connections

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ABSTRACT

We show how to construct a topological quantum field theory which corresponds to a given moduli space. We apply this method to the case of flat gauge connections defined over a Riemann surface and discuss its relations with the Chern-Simons theory and conformal field theory. The case of the SO(2,1) group is separately discussed. A topological field theory is linked to the moduli space of "self-dual" connections over Riemann surfaces. Another relation between the Chern-Simons theory and topological quantum field theory in three dimensions is established. We present the theory which corresponds to three dimensional gravity. Expressions for the Casson invariants are given. Possible generalizations are briefly discussed.

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1. Introduction

Generally covariant field theories have observables which are metric independent. Hence they are global invariants. Recently, a new class of such theories, the so called topological quantum field theories (TQFT), were introduced by E.Witten. Originally they were affiliated with Yang-Mills instantons (TYM),[1] sigma models (TSM)[2] , and gravity (TG)[3] . Later on they enveloped other domains of physical systems[4−7] . The main question is obviously whether the TQFT's probe some physical phenomena or are they merely mathematical tools to study topological properties of certain bundles? The answer to this question is two-fold: (i) The observables of the TQFT span the cohomology ring on certain moduli spaces. These moduli spaces may be intimately related to physics. An example familiar to string theorists is the moduli space of Riemann surfaces. Another example is the moduli space of instantons. (ii) The possibility that the TQFT's describe a generally covariant phase which eventually undergoes a spontaneous symmetry breakdown whereby ordinary gravity emerges[1,7]. This scenario must presumably be related to some new mechanism of symmetry breaking since it involves a passage from a system with finitely many degrees of freedom to one with infinitely many. Other conjectures like possible connections to strings above the Hagedorn temperature[8] , to the scattering of strings at large angles and very large energies[9] and to higher dimensional extended objects[1], were also proposed. In this work we follow the first direction.

The main feature of the TQFT's is the "topological symmetry" which is the largest local symmetry possible for the fields that describe the system. This symmetry is responsible for gauging away any dependence on local properties. The classical action does not play any role and can be taken to be zero or a topological number. The quantum Lagrangian is derived via BRST gauge fixing of the topological symmetry and related "ghost symmetries"[11][12] . The observables of the theory, which are expectation values mainly of the ghost fields, can be expressed as an integral of closed forms on some moduli space. Can one write down a TQFT which corresponds to any given moduli space? In this paper we present a general prescription for the building of such a TQFT.

An older member in the family of generally covariant field theories is the Chern-Simons (CS) action in three dimensions. Just as for the TQFT, here too the general covariance is not achieved by functionally integrating over all possible metrics, but by an inherent metric independence. The CS theory entered recently a renaissance period due to a sequence of works by E. Witten. In those works links were derived between the CS theory, knot theory and the Jones polynomials[13] , the theory of rational conformal field theories in two dimensions and the theory of Einstein gravity in three dimensions[14][15][6](for the CS of ISO(2,1) group). It was shown[13]that the moduli space of flat gauge connections (MSFC) in two dimension is the phase space of the three dimensional CS theory and is related to the space of "conformal blocks" of the corresponding conformal field theory. This naturally invites the construction of a TQFT for the MSFC. Applying the general procedure mentioned above to this particular moduli space, is a main topic of the present work. A BRST quantization of the CS action in the large k limit, does not lead to the MSFC but to a modified moduli space which involves the Yang-Mills ghosts[13][15]. It turns out that the same moduli space emerges in the TQFT of flat connections in three dimensions. In a special situation of this model, the partition function is a field theoretical description of the Casson invariants[15]. When the group is taken to be an IG† group, the TQFT is identical to the super-IG CS theory for this group.

Moore and Seiberg[16] conjectured that all chiral algebras of rational conformal field theories arise from the quantization of CS theories for some compact gauge groups, namely, the quantization of the associated MSFC. Can one describe other two dimensional field theories in terms of certain moduli spaces of gauge connections? One obvious generalization is to the non-compact groups. Another possibility is to describe moduli spaces of some connections which in some limits turn into flat connections. This may correspond to integrable two dimensional sys-

† For the definition of the IG group see section 5

tems which are conformally invariant at their critical points. An example of such a scenario is worked out in the case of the moduli space of "self dual" connections (MSSDC).

A simple discussion of the relevance of moduli spaces to physical systems is presented in section 2. We then show how to derive a TQFT with observables which correspond to a given moduli space. The TQFT is constructed via a gauge fixing of a local "topological symmetry" and a related "ghost symmetry". These ideas are demonstrated with some examples. Some general properties which are shared by most TQFT's are then summarized. Section 3 is devoted to a review of the relations between the three dimensional CS theory, the Moduli space of flat connections and conformal field theories. The equivalence of the ISO(2,1) CS theory and three dimensional gravity is also discussed. In section 4, we apply the procedure of section 2 and write down a TQFT which corresponds to the MSFC in two dimensions. The BRST algebra, and non-trivial global invariants are presented. We comment on the issue of the relation between the $SO(2,1)$ TQFT and the moduli space of Riemann surfaces. Finally, the generalization to the case of MSSDC is discussed. In section 5, we add one more dimension and discuss the differences between the TQFT's in two and three dimensions. The field theory formulation of the Casson invariants is written down. Using the group ISO(2,1) we get the TQFT which is associated with three dimensional gravity. The super-IG CS action for the group IG, is shown to be identical to the TQFT of the group IG. A generalization to higher space-time dimensions and to higher forms is briefly discussed at the end of that section. We summarize, make some concluding remarks and discuss some open questions in section 6 . Flat connections are the topic of the appendix where we present a short summary of the geometrical properties of the orbit space, write down a parametrization of flat connections and describe the MSFC as well as the MSSDC.

2. Topological Quantum Field Theories and Moduli Spaces

TQFT's such as the TYM[1], TSM[2], TG and others are all characterized by some observables- global invariants- which are related to the cohomology ring on certain moduli spaces. A natural question is whether for any given moduli space one can construct a corresponding TQFT. Here, we show how to construct TQFT's which correspond to some given moduli spaces.

The connection between moduli spaces and physical systems can be described simply in the following way. Assume that a physical system is defined by a set of fields Φ_i on a d dimensional space-time manifold M, and a certain local symmetry G under which the fields Φ_i transform in some representation of the group G . Mathematically, a certain bundle is defined over M. Very often we are interested not in the whole space of possible Φ_i configurations but in a particular subset which can be characterized by

$$\{\Phi_i^0 | F(\Phi_i^0) = 0\}, \tag{2.1}$$

where $F(\Phi_i)$ is a given functional of the fields Φ_i. The condition (2.1) can be, for example, the Euler-Lagrange equations of the action describing the physical system. We now perturb a given configuration in this subspace and demand that the perturbation does not take Φ_i out of this subspace, namely:

$$F(\Phi_i^0 + \delta\Phi_i) = 0 \qquad \rightarrow \qquad \frac{\delta F}{\delta \Phi_i}\delta\Phi_i = 0. \tag{2.2}$$

We want further to mod out from the possible variations, $\delta\Phi_i$, those redundent ones which are the transformation of Φ_i under G. We, therefore, choose a gauge slice by imposing a gauge condition

$$G_{GF}(\Phi_i^0 + \delta\Phi_i) = 0 \qquad \rightarrow \qquad \frac{\delta G_{GF}}{\delta \Phi_i}\delta\Phi_i = 0. \tag{2.3}$$

As for solutions to the equations (2.2-2.3) there are two possibilities (i) No non-trivial solutions, then the Φ_i^0 configurations are isolated. (ii) There are solutions and then these solutions span the moduli space, $\mathcal{M}$, of configurations fulfilling (2.1) modulo gauge transformations. For finite deformations we want to integrate $\delta\Phi_i$ but there may be obstructions[1,2] to the integration of the infinitesimal deformations. Regarding the solutions of (2.2-2.3) as the kernel of an operator $\bar{D}$ acting on $\delta\Phi_i$, then the obstructions are given by the cokernel of this operator. Therefore, the dimension of the moduli space is the number of solutions minus the number of obstructions which is :

$$dim\mathcal{M} = dim(Kernel\bar{D}) - dim(coKernel\bar{D}) = index\bar{D} \tag{2.4}$$

We demonstrate the statements made above in table 1† for the moduli spaces which are related to various physical systems: (i) Yang Mills instantons in four dimensions[1,12], (ii) World sheet instantons in two dimensions[2], (iii) two-torus[4], (iv) flat SO(2,1) connections[17] (which is equivalent to Riemann surfaces with $g > 1$)* and (v) (1,1) forms on Calabi-Yau manifolds.

† The notations in the table follow references:[1,12], [2,4], and[17]

∗ For more details see section 4.

Configuration	G-Symmetry	Conditions on $\delta\Phi_i$	Moduli Space
A_α : non-abelian gauge fields in four dim.	non-abelian gauge symmetry	$D_{[\alpha}\delta A_{\beta]} + \epsilon_{\alpha\beta\gamma\delta}D^{[\gamma}\delta A^{\delta]} = 0$ $D_\alpha \delta A^\alpha = 0$	Yang Mills instantons
x^i : coordinates on symplectic manifold	world-sheet reparametrization	$D_\alpha \delta x^i + \epsilon_{\alpha\beta}J^i_j D^\beta \delta x^j = 0$ J: complex structure	world sheet instantons
$g_{\alpha\beta}$: metric on torus	world sheet reparametrization	$\partial_z\partial_{\bar{z}}(g^{z\bar{z}}\delta g_{z\bar{z}}) = 0$ $g_{\bar{z}\bar{z}} = g_{zz} = 0$	torus
$(e_{\alpha a},\ \omega_\alpha)$: world sheet SO(2,1) connections	SO(2,1) gauge symmetry	$\partial_{[\alpha}\delta\omega_{\beta]} + \epsilon_{ab}e^a_{[\alpha}\delta e^b_{\beta]} = 0$ $(\tilde{D}_{[\alpha}\delta e_{\beta]})^a + \epsilon^{ab}e_{b[\alpha}\delta\omega_{\beta]} = 0$ $\tilde{D}^{ab}_\alpha = \delta^{ab}\partial_\alpha + \epsilon^{ab}\omega_\alpha$	Riemann surfaces of $g > 1$
$g_{i\bar{j}}$: metric on Kahler Manifold	diffeomorphism on Khaler manifold	$\partial_i\partial_{\bar{j}}(\frac{\delta g}{g}) = 0$ $g = det(g_{i\bar{j}})$	(1,1) forms Calabi-Yau Manifold

Table 1- *Examples of moduli spaces*

The basic idea of the use of TQFT to explore the moduli spaces is to formulate a field theory which is invariant under an additional local "topological symmetry"[10-12] of the form $\delta\Phi_i = \Theta_i(x)$ where Θ_i has the same properties as Φ_i under the Lorentz and G transformations but may differ in boundary conditions. The form of the original action is not important as long as it is invariant under the topological symmetry. In general, the Lagrangian is taken to be zero up to topological terms and up to eliminating auxiliary fields. In the case that the configurations, Φ^0_i, are characterized by a topological number which can be expressed as a d dimensional integral, it makes sense to take the later as the action. This will imply some boundary condition on the local parameter of the topological symmetry[12].

Quantization of the TQFT is performed by using the BRST method. $\epsilon\Psi_i$ is now replacing Θ_i where ϵ is an anti-commuting global parameter and Ψ_i is an anti-commuting ghost. The gauge-fixing and Faddev-Popov Lagrangians are derived by BRST variation of of a "gauge condition" $\mathcal{Z}^{(1)}$:

$$\mathcal{L}^{(1)}_{(GF+FP)} = \hat{\delta}^{(1)}\mathcal{Z}^{(1)} = \hat{\delta}^{(1)}[\bar{\Psi}F(\Phi_i)] = BF(\Phi_i) - \bar{\Psi}\hat{\delta}[F(\Phi_i)]. \tag{2.5}$$

Here $\delta_{BRST} = i\epsilon\hat{\delta}$, $\bar{\Psi}$ is an anti-ghost in a representation of the group G and the Lorentz group such that $\bar{\Psi}F(\Phi_i)$ is a singlet under both groups and B is the associated auxiliary field. The Euler-Lagrange equation for $\bar{\Psi}$ leads to an equation for Ψ_i which is the same as eqn. (2.2) for $\delta\Phi_i$. The Lagrangian (2.4) is further invariant under a local "ghost symmetry". The origin of this symmetry is the following: $\mathcal{Z}^{(1)}$ is obviously invariant under the G symmetry, thus transformations that leave Φ_i and $\bar{\Psi}_i$ inert and transform Ψ_i and B in the same way as Φ_i and $\bar{\Psi}$ transform under G, leave (2.5) invariant. In general, one can replace $\mathcal{Z}^{(1)}$ by $\mathcal{Z}^{(1)'} = \bar{\Psi}(F(\Phi_i)+\alpha B)$ where α is an arbitrary parameter. For $\alpha \neq 0$ the "ghost symmetry" mentioned above is not a symmetry. However, by adopting the "ghost symmetry" transformation, for the variation of B in $\mathcal{Z}^{(1)'}$ the resulting $\mathcal{L}^{(1)'}$ is invariant again under a "ghost symmetry"[18] . We thus use here the $\alpha = 0$ gauge. In the appendix, we briefly state the inability to gauge fix a Yang-Mills symmetry in a global way (in the space of connections). This phenomena, known as the Gribov ambiguity, shows up also in the gauge fixing of the topological symmetry. Therefore, we can choose a gauge slice only locally around a given Φ_i^0. The implications of this ambiguity will be addressed elsewhere[19] .

To fix the "ghost symmetry" we introduce a commuting "ghost for ghosts" field ϕ and a its anti-ghost $\bar{\phi}$. The BRST gauge fixing Lagrangian now has the following form:

$$\mathcal{L}^{(2)}_{(GF+FP)} = \hat{\delta}^{(2)}\mathcal{Z}^{(2)} = \hat{\delta}^{(2)}[\bar{\phi}G_{GF}(\Psi_i)], \tag{2.6}$$

where $\hat{\delta}^{(2)}$ is the sum of the $\hat{\delta}^{(1)}$ and the BRST transformations associated with the ghost symmetry. $G_{GF}(\Phi_i)$ is the gauge condition of (2.3). It is now obvious that the equation of motion of the combined action will require Ψ_i to obey eqn. (2.3). The equations for Ψ_i are therefore identical to those which define the moduli space (2.2-2.3). The condition for having a moduli space $\mathcal{M}$, thus, translate into a condition of having ghost zero modes.

An important question about our construction is whether a different gauge fixing can yield different observables[19]. This question is under a current investigation . Here we make the following remarks about possible circumstances where different theories emerge from what might be considered an unusual dependence on the gauge fixing, but infact is an outcome of differences in the original systems. (i) Following the condition to have a moduli space, mentioned above, gauge fixings which do not accomodate ghost zero modes are different from those which have them. But in fact the two possibilities cannot coexist for a given topology of the bundle. (ii) In case that "different boundary" conditions are imposed on the topological symmetry (For example TYM in four dimensions there are boundary conditions[1012] whereas the MSFC over the same space-time there are no boundary conditions). (iii) In case that different fields are accompaning the different gauge fixings (For example the TQFT which correspond to the MSFC and MSSDC in section 4).

We now describe this construction for the examples of above in table 2.

Topological Symmetry	Gauge Fixing	Ghost Symmetry	Equations of Ψ
$\hat{\delta}A_\alpha = \psi_\alpha$	$\hat{\delta}[\bar{\psi}^{\alpha\beta}(F_{\alpha\beta} + \tilde{F}_{\alpha\beta})]$	$\hat{\delta}\psi_\alpha = iD_\alpha\phi$ $\hat{\delta}B^{\alpha\beta} = i[\bar{\psi}^{\alpha\beta}, \phi]$	$D_{[\alpha}\psi_{\beta]} + \epsilon_{\alpha\beta\gamma\delta}D^{[\gamma}\psi^{\delta]} = 0$ $D_\alpha\psi^\alpha = 0$
$\hat{\delta}x^i = \psi^i$	$\hat{\delta}[\bar{\psi}^\alpha_i(D_\alpha x^i + \epsilon_{\alpha\beta}J^i_jD^\beta x^j - B^i_\alpha)]$ $\bar{\psi}^i_\alpha = \epsilon_{\alpha\beta}J^i_j\bar{\psi}^\beta_j$		$D_\alpha\psi^i + \epsilon_{\alpha\beta}J^i_jD^\beta\psi^j = 0$
$\hat{\delta}g_{\alpha\beta} = \psi_{\alpha\beta}$	$\hat{\delta}[\bar{\psi}\sqrt{g}R^{(2)}]$	$\hat{\delta}\psi_{\alpha\beta} = D_{(\alpha}\phi_{\beta)}$ $\hat{\delta}B = \phi^\alpha D_\alpha\bar{\psi}$	$D_\alpha D^\alpha\psi = 0$ $\psi = \psi^\alpha_\alpha$
$\hat{\delta}e_{\alpha a} = \psi_{\alpha a}$ $\hat{\delta}\omega_\alpha = \tilde{\psi}_\alpha$	$\hat{\delta}[\bar{\tilde{\psi}}(\epsilon^{\alpha\beta}\partial_\alpha\omega_\beta + det\ e)]$ $+\epsilon^{\alpha\beta}\bar{\psi}^a D_\alpha e_{\beta a}]$	$\hat{\delta}\psi_\alpha = iD_\alpha\phi$ $\hat{\delta}B = i[\bar{\psi}, \phi]$	$\partial_{[\alpha}\tilde{\psi}_{\beta]} + \epsilon_{ab}e^a_{[\alpha}\psi^b_{\beta]} = 0$ $(\tilde{D}_{[\alpha}\psi_{\beta]})^a + \epsilon^{ab}e_{b[\alpha}\tilde{\psi}_{\beta]} = 0$
$\hat{\delta}g_{i\bar{j}} = \psi_{i\bar{j}}$	$\hat{\delta}[\bar{\psi}^{i\bar{j}}\partial_i\partial_{\bar{j}}log\ g]$		$\partial_i\partial_{\bar{j}}(\psi) = 0$ $\psi = \psi_{i\bar{j}}g^{i\bar{j}}$

Table 2- *the corresponding TQFTs.*

Several remarks† related to the table are in order:

(i)There are no ghost symmetries in the second example because we took $\alpha = -1$, and in the last example because the diffeomorphism symmetry is fixed in $\mathcal{Z}^{(1)}$.

(ii) In the examples given in the tables above there is no further local symmetry in addition to the original G symmetry, but there are some cases where there still are further stages of ghost symmetry. Such a case will be considered in section 5.

(iii) In some works the gauge fixing of the original topological symmetry, the ghost symmetry and the local G symmetry is performed simultaneously[10,11,18]. Using several stages of gauge fixing better describes the construction of the space of ghost configurations which is identical to the desired moduli space.

† In the fourth row, we take for $\psi_\alpha = \psi^a_\alpha P_a + \tilde{\psi}_\alpha J$ and similarily for $\phi, \bar{\psi}$ and B see eqn. (3.8).

(iv) One can in general add some interaction terms to the action. Those terms will obviously modify the equation of motion for Ψ[10−12]. However, as explained below, due to an independence on a "coupling constant", the equation of above are still valid.

The BRST algebra that we have at the present stage is not nilpotent but rather it is closed up to a G transformation, $\hat{\delta}_G$, with the ghost for ghost ϕ as the parameter of transformation. For example $\hat{\delta}^2\Phi_i = \hat{\delta}_G\Phi_i$. The correspondence between the TQFT and the related moduli spaces includes the obstruction as well. Recall that the dimension of the moduli space is equal to the index of the operator defined in (2.2) and (2.3). In the TQFT the kernel corresponds to the Ψ zero modes. The cokernel is given by the zero modes of $\bar{\Psi}$ and $\hat{\delta}\bar{\phi}$. Thus the number of obstructions is given by the number of the latter zero modes. The difference between the number of Ψ_i zero modes and the number of $(\bar{\Psi},\ \hat{\delta}\bar{\phi})$ zero modes, is the index of an operator of the TQFT which is equivalent to $\bar{D}$.

The TQFT which we described above has some general properties which are shared by most of the TQFTs:

(1) The correlation functions of BRST invariant operators are independent of arbitrary variations of the metric[1].

$$\delta_{g_{\alpha\beta}} < \mathcal{O} >= \delta_{g_{\alpha\beta}} \int DX \mathcal{O} e^{i\int d^dx\hat{\delta}\mathcal{Z}} = \int DX e^{i\int d^dx\hat{\delta}\mathcal{Z}} \mathcal{O}\hat{\delta}[\delta_{g_{\alpha\beta}} \int d^dx\mathcal{Z}] = 0, \tag{2.7}$$

where DX is the measure, $\mathcal{O}$ is an operator which is a BRST scalar and is independent on the metric. We used here the fact that a vev of any BRST transformation is zero. Note that, even though the original action is metric independent, a metric on M was introduced in defining scalar products in the gauge fixing, so that the result (2.7) is not a trivial one.

The meaning of correlation function here is an expectation of the product of the operators since it is independent on the points on M where the operators are put[1]

(2) So far we considered only configuration which minimize the action. In particular Φ_i^0 and Ψ_i configurations which are solutions to eqn.(2.2-2.3). This is justified only if the path integral is dominated by those configurations. As for Φ^0, this is obvious since this was the gauge fixing we used. As for the rest of the fields we can modify the BRST transformations $\hat{\delta} \rightarrow \hat{\delta}' = \kappa\hat{\delta}$ such that the $\mathcal{L} \rightarrow \kappa\mathcal{L}$. In the same way we showed the independence on the variation of $g_{\alpha\beta}$ it is straightforward to see that correlation functions are also κ independent. Now in the large κ limit it is obvious that the path integral is dominated by the minima of the action. Notice that both (1) and (2) are related to the fact that the action is a BRST variation of some operator since the original action is irrelevant. This property, obviously, is not shared by ordinary gauge fixed actions.

(3) Due to the BRST symmetry, the fermionic determinant is equal to the bosonic up to a sign[1]. Therefore, in the case of no ghost zero modes $dim\mathcal{M} = 0$, the partition function is given by $Z = \Sigma_j(-1)^{S_j}$ where the sum is over all isolated $\Phi_i^{0(j)}$ configurations and S_j is the sign of the ratio of determinantes at the (j) configuration.

(4) TQFT which corresponds to a moduli space. ($dim\mathcal{M} \neq 0$) means that there are fermion zero modes for the Ψ system. Therefore the only non-trivial expectation values are of operators which can soak up those zero modes. This condition translates to a requirement that the ghost number of the operator will equal $dim\mathcal{M}$. As in any BRST Lagrangian we have here too a conserved global U(1) ghost number symmetry. In particular, the partition function vanishes in this case.

(5) It was shown by Witten[1] in the case of the TYM, that the BRST charge is in fact an exterior derivative on the moduli space. Related to this is the fact that an expectation of an operator has the form of an integral over the moduli space of a closed form on this space. These properties apply in the present case as well.

$$< \mathcal{O} >= \int da_1....da_n \; d\psi_1....d\psi_n \; \omega_{i_1...i_n} \; \psi_{i_1}...\psi_{i_n} = \int_{\mathcal{M}} \omega \tag{2.8}$$

where $da_i, d\psi_i$ denote the bosonic and fermionic zero modes respectively and

$\omega_{i_1 \ldots i_n} da^{i_1} \ldots da^{i_n} = \omega$ is an n form on $\mathcal{M}$

(6) Starting from a polynomial of the ghost for ghost $W_0 = Tr[\phi^k]$ which is invariant under the BRST and G symmetries, is independent on the metric and is not BRST trivial, one can generate a hierarchy of global invariants I_i with $i = 0, \ldots d$ (This conditions are not fulfilled in the case of (TG)[17]). The invariants are given by:

$$dW_i = \hat{\delta} W_{i+1}, \qquad I_i = \int_{\gamma_i} W_i, \tag{2.9}$$

where γ_i is a non-trivial i^{th} homology cycle. Global invariants which correspond to non-trivial cohomologies on the moduli space. are expectation values of $< \Pi_i I_i >$ such that $\sum_i U(1)_i = dim\mathcal{M}$. In fact the global invariants W_i are mappings from closed forms on M to closed forms on $\mathcal{M}$. It is thus clear why the global invariants can be sensors only for topological properties on $\mathcal{M}$ but not local ones.

So far we ignored the necessity to gauge fix the G symmetry prior to any path integral computations. The gauge fixing and Faddev-Popov associated actions are also of the form: $\mathcal{L}^{(3)}_{(GF+FP)} = \hat{\delta}_T[\bar{c} G_{GF}(\Phi)]$ where $\delta_T = \hat{\delta} + \hat{\delta}_G$ and $\bar{c}$ is a new anti-ghost. In general the equation of motion which correspond to $\bar{c}$ may impose conditions on Ψ_i which are incompatible with eqn. (2.2-2.3). However, it turns out that for local symmetries like gauge symmetries and diffeomorphisms (as can be checked in the examples of above) they are compatible. There are some other implications from the last stage of gauge fixing. Some of the global invariants which are BRST locally non-trivial become trivial under the BRST combined charge $\hat{\delta}_T = \hat{\delta} + \hat{\delta}_G$ and vice versa. This will be further discussed in section 4, for the moduli space of flat connections. Another way to gauge-fix the G symmetry for the case of the TYM, was introduced in reference[20] . $\mathcal{L}^{(3)}$ is taken to be $\hat{\delta}_G$ rather then $\hat{\delta}_T$ of the functional given above. The quantum action is not invarinat under $\hat{\delta}^{(2)}$ but the global invariants are.

3. Chern-Simons Theory, Flat Connections and Conformal Field Theory

An application of the general procedure described in the previous section to the case of the moduli space of flat non-abelian gauge connections MSFC will be presented in sections 4,5. Here we summarize the relation between the MSFC, the three dimensional Chern-Simons theory, and conformal field theory[13]. The connection between the MSFC of the group ISO(2,1) and three dimensional Einstein gravity is also briefly discussed following ref.[14].

The Chern-Simons (CS) action in three dimensions is

$$S_{CS} \;=\; \frac{k}{8\pi}\int\limits_{M} Tr[A\wedge dA + \frac{2}{3}A\wedge A\wedge A] \tag{3.1}$$

where A is a non-abelian connection of a compact group G, M is a closed compact three dimensional manifold, and the trace is taken in the representation of A. The coefficient k has to be an integer in order that $e^{iS_{CS}}$ will be single valued, since S_{CS} is not invariant under homotopically non-trivial gauge transformations (for groups with $\pi_3(G)=Z)$[21] . The classical configurations are flat three-dimensional gauge fields, namely those fields for which $F=0$. In the case that M has a boundary, choosing one component of A to vanish on the boundary ∂M and reducing the gauge transformation to the identity on ∂M guarantees that the equation of motion is not modified[16].

The quantization mechanism proposed by Witten[13] for an arbitrary three manifold M was to chop M into pieces, quantize on each piece and glue them back. Any such piece was taken to be a $\Sigma\times R$ where Σ is a Riemann surface and the time direction is along the real line R. With this assignment of the time it is natural to gauge fix the Yang-Mills symmetry in the gauge $A_0=0$. The action then takes the form:

$$S \;=\; \frac{k}{8\pi}\int\limits_{\Sigma\times R_1} Tr[A'\wedge\frac{\partial}{\partial t}A'dt] \;=\; \frac{k}{8\pi}\int dt\int\limits_{\Sigma} d^2x\epsilon^{\alpha\beta}Tr[A'_\alpha\frac{\partial}{\partial t}A'_\beta] \qquad (3.2),$$

where $A' = A_\alpha dx^\alpha$ is a connection on Σ. This Lagrangian, linear in the time derivative, leads to the Poisson bracket:

$$\{A^a_\alpha(x), A^b_\beta(y)\} = \frac{8\pi}{k}\epsilon_{\alpha\beta}\delta^{ab}\delta(x-y) \tag{3.3}$$

The "Gauss law" constraint is $F = \epsilon^{\alpha\beta}F_{\alpha\beta} = 0$. Witten advocates in [13] to first impose the constraint and then quantize the system.† Therefore,

The phase space of the CS theory in the $A_0 = 0$ is the moduli space of flat connections on Σ. The role of the flat connections on Σ is apparent also if instead of first gauge fixing, we first integrate in the path integral over A_0:

$$Z = \int DA_0 DA' e^{iS_{CS}} = \int DA'\delta(F_\Sigma)e^{i\frac{k}{8\pi}\int_{\Sigma\times R_1} Tr[A'\wedge\frac{\partial}{\partial t}A'dt]}. \tag{3.4}$$

The delta function implies the projection onto flat connections on Σ.

Instead of gauge fixing in the Weyl gauge we can again use a covariant gauge and apply the BRST procedure of the last section:

$$\mathcal{L}_{(GF+FP)} = \hat{\delta}\mathcal{Z} = \hat{\delta}(\bar{c}D_\alpha A^\alpha) = B_c D_\alpha A^\alpha - \bar{c}D_\alpha D^\alpha c \tag{3.5}$$

We decompose the gauge field to a classical and quantum parts $A = A^{cl} + A^q$. The gauge fields in (3.5) are A^q apart from the covariant derivative which is taken with respect to the classical background. c, $\bar{c}$ and B_c are the Yang-Mills ghost, anti-ghost and auxiliary fields respectively The covariant derivative is also with respect to the metric that was introduced in the scalar products in (3.5). The Euler Lagrange equations of the combined system (3.1) and (3.5) are now not of

† In it was shown that for the CS theory quantization and the application of the constraints are non-commuting operations. The expectation value of an abelian Wilson loop in flat space-time is different in the two procedures of calculation by a universal vacuum holonomy factor.

flat three dimensional configurations but rather

$$ {}^*F + DB_c \; = \; \frac{1}{2}\epsilon^{\alpha\beta\gamma}F_{\alpha\beta} + D^\gamma B_c = 0. \tag{3.6} $$

This point will be addressed in the TQFT construction of the three dimensional MSFC in section 5.

The gauge fixing that led to (3.5) can be done only locally (see appendix) around a given flat connection which we denote by $A_f^{(i)}$. The contribution to the path integral from the region in A space around the flat connection,$\mu(A_f^{(i)})$, was calculated by A. Schwarz[23] , for the abelian case. Following the later, Witten[13]derived $\mu(A_f^{(i)})$ for non-abelian CS theories in the large k limit:

$$ \mu(A_f^{(i)}) = e^{i(\frac{k+C_2}{2})S_{SC}(A_f^{(i)})}\frac{|\sqrt{detL_-}|}{det\triangle}, \tag{3.7} $$

where $\triangle$ is the Laplacian, C_2 is the second casimir operator of G in the adjoint representation. The determinants can be defined using a zeta function regularization[23]. The operator L_- is the restriction to odd forms of the self adjoint operator ${}^*D + D^*$ which is the kinetic operator for the (A^q, B_c) system. The ratio of the determinants was shown to be the Ray-Singer torsion of the flat connection $A_f^{(i)}$ Thus it is a topological invariant. The phase factor is also metric independent.† Similar to the discussion following (2.3) there are two possibilities for summing up the contributions of the $\mu(A_f^{(i)})$ factors depending on M and the flat bundle E_f. One case is that there are only finitely many isolated flat connections $(H^1(M, E_f) = 0)$ in this case the partition function of the CS theory in the large k limit is given by $Z = \Sigma_{(i)}\mu(A_f^{(i)})$. For a non-trivial first cohomology the sum has to be replaced by an integral over moduli space.* The observables of the CS theory are the Wilson lines, $W_R(\gamma) \; = \; PTr_R \; e^{\int_\gamma A}$, where R is the representation of the group, γ is a

† In fact the expression in (3.7) has to be multiplied by an additional phase factor which is not metric independent[13].

∗ It was assumed here that the $B_c, c\,\bar{c}$ do not have zero modes which means $H^0(E_f, M) = 0$.

loop and P denotes a path ordering. Witten[13] proved that correlation functions of the Wilson lines are indeed topological invariants and that they correspond to the Jones polynomials.

A specially interesting case of a CS theory is the one for the non-compact group ISO(2,1) (k is not quantized) which was shown to be equivalent to the Einstein gravity in three dimensions[14]. The CS action now has the form:

$$S_{SC} = \int_M \epsilon^{\alpha\beta\gamma}[e_{\alpha a}(\partial_\beta \omega^a_\gamma - \partial_\gamma \omega^a_\beta + e^{abc}\omega_{\beta b}\omega_{\gamma_c})] \tag{3.8}$$

where the ISO(2,1) gauge field is parametrized as $A_\alpha = e^a_\alpha P_a + \omega^a_\alpha J_a$ with P_a being the translation generators, and J_a is connected to the Lorentz generator J_{ab} via $J_a = \frac{1}{2}\epsilon_{abc}J^{bc}$. The classical configurations here are the flat SO(2,1) and flat ISO(2,1) connections which are (for the case of infinitely many flat connections) the total space of a tangent bundle over the moduli space of flat SO(2,1) connections. Integrating over e in the path integral obviously introduces a delta function $\delta(F)$ for F the $SO(2,1)$ field strength. This delta function will again be modified to an expression like (3.6) (with F and B_e the Lagrange multiplier of $D_\alpha e^\alpha = 0$) once we gauge fix around flat ω connection. The absolute value of the partition function is now given by the sum or the integral over the Moduli space of flat ISO(2,1) connections of the ratio of determinants[15] $\frac{(det\triangle)^2}{|det' L_-|}$ where det' denotes the product of the non-zero eigenvalues.

3.1. CS theory and Conformal Field Theory

The canonical quantization in the Weyl gauge is a quantization of a compact finite dimensional phase-space (since this is the property of the MSFC as is clarified in the appendix). Using the holomorphic polarization $A = A_1 + iA_2$, $\bar{A} = A_1 - iA_2$ the Hilbert space $\mathcal{H}$ is a space of holomorphic sections of a line bundle on $\mathcal{M}$ provided there is a Kahler structure on $\mathcal{M}$. To get such a structure Witten introduced[13] a complex structure J on Σ. This turned the MSFC into $\mathcal{M}_\mathcal{J}$ a complex manifold which is the moduli space of holomorphic vector bundles on Σ which are topologically trivial and have for the structure group the complexification of the gauge group G. The symplectic form is the first Chern class of L^k the k^{th} tensor power of the determinant line bundle. The Hilbert space denoted now as $\mathcal{H}_\Sigma^{(J)}$ should be independent of J, which means that the vector bundle $\mathcal{H}_\Sigma^{(J)}$ on $\mathcal{M}$ must be a flat bundle. Flat vector bundles on the moduli space of the complex structure are very essential in conformal field theories[24] In the CFT the analog of $\mathcal{H}_\Sigma^{(J)}$ is the space of the solutions of the Ward identities for the descendents of the identity, the space of conformal blocks. This is the source of the statement that[13]

The quantum Hilbert space of the three dimensional CS theory is identical to the space of conformal blocks in rational CFT.

For the case that M has a boundary, Moore and Seiberg[16]† derived an explicit expression for the partition function which was identical to the one of a two dimensional WZW theory with the sigma term on ∂M. In particular, for Σ taken to be a disk, they showed that the basic representations of the loop group emerge. For the case of an annulus, due to the holonomy, the other integrable representations may be derived.

4. TQFT Construction of MSFC in Two Dimensions

After the long detour, we present now the TQFT which correspond to the moduli space of flat connections of a given non-abelian* group G over a Riemann surface Σ. The construction of the theory follows the lines described in section 2.

Starting with the gauge fields A of the group G which are connections on the principle bundle P (see appendix) we invoke now the "topological symmetry" $\delta A(x) = \theta(x)$. Since we have in mind to project onto flat gauge configurations which are not characterized by a topological number which is expressible as a two

† Upon completion of this work we received ref.[25] where the connections between the CS theory and the conformal field theory are further elaborated.

* The abelian case was presented in ref. [17].

dimensional action, we take the action to be zero (modulo possible auxiliary fields that can be eliminated). The gauge fixing procedure will involve three stages of gauge fixings as follows:

$$\begin{aligned}\mathcal{L}^{(1)} = \mathcal{L}^{(1)}_{GF+FP} &= \frac{1}{2}\hat{\delta}^{(1)}Tr[\bar{\psi}\wedge F] = \frac{1}{2}\hat{\delta}^{(1)}Tr[\epsilon^{\alpha\beta}\bar{\psi}F_{\alpha\beta}] \\ &= \frac{i}{2}Tr[B\wedge F - i\bar{\psi}\wedge D\psi] = \epsilon^{\alpha\beta}Tr[\frac{i}{2}BF_{\alpha\beta} - i\bar{\psi}D_\alpha\psi_\beta]\end{aligned} \tag{4.1}$$

$$\begin{aligned}\mathcal{L}^{(2)} = \mathcal{L}^{(2)}_{GF+FP} &= \sqrt{g}\hat{\delta}^{(2)}Tr[-\frac{i}{2}(\bar{\phi}D_\alpha\psi^\alpha + i\bar{\psi}B)], \\ &= \sqrt{g}Tr[\frac{1}{2}\bar{\phi}D_\alpha D^\alpha\phi - i\eta D_\alpha\psi^\alpha - \frac{i}{2}\bar{\phi}[\psi_\alpha\ ,\psi^\alpha] + \frac{1}{2}B^2 - \frac{i}{2}\bar{\psi}[\phi,\bar{\psi}]].\end{aligned} \tag{4.2}$$

$$\mathcal{L}^{(2)'} = \frac{i}{4}\sqrt{g}\hat{\delta}^{(2)}Tr(\bar{\phi}[\phi,\eta]) = \frac{i}{4}\sqrt{g}Tr(2\eta[\phi,\eta] - \frac{i}{2}\bar{\phi}[\phi,[\phi,\bar{\phi}]]) \tag{4.3}$$

$$\mathcal{L}^{(3)} = \mathcal{L}^{(3)}_{GF+FP} = \sqrt{g}\hat{\delta}[-iTr(\bar{c}(\partial_\alpha A^\alpha + ib)] = Tr[\bar{c}\partial_\alpha D^\alpha c - ib\partial_\alpha A^\alpha + b^2 - i\bar{c}D_\alpha\psi^\alpha] \tag{4.4}$$

The BRST variations $\hat{\delta}^{(1)}, \hat{\delta}^{(2)}$ and $\hat{\delta}$ correspond to the gauge fixing of the "topological symmetry", the ghost symmetry and the Yang-Mills symmetry respectively. $\mathcal{L}^{(2)'}$ is an additional BRST and gauge invariant renormalizable term that can be added[12]. The various BRST transformations are: $\hat{\delta}^{(1)}A_\alpha = \psi_\alpha$, $\hat{\delta}^{(1)}\bar{\psi} = B$ and the rest of the fields are inert under $\hat{\delta}^{(1)}$, $\hat{\delta}^{(2)} = \hat{\delta}^{(1)} + \hat{\delta}_{gh}$ where the only fields that transform under $\hat{\delta}_{gh}$ are $\hat{\delta}_{gh}\psi_\alpha = i(D_\alpha\phi)$ $\hat{\delta}_{gh}B = i[\bar{\psi},\phi]$ and $\hat{\delta} = \hat{\delta}^{(2)} + \hat{\delta}_G$ which is the total BRST variation is given by:

$$\begin{gathered}\hat{\delta}A_\alpha = \psi_\alpha + iD_\alpha c,\ \ \hat{\delta}\psi_\alpha = iD_\alpha\phi - i\{c,\psi_\alpha\}, \\ \hat{\delta}\bar{\psi} = B - i\{c,\bar{\psi}\},\ \ \hat{\delta}B = -i[\phi,\bar{\psi}] - i[c,B], \\ \hat{\delta}c = -\phi - \frac{i}{2}\{c,c\},\ \ \hat{\delta}\phi = -i[c,\phi], \\ \hat{\delta}\bar{\phi} = 2\eta - i[c,\bar{\phi}],\ \ \hat{\delta}\eta = \frac{i}{2}[\phi,\bar{\phi}] - i\{c,\eta\}, \\ \hat{\delta}\bar{c} = b,\ \ \hat{\delta}b = 0.\end{gathered} \tag{4.5}$$

As for the algebra of the BRST charges, $\hat{\delta}^{(2)}$ is not nilpotent, however, it is closed up to a G gauge transformation with ϕ as the parameter of the transformation. The total BRST variation is generated by a charge which is nilpotent, $\hat{\delta}^2 = 0$. The total Lagrangian $\mathcal{L} = \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \mathcal{L}'^{(2)} + \mathcal{L}^{(3)}$ is invariant under a U(1) global ghost number symmetry under which the fields $(A, \psi,\ \bar{\psi},\ B,\ c, \phi,\ \bar{\phi},\ \eta,\ \bar{c},\ b)$ carry the charges (0,1,-1,0,1,2,-2,-1, -1,0) respectively. As usual the auxiliary fields B and b can be elminated leading to the terms $F^2 + \frac{1}{2}(\partial_\alpha A^\alpha)^2$ in the final action. By redefining the fields $\psi,\ B, \phi$ and η, the BRST algebra can be recast in the form of a locally vanishing cohomology algebra[18]. An interesting geometrical interpretaion of the BRST transformation was given in[10]. Following the later reference, the BRST transformations of $A,\ \psi, c,\ \phi$ in eqn. (4.5) can be rewritten as $\tilde{d}\tilde{A} + \frac{1}{2}[\tilde{A}, \tilde{A}] = \tilde{F}$ and the associated Bianchi identity $\tilde{D}\tilde{F} = 0$ where $\tilde{d} = d + \hat{\delta}$, $\tilde{A} = A + ic$ and $\tilde{F} = F + \psi + i\phi$. The object $\tilde{A}$ is a connection on the space $P \times \mathcal{A}/\mathcal{G}$. The orbit space $\mathcal{A}/\mathcal{G}$ is described in the appendix.

Note that the Euler Lagrange equation of $\bar{c}$ is compatible with the kernel of the operator $\bar{D}$, deduced from the equation of motion of $\bar{\psi}$ and η, which determines the classical ψ configurations.

$$D_{[\alpha}\psi_{\beta]} = 0 \qquad D_\alpha\psi^\alpha = 0. \tag{4.6}$$

These equations are identical to (A.4). As explained in section 2, the obstruction to the solutions of this system is the co-kernel of $\bar{D}$ and therefore:

$$index\bar{D} = \#(\psi\ zero\ modes) - \ \#(\bar{\psi}, \eta\ zero\ modes) = (2g-2)DimG \tag{4.7}$$

where the underlaying space-time is a genus g Riemann surface. This obviously matches the dimension given in (A.5). In the case that the Riemann surface $g > 0$ is punctured in n points then instead of $2g$ we have $2g + n$ in eqn. (4.7).

Let us now proceed and discuss the observables of the theory. As was explained in section 2, the invariants are metric independent and hence topological. Furthermore, from (2.7), it follows that the invariants can be expressed as integrals over closed forms on $\mathcal{A}_f/\mathcal{G}$. Such invariants $I_i^{(i,l)}$, $i = 0, 1, 2$ (the pair of superscripts are the degrees of the form on M and $\mathcal{A}_f/\mathcal{G}$ respectively) obey the following properties:

$$
\begin{aligned}
I_i^{(i,l)} &= \int_{\gamma_i} W_i^{(i,l)} \\
\hat{\delta}^{(2)} W_i^{(i,l-i)} &= dW_{(i-1)}^{(i-1,l+1-i)} \quad \hat{\delta}^{(2)} W_0^{(0,l)} = 0 \quad dW_2^{(2,l-2)} = 0
\end{aligned}
\tag{4.8}
$$

For BRST invariance the BRST variation of W_i has to be globally an exact form on M. In a complete analogy to the way Witten introduced the invariants of the Donaldson theory[1] we present now the global invariants of our theory

$$
\begin{aligned}
I_0 &= W_0^{(0,2k)} = Tr(\phi^k), \\
I_1 = \int_\gamma W_1^{(1,2k-1)} &= k \int_\gamma Tr(\phi^{k-1}\psi) = k \int_\gamma Tr(\psi_\alpha \phi^{k-1} dx^\alpha), \\
I_2 = \int_\Sigma W_2^{(2,2k-2)} &= k(k-1) \int_\Sigma Tr(\phi^{k-2}\psi \wedge \psi) = \\
& \qquad k(k-1) \int_\Sigma Tr(\phi^{k-2}\psi_\alpha \psi_\beta dx^\alpha \wedge dx^\beta),
\end{aligned}
\tag{4.9}
$$

where $k = 2, ...r$, r being the rank of the group G, and γ is a non-trivial cycle. Terms which depend on F were obviously ommited. Thus the only forms are those of degree 2 up to $2r$ on $\mathcal{A}_f/\mathcal{G}$, whereas the dimension of $\mathcal{A}_f/\mathcal{G}$ is $(2g-2)dimG$. Notice that whereas W_0 and W_2 are in a non-trivial cohomology class W_1 is locally in a trivial cohomology class since $W_1 = \hat{\delta}^{(2)}[kTr(\phi^{k-1}A)]$. Recall that the non-trivial topologically invariant correlation functions are only those that can absorb the fermion zero modes and hence are expectation values of operators whose ghost number is equal to the index of $\bar{D}$, namely following eqn. (A.8) equal to the dimension of $\mathcal{A}_f/\mathcal{G}$. This is a strong constraint which leaves only a small number of non-vanishing invariants. For example, it is obvious that for any g $< I_i >= 0$, for powers of I_1 $< I_1^n >= 0$ for odd n, $< I_0^n >= 0$ for even g, etc. For SU(2), the possible invariants are: $< I_0 I_2 >$ for $g = 2$ and $< I_0^3 >$, $< I_1^4 >$, $< I_2^6 >$ and $< (I_0 I_2)^2 >$ for $g = 3$.

So far our discussion of the observables followed the one of reference [1] where computations were considered prior to the gauge fixing of the gauge symmetry. (The BRST transformation in (4.8) is $\hat{\delta}^{(2)}$ rather than $\hat{\delta}$.) The question is whether the third stage of gauge fixing can alter the previous results. Since the observables in (4.9) are both $\hat{\delta}^{(2)}$ and gauge invariant, they are also $\hat{\delta}$ invariant. However, the issue of triviality[20] under the total BRST cohomology is different for the $\hat{\delta}$ and $\hat{\delta}^{(2)}$ operators. The various W_i can now be written as a sum of an exact form on M and $\mathcal{A}_f/\mathcal{G}$ in the following way:

$$\begin{aligned} W_0^{(0,4)} &= \hat{\delta}Tr(-c\phi + \frac{i}{3}c^3) \\ W_1^{(1,3)} &= \frac{1}{2}[\hat{\delta}Tr(-A\phi - \frac{i}{3}c^2 A + c\psi) + dTr(\frac{1}{3}c^3 - ic\phi)] \\ W_0^{(2,2)} &= \hat{\delta}Tr(icDA + A\psi - \frac{i}{3}A^2 c + iAD\phi + iAdc) \\ &+ dTr(-iA\phi + ic\psi + \frac{2}{3}Ac^2 - icdc + ic\psi - cD\phi) \end{aligned} \tag{4.10}$$

where for simplicity we took here $k = 2$. This can be verified easily by using the generalized curvature[10], $\tilde{F}$, defined above and realizing that the global invariants W_i are the $(0,4),(1,3),(2,2)$ components of

$$Tr(\tilde{F}\wedge\tilde{F}) = \tilde{d}[Tr(\tilde{A}\wedge\tilde{d}\tilde{A} + \frac{2}{3}\tilde{A}\wedge\tilde{A}\wedge\tilde{A})] \tag{4.11}$$

The last equation is borrowed from the TYM in four dimensions and that is the reason for the three dimensional CS term in the r.h.s of (4.11). Nevertheless we can still use this formula for the present two dimensional case but now obviously the (4,0) and (3,1) components of both sides of the equation vanish. Moreover we can use (4.11) to clarify the issues of invariance under the BRST and the BRST triviality. Following (4.8) the BRST variations of the various W_i are given by exterior derivatives on M of W_{i-1} which according to (4.11) are derivatives of components of a second Chern class. Therefore these relations are also globally valid. On the other hand (4.10) tells us that the W_i are given by a combination of the exterior derivatives on both M and $\mathcal{A}_f/\mathcal{G}$ of some functional of the connections over those spaces. Thus in general these local properties cannot be simply extended into global properties (see appendix). The I_i are therefore BRST invariant but not trivial. So far we only pointed out that the correlation functions are not necessarily trivial. Very little has been done on the explicit calculation of the observables in all the TQFT that were proposed so far. A derivation of the observables for the topological quantum mechanical case was given in reference[26] using Morse theoy. In [17] we showed that $< \prod^{dimG(g-1)} \int_\Sigma Tr(\psi\wedge\psi) >\neq 0$ and that it is independent on the conplex structure on Σ. This issue of field theoretical calculation of the observables of the TQFT in general and in particular for those related to the MSFC and the MSSDC is under current investigation[19].

In conformal field theories for which there is a current algebra, most of the interesting results are related to the level, k, of the associated Kac-Moody algebra. In the picture of the MSFC the level correspond to the symplectic structure on $\mathcal{M}$ via eqn. (3.3). The observables of the TQFT depend solely on the topological properties of the moduli space and hence are independent on the sympletic structure. Hence the correlation fuction of the TQFT should be associated with properties of the conformal field theories which apply for any allowed k. It is not clear to us what are those properties. (Recall that the TQFT's of here are meaningful mainly for $g > 1$). in the CFT.

An especially interesting case is when the gauge group is $SO(2,1)$. This was discussed by us in reference [17]. The flat connection condition in this case takes the form:

$$\begin{aligned} D_{[\alpha}e^a_{\beta]} =&\partial_{[\alpha}e^a_{\beta]} + \omega_{[\alpha}e^a_{\beta]} = 0,\\ \partial_{[\alpha}\omega_{\beta]} =& - det(e), \end{aligned} \tag{4.12}$$

where $(e_{\alpha a},\ \omega_\alpha)$ are the two dimensional analogs of the fields defined in eqn. (3.8). The first equation in (4.12) states that $e_{\alpha a}$ is covariantly conserved and the second, when translated to metric notation, means that the curvature scalar $R = -1$. This condition of a constant negative curvature metric is characteristic of a Riemann surface with $g \geq 2$. We therefore concluded that the MSFC for the SO(2,1) group is equivalent to the moduli space of reimann surfaces with $g \geq 2$. Similarly, the MSFC of ISO(1,1) and SO(1,2) correspond to the torus and the sphere respectively. Inserting $dimG = 3$ in eqn (4.7) one obtains $dim\ \mathcal{A}_f/\mathcal{G} = 6g-6$ which is known to be the dimension of the moduli space of Riemann surfaces. The question is, however, whether the MSFC corresponds to the moduli space or the Teichmuler space. The Teichmuller space $\mathcal{M}_\tau$ and the moduli space $\mathcal{M}_g$ are related in the following way: $\mathcal{M}_g = \frac{\mathcal{M}_\tau}{\Gamma_g}$ where Γ_g is the mapping class group. The latter is the group of diffeomorphisms modulo diffeomorphisms which are connected to the identity. In the SO(2,1) MSFC the equivalence classes are defined modulo $SO(2,1)$ gauge transformations which include the transformations $\Sigma \to G$ that are not continiously connected to the identity. Thus the question is whether the transformations of $e_{\alpha a}$, ω_α defined over a genus g Σ under the later transformation, correspond to transformations of the metric under the maping class group. The answer is negative. On the other hand it is known that $\mathcal{M}_\tau$ is topologically trivial. In [17] some non-trivial cohomologies were identified. In particular $W_2^{(2,2)}$, which leads to the global invariant $< (I_2)^{3g-3} >$, was believed to correspond to the first Mumford class on $\mathcal{M}_g$. Hence the precise identification of the TQFT of the $SO(2,1)$ MSFC is still not completely clear to us

Two last remarks concerning the SO(2,1) group. Following the general discussion in section 3.1 the MSFC for the SO(2,1) is related to the $SO(2,1)$ conformal field theory which was analyzed[27] recently. It was found that all unitary representations of N=2 superconformal algebra ($c > 3$) may be obtained form representations of SO(2,1) current algebra by substructing and then adding free boson. The CS formulation of 3 dimensional gravity on $\Sigma \times R$ once quantized in the Weil gauge (or after integrating out e_0, w_0) is related to the moduli space of flat $SO(2,1)$ flat connections.

4.1. The TQFT which correspond to the MSSDC

The generalization of the MSFC to the MSSDC is introduced briefly in section (A.4) of the appendix. THe "self-dual" fields on Σ are the fields which emerge from dimensional reduction of self-dual Yang-Mills gauge fields in four dimensions. The motivation for this generalization, as was explained in the introduction, is the conjecture that the MSSDC might correspond to some field theories which in a certain limit (which correspond to the limit $\rho \to 0$ in the present case) turn into conformal field theories. Here we outline the construction of the TQFT of the MSSDC. One approach is to follow the steps taken for the MSFC and replace the condition $F = 0$ with eqn. (A.8). Another route is to dimensionally reduce the construction of the TYM with the following renamings $\rho_1 = A_3, \quad \rho_2 = A_4 \; \psi_\rho = \psi_3 + i\psi_4 \; \bar{\psi}_\rho = \bar{\psi}_3 + i\bar{\psi}_4, \quad B_\rho = B_{13} + iB_{14}$ and $\partial_3 = \partial_4 = 0$. Using this notation in the derivation of the TYM it is straightforward to verify that the classical configurations of ψ, ψ_ρ are the solutions to equation (A.9) where ψ is replacing δA and ψ_ρ is replacing $\delta\rho$.

The invariants of the present TQFT are different from those of the MSFC. Again the easiest way to derive the observables is to dimensionally reduce the global invariants of the TYM. The set of observables for $k = 2$ are given in the generalized second Chern class given in (4.11). If we reduce the four dimensional forms we have in general $W^{(i,l)} \to \Sigma_j W'^{(i-j,l)}$ where appriori i=0,...,4 provided that $i-j = 0, 1, 2$. This leads to the observables given in (4.9) plus possible additional ones. For example for the case of k=2 one gets the invariant $W_0'^{(0,3)} = Tr[\phi \; Re(\psi_\rho)]$.

5. TQFT of Flat Connections in Three and Higher Dimensions

The field theoretical description of the MSFC in three and higher dimensions admits some new features in comparison to the two dimensional case. In particular new "ghost symmetries" emerge. We address these features in this section as well as the relation to other generally covariant theories.

The basic field here again is a connection A on a principle bundle whose base space is now a three dimensional manifold. We assume invariance under a "topological symmetry" which is identical to the one in section 4. The quantum action follows the gauge fixing of the topological symmetry choosing a flat connection gauge condition:

$$\begin{aligned}\mathcal{L}^{(1)}_{GF+FP} &= \hat{\delta}^{(1)}\mathcal{Z}^{(1)} = \hat{\delta}\frac{i}{2}Tr[\bar{\psi}\wedge F] = \frac{i}{2}Tr[B\wedge F - i\bar{\psi}\wedge D\psi] \\ &= \hat{\delta}Tr[\frac{i}{2}\epsilon^{\alpha\beta\gamma}\bar{\psi}_\alpha F_{\beta\gamma}] = \epsilon^{\alpha\beta\gamma}Tr[\frac{i}{2}B_\alpha F_{\beta\gamma} - i\bar{\psi}_\alpha D_\beta\psi_\gamma]\end{aligned} \tag{5.1}$$

where now $\bar{\psi}\in\omega^1$ and $B\in\omega^1$ (see appendix). Unlike the functional $\mathcal{Z}^{(1)}$ in the two dimensional case here $\mathcal{Z}^{(1)}$ is invariant under an additional local symmetry $\delta\bar{\psi} = D\theta$. This leads (as for the Yang-Mills invariance of $\mathcal{Z}^{(1)}$) to two additional symmetries of (5.1). So altogether the action is invariant under the following three ghost symmetries (written in the BRST formulation):

$$\begin{aligned}&\hat{\delta}^{(2)'}\psi_\alpha = iD_\alpha\phi \qquad \hat{\delta}^{(2)''}\bar{\psi}_\alpha = iD_\alpha\phi^{(1)} \qquad \hat{\delta}^{(2)'''}B_\alpha = iD_\alpha\psi_B \\ &\hat{\delta}^{(2)'}B_\alpha = i[\bar{\psi}_\alpha,\phi] \qquad \hat{\delta}^{(2)''}B_\alpha = i[\psi_\alpha,\phi^{(1)}],\end{aligned} \tag{5.2}$$

and the non-Abelian gauge symmetry. As was mentioned in section 2 one can use $\bar{\psi}\wedge(F+\alpha^{(1)*}B)$ instead of $\mathcal{Z}^{(1)}$ given in (5.1) and then a ghost symmetry can be realized once the BRST transformation of B is changed. We fix the three ghost symmetries as follows:

$$
\begin{aligned}
\mathcal{L}^{(2)} = \mathcal{L}^{(2)}_{GF+FP} &= \sqrt{g}\hat{\delta}^{(2)}Tr[\frac{-i}{2}(\bar{\phi}D_\alpha\psi^\alpha + \bar{\phi}^{(1)}D_\alpha\bar{\psi}^\alpha + i\bar{\psi}_\alpha B^\alpha)], \\
&= \sqrt{g}Tr[\frac{1}{2}\bar{\phi}D_\alpha D^\alpha\phi - i\eta D_\alpha\psi^\alpha - \frac{i}{2}\bar{\phi}[\psi_\alpha\ ,\psi^\alpha] \\
&+ \frac{1}{2}\bar{\phi}^{(1)}D_\alpha D^\alpha\phi^{(1)} - i\eta^{(1)}D_\alpha\bar{\psi}^\alpha - \frac{i}{2}\bar{\phi}[\psi_\alpha\ ,\psi^\alpha] \\
&+ \frac{1}{2}(B_\alpha B^\alpha + i\bar{\psi}^\alpha([\bar{\psi}_\alpha,\phi] + [\psi_\alpha,\phi^{(1)}] + D_\alpha\psi_B))]
\end{aligned}
\tag{5.3}
$$

where $\hat{\delta}^{(2)} = \hat{\delta}^{(2)'} + \hat{\delta}^{(2)''} + \hat{\delta}^{(2)'''}$. The new ghosts just as the old are all in the adjoint representation of the group G. The ghost numbers of the new ghosts $\phi^{(1)}$, $\bar{\phi}^{(1)}$ and $\eta^{(1)}$ are (0,0,1) respectively. We can add interaction terms similar to those introduced in the case of four dimensional instantons[12]by replacing $\bar{\phi}D_\alpha\psi^\alpha$ in equation (5.3) with $\bar{\phi}(D_\alpha\psi^\alpha - \frac{1}{2}[\phi,\eta])$ and thus generate terms of the form $\frac{-i}{2}\phi[\eta,\eta] - \frac{1}{4}[\phi,\bar{\phi}]^2$. Similar treatment for the SUSY-Yang-Mills-Higgs was presented in ref. [6]. If we neglect such terms and the interaction terms in (5.3) the Euler-Lagrange equations for $\bar{\psi}$ that follow from $\mathcal{L} = \mathcal{L}^{(1)} + \mathcal{L}^{(2)}$ are

$$
\epsilon^{\alpha\beta\gamma}D_\beta\psi_\gamma = -D^\alpha\eta^{(1)} \qquad\qquad D_\alpha\psi^\alpha = 0
\tag{5.4}
$$

Eventhough the first equation looks different from the condition on δA_α derived from the of preservation the flatness property, it is in fact identical to the latter due to the Yang-Mills ghosts. The "linearization" of equation (3.6) is the same as (5.4) with ψ replacing δA. Since both the two dimensional and three dimensional TQFT's of the MSFC are claimed to correspond to the three dimensional Chern-Simons theory, do we mean that the two TQFT are equivalent? The answer is, of course, no, except for some special cases. For $M = \Sigma \times R$, as is shown below the MSFC over M is the same as over Σ, and therefore the topological field theories are also the same.

We proceed now to examine the BRST algebra. The fields A, ψ, ϕ c and $\bar{c}$ have the same algebra as in the two dimensional case, eqn. (4.5). The rest of the ghosts transform as follows:

$$\begin{array}{ll} \hat{\delta}^{(2)}\bar{\psi}_\alpha = B_\alpha + iD_\alpha\phi^{(1)} & \hat{\delta}^{(2)}B_\alpha = iD_\alpha\psi_B + i[\bar{\psi}_\alpha, \phi] + i[\psi_\alpha, \phi^{(1)}] \\ \hat{\delta}^{(2)}\phi^{(1)} = -\psi_B & \hat{\delta}\psi_B = -i[\phi^{(1)}, \phi] \\ \hat{\delta}^{(2)}\bar{\phi}^{(1)} = 2\eta^{(1)} & \hat{\delta}\eta^{(1)} = \frac{i}{2}[\bar{\phi}^{(1)}, \phi]. \end{array} \tag{5.5}$$

The algebra is closed up to a G-transformation with ϕ as the parameter of the transformation. Just as in section 4, nilpotency of the algebra is achieved when the Yang-Mills symmetry is fixed.

In the case where there are no ghost zero modes and hence the moduli space has a zero dimensionality, then the partition function is a topological invariant. As explained in section 2, the contribution to the partition function is given in such a case by the ratio of the fermionic over the bosinic determinants This ratio is equal to ± 1. Thus the partition function is $Z = \Sigma_j (-1)^{s_j}$ where S_j is the sign of the ratio at any isolated "flat" connection. As was discussed in ref. [15] this global invariant correspond to the Casson invariant in the mathematical literature.

Most of the discussion of the global invariants of the TQFT of flat connections on two dimensions applies also to the present case . The only differences are the following:

(i) Starting with the head of the hierarchy $W_0^{(0,2k)} = Tr\phi^k$ one can generate here a three form on M, $W_3^{(3,2k-3)} = \frac{1}{6}k(k-1)(k-2)Tr(\phi^{k-3}\psi\wedge\psi\wedge\psi)$ with the corresponding BRST global invariant $I_3 = \int_M W_3$. (ii) Obviously the dimensions of $\mathcal{A}_f/\mathcal{G}$ is now different then in eqn. (4.7). Therefore the requirement on the ghost number of the operators in the correlation functions has to be respectively changed. Here are some examples: (a) for $M = \Sigma\times R$, since $\pi_1(M) = \pi_1(\Sigma)$, the dimension of $\mathcal{M}$ is the same as the one for Σ. (b) For a "handle body", which is a three maniflod whose surface is a Riemann surface Σ, the dimension is half the one on Σ namely $dim\mathcal{M} = dimG(g-1)$ for $g > 1$. (iii) Notice that the new ghost fields $\phi^{(1)}$, $\bar{\phi}^{(1)}$, ψ_B and $\eta^{(1)}$ are all non-singlets under $\hat{\delta}^{(2)}$ and thus cannot generate a new hierarchy of invariants.

The TQFT which is related to the three dimensional gravity is obviously the one with $G = ISO(2,1)$ for the case of zero cosmological constant and $SO(3,1)$ and $SO(2,2)$ for the cases of de-Sitter and anti de-Sitter spaces respectively. To get the TQFT which correspond to the Poincare symmetry, we just have to substitute F in eqn. (5.1) with

$$F_{\alpha\beta} = [D_\alpha, D_\beta] = D_{[\alpha} e^a_{\beta]} P_a + (\partial_{[\alpha} \omega^c_{\beta]} + \epsilon^{abc} e_{\omega a} e_{\omega b}) J_c, \tag{5.6}$$

which is the field strength that correspond to the ISO(2,1) connection. In addition we parametrize Φ which stands for any of the fields in (5.1) and (5.3) by: $\Phi = \Phi^a P_a + \tilde{\Phi}^a J_a$. The ISO(2,1) generators P_a and J_a obey: $[P_a, P_b] = 0$, $[P_a, J_b] = \epsilon_{abc} P^c$ and $[J_a, J_b] = \epsilon_{abc} J^c$. The trace is taking now as follows $Tr(P_a P_b) = Tr(J_a J_b) = 0$ and $Tr(J_a P_b) = \delta_{ab}$. The flat $ISO(2,1)$ is identical to the total space of the tangent bundle of the SO(2,1) MSFC[15]. In a similar way the ISO(2,1) MSFC modified by the Poincare ghosts $F = {}^*D\eta^{(1)}$ is the total space of the tangent space of the corresponding $SO(2,1)$ space. The TQFT thus probes the topology of this total space. This space is identical to the one derived from the Euler-Lagrange equations of the quantum (including the ghost and gauge fixing terms) ISO(2,1) CS theory.

As a generalization of the $ISO(2,1)$, one may consider a semidirect product $\mathbf{g} \times G$ of the group G and its Lie algebra g denoted by IG. This means simply that the algebra of IG is given by the generators (P_a, T_a) $a = 1,..dimG$ with the following commutation relations $[P_a, P_b] = 0$, $[P_a, T_b] = f_{abc} P^c$ and $[T_a, T_b] = f_{abc} T^c$. In [15] Witten presented a "twisted" N=2 supersymmetrization of the IG group. Denoting by θ_1 and θ_2 the fermionic coordinates, a connection A^s on the super IG bundle was written as $A^s = A + \theta_1 \psi + \theta_2 \bar{\psi} + \theta_1 \theta_2 B$. Substituting this connection to the Chern-Simons form one gets:

$$S^s_{CS} = \int_M \int d\theta_1 d\theta_2 Tr[A^s \wedge dA^s + \frac{2}{3} A^s \wedge A^s \wedge A^s] = \int_M Tr[B \wedge F - \bar{\psi} \wedge \psi] \tag{5.7}$$

which is identical to the first stage of gauge fixing of the topological symmetry in

(5.1). Thus the CS super IG theory is in fact exactly the same as the TQFT for the MSFC which corresponds to the group IG.

5.1. TQFT construction of MSFC in higher dimensions

Higher dimensional space-time manifolds M with $\pi_1(M) \neq 0$ have non-trivial flat connections (see appendix) and hence one can write down TQFT which correspond to the MSFC over M. The TQFT in three dimensions has a larger ghost symmetry than the two dimensional case, and correspondingly a larger set of ghosts. There was no need to introduce further generations of "ghosts for ghosts for ghosts...". It turns out however, that for $dim\ M > 3$ one faces such situation. This can be realized very easily. The gauge fixing is again introduced with a Lagrangian of the form (5.1), but now $\bar{\psi}$ and $B \in \omega^{D-2}$, and correspondigly, $\phi^{(1)}$ and ψ_B are D-3 forms (for $dimM = D$). By construction the gauge fixing lagrangian is invariant under $\hat{\delta}\phi^{(1)} = D\psi^{(1)}$ with $\psi^{(1)} \in \omega^{D-4}$ and similarly for ψ_B. This invariance will repeat itself until one reaches ghost for ghost which are zero forms. There are thus $D-2$ (for $D > 2$) generations of ghost symmetry. Note, however, that the classical configurations of ψ are still of the form of (5.4) and hence correspond to the MSFC in the presence of Yang-Mills ghosts. The bosonic part of $\mathcal{L}^{(1)}$ which again has the form of $Tr[B \wedge F]$ where B is a D-2 form, is identical to the covariant theories introduced in[28] .

As mentioned in the introduction, a straightforward generalization to the the case of MSFC in two dimensions is the moduli space that emerges for the MSFC in higher dimensions. In general dimensional reduction of $F = 0$ in D dimensions leads $F = 0$, $D\rho_i = 0$ and $[\rho_i, \rho_j] = 0$ in D' dimensions with ρ_i being real scalar fields in the adjoint representation and $i, j = 1, ..., (D - D')$. For example in analogy to the MSSDC one gets for the dimensional reduction from four to two dimenisions $F = 0$ and $D\rho_1 = D\rho_2 = [\rho_1, \rho_2] = 0$.

A different generalization in higher dimensions is when the gauge connection is replaced by a d dimensional form whose corresponding field strenght F which is a form of degree $d+1$ vanishes. Introducing the largest possible transformation of A, namely the topological symmetry will lead to the quantum action by applying the procedure of section 2.

6. Summary and Conclusions

The main message of this work is that it is straightforward to attach to any moduli space its TQFT partner. By invoking the "topological symmetry" it is possible to project onto a space of configurations which admit some topological properties. The zero modes of the ghosts, which appear in the gauge fixing of the topological symmetry, serve as the coordinates of the moduli space. In ordinary field theories the observables are expectation values of fields which describe the system, such as gauge fields, matter fields, etc. In TQFT most of the observables are expectations of ghosts (in the case of the MSFC only ghosts appear in the observables). This phenomena may seem as indicating some inconsistency of the theory. But it is in fact natural that the ghosts play the important role in the observables since they span the moduli space whose topological properties are probed by the observables.

Self-dual gauge bundles were found in the work of Donaldson to be very effective for the investigation of the topological properties of four manifolds. It turns out that flat gauge bundles can play an important role when defined over Riemann surfaces. E. Witten has shown[13] that rational conformal field theories are intimately related to quantization of a MSFC where the level of current algebra representation is related to the symplectic structure on the moduli space. This is exactly what emerges in the quantization of the three dimensional CS theories over $\Sigma \times R$. TQFT's which correspond to the MSFC over Σ probe the topology of this space which is the space of conformal blocks. It is insensitive to the level. Thus any information from the TQFT about the RCFT (for $g > 1$) should apply to any level. The observables of the CS theories are the Wilson lines which are derived by taking the trace of gauge holonomies. The equivalence classes of flat connections over any manifold are determined by homeomorphisms of mappings from the fundamental group of Σ to the group G. This is the source of the relations between the MSFC and its TQFT partner and the CS and its RCFT relatives. Using a covariant gauge in the CS theory (for example when M is not $\Sigma \times R$) leads to a moduli space of gauge connections in three dimensions which depend on the Yang-Mills ghosts. The same moduli space appears in the construction of a TQFT for flat connections in three dimensions. Quantum gravity in four dimensions is one of the most important puzzles of field theory. However, it turns out that in three dimensions Einstein theory is solvable once it is formulated as a CS of the isometry group of the space-time. Therefore even in this case the flat gauge configurations are important and presumably the related TQFT may be helpful.

TQFT is a bridge between quantum field theory and topology. Two directions are possible on this bridge. One suggests using field theoretical techniques for the calculations of topological properties, and the other is to apply topological methods to describe physical systems. Apparently both directions deserve some new developments. It is not clear that TQFT's can lead to direct descriptions of a physical systems[7] and on the other side, thus far, field theoretical techniques have yet to yield new topological results. Other interesting open questions are under current investigation: (i) TQFT as the theory of Gribov ambiguities. (ii) The dependence of the quantum actions on the gauge condition that is imposed on the topological symmetry. (iii) The passage from local trivial BRST cohomology to a global one. (iv) The implications of the results form TQFT's of MSFC on conformal field theories.

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APPENDIX A

Geometrical Properties of Flat Connections

The MSFC, the CS theory and rational conformal field theory are related to each other as was explained in the last section. Thus, geometrical properties of the MSFC may shed light on CS and the conformal field theories. In this section before, describing the TQFT which correspond to the MSFC, we summarize several geometrical features of the MSFC. We start with some geometrical properties of the space of gauge connections and the orbit space, proceed to the parametrization of flat connections and then present some information from the mathematical literature on the MSFC and the MSSDC. The reader who is interested mainly in the TQFT, may skip this section.

A.1. Geometrical Properties of the Orbit Space

Consider a gauge theory† defined over a compact, boundryless space-time M. The gauge group is a compact Lie group G, with the lie algebra $\mathbf{g}$. The gauge fields A are connections on a principal bundle P(M,G). One can construct two additional bundles: $P' = P \times_{Ad} \mathbf{g}$ which is a vector bundle a fiber $\mathbf{g}$ with the adjoint action of G on $\mathbf{g}$, and $P'' = P \times_{ad} G$ where the group is the bundle with the adjoint action. We denote by ω^p forms of degree p on M which take their values in P'. We define the scalar product in ω^p using the Hodge * operator $\forall\omega_1 \in \omega^p\ \forall\omega_2 \in \omega^p\ \ (\omega_1, \omega_2) = \int_M Tr(\omega_1 \wedge^* \omega_2)$. A gauge transformation $g \in \mathcal{G}$ of the connection A is given locally by the well known expression:

$$A \rightarrow A^g = A + g^{-1}Dg \tag{A.1}$$

where D is the covariant derivative. This transformation is in an automorphism on P which induces the identity mapping on the base space. Note that the difference of two connections $A_\tau = A - A'$ is not a connection since its gauge transformation

† We follow here ref.[29]

is $A_\tau \to A_\tau^g = g^{-1} A_\tau g$. Thus, $A_\tau \in \omega^1$. Therefore the space of gauge connections $\mathcal{A}$ is not a vector space but an affine space modelded on ω^1. This property leads to the inability to define connections in a global way. This is significant for the proof of the non-triviality of the global invariants of TQFT's as is expalined in section 4. The infinitesimal gauge transformation is $A \to A + D\xi, \quad \xi \in \omega^0$. This gives the elements of $T_A(\mathcal{A})$ the tangent space to $\mathcal{A}$ at A (tangent to the fiber through A) which span the vector space of vertical vectors $V_A(\mathcal{A})$ at A. A metric on $\mathcal{A}$ is given by the scalar product (,) on $\omega^1 \approx T_A(\mathcal{A})$. This is a gauge invariant flat metric since it is independent of A.

Gauge fixing means choosing one representative from each equivalence class (orbits) namely drawing a surface in $\mathcal{A}$ which cuts all orbits once. This can be done only <u>locally</u> around a given connection A_0 in the following way: We define a subspace $\mathcal{A}_0$ around A_0 by requiring that any point on it (apart from A_0) is orthogonal to the orbit through A_0 namely for $A_\tau = A - A_0$ we demand $D^*_{A_0} A_\tau = 0$ where D^* is the covariant divergence which is related to the covariant derivative by $\forall \omega_1 \in \omega^p, \forall \omega_2 \in \omega^{(p-1)} (\omega_1, D\omega_2) = (D^* \omega_1, \omega_2)$. $\mathcal{A}_0$ is the affine space generated by H_{ω_0}, which is the space of solutions of the horizontality condition $D^* A_\tau = 0$. To introduce a connection on $\mathcal{A}$ one defines $\chi_A = \triangle_A^{-1} D_A^*$ where $\triangle_A^{-1}$ denotes the inverse of the laplacian $\triangle_A = D_A^* D_A$. $\chi_A : \omega^1 \to \omega^0$ is a 1-form on $\mathcal{A}$ with values in the Lie algebra of $\mathcal{G}$. The tangent space can now be split into two orthogonal subspaces $T_A(\mathcal{A}) = H_A \bigoplus V_A$. H_A is a horizontal subspace which is the kernel of χ_A. There is a horizontal projection $\Pi_A = 1 - D_A \chi_A$.

The space of non-equivalent gauge connections is obtained by quotienting the space $\mathcal{A}$ by the action of $\mathcal{G}$. The quotient space $\mathcal{A}/\mathcal{G}$ which is a manifold modulo certain restrictions[29] is often called the orbit space. We denote by ρ the projection $\rho : \mathcal{A} \to \mathcal{A}/\mathcal{G}$. A scalar product in the tangent space $T_a(\mathcal{A}/\mathcal{G})$ at a point $a \in \mathcal{A}/\mathcal{G}$ is defined as the one induced by (,). The metric on $\mathcal{A}/\mathcal{G}$ can be defined as follows. At a point A on the fiber $\rho^{-1}(a)$ above a, $X, Y \in T_a(\mathcal{A}/\mathcal{G})$ have horizontal lifts τ_X, τ_y then the metric on $\mathcal{A}/\mathcal{G}$ is given by $g(X,Y) = (\tau_X, \tau_Y)$. This metric is independent on the choice of A in $\rho^{-1}(a)$. If we denote by α_X, α_Y the coordinates of X and Y such that $D^*_{A_0}\alpha_X = D^*_{A_0}\alpha_Y = 0$ then the metric can be rewritten as $g(X,Y) = (\Pi_A\alpha_X, \Pi_A\alpha_Y)$. It can be shown[29] that the Lagrangian of the Yang-Mills theory has the following form $L = \frac{1}{2}(\Pi_A\dot{A}, \Pi_A\dot{A}) + V$. Thus the kinetic term is constructed with a metric in the orbit space.

A.2. Flat Connections

Flat connections A_f are connections for which the associated 2-form F is zero. Geometrically these are connections which correspond to a zero curvature on the principal bundle. Physically of course those are gauge potentials with zero field-strength. As is well known quantum mechanically the filed strength does not completely describe electromagnetic effects and their non-abelian generalizations.[30] The most important physical system that admits such structure is the Aharonov-Bohm effect[31] In this system an electron beam is affected by the electromagnetic potential in a doubly connected region $\mathbf{R^3} - \{l\}$ (or in fact $\mathbf{R^2} - \{0\}$ after dropping the irrelevant direction parallel to the flux line $\{l\}$) where the field-strength is zero. An interference pattern is observed which depends on the phase factor $exp(\int_\gamma A)$ around γ which is an unshrikable loop around the flux line. The Yang-Mills analog of the Aharonov-Bohm effect is dominated by the holonomy

$$\varphi_h = P[exp(\int_\gamma A)] = P[exp(\int_\gamma A_\alpha dx^\alpha)] \tag{A.2}$$

where P denotes path ordering It transforms under the gauge transformation (A.1) as follows: $\varphi_h \rightarrow g^{-1}(x_0)\varphi_h g(x_0)$ where x_0 is a point on γ. The Wilson line mentioned above is just the trace of φ_h in a given representation. So that φ is determined by the gauge fields up to conjugation.

Gauge equivalence classes of flat connections for any manifold M correspond to equivalence classes of homomorphism

$$\phi : \pi_1(M) \rightarrow G \tag{A.3}$$

up to conjugation.

Since we will be interested mainly in Riemann surfaces Σ, we proceed now to discuss the parametrization of flat connections A_f on Σ. On the sphere with no punctures it is straightforward to check that $F = 0$ correspond to the trivial "pure gauge" $A = h^{-1}\partial h \quad \bar{A} = h^{-1}\bar{\partial} h \quad h \in G$. For higher genus or in presence of punctures on the sphere the flat connections are determined, following (A.3) by the holonomies ϕ_h given in (A.2) in Σ. Consider the parallel transport[32] $\varphi(\xi) = P[exp \int_{\xi_0}^{\xi} A_f]$ which is a map from the universal covering of Σ to G. The fundamental group $\pi_1(\Sigma)$ acts on φ in the following way: $\varphi(p\xi) = \varphi_h(p)\varphi(\xi)$ where $\pi_1(\Sigma) \ni p \rightarrow \varphi_h(p) \in H$ is the holonomy of A_f. A_f is given by

$$A_f = \varphi^{-1}\partial\varphi, \tag{A.4}$$

up to the left multiplication of φ by a constant element of G^C which is the complexification of G. The conjugacy classes of the holonomy representations φ_h of $\pi_1(\Sigma)$ can be used to parametrize the flat connections. For the torus, the two commuting holonomies around the two homology cycles $\varphi(z+1) = \varphi_h^1\varphi(z) \quad \varphi(z+\tau) = \varphi_h^2\varphi(z)$ where $\tau = \tau_1 + i\tau_2$ is the period of the torus can be written, using conjugation, as $\varphi_h^1 = e^{-2\pi\theta_1} \quad \varphi_h^2 = e^{-2\pi\theta_2}$ In the last expressions θ_1 , and θ_2 are specific elements of the Cartan sub-Algebra[32]. We can then express A_f in terms of $\varphi = exp[\frac{\pi}{\tau_2}(\bar{\tau}z - \tau\bar{z})\theta_1 - (z - \bar{z})\theta_2]$. Substituting $\varphi = \varphi' h$, eqn. (A.4) now has the form $A = h^{-1}\partial h + h^{-1}M\omega h$ where $M\omega = \varphi'^{-1}\partial\varphi'$ is a Cartan sub-algebra one form. This expression can be generalized to higher genus as well. Denoting by ω_i and $\bar{\omega}_j$ the abelian differential around the a and b cycles, we now have: $A = h^{-1}\partial h + h^{-1}M^i\omega_i h$ and similarily for $\bar{A}$ where M and $\bar{M}$ are now in the Lie algebra with $[M, \bar{M}] = 0$. (This expression is of flat connection prior to moding out the gauge transformations.)

A.3. On the MSFC over Riemann surfaces

MSFC, the space of flat connections on a fixed principle bundle $P(\Sigma, G)$ modulo the group of gauge transformations, is determined by the equations:

$$\begin{aligned} D\delta A = 0 \qquad & D^*\delta A = 0 \\ D_{[\alpha}\delta A_{\beta]} = 0 \qquad & D_\alpha \delta A^\alpha = 0. \end{aligned} \tag{A.5}$$

Here $\delta A \in \omega^1(\Sigma, P')$ is an infinitesimal variation of the connection. This equation is of course a special case of the general discussion of section 2. In the mathematical literature[33][34] eqn. (A.5) is referred to as the linearization of the flat connection equations. An elliptic complex related to (A.5) is constructed and an application of the Atiyha-Singer theorem yields the dimension of the linearization. Then using slice theorems it is shown that the MSFC is a manifold with ~~the~~ same dimension. The flat connections generate a flat bundle $E_f(\Sigma, G)$ over Σ. According to eqn.(A.5) the flat connections can be classified in two cases: (i) No non-trivial solutions to (A.5) then the space of flat connections consist of finitely many isolated points that cannot be connected by a finite deformation. This is the situation when the fundamental group of M, $\pi_1(\Sigma)$, is a finite group. (ii) In case that $H^1(\Sigma, P) \neq 0$ then the flat connections are not isolated but are in a moduli space of gauge inequivalent flat connections, $\mathcal{A}_f/\mathcal{G}$.

The space dimensions of the moduli space for a Riemann surface of genus $g > 1$ is given by:

$$dim\mathcal{M} \;=\; (2g-2) \times dimG \tag{A.6}$$

where $dimG$ is the dimension of the group G. This can be easily shown in the following way: There are $2g$ homology cycles denoted by a_i and b_i with $i = 1, ..., g$. Correspondingly there are $2g$ holonomies of the connections denoted by $(\varphi^1_h)_i$ and $(\varphi^2_h)_i$. These holonomies have to obey the restriction

$(\varphi^1_h)_1(\varphi^2_h)_1(\varphi^1_h)_1^{-1}(\varphi^2_h)_1^{-1}...(\varphi^1_h)_g(\varphi^2_h)_g(\varphi^1_h)_g^{-1}(\varphi^2_h)_g^{-1} \;=\; 1$. There is a further redundancy related to conjugation. Thus eqn. (A.6) is verified.

Other geometrical properties of $\mathcal{A}_f/\mathcal{G}$ are briefly summarized: (i)$\mathcal{A}_f/\mathcal{G}$ inherits a symplectic structure from the symplectic structure of $\mathcal{A}/\mathcal{G}$ the orbit space. (ii)$\mathcal{A}_f/\mathcal{G}$ is a compact space (with some singularities). Its volume, with the natural symplectic volume element, is finite. (iii) $\mathcal{A}_f/\mathcal{G}$ is connected and simply connected. (iv) There is a way to obtain a Kahler structure on $\mathcal{A}_f/\mathcal{G}$. Taking a complex structure J on Σ, namely turning Σ into a Riemann manifold $\mathcal{A}_f/\mathcal{G}$ can be regarded as the moduli space of holomorphic vector bundles which are topologically trivial and have for the structure group the complexification of G. (v) $\mathcal{A}_f/\mathcal{G}$ for the case that a complex structure was introduced is a complex Kahler variety.

A.4. The moduli space of "self dual" connections

A generalization of the MSFC over a Riemann surface is the moduli space of "self dual" connections which is the space of solutions of the equations

$$F - \frac{i}{2}[\rho, \rho^*] = 0 \qquad D_{\bar{z}}\rho = 0, \tag{A.7}$$

modulo gauge transformations. Here $F = \epsilon^{\alpha\beta}F_{\alpha\beta}$ and ρ is a complex two dimensional scalar field in the adjoint representation (mathematically $\rho \in \omega^0(\Sigma,\ P' \times \mathcal{C})$)and $D_{\bar{z}} = D_1 - iD_2$. The source of the name "self dual" is the fact that eqn. (A.8) is just a dimensional reduction of the four dimensional self-dual configurations $F = \ {}^*F$ with $\partial_3 = \partial_4 = 0$ and $\rho = A_3 + iA_4$. Obviously by setting $\rho = 0$ the case of flat connection is recovered. The moduli space which is defined by the following linearizing equations

$$\begin{aligned} D\delta A - \frac{i}{2}([\delta\rho, \rho^*] + [\rho + \delta\rho^*] &= 0 \\ D_{\bar{z}}\delta\rho + [\delta A_{\bar{z}}, \rho]) &= 0 \\ D^*\delta A + Re[\rho^*, \delta\rho] &= 0 \end{aligned} \tag{A.8}$$

was investigated thoroughly by Hitchin [34].

Here are some of the geometrical properties of the MSSDC:

(i) The dimension of the MSSDC is $dim\ \mathcal{M} = 4(g-1) \times dim\mathrm{G}$. (ii) $\mathcal{M}$ is non-compact. (iii) $\mathcal{M}$ is connected and simply connected. (iv) $\mathcal{A}_f/\mathcal{G}$ has a natural metric which is complete. This is a hyperkhaler metric which means that the metric is khalerian with respect to three complex structures which satisfy the algebraic identities of the quaternions. (v) The Betti numbers b_i of the $\mathcal{M}$ vanish for $i > 6g-6$. The expressions for the non vanishing Betti numbers are given in ref.[34].

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