TOPOLOGICAL REPRESENTATION OF SEMIGROUPS

A. B. PAALMAN - DE MIRANDA

Amsterdam

1. Introduction

J. de Groot has proved in [3] that for every group G one can find a connected metric space M such that the group of all autohomeomorphisms of M is isomorphic to $G : G \simeq A(M)$.

To represent semigroups in a similar way, we must replace the group of auto-homeomorphisms by a suitable semigroup of continuous mappings. The aim of this note is to prove that every semigroup S with identity element can be represented by the semigroup $Q(M)$ of all quasi-local homeomorphisms of a metric space M into itself.

Let X, Y be topological spaces. A mapping $f : X \to Y$ is called a *quasilocal homeomorphism* if f is continuous and if for each open set $O \subset X$ there exists an open set V, $V \subset O$ such that $f \mid V$ is a homeomorphism of V onto $f(V)$.

The proof of the theorem is essentially a modification of the proof for groups by J. de Groot in [3].

The semigroup $Q(M)$ of all quasi-local homeomorphisms seems to be the most suitable to replace the group of all autohomeomorphisms $A(M)$. We prove in section 4 the existence of a semigroup S such that there is no Hausdorff-space H such that S is isomorphic to the semigroup of all local homeomorphisms of H into itself. Neither can S be isomorphic to the semigroup of all open continuous mappings of H into itself. $f : X \to Y$ is a local homeomorphism if for each $x \in X$ there exists an open set O, $x \in O$ such that $f \mid O$ is a homeomorphism of O onto $f(O)$.

Analogous problems were treated by Z. Hedrlín and A. Pultr [6] and by L. Bukovský, Z. Hedrlín and A. Pultr [1]. In [6] the following theorem was proved. Let S be a semigroup with identity element, then there exists a T_0-space T such that S is isomorphic to the semigroup of all local homeomorphisms of T into itself.

In [1] it has been shown that every semigroup with identity element may be represented by the semigroup of all "quasi-coverings" of a Hausdorff space into itself. The "quasi-coverings" however are rather special mappings.

Let for instance X be the subset of the real line R consisting of the point 0 and all x, $x \geqq 1$. $X = \{x \mid x \in R,\ x = 0 \text{ or } x \geqq 1\}$.

Let $f : X \to X$ and $g : X \to X$ be defined respectively by

$$f(x) = \begin{cases} x & \text{if } x = 0 \\ 2x & \text{if } x \neq 0 \end{cases}, \qquad g(x) = \begin{cases} 1 & \text{if } x = 0 \\ 2x & \text{if } x \neq 0 \end{cases}.$$

Both f and g are homeomorphisms of X into X, f however is a quasicovering of $f(X)$ but g is not a quasi-covering of $g(X)$.

2. Graph-representations

Let S be a semigroup with identity element e and $\{s_\alpha\}$ a system of generators of S. We now construct the Cayley-graph S' of S. S' is a coloured, directed graph such that each element $a \in S$ is represented by one vertex v_a of S'. Two vertices v_a and v_b are joined by an edge with "colour" s_α directed from v_a to v_b whenever $b = s_\alpha a$. S' is clearly connected (if $a = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_n}$, then v_e and v_a are joined by a path along a set of consecutively adjacent edges with colour respectively $s_{\alpha_n}, s_{\alpha_{n-1}}, \dots, s_{\alpha_2}, s_{\alpha_1}$). With each $a \in S$ we associate the inner right translation ϱ_a

$$\varrho_a : x \to xa \quad \text{for all} \quad x \in S .$$

When applying products of mappings from the left to the right

$$(x)\, \varrho_a \cdot \varrho_b = (x \varrho_a)\, \varrho_b$$

we see that S is homeomorphic to its regular representation S_r. This representation is faithful since S contains an identity element: $S \simeq S_r$. Furthermore it can easily be seen that S_r is isomorphic to the semigroup of all transformations of the graph S' into itself which are colour and orientation preserving.

If S is a semigroup with cancellation then all such transformations are one-to-one mappings of S' into itself.

From S' we now construct an (uncoloured) directed graph S^* such that the semigroup of all endomorphisms $E(S^*)$ of S^* is isomorphic to S. For countable semigroups this has been done first by the author [7], for semigroups with cardinality less than the first unaccessible cardinal by Z. Hedrlín and A. Pultr [5] and for arbitrary semigroups by P. Vopěnka, A. Pultr and Z. Hedrlín [8]. They constructed for any cardinal $\mathfrak{m}$ a directed graph X such that the identity transformation is the only endomorphism of X and such that the cardinal of the set of vertices of X is equal to $\mathfrak{m}$.

The construction of S^* given here is different from the one in [5], since the rigid graph X plays a completely different role.

Construction. Let S' be the Cayley-graph of S and let $\mathfrak{m}$ be the cardinal of the set of generators $\{s_\alpha\}$ of S. We assume $\mathfrak{m} \geqq 3$ (the case of semigroups of order < 3 can be treated separately in a simple way). Let D be the rigid graph constructed in [8], where $D = \{\beta \mid \beta \leqq \omega_\xi + 1,\ \omega_\xi$ the least ordinal with card $\omega_\xi = \mathfrak{m}\}$. Finally let ϕ be a one-to-one mapping of the set $\{s_\alpha\}$ onto D.

Suppose that a directed edge with colour s_α leads from vertex v_a to v_b. Replace the edge in S' by a graph (D, α, a, b) defined as follows: edges $(v_a, p^\alpha_{a,b})$, $(p^\alpha_{a,b}, v_b)$,

$(p^{\alpha}_{a,b}, \phi(s_{\alpha}))$ and furthermore D. We do this for every edge of S', but we take care that all graphs (D, α, a, b) are disjoint with the possible exception of their vertices v_a and v_b. In this way S' is transformed into a graph S^*.

Theorem 1. $E(S^*) \simeq S$.

Proof. Let $f \in E(S^*)$ and let $D^{\alpha}_{a,b}$ be the copy of D contained in the subgraph (D, α, a, b) of S^*.

We first prove that $f(D^{\alpha}_{a,b}) \subset D^{\gamma}_{c,d}$ for some γ, c and d.

Since $D^{\alpha}_{a,b}$ contains the edges

$$(0^{\alpha}_{a,b}, 1^{\alpha}_{a,b}), \ (0^{\alpha}_{a,b}, 2^{\alpha}_{a,b}) \quad \text{and} \quad (1^{\alpha}_{a,b}, 2^{\alpha}_{a,b})$$

it follows that $f(0^{\alpha}_{a,b})$ cannot be a vertex of the form v_a or $p^{\alpha}_{a,b}$ of S^*. Hence $f(0^{\alpha}_{a,b}) \subset$ $\subset D^{\gamma}_{c,d}$ for some γ, c and d.

If $\beta^{\alpha}_{a,b} \in D^{\alpha}_{a,b}$, then there is a finite chain of directed edges connecting $0^{\alpha}_{a,b}$ and $\beta^{\alpha}_{a,b}$. From this it follows that $f(\beta^{\alpha}_{a,b}) \in D^{\gamma}_{c,d}$, hence $f(D^{\alpha}_{a,b}) \subset D^{\gamma}_{c,d}$.

From the rigidity of D it follows that $f(\beta^{\alpha}_{a,b}) = \beta^{\gamma}_{c,d}$.

We next prove that $f(p^{\alpha}_{a,b}) = p^{\gamma}_{c,d}$.

Since $p^{\alpha}_{a,b}$ is connected with $\phi(s_{\alpha})^{\alpha}_{a,b}$, we have $f(p^{\alpha}_{a,b}) = p^{\gamma}_{c,d}$ which implies $\gamma = \alpha$ or $f(p^{\alpha}_{a,b}) \in D^{\gamma}_{c,d}$.

In this case $f(p^{\alpha}_{a,b}) = \beta^{\gamma}_{c,d}$ for some $\beta \in D$ $\beta < \phi(s_{\alpha})$. Now let α' be chosen so that $\phi(s_{\alpha'}) = \beta$, and let $q = s_{\alpha'}b$. Then it follows from the construction of S^* that $f(v_b) \in D^{\gamma}_{c,d}$, hence $f(p^{\alpha'}_{b,q}) \in D^{\gamma}_{c,d}$ and this implies $f(\phi(s_{\alpha'})^{\alpha'}_{b,q}) = \phi(s_{\alpha'})^{\gamma}_{c,d} \in D^{\gamma}_{c,d}$.

From the construction of D it then follows that $\beta < \phi(s_{\alpha'})$ a contradiction.

Thus each vertex of the form $p^{\alpha}_{a,b}$ of S^* is mapped onto a vertex of the form $p^{\alpha}_{c,d}$. From this it follows that each vertex of the form v_a is mapped onto a vertex of the form v_b.

It can now easily be seen that $E(S^*)$ is isomorphic to the semigroup of all transformations of S' into itself which are colour and orientation preserving. Hence $E(S^*) \simeq S$.

If S is a semigroup with cancellation then each transformation $f \in E(S^*)$ is one-to-one.

3. Quasi-local homeomorphisms

Similarly as in [3] we shall replace every edge of S^* by mutually homeomorphic topological spaces P and introduce a topology in the resulting set such that a space M will be obtained satisfying the following condition:

$$Q(M) \simeq S .$$

An example of a Peano curve P which is rigid under topological transformations of P into P was given in [2]. We briefly mention its construction.

Consider a circle C^1 in the plane and let $\{a_i^k\}_{i,k}$ be a double sequence of distinct natural numbers >2. Let $\{p_i^1\}$ be a countable everywhere dense subset of C^1. Affixe to each p_i^1 a chain C_i^1 of a_i^1 links, contained in the interior of C^1 (p_i^1 excepted) and mutually disjoint. Next we take a countable dense subset $\{p_i^2\}$ on the union of all C_i^1 such that each p_i^2 is of order two. Affixe to each p_i^2 a chain C_i^2 of a_i^2 links contained in the interior of that link to which p_i^2 belongs, and such that all new chains are mutually disjoint. Proceed by induction; we take care that the diameters of the C_i^k tend to zero, and take the closure P of the countable number of chains obtained in this manner. We remark that P is not rigid for topological transformations of P into P only, but also for quasi-local homeomorphisms.

Let f be a quasi-local homeomorphism and let $\{p_i^k\}^*$ be the set of all points p_i^k such that there is an open set O, $p_i^k \in O$ with $f \mid O$ a homeomorphism. The set $\{p_i^k\}^*$ is everywhere dense in P. Since the p_i^k are the only points of maximal order (order 6) in P, the set $\{p_i^k\}^*$ is mapped into the set $\{p_i^k\}$. To each p_i^k is affixed a chain of a_i^k links, all a_i^k distinct. This implies that $f(p_i^k) = p_i^k$ for all $p_i^k \in \{p_i^k\}^*$. Since $\{p_i^k\}^*$ is dense in P, f is the identity transformation.

Now let a and b be two points on the circle C^1 of order two. Each directed edge $\alpha = \overrightarrow{(x_1, x_2)}$ of S^* is replaced by a copy P_α of P, a replacing x_1 and b replacing x_2. We take care that all P_α are disjoint with the possible exception of the points a and b.

Into the union of all P

$$M = \bigcup_\alpha P_\alpha$$

we introduce a metric in the same way as in [3].

Theorem 2. *Let S be a semigroup with identity element. Then there exists a connected metric space M such that S is isomorphic to the semigroup of all quasi-local homeomorphisms of $M : S \simeq Q(M)$.*

Proof. Let M be the metric space, obtained from the graph S^*. M is clearly connected.

If $f^* \in E(S^*)$, then it can easily be seen that f^* can be extended to a quasi-local homeomorphism f of M into M.

Now let f be a quasi-local homeomorphism of M into M. We shall prove that f maps every copy of P identically onto a copy of P. Let P_α be such a copy of P. P_α is compact and connected, hence $f(P_\alpha)$ is compact, which implies $f(P_\alpha) \subset \bigcup_{i=1}^{n} P_{\beta_i}$. Let $\{p_i^k\}^*$ be the set of all points $p_i^k \in P_\alpha$ such that there is an open set O, $p_i^k \in O$ with $f \mid O$ a homeomorphism. Then $\{p_i^k\}^*$ is mapped into the set of all points of maximal order in $\bigcup_{i=1}^{n} P_{\beta_i}$ together with the set of endpoints $\{a_{\beta_i}, b_{\beta_i}\}_{i=1}^{n}$.

Let $\{p_i^k\}^1 \subset \{p_i^k\}^*$ be the set of all points which are mapped into the set of all points of maximal order in $\bigcup_{i=1}^{n} P_{\beta_i}$. Then $\{p_i^k\}^1$ is everywhere dense in P_α, and it is not difficult to see that each point $p_i^k \in \{p_i^k\}^1$ is mapped onto the corresponding point p_i^k contained in one of the P_{β_i}. From this it follows that every point $x \in P_\alpha$ is mapped onto a corresponding point x contained in one of the P_{β_i}.

Since we have chosen the endpoints a and b of P to be points of order two and since S^* contains no trivial cycles of order two it follows that P_α is mapped identically on another copy P_β of P.

Hence f permutes the P_α's among themselves, and we may conclude from theorem 1 that $S \simeq E(S^*) \simeq Q(M)$.

Corollary. *Let S be a semigroup with cancellation, with identity element. Then there is a connected metric space M such that S is isomorphic to the semigroup of all homeomorphisms of M into M.*

The proof follows easily from the fact that in this case each transformation $f^* \in E(S^*)$ is one-to-one.

Theorem 3. *Let S be a semigroup with identity element. Then there exists a connected compact Hausdorff space H such that S is isomorphic to $Q(H)$.*

Proof. Let M be the metric space such that $S \simeq Q(M)$, and let H be the Čech-Stone compactification of M. Let f be a quasi-local homeomorphism of M into M and βf its extension to H. Since M contains an open dense subset such that every point of this set has a neighbourhood with compact closure, it follows that for every open set $O \subset H$ there is an open set V, $V \subset O$ such that $V \subset M$. This together with the fact that βf is continuous implies that βf is a quasi-local homeomorphism of H.

Now let g be an element of $Q(H)$. As g is a quasi-local homeomorphism there is for every open set $O \subset H$ an open set $V \subset M$ such that $g \mid V$ is a homeomorphism.

Since M is metric, it satisfies the first axiom of countability and for every point $x \in V$ there is a countable sequence of different points $x_n \in V$ converging to x, hence $g(V) \subset M$. Next let x be an arbitrary point of M, then there exists a sequence $\{x_n\}$, $x_n \in M$, $x_n \to x$ such that $g(x_n) \in M$. From the continuity of g it follows that $g(x_n) \to g(x)$ and hence $g(x) \in M$.

Thus $g(M) \subset M$ and g restricted to M is a quasi-local homeomorphism of M into itself. From this follows easily

$$Q(H) \simeq Q(M), \quad \text{so} \quad Q(H) \simeq S.$$

Corollary. *Let S be a semigroup with cancellation and identity element. Then there is a connected compact Hausdorff space H such that S is isomorphic to the semigroup $T(H)$ of all topological transformations of H into H. Moreover $T(H) = Q(H)$.*

4. Local homeomorphisms and open continuous mappings

Let S be the semigroup $\{e, a, b\}$ with identity element e and multiplication defined by $ab = ba = aa = bb = a$.

Let H be a Hausdorff space and $L(H)$ the semigroup of all local homeomorphisms of H into itself.

$O(H)$ will denote the semigroup of all open continuous mappings of H into H.

Theorem 4. *There is no Hausdorff space H such that S is isomorphic to $L(H)$.*

Proof. Let S be isomorphic to $L(H)$. Then $L(H) = \{\varepsilon, f, g\}$ with ε the identity mapping and f and g local homeomorphisms such that $fg = gf = ff = gg = g$. Let A be the subset of H such that for each $a \in A$ $f(a) = g(a)$. Then A is closed. $A \neq H$ and $A \neq \emptyset$ since for each point $b \in f(H)$ we have $f(b) = g(b)$. We now prove that A is open. Let $p \in \overline{H \setminus A}$, $p \in A$. Let O be a neighbourhood of $f(p) = g(p)$ such that f is a homeomorphism on O.

Let V be a neighbourhood of p such that $f(V) \subset O$ and $g(V) \subset O$. Since $p \in \overline{H \setminus A}$, there is a point $x \in H \setminus A$, $x \in V$. Then it follows that $f(x) \neq g(x)$ and both $f(x)$ and $g(x)$ are contained in O.

Since $ff = fg$ we have $f(f(x)) = f(g(x))$ and hence f is not one-to-one on O, a contradiction.

Thus A is open and closed.

Now let ϕ be the mapping defined by

$$\phi(x) = \begin{cases} x & \text{for } x \notin A \\ g(x) & \text{for } x \in A \end{cases}$$

It is clear that ϕ is a local-homeomorphism of H. Since $g(H) \subset f(H) \subset A$, we have $\phi \neq f$, $\phi \neq g$. Furthermore for each $x \notin A$ we have $f(x) \notin g(H)$, since otherwise $f(x) = g(y)$ and hence $gf(x) = g(x) = gg(y) = g(y) = f(x)$. Thus $g(H) \neq f(H)$. Since $\phi(A) = g(A) = g(H) \neq A$, we have $\phi \neq \varepsilon$. This however is contradictory to the fact that each local homeomorphism ϕ of H is contained in $L(H)$.

Theorem 5. *There is no Hausdorff space H such that S is isomorphic to $O(H)$.*

Proof. Let $O(H) = \{\varepsilon, f, g\}$ with ε the identity and f and g open continuous mappings such that $fg = gf = ff = gg = g$. If $A = \{x \mid x \in H, f(x) = g(x)\}$, then $A \neq \emptyset$ and A is closed. Furthermore $g(H) \subset f(H) \subset A$, $f(H)$ and $g(H)$ open. Let $p \in \overline{H \setminus A}$, $p \in A$, then $f(p) = g(p) \in g(H)$ and hence there is an open set V, $p \in V$ such that $f(V) \subset g(H)$. Let $x \in H \setminus A \cap V$. Then $f(x) \in g(H)$ and hence $f(x) = g(y)$. Thus $g(f(x)) = g(x) = g(g(y)) = g(y) = f(x)$. From this it follows that $x \in A$, a contradiction.

The set $A = \{x \mid x \in H, f(x) = g(x)\}$ is an open and closed set. In the same way as in the proof of Theorem 4 we now can construct an open continuous mapping ϕ such that $\phi \notin O(H)$.

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