TOPOLOGICAL SEMIGROUPS AND FIXED POINTS

BY
THEODORE MITCHELL[1]

1. Introduction

In this paper, we consider four fixed point properties that a topological semigroup S might conceivably possess.

($F1$) Whenever S acts on a compact Hausdorff space Y, where the map $S \times Y \to Y$ is jointly continuous, then Y contains a common fixed point of S.

($F2$) Whenever S acts affinely on a convex compact subset Y of a locally convex linear topological space, where the map $S \times Y \to Y$ is jointly continuous, then Y contains a common fixed point of S.

($F3$) Whenever S acts on a compact Hausdorff space Y, where the map $S \times Y \to Y$ is separately continuous, then Y contains a common fixed point of S.

($F4$) Whenever S acts affinely on a convex compact subset Y of a locally convex linear topological space, where the map $S \times Y \to Y$ is separately continuous, then Y contains a common fixed point of S.

For each of these four properties, we investigate the question as to whether there exists some subspace of $C(S)$ whose left amenability (or whose extreme left amenability) is equivalent to the specified (Fi). It is shown that for each of the (Fi), there does indeed exist such an associated subspace of $C(S)$; in fact, a total of three spaces will suffice to characterize the four properties in this manner. Let $f \in C(S)$, and define $\theta_f : S \to C(S)$ by $\theta_f\, s = l_s\, f$ for $s \in S$. Then we will say $f \in LUC(S)$ ($f \in WLUC(S)$) $\{f \in LMC(S)\}$ if the map θ_f is continuous when $C(S)$ is given the supremum norm topology (w-topology) {weak topology induced by the multiplicative means on $C(S)$}. The following are shown in Sections 3 and 4

THEOREM 1. *S satisfies ($F1$) iff $LUC(S)$ has a multiplicative left invariant mean.*

THEOREM 2. *S satisfies ($F2$) iff $LUC(S)$ has a left invariant mean.*

THEOREM 3. *S satisfies ($F3$) iff $LMC(S)$ has a multiplicative left invariant mean.*

THEOREM 4. *S satisfies ($F4$) iff $WLUC(S)$ has a left invariant mean.*

The concept of characterizing fixed point properties of topological semi-

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groups in terms of left invariant means was introduced by M. M. Day in [3]. Let Y be a convex compact subset of a locally convex space, and let $\alpha(Y)$ denote the semigroup of affine continuous self-maps of Y, where $\alpha(Y)$ is given the topology of pointwise convergence. In [3], the property $(F4)$ was studied in an equivalent form, $(F4')$, that we list below:

$(F4')$ For each convex compact subset Y of each locally convex space, and for each continuous homomorphism $\lambda : S \to \alpha(Y)$, there is in Y a common fixed point of λS.

(The equivalence of $(F4)$ to $(F4')$ can be verified by noting that the separately continuous affine actions $\Lambda : S \times Y \to Y$ can be placed in 1-1 correspondance with the continuous homomorphisms $\lambda : S \to \alpha(Y)$ by the formula $\Lambda(s, y) = \lambda(s)y$.) It was shown in [3] that if $C(S)$ has a left invariant mean, then S satisfies $(F4)$. The converse was shown [3, Theorem 1] for the case where S is discrete. Later, Day proved in [4] the equivalence of the left amenability of $C(S)$ to a fixed point property that is formally stronger than $(F4)$, thus pivoting his generalization of [3, Theorem 1] about $C(S)$ rather than $(F4)$. In Theorem 4, we obtain a generalization centered around $(F4)$ instead.

H. Furstenberg [7] investigated the class of topological groups that satisfy $(F2)$, which attribute he called "the fixed point property". (Because of the plethora of fixed point properties considered in this paper, we do not adhere to this nomenclature.) N. W. Rickert [16, Theorem 4.2] proved that a topological group G satisfies $(F2)$ iff $LUC(G)$ has a left invariant mean. Theorem 2 generalizes this result to a topological semigroup S, without benefit of a uniformity on S.

Theorems 1 and 3 were previously shown for discrete S in [10, Theorem 1].

In Section 5, we discuss the relationship of the spaces $LUC(S)$, $WLUC(S)$, and $LMC(S)$ in various special cases. Our principal result in this section is that if S is a locally compact group, then the three spaces mentioned above coincide (Theorem 7).

I am greatly indebted to G. Itzkowitz for several stimulating discussions, and in particular, for the suggestion that the method used to prove Theorem 1 could possibly be modified to obtain Theorem 2. I am further indebted to L. N. Argabright (private correspondance) for asking several questions regarding fixed point properties and amenability; the consideration of these questions was instrumental in leading me to the work of Section 4. I am also grateful to the referee for several helpful comments.

2. Preliminaries

A *topological semigroup* is a semigroup with a Hausdorff topology in which the product st is separately continuous. (J. Berglund and K. Hofmann [2] call this a semitopological semigroup.) A *topological group* is a group with a Hausdorff topology for which the product st^{-1} is jointly continuous.

Let S be a semigroup, $m(S)$ the space of all bounded real-valued functions on S, where $m(S)$ has the supremum norm. For $s \in S$, the *left translation* l_s {*right translation* r_s} of $m(S)$ by s is given by $(l_s f)s' = f(ss')$ $\{(r_s f)s' = f(s's)\}$, where $f \in m(S)$ and $s' \in S$. Let X be a subspace of $m(S)$, then X is *left* {*right*} *translation-invariant* if $l_s X \subseteq X$ $\{r_s X \subseteq X\}$ for all $s \in S$. If X is both left and right translation-invariant, then X is called *translation-invariant.*

Let X be a left translation-invariant closed subspace of $m(S)$ that contains e, the constant 1 function on S. An element $\mu \in X^*$ is a *mean* on X if $\| \mu \| = \mu(e) = 1$. A mean μ on X is *left invariant* if $\mu(l_s f) = \mu(f)$ for all $f \in X$ and $s \in S$. If X is also an algebra (with the pointwise product), a mean μ on X is *multiplicative* if $\mu(f) \cdot \mu(g) = \mu(f \cdot g)$ for all $f, g \in X$.

When Y is a topological space, $C(Y)$ denotes the space of all bounded real-valued continuous functions on Y, where $C(Y)$ has the supremum norm. If Y is a convex subset of a linear topological space, then $A(Y)$ designates the Banach space of all affine $f \in C(Y)$.

Let S be a semigroup, X a translation-invariant closed subalgebra of $m(S)$ that contains the constant functions, and Y a compact Hausdorff space. Suppose that $s \to \lambda s$ is a representation of S by continuous self-maps of Y. For each $y \in Y$, define a map $Ty : C(Y) \to m(S)$ by $(Tyh)s = h((\lambda s)y)$ for $h \in C(Y)$ and $s \in S$. We say λ is a *D-representation of S, X on Y* if

$$\{y \in Y;\ Ty(C(Y)) \subseteq X\}$$

s dense in Y. It was shown in [11, Theorem 1, p. 118] that X has a multiplicative left invariant mean (in other words, X is *extremely left amenable*) iff for every compact Hausdorff space Y, and every D-representation λ of S, X on Y, there is in Y a common fixed point of the family λS.

L. N. Argabright [1] has shown that concepts analogous to those indicated in the paragraph above can be usefully constructed for affine representations of semigroups. Let S be a semigroup, X a translation-invariant closed subspace of $m(S)$ that contains the constant functions, and let Y now be a compact convex subset of a locally convex space. Suppose that $s \to \lambda s$ is now a representation of S by continuous affine self-maps of Y. We say λ is a *D-representation of S, X on Y by continuous affine maps* if $\{y \in Y;\ Ty(A(Y)) \subseteq X\}$ is dense in Y, where Ty is defined as before. Then Argabright [1, Theorem 1, p. 128] has shown that X has a left invariant mean (in other words, X is *left amenable*) iff for every compact convex subset Y of a locally convex space, and for every D-representation λ of S, X on Y by continuous affine maps, there is in Y a common fixed point of the family λS. The subsequent work leans heavily on Argabright's results.

3. Jointly continuous actions

Let S be a topological semigroup. The space of *left uniformly continuous* functions on S, designated by $LUC(S)$, is the set of those $f \in C(S)$ such that for each $s \in S$, if $s(\gamma) \to s$, then $l_{s(\gamma)} f \to l_s f$ uniformly. When S is a topological group, then $LUC(S)$ coincides with the space of all $f \in C(S)$ which are left uniformly continuous on S in the usual sense. This provides some justification for the use of the term "uniformly" in the definition above, although no uniformity is invoked in the general case. Unlike the case of topological groups, if S' is a dense sub-semigroup of a topological semigroup S, an $f' \in LUC(S')$ need not be extendible to an $f \in LUC(S)$. (This is illustrated at the end of Section 5.)

The space $LUC(S)$ was introduced jointly by G. Itzkowitz and the author [12], [9]; and independently by I. Namioka [13]. In [13], Namioka also considered left invariant means on $LUC(S)$ and obtained fixed point theorems concerning the linear action of S on certain Banach spaces. His results neither include, nor are included in, the theorems obtained below.

We list the following lemma for later use; the proof is given in [13, Lemmas 1.1 and 1.2].

LEMMA 1. *Let S be a topological semigroup. Then $LUC(S)$ is a translation-invariant closed sub-algebra of $m(S)$ that contains the constant functions.*

In what follows, recall that a topological semigroup S is assumed throughout to have a separately continuous product. An *action* of S on a topological space X is a map $S \times X \to X$ that satisfies $(s_1 s_2)x = s_1(s_2 x)$ for all $s_1, s_2 \in S$ and all $x \in X$. In this section, we will consider only those actions for which the map $S \times X \to X$ is jointly continuous.

In view of Lemma 1, it is meaningful to speak of a multiplicative left invariant mean on $LUC(S)$. The next theorem generalizes a result [10, Theorem 1, p. 196] concerning discrete semigroups.

THEOREM 1. *Let S be a topological semigroup. Then the following properties are equivalent:*

($P1$) $LUC(S)$ has a multiplicative left invariant mean.

($F1$) Whenever S acts on a compact Hausdorff space Y, where the map $S \times Y \to Y$ is jointly continuous, then Y contains a common fixed point of S.

Proof. ($P1$) $\Rightarrow$ ($F1$). For a specific $y \in Y$, define the two maps

$$Ty : C(Y) \to m(S) \quad \text{and} \quad Vy : S \to Y$$

by $(Tyh)s = h(sy)$ and $Vys = sy$ for $h \in C(Y)$ and $s \in S$. It follows that $(Tyh)s = h(sy) = (hVy)s$; thus $Tyh \in C(S)$ since hVy is the composition of two continuous functions. (This part of the argument is also valid for separately continuous actions of S on Y.) Designate Tyh by f. Then for s, $t \in S$, we have $(l_s f)t = f(st) = (Tyh)st = h(sty)$.

We wish to show that $f \in LUC(S)$. Suppose not; then there exists an $s \in S$ and a net $\{s(\gamma)\}$ in S such that $s(\gamma) \to s$, but $l_{s(\gamma)} f$ does not converge uniformly to $l_s f$. Hence there exists some real number $a > 0$, a subnet $\{s(\delta)\}$ of $\{s(\gamma)\}$, and a net $\{t(\delta)\}$ in S such that

$$| h(s(\delta)(t(\delta)y)) - h(s(t(\delta)y)) | \geq a, \qquad \text{for all } \delta.$$

Denote $t(\delta)y$ by $y(\delta)$. By compactness of Y, the net $\{y(\delta)\}$ has a subnet

$\{y(\eta)\}$ that converges to some $y' \in Y$. But by continuity of h and the joint continuity of the action of S on Y, we have

$$0 < a \leq \lim_\eta | h(s(\eta)y(\eta)) - h(sy(\eta)) | = | h(sy') - h(sy') | = 0,$$

a contradiction. Hence $f \in LUC(S)$, so the action of S on Y is a D-representation of S, $LUC(S)$ on Y since $y \in Y$ was chosen arbitrarily. The property $(F1)$ now follows from $(P1)$ by use of [11, Theorem 1] and Lemma 1.

$(F1) \Rightarrow (P1)$. Choose the compact Hausdorff space Y to be the set of all multiplicative means on $LUC(S)$, where Y is given the w^*-topology of $LUC(S)^*$. Define an action of S on Y by $s\mu = l_s^* \mu$ for $s \in S$ and $\mu \in Y$. For given $s \in S$ and $\mu \in Y$; let $\{s(\gamma)\}$ and $\{\mu(\delta)\}$ be nets in S and Y, respectively, satisfying $s(\gamma) \to s$ and $\mu(\delta) \to \mu$. Then for any $f \in LUC(S)$, it follows that

$$\begin{aligned} 0 &\leq \lim_{\gamma,\delta} | (s(\gamma)\mu(\delta))f - (s\mu)f | \\ &= \lim_{\gamma,\delta} | (\mu(\delta)(l_{s(\gamma)} f - l_s f)) + ((\mu(\delta) - \mu)l_s f) | \\ &\leq \lim_\gamma \| l_{s(\gamma)} f - l_s f \| + \lim_\delta | (\mu(\delta) - \mu) l_s f | = 0. \end{aligned}$$

Hence $s(\gamma)\mu(\delta) \to s\mu$, so the action of S on Y is jointly continuous. By $(F1)$, there exists $\mu_0 = s\mu_0 = l_s^* \mu_0$ for all $s \in S$, which shows $(P1)$.

An action of a topological semigroup S on a convex subset X of a (real) linear topological space is *affine* if for all $s \in S$, and $x_1, x_2 \in X$, and for all real a such that $0 \leq a \leq 1$, it follows that $s(ax_1 + (1 - a)x_2) = asx_1 + (1 - a)sx_2$. Theorem 2 below is a generalization of a result of N. W. Rickert [16, Theorem 4.2, p. 227] on topological groups.

THEOREM 2. *Let S be a topological semigroup. Then the following properties are equivalent*:

$(P2)$ *$LUC(S)$ has a left invariant mean.*

$(F2)$ *Whenever S acts affinely on a convex compact subset Y of a locally convex linear topological space, where the map $S \times Y \to Y$ is jointly continuous, then Y contains a common fixed point of S.*

Proof. $(P2) \Rightarrow (F2)$. For a specific $y \in Y$, define $Ty : C(Y) \to m(S)$ by $(Tyh)s = h(sy)$ for $h \in C(Y)$ and $s \in S$. The proof of Theorem 1 yields $TyC(Y) \subseteq LUC(S)$; hence $TyA(Y) \subseteq LUC(S)$. This means that the action of S on Y is a D-representation of S, $LUC(S)$ on Y by continuous affine maps; thus $(F2)$ follows by the result of Argabright [1, Theorem 1] mentioned previously, and by Lemma 1.

$(F2) \Rightarrow (P2)$. Choose the compact convex set Y to be the space of all means on $LUC(S)$, where Y is given the w^*-topology of $LUC(S)^*$. Define the affine action of S on Y by $s\mu = l_s^* \mu$ for $s \in S$ and $\mu \in Y$. The argument used in the proof of Theorem 1 can be used to show that $S \times Y \to Y$ is jointly continuous; hence by $(F2)$, there exists $\mu_0 \in Y$ such that $\mu_0 = s\mu_0 = l_s^* \mu_0$ for all $s \in S$, which shows $(P2)$.

We remark that it is tempting to define, as given in [13], a topological semigroup S to be left amenable if and only if $LUC(S)$ has a left invariant mean. Certainly, Theorem 2 would offer justification for this terminology. However in the light of Theorem 4 in the next section, it is not clear that $LUC(S)$ should be preferred above all other subspaces of $C(S)$ for this distinction.

4. Separately continuous actions

Theorems 1 and 2, which are concerned with the jointly continuous actions of a topological semigroup S, do not represent the only possible ways of generalizing the discrete semigroup cases. It is equally valid to consider the class of separately continuous actions of S, and to inquire whether there exists some subspace of $C(S)$ which plays the same role for this class that $LUC(S)$ did for the jointly continuous actions of Section 3. The answer appears to be that each of the respective analogues of Theorem 1 and Theorem 2 requires such a space; only in special cases have we been able to show that these two subspaces of $C(S)$ coincide. The first of these spaces, $LMC(S)$, is introduced below.

Let S be a topological semigroup. The space of *left multiplicatively continuous* functions on S, designated by $LMC(S)$, is the set of those $f \in C(S)$ such that for each multiplicative mean μ on $C(S)$ and each $s \in S$, if $s(\gamma) \to s$, then $\mu(l_{s(\gamma)} f) \to \mu(l_s f)$.

LEMMA 2. *Let S be a topological semigroup. Then $LMC(S)$ is a translation-invariant closed sub-algebra of $m(S)$ that contains the constant functions.*

Proof. By a straightforward computation, $LMC(S)$ can be shown to be a closed linear subspace of $m(S)$ that contains e. For the rest, let $s, t \in S$, $s(\gamma) \to s$, and let μ be a multiplicative mean on $C(S)$. If $f \in LMC(S)$, then

$$\lim_\gamma (\mu(l_{s(\gamma)}(r_t f))) = \lim_\gamma ((r_t^* \mu)(l_{s(\gamma)} f)) = (r_t^* \mu)(l_s f) = \mu(l_s(r_t f)).$$

Also,

$$\begin{aligned} \lim_\gamma (\mu(l_{s(\gamma)}(l_t f))) &= \lim_\gamma (\mu(l_{ts(\gamma)} f)) \\ &= \mu(l_{ts} f) = \mu(l_s(l_t f)); \end{aligned}$$

hence $LMC(S)$ is translation-invariant. Now let $f, g \in LMC(S)$. By use of the multiplicative property of μ, we obtain

$$\begin{aligned} \lim_\gamma (\mu(l_{s(\gamma)}(f \cdot g))) &= \lim_\gamma (\mu(l_{s(\gamma)} f) \cdot \mu(l_{s(\gamma)} g)) \\ &= \mu(l_s f) \cdot \mu(l_s g) = \mu(l_s(f \cdot g)), \end{aligned}$$

so $f \cdot g \in LMC(S)$, which shows Lemma 2.

The space $LMC(S)$ can also be described in terms of a concept studied in [11], that of a left M-introverted algebra. Let X be a left translation-invariant closed sub-algebra of $m(S)$ that contains the constant functions.

For each $\mu \in X^*$, there is associated a map $\mu_l : X \to m(S)$ given by $(\mu_l f)s = \mu(l_s f)$ for $f \in X$ and $s \in S$. The algebra X is called *left M-introverted* if $\mu_l X \subseteq X$ for every multiplicative mean $\mu \in X^*$. It can be shown (cf. C. R. Rao [15, Theorem 1, p. 190]) that $LMC(S)$ is the unique maximal left M-introverted algebra contained in $C(S)$; however this will not be needed in the material that follows.

THEOREM 3. *Let S be a topological semigroup. Then the following properties are equivalent*:

($P3$) *$LMC(S)$ has a multiplicative left invariant mean.*

($F3$) *Whenever S acts on a compact Hausdorff space Y, where the map $S \times Y \to Y$ is separately continuous, then Y contains a common fixed point of S.*

Proof. $(P3) \Rightarrow (F3)$. Select a specific $y \in Y$, and a specific multiplicative mean μ on $C(S)$. Define a map $Ty : C(Y) \to m(S)$ by $(Tyh)s = h(sy)$ for $h \in C(Y)$ and $s \in S$. Designate Tyh by f; it follows from the proof of Theorem 1 that $f \in C(S)$. Let Q be the natural map of S into $C(S)^*$, where $(Qs)g = g(s)$ for $s \in S$ and $g \in C(S)$. Since QS is w^*-dense in the set of multiplicative means on $LMC(S)$ (see the proof of [5, Corollary 19, p. 276]), there exists a net $\{s(\omega)\}$ in S such that $Qs(\omega) \to \mu$. By compactness of Y, there is a subnet $\{s(\delta)\}$ which satisfies $s(\delta)y \to x$ for some $x \in Y$. So for any $t \in S$, we have

$$\mu(l_t f) = \lim_\delta ((Qs(\delta))l_t f) = \lim_\delta f(ts(\delta)) = \lim_\delta h(t(s(\delta)y)) = h(tx),$$

where the last step follows by continuity of h and the continuity of t on Y. Let $s \in S$, and let $s(\gamma) \to s$. Then

$$\lim_\gamma (\mu(l_{s(\gamma)} f)) = \lim_\gamma h(s(\gamma)x) = h(sx) = \mu(l_s f),$$

where the second equality now uses the continuity of the action of S on x. But μ was an arbitrarily chosen multiplicative mean on $C(S)$, so $f \in LMC(S)$. Hence the action of S on Y is a D-representation of S, $LMC(S)$ on Y since $y \in Y$ also was chosen arbitrarily. Thus, as in the proof of Theorem 1, ($F3$) now follows from [11, Theorem 1] and Lemma 2.

$(F3) \Rightarrow (P3)$. Let the compact Hausdorff space Y be the set of all multiplicative means on $LMC(S)$, where Y has the w^*-topology of $LMC(S)^*$. Define the action of S on Y by $s\mu = l_s^* \mu$ for $s \in S$ and $\mu \in Y$. Each $s \in S$ acts continuously on Y since the adjoint of an operator is w^*-continuous. To show continuity in the other variable, let $s(\gamma) \to s$. Then for each $f \in LMC(S)$,

$$\lim_\gamma ((s(\gamma)\mu)f) = \lim (\mu(l_{s(\gamma)} f)) = \mu(l_s f) = (s\mu)f.$$

Thus $s(\gamma)\mu \to s\mu$, so the action of S on Y is separately continuous. The result now follows from ($F3$).

We come now to the second of the two spaces mentioned at the beginning of this section. Let S be a topological semigroup. The space of *weakly left* *uniformly continuous* functions on S, denoted by $WLUC(S)$, is the set of those $f \in C(S)$ such that for each $s \in S$, if $s(\gamma) \to s$, then $l_{s(\gamma)} f \to l_s$ in the weak topology of $C(S)$. (That is, for every $\mu \in C(S)^*$, $\mu(l_{s(\gamma)} f) \to \mu(l_s f)$.)

The space $WLUC(S)$ was introduced, under a different designation, by C. R. Rao [15]. Let X be a left translation-invariant closed subspace of $m(S)$ that contains the constant functions. The space X is *left introverted* if $\mu_l X \subseteq X$ for every $\mu \in X^*$. Rao showed [15, Theorem 1, p. 190] that $WLUC(S)$ is the unique maximal left introverted subspace of $C(S)$.

LEMMA 3. (Rao) *Let S be a topological semigroup. Then $WLUC(S)$ is a translation-invariant closed subspace of $m(S)$ that contains the constant functions.*

Proof. In the proof of Lemma 2, replace the multiplicative mean μ by any element of $C(S)^*$. Then all except the containment of the pointwise product goes through as before.

The next result is a second generalization of Day's fixed point theorem [3, Theorem 1, p. 586].

THEOREM 4. *Let S be a topological semigroup. Then the following properties are equivalent:*

($P4$) *$WLUC(S)$ has a left invariant mean.*

($F4$) *Whenever S acts affinely on a convex compact subset Y of a locally convex linear topological space, where the map $S \times Y \to Y$ is separately continuous, then Y contains a common fixed point of S.*

Proof. ($P4$) $\Rightarrow$ ($F4$). Let $y \in Y$, $h \in A(Y)$ and let μ be a mean on $C(S)$. Define Ty as in the proofs of Theorems 1–3, and let $f = Tyh$. As before, $f \in C(S)$; now we wish to show that $f \in WLUC(S)$. For this purpose, the finite means on $C(S)$ play the same role that the evaluations Qs did in the proof of Theorem 3. Let Φ be the set of all non-negative real-valued functions ϕ on S for which the set $\{s \in S; \phi(s) > 0\}$ is finite, and such that $\sum_{s \in S} \phi(s) = 1$. Let $q : \Phi \to C(S)^*$ be given by $(q\phi)g = \sum_{s \in S} \phi(s)g(s)$ for $\phi \in \Phi$, $g \in C(S)$; then $q\Phi$ is w^*-dense in the set of means on $C(S)$ (see [3, p. 588]). Let $\{\phi_\omega\}$ be a net in Φ which satisfies $q\phi_\omega \to \mu$; then by compactness of Y, there is a subnet $\{\phi_\delta\}$ for which $\sum_{s \in S} \phi_\delta(s)sy \to x$ for some $x \in Y$. Then for any $t \in S$,

$$\begin{aligned}\mu(l_t f) &= \lim_\delta ((q\phi_\delta)(l_t f)) = \lim_\delta (\textstyle\sum_{s \in S} \phi_\delta(s) f(ts)) \\ &= \lim_\delta (\textstyle\sum_{s \in S} \phi_\delta(s) h(tsy)) = \lim_\delta h(t \textstyle\sum_{s \in S} \phi_\delta(s)sy) = h(tx),\end{aligned}$$

where the fourth equality follows by virtue of the affineness of h and t on Y. By continuing as in the proof that ($P3$) $\Rightarrow$ ($F3$), we obtain the result that if $s \in S$ and $s(\gamma) \to s$, then $\mu(l_{s(\gamma)} f) \to \mu(l_s f)$. Since each $\mu' \in C(S)^*$ can be expressed as a linear combination of two means on $C(S)$, then $f \in WLUC(S)$, so the action of S on Y is a D-representation of S, $WLUC(S)$ on Y by continuous affine maps. Hence ($F3$) follows from Lemma 3 and Argabright's fixed point theorem [1, Theorem 1].

$(F4) \Rightarrow (P4)$. Let the convex compact space Y be the set of all means on $WLUC(S)$, where Y has the w^*-topology of $WLUC(S)^*$. Let the affine action of S on Y be given by $s\mu = l_s^*\mu$ for $s \in S$ and $\mu \in Y$. The proof that $(F3) \Rightarrow (P3)$ can now be repeated to show that the action of S on Y is separately continuous, hence $(P4)$ follows.

5. Special cases

For any topological semigroup S, it is immediate from the definitions of the spaces concerned that $LUC(S) \subseteq WLUC(S) \subseteq LMC(S) \subseteq C(S)$. This inclusion, when inserted into the previous four theorems, yields only obvious results. For example, since $LUC(S) \subseteq LMC(S)$, then $(P3) \Rightarrow (P1)$ (by restriction of the mean to $LUC(S)$), hence $(F3) \Rightarrow (F1)$ by Theorems 1 and 3; a conclusion which hardly required this kind of argument. However, this method gives less trivial results when additional conditions are imposed on S. We list some items, useful for this purpose, below.

(a) Let S be a first-countable topological semigroup, then $WLUC(S) = LMC(S)$.

To indicate the proof; since S is first-countable, we need only show that for $f \in LMC(S)$, if $s(n) \to s$ where $\{s(n)\}$ is a *sequence* in S, then $l_{s(n)} f \to l_s f$ weakly in $C(S)$. J. Rainwater [14, p. 999] has shown the following theorem:

Let N be a normed linear space, $\{f(n)\}$ a bounded sequence in N, and let $f \in N$. If $\mu(f(n)) \to \mu(f)$ for each extreme point μ of the unit ball of N^*, then $f(n) \to f$ weakly.

In the above theorem, let $N = C(S)$. The desired conclusion follows.

(b) If S is a compact topological semigroup, then $LMC(S) = C(S)$.

To see this, let $f \in C(S)$, $s \in S$, and let $s(\gamma) \to s$. Each multiplicative mean μ on $C(S)$ is an evaluation on some $t \in S$, so

$$\mu(l_{s(\gamma)} f) = (l_{s(\gamma)} f)t = f(s(\gamma)t) \to f(st) = \mu(l_s f);$$

thus $f \in LMC(S)$.

(c) Let S be a compact topological semigroup with jointly continuous product. Then $LUC(S) = C(S)$ by Namioka [13, Lemma 1.3], so $LUC(S) = WLUC(S) = LMC(S) = C(S)$.

(d) Let S be a topological group which is complete in an invariant metric, then $LUC(S) = WLUC(S)$.

This is shown in Rao [15, Theorem 2, p. 192]. By combining this with (a), we obtain $LUC(S) = WLUC(S) = LMC(S)$.

(e) If S is a discrete topological semigroup, then $LUC(S) = WLUC(S) = LMC(S) = m(S)$.

With the aid of the above remarks, some consequences of Theorems 1–4 can now be obtained.

THEOREM 5. *Let S be a compact topological semigroup with jointly continuous product. Then $(F1)$ is equivalent to $(F3)$, and $(F2)$ is equivalent to $(F4)$.*

Proof. This follows by remark (c) and Theorems 1–4.

THEOREM 6. *Let S be a topological group which is complete in an invariant metric. Then $(F1)$ is equivalent to $(F3)$, and $(F2)$ is equivalent to $(F4)$.*

Proof. This is obtained by Theorems 1–4 and remark (d). (It is only fair to warn the reader that the only example we know of a topological group that satisfies $(F1)$, is the trivial one-element group.[2] Of course, there are abundant examples of complete metric groups that satisfy $(F2)$, hence $(F4)$.)

If S is a locally compact group, then $LUC(S)$ has a left invariant mean iff $C(S)$ does also (F. P. Greenleaf [8, Theorem 2.2.1]). One could then employ Theorems 2 and 4 to obtain the known result (see [8, Theorem 3.3.5]) that properties $(F2)$ and $(F4)$ are equivalent for such S, but an even stronger result is already known. R. Ellis [6] has shown that if an action of a locally compact group S on a locally compact Hausdorff space Y is separately continuous, then the action is jointly continuous. (Let 1 be the identity element of S; as is noted in [2, p. 36], Ellis' result does not require that $1y = y$ for all $y \in Y$.) This, of course, yields the result that $(P1) \Leftrightarrow (P3)$ and that $(P2) \Leftrightarrow (P4)$ for such an S, however we can obtain something stronger than that by use of Ellis' theorem.

THEOREM 7. *Let S be a locally compact topological group. Then $LUC(S) = WLUC(S) = LMC(S)$.*

Proof. Let Y be the set of all multiplicative means on $LMC(S)$, where Y has the w^*-topology of $LMC(S)^*$. Define an action of S on Y by $s\mu = l_s^*\mu$ for $s \in S$ and $\mu \in Y$. The action is separately continuous by the proof of Theorem 3, hence is jointly continuous by [6, Theorem 1]. Let Q be the natural map of S into Y, where $(Qs)f = f(s)$ for $s \in S$ and $f \in LMC(S)$. Then for $s, t \in S$ and $f \in LMC(S)$, we have

$$(l_s f)t = (l_s^* Qt)f = (sQt)f.$$

[2] By Theorem 1, $(P1)$ is equivalent to $(F1)$. In [10, Theorem 2], the author showed that no non-trivial discrete group satisfies $(P1)$. E. Granirer (*Extremely amenable semigroups II*, Math. Scand., vol. 20 (1967), pp. 93–113, Theorem 3) has shown that there is a large class of topological groups which do not satisfy $(P1)$. This class includes, among others, all (non-trivial) locally compact abelian groups, totally bounded groups, and all additive subgroups of locally convex spaces. Granirer (ibid. p. 103) raises the question of determining which (non-trivial) topological groups satisfy $(P1)$. We ask a slightly simpler form of the same question: are there any such groups at all?

Let $f \in LMC(S)$; we wish to show that $f \in LUC(S)$. Suppose not; then there exists an $s \in S$ and a net $\{s(\gamma)\}$ in S such that $s(\gamma) \to s$, but $l_{s(\gamma)} f$ does not converge uniformly to $l_s f$. Hence there exists some real number $a > 0$, a subnet $\{s(\delta)\}$ of $\{s(\gamma)\}$, and a net $\{t(\delta)\}$ in S such that

$$| (s(\delta)Qt(\delta))f - (sQt(\delta))f | \geq a, \qquad \text{for all } \delta.$$

By compactness of Y, the net $\{Qt(\delta)\}$ has a subnet $\{Qt(\eta)\}$ which converges to some $\mu \in Y$. By the joint continuity of the action of S on Y, we have

$$0 < a \leq \lim_\eta | (s(\eta)Qt(\eta))f - (sQt(\eta))f | = | (s\mu)f - (s\mu)f | = 0,$$

a contradiction. Thus $LMC(S) \subseteq LUC(S)$, which proves Theorem 7, since we already had that $LUC(S) \subseteq WLUC(S) \subseteq LMC(S)$.

We give one further application of Theorems 1–4. In Theorem 8 given below, part (b) and its converse have been shown by Rickert [16, Corollary 4.5, p. 227] for the case where S and T are topological groups. As we note later, this converse to (b) is false if S and T are not thus constrained.

THEOREM 8. *Let S be a dense subsemigroup of T, a topological semigroup.*
(a) *If $LUC(S)$ has a multiplicative left invariant mean, so has $LUC(T)$.*
(b) *If $LUC(S)$ has a left invariant mean, so has $LUC(T)$.*
(c) *If $LMC(S)$ has a multiplicative left invariant mean, so has $LMC(T)$.*
(d) *If $WLUC(S)$ has a left invariant mean, so has $WLUC(T)$.*

Proof.[3] This will be shown for only one of the cases, say (d). The proofs for the other cases use obvious modifications. Let S satisfy $(P4)$, hence also $(F4)$ by Theorem 4. Let T act affinely on a compact convex subset Y of a locally convex space, where the action is separately continuous. The restriction of the action to $S \times Y$ is also a separately continuous affine action, so $Sx = x$ for some $x \in Y$. By continuity of the action of T on x, it follows that the set $\{t \in T;\ tx = x\}$ is closed in T, hence equals T since S is dense in T. So T satisfies $(F4)$, thus the result follows by Theorem 4. (An alternate proof, doubtless preferred by some, can be obtained by defining an action of S on Z, the space of means on $WLUC(T)$, by $s\mu = l_s^* \mu$, for $s \in S$, $\mu \in Z$. We can safely suppress further details.)

The converses to (a)–(d) of Theorem 8 are all false, even if S and T are required to have jointly continuous products. To see this, let S be the free semigroup on two generators, where S has the discrete topology. Let

[3] The referee has commented that if P is the restriction map from $WLUC(T)$ to $WLUC(S)$, and μ is a LIM on $WLUC(S)$, then $P^*\mu$ can be verified to be a LIM on $WLUC(T)$. (A similar remark holds for each of the other parts of Theorem 8.) In turn, we note that the proof of the referee's statement depends upon the fact (developed in the proofs of Theorems 3 and 4) that the action of T on the space of means on $WLUC(T)$ is separately continuous. To highlight the role played by this last property, we list two open problems that occur when the desired form of continuity is lacking. If μ is a LIM on $C(S)$, is $P^*\mu$ a LIM on $C(T)$? If μ is a LIM on $LMC(S)$, is $P^*\mu$ a LIM on $LMC(T)$? (Modify the definition of the restriction map P as appropriate.)

$T = S \cup \{z\}$ be the one-point compactification of S, where $tz = zt = z$ for all $t \in T$. (The topological semigroup T has a jointly continuous product, whereas the corresponding compactification of the free group on two generators has a product that is only separately continuous.) Then the pair S, T can be verified to be a counter-example to all four converses. For by remark (e) of this section, the spaces $LUC(S)$, $WLUC(S)$, and $LMC(S)$ all coincide with the space $m(S)$, which is well known to lack a left invariant mean. But by remark (c), $LUC(T) = WLUC(T) = LMC(T) = C(T)$, for which the evaluation Qz can be verified to be a multiplicative left invariant mean. It is perhaps enlightening to note that $m(S)$ is not the restriction of $C(T)$ to S, and hence, among other items, an $f \in LUC(S)$ need not be extendible to a $g \in LUC(T)$.

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Temple University
Philadelphia, Pennsylvania