J. Math. Kyoto Univ. (JMKYAZ)
38-4 (1998), 781–788

# Topological realization of the integer ring of local field

By

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## 1. Introduction

In the stable homotopy theory the complex cobordism theory $MU$ and its $p$-local wedge summand $BP$ are very important. The Morava $K$-theories $K(n)^*(\ )$ were invented by J. Morava in the early 1970s to understand the complex cobordism theory. In the present, however, from the work of Devinatz, Hopkins and Smith [2], [3], it becomes clear that Morava $K$-theories themselves play a very important and fundamental role in the stable homotopy theory.

Let $p$ be a prime number. We consider in the $p$-local stable homotopy category. Morava $K$-theory $K(n)^*(\ )$ is a periodic cohomology of period $2(p^n-1)$. The coefficient ring is given by

$$K(n)_* = \boldsymbol{F}_p[v_n, v_n^{-1}],\ |v_n| = 2(p^n-1)$$

where $v_n$ is the Hazewinkel generator. Let $\widehat{K(n)}$ be the $p$-adic Morava $K$-theory spectrum whose coefficient ring satisfies

$$\widehat{K(n)}_* = \mathcal{O}_K[u, u^{-1}],\ |u| = 2$$

where $K$ is the degree $n$ unramified extension of the $p$-adic number field $\boldsymbol{Q}_p$ and $\mathcal{O}_K$ is its integer ring. To simplify gradings, we use a formal $(p^n-1)$-th root $u$ of $v_n$. It is known that the associated formal group law is the Lubin-Tate one [4]. Therefore $\widehat{K(n)}$ has intimate relation with the local class field theory. For example, the homotopy group of the Tate spectrum $t_{\boldsymbol{Z}/p}\widehat{K(n)}$ is the degree $p^n-1$ totally ramified abelian extension of $K$.

In this paper, as one aspect of this relation, we shall topologically realize the totally ramified abelian extensions of $\mathcal{O}_K$ which appear in the local class field theory by the method of Lubin-Tate formal group law. The main result (Theorem 3.3) is saying that we can construct a sequence of ring spectra and ring spectrum maps

$$\widehat{K(n)}(0) \xrightarrow{F_0} \widehat{K(n)}(1) \xrightarrow{F_1} \widehat{K(n)}(2) \xrightarrow{F_2} \cdots$$

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Supported by JSPS Research Fellowships for Young Scientists.
Received Feburuary 13, 1998.

whose homotopy group is isomorphic to the tower of totally ramified abelian extensions of $\mathcal{O}_K$, by using the classifying spaces of cyclic groups and the stable transfer maps. Then we shall show that there is a similarity between the ring spectrum automorphisms of the spectra constructed above and the Galois theory.

Now we consider the ring spectrum maps from $BZ/p^r{}_+$ to $\widehat{K(n)}\otimes\mathcal{O}_L$ where $L$ is a finite extension of the quotient field of $\widehat{K(n)}_0$ and $\mathcal{O}_L$ is its integer ring. Kordzaya and Nishida [5] proved that the group of such ring spectrum maps is isomorphic to $(Z/p^r)^n$ if $L$ is sufficiently large. Since the quotient field $L_r$ of $\widehat{K(n)}(r)_0$ contains the primitive $p^r$-th roots of unity, it is the minimum splitting field of $\widehat{K(n)}^0(BZ/p^r)$ in the sense of [5]. Hence, by the result of [5], we see that the grouplike elements of $\widehat{K(n)}(r)^0(BZ/p^r)$ induce the ring spectrum maps from $BZ/p^r{}_+$ to $\widehat{K(n)}\otimes\mathcal{O}_L$.

I would like to thank Professor Goro Nishida for suggestion of this work and many helpful conversations.

## 2. Totally ramified extension of $\mathcal{O}_K$

Let $\widehat{K(n)}^*(\ )$ be the $p$-adic Morava $K$-theory. Using periodicity, we can consider that $\widehat{K(n)}^*(\ )$ is graded by $Z/2$. Then we obtain a formal group law over $\widehat{K(n)}_0=\mathcal{O}_K$. By the Lubin-Tate theory [4], we can choose an orientation class $x\in\widehat{K(n)}^0(CP^\infty)$ such that

$$[p](x)=px+x^{p^n}.$$

We recall that

$$\widehat{K(n)}^0(BZ/p^r)\cong\mathcal{O}_K[[x]]/([p^r](x)).$$

Hence $\widehat{K(n)}^0(BZ/p^r)$ is a finitely generated free module over $\mathcal{O}_K$.

We consider the following decomposition of $[p^r](x)$:

$$[p^r](x)=x\cdot\frac{[p](x)}{x}\cdots\frac{[p^r](x)}{[p^{r-1}](x)}$$

where $\frac{[p^{i+1}](x)}{[p^i](x)}$ are so called Eisenstein polynomials. Let

$$L_i=K[x]/\left(\frac{[p^i](x)}{[p^{i-1}](x)}\right)$$

and

$$\mathcal{O}_{L_i}=\mathcal{O}_K[x]/\left(\frac{[p^i](x)}{[p^{i-1}](x)}\right).$$

By the local class field theory, $L_i$ is the degree $p^{n(i-1)}(p^n-1)$ totally ramified abelian extension of $K$ and $\mathcal{O}_{L_i}$ is its integer ring. Then we have an epimorphism:

$$\widehat{K(n)}^0(BZ/p^r)\to\mathcal{O}_{L_r}.$$

In this section we shall topologically realize this epimorphism.

First we recall the well-known fact about the multiplicative property of transfer (cf.[1]). Let $\pi: E \to B$ be a finite covering and let $\tau: B_+ \to E_+$ be the corresponding transfer where $(\ )_+$ denote the suspension spectrum of the pointed space with disjoint base point.

**Lemma 2.1.** *Let $h$ be a multiplicative cohomology theory. Then $\tau^*(y \cup \pi^*(x)) = \tau^*(y) \cup x$ for all $x \in h^*(B)$, $y \in h^*(E)$.*

Let $\tau_r: BZ/p^r{}_+ \to BZ/p^{r-1}{}_+$ be the transfer associated with the inclusion $Z/p^{r-1} \subset Z/p^r$. We consider the homomorphism

$$\tau_r^*: \widehat{K(n)}^0(BZ/p^{r-1}) \to \widehat{K(n)}^0(BZ/p^r).$$

**Lemma 2.2.** $\tau_r^*(1) = \dfrac{[p^r](x)}{[p^{r-1}](x)}$.

*Proof.* We prove this by induction on $r$. For $r=1$, let $t(x) \in \widehat{K(n)}_*[[x]]$ be a power series such that $t(x) \equiv \tau_1^*(1) \bmod ([p^r](x))$. Then it is easy to see that $t(0)=p$. By Lemma 2.1,

$$0 = \tau_1^*(x) = \tau_1^*(1) \cdot x.$$

Therefore there is a power series $v(x) \in \widehat{K(n)}_*[[x]]$ such that

$$x \cdot t(x) = [p](x) \cdot v(x).$$

Since $\widehat{K(n)}_*[[x]]$ is a domain, we see that

$$t(x) = v(x) \cdot \frac{[p](x)}{x} \quad \text{and} \quad \tau_1^*(1) = v(0) \cdot \frac{[p](x)}{x}.$$

From the fact that the constant term of $\dfrac{[p](x)}{x}$ is $p$, we obtain

$$\tau_1^*(1) = \frac{[p](x)}{x}.$$

Next we assume that the lemma is true for $r-1$. There is a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & Z/p^{r-1} & \to & Z/p^r & \to & Z/p & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & Z/p^{r-2} & \to & Z/p^{r-1} & \to & Z/p & \to & 0. \end{array}$$

Hence we obtain a covering map:

$$\begin{array}{ccc} BZ/p^{r-1} & \overset{\pi_{r-1}}{\to} & BZ/p^{r-2} \\ \downarrow & & \downarrow \\ BZ/p^{r} & \overset{\pi_{r}}{\to} & BZ/p^{r-1} \end{array}$$

where $\pi_{r-1}$ and $\pi_r$ are the maps induced by the projection. By the naturality of the transfer, we obtain a commutative diagram:

$$\begin{array}{ccc} BZ/p^{r-1}{}_+ & \overset{\pi_{r-1}}{\to} & BZ/p^{r-2}{}_+ \\ \uparrow \tau_r & & \uparrow \tau_{r-1} \\ BZ/p^{r}{}_+ & \overset{\pi_{r}}{\to} & BZ/p^{r-1}{}_+ . \end{array} \tag{1}$$

Then $\tau_r^*(1)=\tau_r^*\pi_{r-1}^*(1)=\pi_r^*\tau_{r-1}^*(1)=\pi_r^*\left(\frac{[p^{r-1}](x)}{[p^{r-2}](x)}\right)=\frac{[p^r](x)}{[p^{r-1}](x)}$.

**Remark 2.3.** *The fact that* $\tau^*{}_1(1)=\frac{[p](x)}{x}$ *appears in Kriz's paper* [6].

Let $F(p^r)\to BZ/p^r{}_+\to BZ/p^{r-1}{}_+$ be a cofibre sequence.

**Lemma 2.4.** *There is an exact sequence*:

$$0\to\widehat{K(n)}^0(BZ/p^{r-1})\overset{\tau_r^*}{\to}\widehat{K(n)}^0(BZ/p^r)\to\widehat{K(n)}^0(F(p^r))\to 0.$$

*Proof.* It is enough to prove that $\tau_r^*$ is injective. Let $a\in\operatorname{Ker}\tau_r^*$. There is a power series $t(x)\in\mathcal{O}_K[[x]]$ such that $t(x)\equiv a \bmod ([p^{r-1}](x))$. Let $b\in\widehat{K(n)}^0(BZ/p^r)$ be the reduction of $t(x)$. By Lemma 2.1 and Lemma 2.2, $0=\tau_r^*a=\tau_r^*(1)\cdot b$ $=b\cdot\frac{[p^r](x)}{[p^{r-1}](x)}$. Hence there is a power series $v(x)\in\mathcal{O}_K[[x]]$ such that

$$t(x)\cdot\frac{[p^r](x)}{[p^{r-1}](x)}=v(x)\cdot[p^r](x).$$

This implies $t(x)=v(x)\cdot[p^{r-1}](x)$ and $a=0$. This completes the proof.

Using this lemma, we can regard $\widehat{K(n)}^0(BZ/p^{r-1})$ as a submodule of $\widehat{K(n)}^0(BZ/p^r)$. By Lemma 2.1, we see that $\widehat{K(n)}^0(BZ/p^{r-1})$ is an ideal of $\widehat{K(n)}^0(BZ/p^r)$. Therefore $\widehat{K(n)}^0(F(p^r))$ has the induced ring structure.

**Theorem 2.5.** $\widehat{K(n)}^0(F(p^r))\cong\mathcal{O}_{L_r}$.

*Proof.* This follows from Lemma 2.2 and Lemma 2.4.

**Remark 2.6.** Let $E^*(\ )$ be a complex oriented cohomology theory. We consider the transfer $\tau_r^*: E^*(BZ/p^{r-1})\to E^*(BZ/p^r)$. Then in the same way we can show that $\tau_r^*(1)=\dfrac{[p^r](x)}{[p^{r-1}](x)}$. Furthermore, if $E^*(BZ/p^r)\cong E_*[[x]]/([p^r](x))$, then

$$E^*(F(p^r))\cong E_*[[x]]/\left(\frac{[p^r](x)}{[p^{r-1}](x)}\right).$$

Now we consider the relation between $\widehat{K(n)}^0(F(p^r))$ and $\widehat{K(n)}^0(F(p^{r+1}))$. From the commutative diagram (1), we obtain a spectrum map $f_r: F(p^{r+1})\to F(p^r)$ which commutes the following diagram:

$$\begin{array}{ccccc} F(p^{r+1}) & \to & BZ/p^{r+1}{}_+ & \xrightarrow{\tau_{r+1}} & BZ/p^r{}_+ \\ \downarrow f_r & & \downarrow \pi_{r+1} & & \downarrow \pi_r \\ F(p^r) & \to & BZ/p^r{}_+ & \xrightarrow{\tau_r} & BZ/p^{r-1}{}_+ . \end{array}$$

**Proposition 2.7.** *The induced homomorphism*

$$f_r^*:\widehat{K(n)}^0(F(p^r))\to\widehat{K(n)}^0(F(p^{r+1}))$$

*is the degree $p^n$ totally ramified abelian extension.*

*Proof.* We consider the homomorphism

$$\pi_{r+1}^*:\widehat{K(n)}^0(BZ/p^r)\to\widehat{K(n)}^0(BZ/p^{r+1}).$$

Then $\pi_{r+1}^*(x)=[p](x)$. Hence $f_r^*(x)=[p](x)$. The proposition thus follows from the local class field theory.

## 3. Cohomology theory $\widehat{K(n)}(r)^*(\ )$

Let $\widehat{K(n)}(r)$ be the function spectrum $F(F(p^r),\widehat{K(n)})$. In particular we define $\widehat{K(n)}(0)=\widehat{K(n)}$. We recall that there are spectrum maps

$$f_r: F(p^{r+1})\to F(p^r).$$

We define

$$f_0: F(p)\to BZ/p_+\xrightarrow{j} S^0$$

where $j$ is the pinch map. Let $F_r$ be the spectrum map

$$F(f_r, id): \widehat{K(n)}(r) \to \widehat{K(n)}(r+1).$$

In this section we show that $\widehat{K(n)}(r)$ are ring spectra and $F_r$ ring spectrum maps. Then we show that there is a similarity between the ring spectrum automorphism of $\widehat{K(n)}(r)$ and the Galois theory.

Let $\widehat{K(n)}(r)^*(\ )$ be the cohomology theory represented by $\widehat{K(n)}(r)$. We recall that $\widehat{K(n)}^*(F(p^r))$ is a finitely generated free module over $\widehat{K(n)}_*$. Hence if $X$ is a CW complex, then there is an isomorphism:

$$\begin{aligned}\widehat{K(n)}(r)^*(X) &\cong \widehat{K(n)}^*(X) \otimes_{\widehat{K(n)}_*} \widehat{K(n)}^*(F(p^r)) \\ &\cong \widehat{K(n)}^*(X) \otimes_{\mathcal{O}_K} \mathcal{O}_{L_r}.\end{aligned}$$

Using this isomorphism we can define a natural ring structure on $\widehat{K(n)}(r)^*(\ )$. Hence we obtain the following lemma.

**Lemma 3.1.** *$\widehat{K(n)}(r)$ are ring spectra.*

Let $F_{r*}$ be the natural transformation defined by $F_r$. By the definition of the multiplicative structure of $\widehat{K(n)}(r)^*(\ )$, we see that $F_{r*}$ are multiplicative natural transformations. Hence we obtain the following lemma.

**Lemma 3.2.** *$F_r: \widehat{K(n)}(r) \to \widehat{K(n)}(r+1)$ are ring spectrum maps.*

Therefore we obtain the following theorem.

**Theorem 3.3.** *There is a sequence of ring spectra and ring spectrum maps:*

$$\widehat{K(n)}(0) \xrightarrow{F_0} \widehat{K(n)}(1) \xrightarrow{F_1} \widehat{K(n)}(2) \xrightarrow{F_2} \cdots.$$

*The homotopy group of this sequence is the tower of the totally ramified abelian extensions of $\mathcal{O}_K$:*

$$\mathcal{O}_K \subsetneq \mathcal{O}_{L_1} \subsetneq \mathcal{O}_{L_2} \subsetneq \cdots.$$

Let $r > s \geq 0$. By a $\widehat{K(n)}(r)$-isomorphism over $\widehat{K(n)}(s)$, we mean a ring spectrum map $\Phi: \widehat{K(n)}(r) \to \widehat{K(n)}(r)$ which is a homotopy equivalence and satisfies the following commutative diagram:

$$\begin{array}{ccc} \widehat{K(n)}(r) & \xrightarrow{\Phi} & \widehat{K(n)}(r) \\ & \nwarrow \quad \nearrow & \\ & \widehat{K(n)}(s). & \end{array}$$

Let $Aut(\widehat{K(n)}(r,s))$ denote the set of $\widehat{K(n)}(r)$-isomorphisms over $\widehat{K(n)}(s)$. Then $Aut(\widehat{K(n)}(r,s))$ has a group structure with respect to the composition.

**Theorem 3.4.** $Aut(\widehat{K(n)}(r,s)) \cong Gal(L_r/L_s)$.

*Proof.* This isomorphism follows from the facts that there are multiplicative isomorphisms:

$$\widehat{K(n)}(r)^*(\ ) \cong \widehat{K(n)}^*(\ ) \otimes_{\mathcal{O}_K} \mathcal{O}_{L_r}$$
$$\widehat{K(n)}(s)^*(\ ) \cong \widehat{K(n)}^*(\ ) \otimes_{\mathcal{O}_K} \mathcal{O}_{L_s}$$

and $F_{r-1*} \circ \cdots \circ F_{s*}$ is induced by the inclusion: $\mathcal{O}_{L_s} \subsetneq \mathcal{O}_{L_r}$.

Now we consider the ring spectrum maps $BZ/p^r{}_+ \to \widehat{K(n)}(r)$. Let $L$ be an extension of $K$. We set $\widehat{K(n)}^0(BZ/p^r)_L = \widehat{K(n)}^0(BZ/p^r) \otimes L$. According to [5], we say that $L$ is a splitting field of the Hopf algebra $\widehat{K(n)}^0(BZ/p^r)$ if $L$ is a splitting field of both $K$-algebras $\widehat{K(n)}^0(BZ/p^r)_K$ and $\widehat{K(n)}^0(BZ/p^r)_K^*$ where we regard $\widehat{K(n)}^0(BZ/p^r)_K^* = \mathrm{Hom}_K(\widehat{K(n)}^0(BZ/p^r)_K, K)$ as $K$-algebra by means of the coalgebra structure of $\widehat{K(n)}^0(BZ/p^r)$. It was proved by Kordzaya and Nishida [5] that the group of the ring spectrum maps $BZ/p^r_+ \to \widehat{K(n)}^0 \otimes \mathcal{O}_L$ is isomorphic to $(Z/p^r)^n$ if $L$ is a splitting field. From the $K$-algebra structure of $\widehat{K(n)}^0(BZ/p^r)_K$ we note that every splitting field must contain the quotient field $L_r$ of $\widehat{K(n)}(r)_0$. Let $V_r$ be the set of all $K$-algebra homomorphisms from $\widehat{K(n)}^0(BZ/p^r)_K$ to the algebraic closure $\bar{K}$ of $K$. Then $V_r$ is a group isomorphic to $(Z/p^r)^n$. There is an isomorphism as Hopf algebras:

$$\widehat{K(n)}^0(BZ/p^r)_{\bar{K}} \cong \bar{K}[V_r]^*$$

where $\bar{K}[V_r]^*$ is the dual Hopf algebra of the group ring $\bar{K}[V_r]$.

**Lemma 3.5.** *The quotient field $L_r$ of $\widehat{K(n)}(r)_0$ contains all the $p^r$-th roots of unity.*

*Proof.* Let $Q_p^r$ be the cyclotomic field of $p^r$-th roots of unity over $Q_p$ and $E = Q_p^r \cdot K$. We note that $E$ is a finite abelian extension over $K$. Let $N(\ )$ denote a norm group. Then

$$N(L_r/K) = \langle p \rangle \times (1 + p^r \mathcal{O}_K)$$
$$N(Q_p^r/Q_p) = \langle p \rangle \times (1 + p^r Z_p).$$

By local class field theory, there is a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
1 & \to & N(E/K) & \to & K^{\times} & \to & Gal(E/K) & \to & 1 \\
 & & \downarrow & & \downarrow N & & \downarrow \cong & & \\
1 & \to & N(Q_p^r/Q_p) & \to & Q_p^{\times} & \to & Gal(Q_p^r/Q_p) & \to & 1
\end{array}$$

where the middle vertical arrow is the norm map and the right vertical arrow is an isomorphism. So we see that $N(E/K) \supset N(L_r/K)$. Then the lemma follows from the fundamental theorem in local class field theory.

Therefore we obtain the following theorem.

**Theorem 3.6.** *The quotient field $L_r$ of $\widehat{K(n)}(r)_0$ is the unique minimal splitting field of $\widehat{K(n)}^0(BZ/p^r)$. If $L$ is any splitting field, then the ring spectrum map $BZ/p^r{}_+ \to K(n)\otimes\mathcal{O}_L$ factors through the ring spectrum map $BZ/p^r{}_+ \to \widehat{K(n)}(r)$.*

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