Topological states of matter and noncommutative geometry

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August 2015

A thesis submitted for the degree of Doctor of Philosophy
of the Australian National University

Declaration

The work in this thesis is my own except where otherwise stated.

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Abstract

This thesis examines topological states of matter from the perspective of noncommutative geometry and KK-theory. Examples of such topological states of matter include the quantum Hall effect and topological insulators.

For the quantum Hall effect, we consider a continuous model and show that the Hall conductance can be expressed in terms of the index pairing of the Fermi projection of a disordered Hamiltonian with a spectral triple encoding the geometry of the sample's momentum space. The presence of a magnetic field means that noncommutative algebras and methods must be employed. Higher dimensional analogues of the quantum Hall system are also considered, where the index pairing produces the 'higher-dimensional Chern numbers' in the continuous setting.

Next we consider a discrete quantum Hall system with an edge. We show that topological properties of observables concentrated at the boundary can be linked to invariants from a boundary-free model via the Kasparov product. Hence we obtain the bulk-edge correspondence of the quantum Hall effect in the language of KK-theory.

Finally we consider topological insulators, which come from imposing (possibly anti-linear) symmetries on condensed-matter systems and studying the invariants that are protected by these symmetries. We show how symmetry data can be linked to classes in real or complex KK-theory. Finally we prove the bulk-edge correspondence for topological insulator systems by linking bulk and edge systems using the Kasparov product in KKO-theory.

Acknowledgements

This research has been supported by an Australian Postgraduate Award and ANU Supplementary Scholarship. I thank the Australian Research Council and the Australian National University for their assistance.

My foremost thanks go to my supervisors, Alan Carey and Adam Rennie. This thesis would simply not exist if it weren't for their support and encouragement. I thank them for their time, their patience and their invaluable assistance with the final editing of this thesis.

I'd also like to thank my fellow students, Koen van den Dungen, Iain Forsyth and Andreas Andersson, who have helped me throughout my candidature and provided an enjoyable environment to work in.

This work has benefited greatly from discussions with Guo Chuan Thiang, Hermann Schulz-Baldes, Johannes Kellendonk, Julian Grossmann and Giuseppe De Nittis.

Much of this thesis was completed at the University of Wollongong. I acknowledge and thank the School of Mathematics and Applied Statistics for allowing me to work at UOW. I also thank Adam for his assistance in making the transition as smooth as possible and the operator algebra group for making me feel welcome.

During my candidature, I was fortunate enough to attend the trimester on 'Noncommutative geometry and its applications' at the Hausdorff Research Institute for Mathematics in September-October 2014. I thank the Hausdorff institute for their financial support. I'd also like to thank the Advanced Institute for Materials Research at Tohoku University for the invitation and financial support to attend the mini-workshop on 'Topological states and noncommutative geometry' in March 2015.

I thank Hermann Schulz-Baldes for allowing me to visit Friedrich-Alexander Universität Erlangen-Nürnberg in October-November 2014. The work and discussions during my visit have been of great use to this work.

I'd also like to thank my family for their long-distance support and my friends for ensuring that the last few years have been fun. These include (but are not limited to) Alex, Kowshik, Michael, Padarn, Ziping, James, Andrew and Mitch.

As a final note, I am immensely grateful to my brother Scott who lent me a computer when mine failed at a critical juncture.

Contents

Notation and terminology

We will generally denote by $\mathcal{H}$ a separable Hilbert space (usually complex but possibly real). Given C^*-algebras A and B we use the script lettering $\mathcal{A}$ and $\mathcal{B}$ to denote dense $*$-subalgebras.

Notation

E_B	Right Hilbert C^*-module over B
$(\cdot \mid \cdot)_B$	B-valued inner product over C^*-module E_B.
$\mathcal{E}_{\mathcal{B}}$	Pre-C^*-module over the dense $*$-subagebra $\mathcal{B}$
$\langle \cdot\, , \cdot \rangle$	Hilbert space inner-product.
$\mathrm{End}_B(E)$	Set of adjointable acting on a right-B C^*-module E_B.
$\Theta_{e,f}$	Rank-1 operator on a Hilbert C^*-module E_B, $\Theta_{e,f}g = e \cdot (f\|g)_B$.
$\mathrm{End}^0_B(E)$	Space of compact adjointable operators on Hilbert C^*-module E_B.
$(\mathcal{A}, \mathcal{H}, D, \gamma)$	Even spectral triple (odd if there is no γ).
$(\mathcal{A}, E_B, D, \gamma)$	Unbounded Kasparov A-B module with grading γ.
(A, E_B, F, γ)	Kasparov A-B module with grading γ.
$\hat{\otimes}$	$\mathbb{Z}_2$-graded tensor product.
$\mathcal{E} \otimes_B \mathcal{F}$	Balanced tensor product of a right B-module $\mathcal{E}_B$ with left B-module ${}_B\mathcal{F}$.
$[\eta]\hat{\otimes}_B[\lambda]$	Internal Kasparov product of classes represented by Kasparov modules η and λ over the algebra B.
$C\ell_{r,s}$	Real Clifford algebra with $r+s$ generators (r generators square to $+1$, s generators square to -1).

$\mathbb{C}\ell_n$	Complex Clifford algebra with n generators.
$\bigwedge^* V$	Exterior algebra of a vector space V.
A^{op}	Opposite algebra of an algebra A, where $(ab)^{\mathrm{op}} = b^{\mathrm{op}} a^{\mathrm{op}}$.
H	Hamiltonian operator.
P_μ	Fermi projection of Hamiltonian, $P_\mu = \chi_{(-\infty,\mu]}(H)$.
$[H^G]$	KK-class of a Kasparov module coming from a Hamiltonian H that is compatible with the symmetry group G.

Chapter 1

Introduction

1.1 Motivation

Solid state and condensed matter physics have been responsible for some of the vast technologial advancements that have occurred over the last half-century or more. Behind these advances is a well-established theory laid down by theoretical physicists and mathematicians alike. Both theory and experimental discovery fuel technical advancement and the field continues to be a dynamic area of research.

A useful approach to understanding condensed matter systems is to consider what symmetries the system possesses. One can then interpret the physical properties and phenomena as the 'spontaneous symmetry breaking' of the condensed matter system [Str05]. For example, a ferromagnet has a north and south pole, which can be expressed as a breaking of rotational symmetry.

For many years, it was assumed that *all* physical properties could be explained by the spontaneous symmetry breaking of a system. This was found to be incorrect in 1980 with the discovery of the integer quantum Hall effect by von Klitzing et al. [vKDP80].* The quantised and stable Hall conductance that characterises the effect did not come from any symmetry of the quantum Hall system being broken. The quantum Hall effect was not theoretically predicted, and so lead to new avenues of theoretical research in order to account for the phenomena.

Somewhat unexpectedly, a physically reasonable but still mathematically valid explanation was found via Alain Connes' noncommutative geometry (we will conduct a more thorough historical overview of the quantum Hall effect in Section 1.2.1 and and Chapter 3). The French mathematician Jean Bellissard adapted Connes' immense machinery to study the quantum Hall problem. Bellissard showed that while no sym-

*It should be noted that we will always mean the *integer* quantum Hall effect. The fractional quantum Hall effect, a many-body problem still without a mathematically sound explanation, lies outside the scope of this work.

metries were being broken, the physical effect could be explained by linking the Hall conductance to (noncommutative) bundles over the topologically non-trivial momentum space of the system [BvS94]. This was the first example of the properties of a condensed matter material being determined by purely topological notions, i.e., a topological state of matter.

For the ten or so years that followed Bellissard's work, the quantum Hall effect was something of an isolated curiosity, with no other experimentally verifiable physical systems displaying comparable properties. This changed in 2005 with the theoretical prediction and subsequent experimental verification of the quantum spin-Hall effect. Very roughly speaking, the quantum spin-Hall effect is the quite remarkable behaviour where a material behaves as an insulator in its interior, but possesses a robust (spin-oriented) current along the sample's surface/edge. The effect caused a great deal of interest from the condensed matter physics community and other similar materials were soon predicted and discovered. Such materials are collectively termed topological insulators and there are now many papers discussing theory, experiment and potential applications of topological insulators to, amongst other areas, quantum computing.

The physical explanation of topological insulators is similar to the quantum Hall effect in that, as the name suggests, topological considerations are thought to play a central role. However, a mathematically rigorous yet still physically reasonable explanation of such systems is still a work in progress. This issue aside, the discovery of topological insulators has opened the doorway to research, theoretical and experimental, into other types of topological states of matter, their effects and their applications.

Simply speaking, the aim of this thesis is to provide a mathematically concrete explanation of topological insulators and other topological states of matter. Because of Bellissards' success in solving the quantum Hall effect using noncommutative geometry, we also adopt such a framework. In this introduction we shall first provide a brief review of work into this problem, highlighting what still remains unclear in the subject. We then outline the content of this thesis and how it contributes to a more complete understanding of these systems.

1.2 Topological states of matter

We start with a short review of the major contributions to understanding topological insulators and topological states of matter more broadly (a more detailed review on topological insulators is carried out in Chapter 5.1). Because our work is a thesis in mathematical sciences, our review shall be more focused on articles classified as 'mathematical physics' rather than 'theoretical physics', except in the cases of breakthrough physics articles where new concepts and ideas are introduced.

Because the theoretical description of topological insulators builds on ideas first developed in the explanation of the quantum Hall effect, it is important to review the key arguments of how the quantum Hall effect works mathematically. Starting from these ideas, we will then show how looking at quantum Hall systems with an edge lead to constructions that can help explain the edge effects in more general topological insulator systems.

1.2.1 The quantum Hall effect

To briefly review, the quantum Hall effect is the quantisation at very low temperature of the Hall conductance of a material, $\sigma_H = n\frac{e^2}{h}$ with $n \in \mathbb{Z}$. Furthermore, this conductance is stable between 'jumps' in n and the effect can still be observed in samples with impurities and disorder.

Many possible theoretical explanations of the quantum Hall effect appeared after its discovery. Of particular note are the explanations given by Laughlin [Lau81] and Thouless et al. [TKNdN82], which are still widely accepted in the physics community. Both articles are able to show that in suitable circumstances the Hall conductance, σ_H, is quantised.

Briefly, Laughlin's argument uses cylindrical geometry and a clever gauge-invariance trick to show quantisation. However, as [BvS94, Section 2.5] demonstrates, in between the jumps in the Hall conductance, Laughlin's argument can only reproduce the classical formula for the Hall conductance. Hence the argument does not account for the plateau and stability of the Hall conductance in between jumps.

The argument of Thouless and collaborators is that, assuming the magnetic flux through the sample is rational, a principal $U(1)$-bundle can be constructed over the Brillouin zone (momentum space) of the sample, topologically a torus. The authors then build a particular connection on this $U(1)$-bundle and, using the Kubo formula for conductance from statistical mechanics, show that (up to a universal constant) the Hall conductance can be expressed as the integral of the curvature of this connection. One then consults geometric theory to find that the Hall conductance is a pairing of a Chern class and a homology class of the Brillouin zone. Thus it is an integer. Thouless et al.'s result was the first to relate the Hall conductance to topological data and subsequent papers by, amongst others, Kohmoto [Koh85] showed that the quantisation was stable under small amounts of disorder. The geometric 'bundle'-viewpoint was a significant step forward in providing an adequate explanation for the quantisation of Hall conductance, but relied on the physically unrealistic assumption of rational magnetic flux.

Over the course of several papers, whose results are summarised and expanded upon in the review [BvS94], Bellissard and his collaborators were able to overcome the problem of rational magnetic flux. The two main results of Bellissard's work concerning the quantum Hall effect are the quantisation of Hall conductance for rational and irrational flux and the stability of the Hall conductance in between the Landau levels (spectral bands) of the quantum Hall Hamiltonian. The key new ingredient to obtain these results is the construction of the algebra of observables. For the continuous case where the Hilbert space is $\mathcal{H} = L^2(\mathbb{R}^2)$, this algebra is the twisted C^*-algebraic crossed product

$$A = C(\Omega) \rtimes_\theta \mathbb{R}^2,$$

where Ω is encoding the disorder of the system and the action comes from translations twisted by the magnetic flux θ (now without any assumption on its rationality). Bellissard, taking inspiration from Connes, constructs a calculus of sorts on this noncommutative algebra and interprets his construction as a 'noncommutative Brillouin zone'. Using ideas from noncommutative geometry, Bellissard's constructions allow for what can be interpreted as a noncommutative analogue of the Thouless et al. argument. Starting from the Kubo formula and making some (reasonable) physical assumptions, Bellissard derived an expression for the Hall conductance as a pairing of the even K-theory of the observable algebra A with a cyclic 2-cocycle, ϕ. The cocycle ϕ comes from the 'differential structure' on the noncommutative Brillouin zone, namely a dense subalgebra $\mathcal{A}$ of A. By constructing a $(2+1)$-summable Fredholm module whose Chern character is the same as the expression for σ_H (up to a constant), it follows that σ_H is proportional to a Fredholm index and, therefore, quantised.

In many ways, this is only a small part of the story as experiments show that σ_H is quantised but also plateaus in between jumps. By adding disorder space Ω to the algebra of observables, one can also consider states that are 'localised' by the disorder. In physical regions where such states are localised, [BvS94] showed that σ_H is constant. A physical state is not localised if the state corresponds to the continuous spectrum of the Hamiltonian. That is, the Hall conductance jumps when one passes to a higher spectral band but remains constant in the localised region between bands. Hence a system with disorder seems necessary in order to obtain the plateaus of the Hall conductance.

Bellissard's work was also given a more functional analytic interpretation by Avron, Seiler and Simon [ASS94a, ASS94b], who link the Hall conductance to the relative index of projections and charge pumps.

The other explanation of the quantum Hall effect via noncommutative geometry was due to Xia [Xia88], who was able to give a slightly more geometric interpretation of Bellissard's results. Xia showed that the algebra of observables $C(\Omega) \rtimes_\theta \mathbb{R}^2$ could be expressed as a double twisted crossed product. From this observation, an application of the Connes-Thom isomorphism simplifies the K-theory of the observable algebra. This allows the pairing between K-theory and periodic cyclic cohomology to be more easily computed and Xia derives the desired quantisation. The limitation of this argument is that it relied on very specific and quite technical results about smooth crossed products due to Elliott, Natsume and Nest [ENN88], which cannot be easily adapted or generalised to other systems.

1.2.2 Systems with boundaries and bulk/edge states

Bellissard's and Xia's explanations of the quantum Hall effect were a significant advance in understanding how topology can lead to physical properties. There were, however, extensions that one could consider. In the case of a two-dimensional system with magnetic field, one would expect the cyclotronic orbit of the electrons to concentrate at the boundary of the sample, giving rise to an edge current. Currents concentrated at or near the boundary are common in condensed matter systems, superconductors being an important example [Kit04, Chapter 10]. Indeed, it was claimed by Halperin soon after the quantum Hall effect's discovery that the Hall current should be carried along the sample's edge [Hal82]. The models Bellissard and Xia consider do not include boundaries, so we would like to extend their picture to a system with edge.

Following Halperin's suggestion, we consider a system with boundary. One can consider so-called 'bulk' and 'edge' states as quantum states representing states on the interior and boundary of the sample respectively. Given a Hamiltonian H on a system without boundary and a spectral gap $\Delta \subset \mathbb{R} \setminus \sigma(H)$, we then consider the Hamiltonian $\widehat{H}$ on a system with edge. We take the spectral projection of $\widehat{H}$ corresponding to Δ, $P_\Delta(\widehat{H})$. The addition of the boundary to our sample means that $\mathrm{Ran}[P_\Delta(\widehat{H})]$ may be non-zero and we think of elements in this subspace as 'edge states'. The functional $a \mapsto \mathcal{T}(P_\Delta a)$ for $\mathcal{T}$ a trace on the algebra of observables, can also be used to measure properties of observables which we interpret to be concentrated at the edge of a sample.

Of course, we need to extract topological information from observables on edge states, which in turn should be related to our bulk (boundary-free) system. What we refer to is called the *bulk-edge correspondence*, which says that these two topological quantities are, in fact, equal. Considering the case of the quantum Hall effect, our bulk invariant should be the topological invariant found by Bellissard/Xia that gives the Hall conductance.

Articles that consider the quantum Hall bulk-edge correspondence are those by Kellendonk, Schulz-Baldes, Graf and collaborators [SBKR02, KSB04a, KSB04b, EG02, EGS05, KR08]. While both the Kellendonk group and the Graf group prove the existence of a bulk-edge correspondence, their methodologies are quite different. The papers of Kellendonk et al. use a K-theoretic argument while the papers of Graf et al. employ more 'classical' techniques from functional analysis. Because of the success of Bellissard's use of noncommutative topology, we shall focus on Kellendonk et al.'s method. The key idea behind the K-theoretic approach is the six-term exact sequence in K-theory and K-homology. Given an exact sequence of C^*-algebras

$$0 \to B \to C \to A \to 0,$$

we obtain the six-term exact sequence in K-theory and K-homology

$$\begin{tikzcd} K_0(B) \arrow[r] & K_0(C) \arrow[r] & K_0(A) \arrow[d, "\partial"] \\ K_1(A) \arrow[u, "\partial"] & K_1(C) \arrow[l] & K_1(B) \arrow[l] \end{tikzcd} , \qquad \begin{tikzcd} K^0(A) \arrow[r] & K^0(C) \arrow[r] & K^0(B) \arrow[d, "\partial"] \\ K^1(B) \arrow[u, "\partial"] & K^1(C) \arrow[l] & K^1(A) \arrow[l] \end{tikzcd} .$$

The work of [SBKR02, KSB04a, KSB04b, KR08] defines an 'edge algebra of observables', B, which they link to the more well-known bulk observable algebra, A, by an extension

$$0 \to B \to C \to A \to 0. \tag{1.1}$$

This gives rise to six-term exact sequences as above. By showing that the short exact sequence of Equation (1.1) is semi-split, one can say that these sequences are compatible with the index pairing of K-theory and K-homology. More specifically, for $[P] \in K_0(A)$ and $[F] \in K^1(B)$, we have that

$$\langle \partial[P], [F] \rangle = -\langle [P], \partial[F] \rangle, \tag{1.2}$$

where ∂ denotes the relevant boundary map in the six-term exact sequences (see [HR01, Prop 8.7.5]). Note that the left hand side of Equation (1.2) depends solely on the edge algebra B and the right hand side is only dependent on the bulk algebra A. Furthermore, for a correct choice of $[P]$ and $[F]$, the right hand side of Equation (1.2) can be interpreted as the same topological pairing as was used in Bellissard's expression for the Hall conductance. Thus, we can interpret the right hand side as the bulk conductance, $\sigma_b = \sigma_H$, and the left hand side as (the negative of) an edge conductance, σ_e, with $\sigma_H = \sigma_b = \sigma_e$. In other words, both the bulk algebra and the edge algebra give rise to topological data describing the the Hall conductance (and its quantisation): a bulk-edge correspondence.

We should note that Kellendonk et al. do not define their bulk and edge conductance via the index pairing of K-theory and K-homology, but instead by the pairing of K-theory with periodic cyclic cohomology. Under certain conditions (which hold for the quantum Hall effect), these two pairings coincide. In other examples and settings (including many topological insulator systems), this is no longer true.

The viewpoint that we take in this thesis is that Kasparov's KK-theory provides the fundamental framework required to understand the topological invariants and bulk-edge correspondence of condensed matter systems. In particular, we avoid the need to pass to periodic cyclic cohomology to compute the quantities of interest. By working in so-called 'unbounded Kasparov theory', our constructions have geometric interpretations and can be easily linked to the underlying physics.

1.2.3 Topological insulators

The term 'topological insulator' can be applied to a range of physical systems. Generally speaking, the term refers to the quantum spin-Hall effect and 3D topological insulators, the main two examples. We will conduct a more thorough review of topological insulators in Chapter 5.1, though we make some basic remarks here.

We will start with the quantum spin-Hall effect as it has the most theory behind it and will inform how other more general systems work. The prediction of the quantum spin-Hall effect is generally attributed to Kane and Mele [KM05]. Kane and Mele considered the bulk/edge picture described by Halperin but imposed time-reversal symmetry on the system (so there is no external magnetic field). The lack of magnetic field means that Hall current vanishes and the net edge current is zero. Instead, Kane and Mele proposed that the electron's spin will now play an important role, with the electrons on the edge splitting into spin-up and spin-down currents travelling in opposite directions. To explain this further, while the topological invariant found by Bellissard is equal to zero, there are finer invariants that are able to detect the presence of the oriented spin current. In particular, Kane and Mele assign a $\mathbb{Z}_2$-number to the quantum spin-Hall system, distinguishing a 'trivial insulator' from one with spin current. While not proved, the authors claim that this $\mathbb{Z}_2$-number is topological in nature and related to the time-reversal invariance, as one can not continuously deform a trivial insulator (topological number 0) to one with a spin current (topological number 1) without breaking time-reversal symmetry, say by turning on an external magnetic field.

The effect was initially predicted in [KM05] to occur in graphene, but graphene is hard to work with experimentally. The effect was later predicted to be found in HgTe [BHZ06], a compound much more usable in a laboratory, and subsequently the quantum spin-Hall effect was experimentally confirmed in [KWB$^+$07].

In order to model a system with time-reversal symmetry, we need to represent the time-reversal involution on the Hilbert space of states of the system. However, this involution is an anti-unitary operator. To adequately incorporate the time-reversal involution into our observable algebra, we must use real or Real C^*-algebras (where the capitalisation makes a difference). The widely held belief, that was only mathematically proved quite recently [FM13, Thi15, GS15], is that quantities protected by time-reversal or other anti-unitary involutions are linked to the real/Real K-theory of the algebra of observables. In particular, the $\mathbb{Z}_2$ invariant of Kane-Mele arises from the group $KO_2(\mathbb{R}) \cong KR_2(\mathbb{C}) \cong \mathbb{Z}_2$.

We would like to use a version of the Kellendonk et al. argument in the setting of the quantum spin-Hall effect to obtain a new bulk-edge correspondence, but there are some obstacles. The pairing used in [SBKR02, KSB04b, KR08] comes from translating the pairing of K-theory and K-homology to a pairing of cyclic homology with cyclic cohomology. However, the equivalence of pairings only works when we are not interested in torsion invariants. Periodic cyclic homology and cohomology can not detect torsion groups. One of the goals of this thesis is to use (unbounded) Kasparov theory to work around this obstacle.

There are, of course, other examples of topological insulators. The 3-dimensional systems are of interest to experimentalists due to their potential applications. The basic idea is the same though: we have a d-dimensional system with some symmetry property and, given the right parameters, we also have an observable concentrated on the edge of the sample (e.g. current) which is 'topologically protected' by the symmetries of the whole system. Topologically protected meaning, as before, that the observable of interest does not change its value unless the symmetry is broken (provided the disorder and impurities in our sample are controlled). See Chapter 5.1 for more on the other insulating systems and their symmetry properties.

1.3 Outline and purpose of this thesis

Our overarching goal for this thesis is to show how Kasparov theory can be used to understand topological states of matter, in particular the bulk-edge correspondence of such systems. This involves showing how Kellendonk et al.'s argument for the bulk-edge correspondence can be expressed in purely K-theoretic terms without mapping into cyclic cocycles. Such a viewpoint can then be generalised to the real picture and applied to topological insulator systems like the quantum spin-Hall effect.

We use Kasparov theory because all the invariants we have discussed in the introduction come from the pairing of K-theory and K-homology, which is a (very) special case of the Kasparov product. The boundary maps of Equation (1.2) are also realisable as Kasparov products, and the whole formalism is flexible enough to deal with complex, real or Real C^*-algebras.

First we outline how index theory, K-theory and K-homology can be understood in terms of unbounded Kasparov theory. We do this in Chapter 2, which summarises the results of interest to us in unbounded Kasparov theory. We also briefly comment on KK-theory for real C^*-algebras as this theory is relatively under-studied but required for anti-linear symmetries. It should be noted that unbounded Kasparov theory is a research area that is still in development. One benefit of the unbounded theory, despite some extra technical details, is that the operators one works with are more geometric in origin and can be explicitly linked to the physics that is being modelled.

In Chapter 3, we outline how our approach applies to the quantum Hall effect without boundary and higher-dimensional systems with magnetic field. Much of Chapter 3 involves a translation of Bellissard's work into our picture. In particular, we aim to show how the topological properties of a Hamiltonian H for a suitable system arise from the index pairing (Kasparov product) of a K-theory class (represented by the

Fermi projection of H for even-dimensional systems) with a particular spectral triple or unbounded Fredholm module that captures the geometry of the Brilllouin zone (momentum space). Part of the work in this chapter (Section 3.3) was performed in collaboration with Prof. Hermann Schulz-Baldes and Dr Giuseppe De Nittis during a visit to Friedrich-Alexander Universität Erlangen-Nürnberg in October-November 2014. The proofs are the author's.

Chapter 4 considers boundaries and the bulk-edge correspondence of the quantum Hall effect. The chapter adapts the work of Kellendonk, Schulz-Baldes and Richter [SBKR02, KSB04b] into Kasparov theory and without the need for cyclic cohomology. This chapter is based on the publication [BCR15], written in collaboration with the author's advisors, Prof. Alan Carey and A/Prof. Adam Rennie. This publication has been accepted in the journal Letters in Mathematical Physics.

Finally, in Chapter 5 we consider the problem of general topological insulators. Our aim for the chapter is two-fold. First to show how previous work on the problem can be expressed in terms of Kasparov theory. Then to show how we can use KK-theory to obtain a bulk-edge correspondence for discrete insulator systems with particular symmetry properties in arbitrary dimension. We note that not all possible topological insulator models fit into our current framework (such as continuous models or those with disorder), though many systems do, including the quantum spin-Hall effect.

To the best of our knowledge, a result analogous to ours has yet to appear in the mathematics literature and will hopefully aid the general understanding of the topological nature of the bulk-edge correspondence. We also note that the results presented here are just a starting point and we are of the opinion that our approach will extend to more complicated systems. We conclude with other possible directions for future work into this area.

Chapter 2

Unbounded Kasparov theory

2.1 Spectral triples and index theory

2.1.1 Basic definitions

In what follows we will assume that the algebras we deal with are separable and nuclear. Much of what we say does not require these assumptions, but they will ease our description of Kasparov theory. We recall the general definition of a spectral triple.

Definition 2.1.1. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by a $*$-algebra represented on a Hilbert space $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ along with a densely defined, self-adjoint operator $D : \mathrm{Dom}(D) \subset \mathcal{H} \to \mathcal{H}$ such that for all $a \in \mathcal{A}$,

1. The commutator $[D, \pi(a)]$ is well-defined on $\mathrm{Dom}(D)$ and extends to a bounded operator on $\mathcal{H}$,
2. The operator $\pi(a)(1 + D^2)^{-1/2}$ is a compact.

If in addition there is an operator γ that commutes with $\pi(a)$ for all $a \in \mathcal{A}$ and anti-commutes with D, we call the spectral triple even. Otherwise, it is odd.

Remark 2.1.2. We see from our definition that if $\mathcal{A}$ is unital and $\pi(1_A) = 1_{\mathcal{H}}$, then $\pi(1)(1 + D^2)^{-1/2} = (1 + D^2)^{-1/2}$ is compact. The more standard definition of a unital spectral triple requires D to have compact resolvent, see for example [GBVF01]. To see the equivalence, we note that $(\lambda - D)^{-1}$ is compact for $\lambda \notin \sigma(D)$ if and only if

$$\left((D - \overline{\lambda})^{-1}(D - \lambda)^{-1}\right)^{1/2} = (D^2 + |\lambda|^2)^{-1/2} \in \mathcal{K}(\mathcal{H}).$$

The resolvent formula then shows that replacing $|\lambda|^2$ by 1 is inessential. We view our definition as a generalisation to non-unital algebras required to handle non-compact examples like the real line, where $D = -i\frac{\mathrm{d}}{\mathrm{d}x}$. One finds that $(1 + D^2)^{-1/2}$ is not a compact operator on $L^2(\mathbb{R})$ whereas $\pi(f)(1+D^2)^{-1/2}$ is compact for $f \in C_c^\infty(\mathbb{R})$ and π

the representation by left multiplication [Sim05, Chapter 4]. Hence the condition that D has compact resolvent is replaced by a relative compactness condition.

Provided the context is clear, we will be sloppy with notation and simply write $\mathcal{A}$ instead of $\pi(\mathcal{A})$.

Proposition 2.1.3 ([BJ83])**.** *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple let F_D be the bounded operator $D(1 + D^2)^{-1/2}$. Then $(A, \mathcal{H}, F_D)$ is a Fredholm module, where A is the C^*-closure of $\mathcal{A}$.*

We will prove this result in the case of more general Kasparov modules in Theorem 2.2.27. Hence we know that spectral triples give rise to classes in K-homology without any assumptions on whether A is unital or not.

Due to the rigidity of C^*-algebras, we often work with 'smooth' subalgebras.

Definition 2.1.4. A $*$-algebra $\mathcal{A}$ is smooth if it is

1. Fréchet, i.e. complete and metrizable such that the multiplication is jointly continuous;

2. Isomorphic to a proper dense $*$-subalgebra $\iota(\mathcal{A})$ of a C^*-algebra A, where $\iota : \mathcal{A} \hookrightarrow A$ is the inclusion map, and $\iota(\mathcal{A})$ is stable under the holomorphic functional calculus. That is, if f is a holomorphic function on a neighbourhood of the spectrum of $a \in \iota(\mathcal{A})$, then $f(a) \in \iota(\mathcal{A})$.

Stability under the holomorphic functional calculus extends to nonunital algebras, since the spectrum of an element in a nonunital algebra is defined to be the spectrum of this element in the one-point unitization, though we must restrict to functions satisfying $f(0) = 0$. Similarly, the definition of a Fréchet algebra does not require a unit.

Proposition 2.1.5 ([Sch92])**.** *If $\mathcal{A}$ is a smooth subalgebra of a C^*-algebra A, then the map induced by the inclusion $\iota_* : K_j(\mathcal{A}) \to K_j(A)$ is an isomorphism.*

Spectral triples quite often contain more than just K-homological data. Hence we introduce extra structure on spectral triples that have the interpretation of a differential structure and measure theory.

Definition 2.1.6. Let $\delta(T) = [(1 + D^2)^{1/2}, T]$ for $T \in \mathrm{Dom}(\delta)$. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is QC^∞ if

$$\mathcal{A},\ [D, \mathcal{A}] \subset \bigcap_{m \geq 0} \mathrm{Dom}(\delta^m).$$

Proposition 2.1.7 ([Ren03])**.** *If $(\mathcal{A}, \mathcal{H}, D)$ is a QC^∞ spectral triple, then $(\mathcal{A}_\delta, \mathcal{H}, D)$ is also a QC^∞ spectral triple, where $\mathcal{A}_\delta$ is the completion of $\mathcal{A}$ in the locally convex topology determined by the seminorms $q_n(a) = \|\delta^n(a)\| + \|\delta^n([D, a])\|$ for $n \geq 0$. Moreover, $\mathcal{A}_\delta$ is a smooth algebra.*

Hence, if we are given a QC^∞ spectral triple $(\mathcal{A}, \mathcal{H}, D)$, we can always take the completion $(\mathcal{A}_\delta, \mathcal{H}, D)$ with $\mathcal{A}_\delta$ smooth.

The notion of dimension of non-unital spectral triples can be quite complicated. A simplification occurs if the $*$-algebra $\mathcal{A}$ is *local*.

Definition 2.1.8. An algebra $\mathcal{A}_c$ has local units if for every finite subset of elements $\{a_i\}_{i=1}^n \subset \mathcal{A}_c$, there exists $\phi \in \mathcal{A}_c$ such that $\phi a_i = a_i \phi = a_i$ for each i. An algebra $\mathcal{A}$ is local if it is Fréchet and there exists a dense ideal $\mathcal{A}_c \subset \mathcal{A}$ with local units.

To aid the reader, we consider the notion of summability for spectral triples over local algebras before looking at the general picture. We can define the dimension of spectral triples using the Schatten ideals $\mathcal{L}^p(\mathcal{H})$ (see [Sim05]) and Dixmier ideals $\mathcal{L}^{(p,\infty)}(\mathcal{H})$ (see [GBVF01, Chapter 7.5]) for $p \geq 1$.

Definition 2.1.9. We say that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with $\mathcal{A}$ local is (p,∞)-summable if $p \geq 1$ and
$$a(1+D^2)^{-1/2} \in \mathcal{L}^{(p,\infty)}(\mathcal{H})$$
for all $a \in \mathcal{A}$.

Proposition 2.1.10 ([Ren04])**.** *Let $(\mathcal{A}, \mathcal{H}, D)$ be a (p,∞)-summable spectral triple with $\mathcal{A}$ local.*

1. *For all s with $1 \leq s \leq p$,*
$$a(1+D^2)^{-s/2} \in \mathcal{L}^{(p/s,\infty)}(\mathcal{H})$$
and for $\mathrm{Re}(s) > p$, $a(1+D^2)^{-s/2}$ is trace-class.

2. *For any Dixmier Trace Tr_ω, the function*
$$a \mapsto \mathrm{Tr}_\omega\Big(a(1+D^2)^{-p/2}\Big)$$
defines a trace on $\mathcal{A}$.

Example 2.1.11. Let $S \to \mathbb{R}^d$ be the (trivial) complex spinor bundle over $\mathbb{R}^d$. Then the triple $\big(C_c^\infty(\mathbb{R}^d), L^2(\mathbb{R}^d, S), \not{D}\big)$ is a smooth, local (d,∞)-summable spectral triple, where $f \in C_c^\infty(\mathbb{R}^d)$ acts by left-multiplication and $\not{D}$ is the Dirac operator acting on sections of the spinor bundle S. Our spectral triple is local as $C_c^\infty(\mathbb{R}^d)$ is a local algebra and $\not{D}$ preserves supports (being a differential operator). This means that if ϕ is a local unit for $f \in C_c^\infty(\mathbb{R}^d)$, then ϕ is also a local unit for $[\not{D}, f] = df$. See [Ren04, Proposition 13, Corollary 14] for a proof of (p,∞)-summability as well as a computation of the result that, for any $f \in C_c^\infty(\mathbb{R}^d)$,
$$\mathrm{Tr}_\omega(f(1+\not{D}^2)^{-d/2}) = \frac{2^{\lfloor d/2 \rfloor}\mathrm{Vol}(S^{d-1})}{d(2\pi)^d}\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x.$$

Spectral triples over non-local algebras require more care and we must turn to the integration and index theory developed in [CGRS12, CGRS14].

2.1.2 Preliminaries on non-unital spectral triples

Here we introduce some of the more sophisticated technology required to deal with general non-unital spectral triples. Some of our definitions can be thought of as analogues of the constructions Connes and Moscovici used to prove the local index formula [CM95], though we note that many are novel. Our brief exposition follows [vdDPR13, Section 2], which in turn is a summary of [CGRS14, Chapter 1, 2]. In order to discuss smoothness and summability for non-unital spectral triples, we need to introduce an analogue of L^p-spaces for operators and weights.

Definition 2.1.12. Let D be a densely defined operator on a Hilbert space $\mathcal{H}$. Then for each $p \geq 1$ and $s > p$ we define a weight φ_s on $\mathcal{B}(\mathcal{H})$ by

$$\varphi_s(T) = \mathrm{Tr}\Big((1+D^2)^{-s/4}T(1+D^2)^{-s/4}\Big)$$

for T a positive operator on $\mathcal{H}$. We define the subspace $\mathcal{B}_2(D,p)$ of $\mathcal{B}(\mathcal{H})$ by

$$\mathcal{B}_2(D,p) = \bigcap_{s>p}\Big(\mathrm{Dom}(\varphi_s)^{1/2}\bigcap(\mathrm{Dom}(\varphi_s)^{1/2})^*\Big).$$

Take $T \in \mathcal{B}_2(D,p)$. The norms

$$\mathcal{Q}_n(T) = \big(\|T\|^2 + \varphi_{p+1/n}(|T|^2) + \varphi_{p+1/n}(|T^*|^2)\big)^{1/2}$$

for $n = 1, 2, \ldots$ take finite values on $\mathcal{B}_2(D,p)$ and provide a topology on $\mathcal{B}_2(D,p)$ stronger than the norm topology.

The space $\mathcal{B}_2(D,p)$ is in fact a Fréchet algebra [CGRS14, Proposition 1.6] and can be interpreted as the bounded square integrable operators.

To introduce the bounded integrable operators, first take $\mathcal{B}_2(D,p)^2$, the span of products in $\mathcal{B}_2(D,p)$, and define the norms

$$\mathcal{P}_n(T) = \inf\left\{\sum_{i=1}^k \mathcal{Q}_n(T_{1,i})\mathcal{Q}_n(T_{2,i}) \,:\, T = \sum_{i=1}^k T_{1,i}T_{2,i},\ T_{1,i}, T_{2,i} \in \mathcal{B}_2(D,p)\right\},$$

where the sums are finite and the infimum is over all possible such representations of T. It is shown in [CGRS14, p12-13] that $\mathcal{P}_n$ are norms on $\mathcal{B}_2(D,p)^2$.

Definition 2.1.13. Let D be a densely defined and self-adjoint operator on $\mathcal{H}$ and $p \geq 1$. We define $\mathcal{B}_1(D,p)$ to be the completion of $\mathcal{B}_2(D,p)^2$ with respect to the family of norms $\{\mathcal{P}_n : n = 1, 2, \ldots\}$.

Definition 2.1.14. A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be finitely summable if there exists $s > 0$ such that for all $a \in \mathcal{A}$, $a(1+D^2)^{-s/2} \in \mathcal{L}^1(\mathcal{H})$. In such a case we let

$$p = \inf\{s > 0 \,:\, \forall a \in \mathcal{A},\ \mathrm{Tr}(|a|(1+D^2)^{-s/2}) < \infty\}$$

and call p the spectral dimension of $(\mathcal{A}, \mathcal{H}, D)$.

Note that $|a|(1+D^2)^{-s/2} \in \mathcal{L}^1(\mathcal{H})$ by the polar decomposition $a = v|a|$; we are not claiming that $|a| \in \mathcal{A}$. For the definition of spectral dimension to have meaning, we require that $\operatorname{Tr}(a(1+D^2)^{-s/2}) \geq 0$ for $a \geq 0$, a fact that follows from [Bik98, Theorem 3]. One finds that for a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ to be finitely summable with spectral dimension p, it is a necessary condition that $\mathcal{A} \subset \mathcal{B}_1(D,p)$ [CGRS14, Proposition 2.17]. This condition is almost sufficient as well [CGRS14, Proposition 2.16].

Definition 2.1.15. Let D be a densely defined self-adjoint operator on $\mathcal{H}$. Set $\mathcal{H}_\infty = \bigcap_{k\geq 0} \operatorname{Dom}(D^k)$. For an operator $T : \mathcal{H}_\infty \to \mathcal{H}_\infty$ we define

$$\delta(T) = [(1+D^2)^{1/2}, T], \quad L(T) = (1+D^2)^{-1/2}[D^2, T], \quad R(T) = [D^2, T](1+D^2)^{-1/2}.$$

One has that (cf. [CM95, CPRS06a])

$$\bigcap_{n\geq 0} \operatorname{Dom}(L^n) = \bigcap_{n\geq 0} \operatorname{Dom}(R^n) = \bigcap_{k,l\geq 0} \operatorname{Dom}(L^k \circ R^l) = \bigcap_{n\geq 0} \operatorname{Dom}(\delta^n).$$

We see that to define $\delta^k(T)$, we require that $T : \mathcal{H}_k \to \mathcal{H}_k$ for $\mathcal{H}_k = \bigcap_{l=0}^k \operatorname{Dom}(D^l)$.

Definition 2.1.16. Let D be a densely defined self-adjoint operator on $\mathcal{H}$ and $p \geq 1$. Then define for $k = 0, 1, \ldots$

$$\mathcal{B}_1^k(D,p) = \left\{ T \in \mathcal{B}(\mathcal{H}) \mid T : \mathcal{H}_l \to \mathcal{H}_l \text{ and } \delta^l(T) \in \mathcal{B}_1(D,p) \ \forall l = 0, \ldots, k \right\}$$

as well as

$$\mathcal{B}_1^\infty(D,p) = \bigcap_{k=0}^{\infty} \mathcal{B}_1^k(D,p).$$

For any k (including ∞), we equip $\mathcal{B}_1^k(D,p)$ with the topology induced by the seminorms

$$\mathcal{P}_{n,l}(T) = \sum_{j=0}^{l} \mathcal{P}_n(\delta^j(T))$$

for $T \in \mathcal{B}(\mathcal{H})$, $l = 0, \ldots, k$ and $n \in \mathbb{N}$.

If we are interested in index theory in the non-compact setting, we need to control the integrability of both functions and their derivatives. The noncommutative analogue of this turns out to be a finitely summable spectral triple but with additional smoothness properties.

Definition 2.1.17. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. We say that $(\mathcal{A}, \mathcal{H}, D)$ is QC^k summable if it is finitely summable with spectral dimension p and

$$\mathcal{A} \cup [D, \mathcal{A}] \subset \mathcal{B}_1^k(D,p).$$

We say that $(\mathcal{A}, \mathcal{H}, D)$ is smoothly summable if it is QC^k summable for all $k \in \mathbb{N}$ or, equivalently, if

$$\mathcal{A} \cup [D, \mathcal{A}] \subset B_1^\infty(D,p).$$

Example 2.1.18. Denote by $W^{\infty,1}(\mathbb{R}^d)$ the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to the seminorms

$$q_n(f) = \max_{|\alpha|\leq n}\|\partial^\alpha f\|_1,$$

where $\alpha \in \mathbb{N}^d$ is a multi-index and $\|\cdot\|_1$ is the Sobolev norm. It is shown in [CGRS14, Chapter 4] that $\big(W^{\infty,1}(\mathbb{R}^d), L^2(\mathbb{R}^d, S), \not{D}\big)$ is a smoothly summable spectral triple with spectral dimension d, where $S \to \mathbb{R}^d$ is the (trivial) spinor bundle and $\not{D}$ the Dirac operator.

Of course, $C_c(\mathbb{R}^d)$ has local units and this example does not require the full non-unital machinery. However, Chapter 4 of [CGRS14] extends such results to general (non-compact) manifolds with strictly positive injectivity radius and whose curvature tensors have covariant derivatives bounded in M.

In Chapter 3, we will find that the crossed-product algebra we use to study continuous quantum systems, $C(\Omega) \rtimes_\theta \mathbb{R}^d$, is *not* a local algebra and so we must use the more general framework.

For a smoothly summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$, we can introduce the δ-φ topology on $\mathcal{A}$ by the seminorms

$$\mathcal{A} \ni a \mapsto \mathcal{P}_{n,k}(a) + \mathcal{P}_{n,k}([D, a]) \tag{2.1}$$

for $n, k \in \mathbb{N}$. We obtain an analogue of Proposition 2.1.7 taking summability into account.

Proposition 2.1.19 ([CGRS14], Proposition 2.20)**.** *Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple that is smoothly summable with spectral dimension p. If $\mathcal{A}_{\delta,\varphi}$ is the completion of $\mathcal{A}$ in the δ-φ topology, then $(\mathcal{A}_{\delta,\varphi}, \mathcal{H}, D)$ is also a smoothly summable spectral triple with spectral dimension p. Moreover, $\mathcal{A}_{\delta,\varphi}$ is a smooth algebra.*

We finish this section with a sufficient and checkable condition of smooth summability of spectral triples.

Proposition 2.1.20 ([CGRS14], Proposition 2.21)**.** *Let $(\mathcal{A}, \mathcal{H}, D)$ be a finitely summable spectral triple of spectral dimension p. If for all $T \in \mathcal{A} \cup [D, \mathcal{A}]$, $k \in \mathbb{N}$ and $s > p$ we have that*

$$(1 + D^2)^{-s/4} L^k(T)(1 + D^2)^{-s/4} \in \mathcal{L}^1(\mathcal{H}),$$

where $L(T) = (1 + D^2)^{-1/2}[D^2, T]$, then $(\mathcal{A}, \mathcal{H}, D)$ is smoothly summable.

2.1.3 The index pairing of K-theory with K-homology

Unital spectral triples are thought of as a noncommutative analogue of compact spin manifolds, whose Dirac operators have, amongst other properties, a well-defined analytic index. In the noncommutative setting, the index pairing occurs on the level of Fredholm modules and K-homology. By Proposition 2.1.3, we have the standard transformation $F_D = D(1+D^2)^{-1/2}$, which takes us from a spectral triple to a Fredholm module, but also presents us with two issues. First, if we want something like $\mathrm{Index}(pF_Dp)$ to be well-defined for some projection p, we require that $F_D^2 = 1$, which is not true in general [CGRS14, Section 2.3]. Second, while the definition of the K-homology group of an algebra is the same regardless of whether the algebra is unital or not, this is not true for K-theory. Therefore, we need to take some care in making sure that what we write down as an index pairing between K-theory and K-homology extends to non-unital algebras.

The solution to the first problem comes from [Con85].

Definition 2.1.21. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. For any $\mu > 0$, define the double of $(\mathcal{A}, \mathcal{H}, D)$ to be the spectral triple $(\mathcal{A}, \mathcal{H} \oplus \mathcal{H}, D_\mu)$, where the operator D_μ and the action of $\mathcal{A}$ is given by

$$D_\mu = \begin{pmatrix} D & \mu \\ \mu & -D \end{pmatrix}, \qquad\qquad a \mapsto = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

for all $a \in \mathcal{A}$. If $(\mathcal{A}, \mathcal{H}, D)$ is graded by γ, then the double is graded by $\hat{\gamma} = \gamma \oplus (-\gamma)$.

Remark 2.1.22. Regardless of the invertibility of D, we have that D_μ is always invertible and so we may take $F_\mu = D_\mu |D_\mu|^{-1}$, which has square 1. It is shown in [Con85] that taking the double of a spectral triple does not change the corresponding K-homology class for any $\mu > 0$. Hence we get a 'normalised' Fredholm module at the cost that our representation is now degenerate. This is not surprising and is characteristic of K-homology (see [HR01, Section 8.3] for more on this).

Let $\mathcal{A}^\sim = \mathcal{A} \oplus \mathbb{C}$ be the minimal unitisation of $\mathcal{A}$. We also need to extend the action of $M_n(\mathcal{A}^\sim)$ to the double $(\mathcal{H} \oplus \mathcal{H}) \otimes \mathbb{C}^n$ in a manner that is compatible with the action of $\mathcal{A}$ on $\mathcal{H} \oplus \mathcal{H}$. Given $b \in M_n(\mathcal{A}^\sim)$ we let

$$\hat{b} = \begin{pmatrix} b & 0 \\ 0 & 1_b \end{pmatrix} \in M_{2n}(\mathcal{H}),$$

where $1_b = \pi^n(b)$ and $\pi^n : M_n(\mathcal{A}^\sim) \to M_n(\mathbb{C})$ is the quotient map coming from the unitisation.

The double construction allows us to write down the pairings in the nonunital case explicitly.

Definition 2.1.23 (Index pairing - odd case)**.** Let $(\mathcal{A}, \mathcal{H}, D)$ be an odd spectral triple with $\mathcal{A}$ separable and u a unitary in $M_n(\mathcal{A}^\sim)$ which represents $[u] \in K_1(\mathcal{A})$. Then with $F_\mu = D_\mu |D_\mu|^{-1}$ and $P_\mu = (1 + F_\mu)/2$, we have

$$\langle [u], [(\mathcal{A}, \mathcal{H}, D)] \rangle = \mathrm{Index}((P_\mu \otimes 1_n)\hat{u}(P_\mu \otimes 1_n))\,.$$

Definition 2.1.24 (Index pairing - even case)**.** Let $(\mathcal{A}, \mathcal{H}, D, \gamma)$ be an even spectral triple with $\mathcal{A}$ separable and e a projection in $M_n(\mathcal{A}^\sim)$ that represents $[e] \in K_0(\mathcal{A})$. Given $P = (1 + \hat{\gamma})/2$ and $P^\perp = 1 - P$, we define $(F_\mu)_+ = P^\perp F_\mu P$. The index pairing is given by

$$\langle [e] - [1_e], [(\mathcal{A}, \mathcal{H}, D, \gamma)] \rangle = \mathrm{Index}((\hat{e}(F_\mu \otimes 1_n)\hat{e})_+)\,.$$

Implicit in these definitions is that $(P_\mu \otimes 1_n)\hat{u}(P_\mu \otimes 1_n)$ and $(\hat{e}(F_\mu \otimes 1_n)\hat{e})_+$ are Fredholm. To see this, we observe that $(P_\mu \otimes 1_n)\hat{u}^*(P_\mu \otimes 1_n)$ and $(\hat{e}(F_\mu \otimes 1_n)\hat{e})_-$ give pseudo-inverses for the operators of interest (see [CGRS14, Section 2.3]). Hence the operators are Fredholm.

2.1.4 The local index formula

Given a unital spectral triple $(\mathcal{A}, \mathcal{H}, D)$ satisfying extra regularity properties, Connes and Moscovici [CM95] found a formula to compute the index pairing of K-theory with K-homology directly using the operator D. This formula is, as such, much more amenable to computations than the abstract index pairing. This result was generalised to semifinite (but still unital) spectral triples in [CPRS06a, CPRS06b] and finally to nonunital semifinite triples in [Ren04, CGRS14]. It is important to note that all local index formulae require summability and smoothness of the spectral triple, which is one of the reasons we define these structures. This may seem like a large restriction, but turns out to be satisfied in our examples.

A simplification of the local index formula occurs when our smooth and summable spectral triple has isolated spectral dimension. To define this notion, we first consider the iterated commutator $T^{(k)}$, where

$$T^{(k)} = \underbrace{[\,D^2, [\,D^2, [\,\ldots [\,D^2, T\,] \ldots]\,]\,]}_{k \text{ times}}$$

and $T^{(0)} = T$.

Definition 2.1.25 ([CPRS06a, CPRS06b])**.** Let $(\mathcal{A}, \mathcal{H}, D)$ be a smoothly summable spectral triple of spectral dimension q. We say that the spectral dimension is isolated if, for any element $b \in \mathcal{B}(\mathcal{H})$ of the form

$$b = a_0[D, a]^{(k_1)} \cdots [D, a_m]^{(k_m)}(1 + D^2)^{-|k| - m/2},$$

where $a_0, \ldots, a_m \in \mathcal{A}$ and $k \in \mathbb{N}^m$ is a multi-index with $|k| = k_1 + \ldots + k_m$, the zeta function

$$\zeta_b(z) = \mathrm{Tr}(b(1 + D^2)^{-z}),$$

has an analytic continuation to a deleted neighbourhood of $z = 0$.

We set some notation. For the multi-index k, let

$$\alpha(k) = \frac{1}{k_1!k_2!\cdots k_m!(k_1+1)(k_1+k_2+2)\cdots(|k|+m)}.$$

We also define $\sigma_{n,j}$ and $\tilde{\sigma}_{n,j}$ by the equalities

$$\prod_{j=0}^{n-1}(z+j) = \sum_{j=1}^{n} z^j \sigma_{n,j}, \qquad \prod_{j=0}^{n-1}(z+j+1/2) = \sum_{j=0}^{n} z^j \tilde{\sigma}_{n,j}$$

and finally define the functional

$$\tau_j(b) = \operatorname*{res}_{z=0} z^j \operatorname{Tr}(b(1+D^2)^{-z}), \quad j=-1,0,1,\ldots.$$

Definition 2.1.26. Suppose that $(\mathcal{A},\mathcal{H},D,\gamma)$ is a smoothly summable spectral triple with spectral dimension p and isolated spectral dimension (if the triple is odd, then $\gamma = 1$). Given, $a_0, a_1, \ldots, a_m \in \mathcal{A}$, the residue cocycle $(\phi_m)_{m=0}^M$ is defined by $\phi_0(a_0) = \tau_{-1}(\gamma a_0)$ and

$$\phi_m(a_0,\ldots,a_m) = \sqrt{2\pi i}\sum_{|k|=0}^{M-m}(-1)^{|k|}\alpha(k)\sum_{j=0}^{|k|+(m-1)/2}\tilde{\sigma}_{(|k|+(m-1)/2),j} \times \tau_j\Big(a_0[D,a_1]^{(k_1)}\cdots[D,a_m]^{(k_m)}(1+D^2)^{-|k|-m/2}\Big),$$

for m odd and

$$\phi_m(a_0,\ldots,a_m) = \sum_{|k|=0}^{M-m}(-1)^{|k|}\alpha(k)\sum_{j=1}^{|k|+m/2}\sigma_{(|k|+m/2),j} \times \tau_{j-1}\Big(\gamma a_0[D,a_1]^{(k_1)}\cdots[D,a_m]^{(k_m)}(1+D^2)^{-|k|-m/2}\Big)$$

for m even.

The residue cocycle requires isolated spectral dimension of our spectral triple. We deal with more general spectral triples using the resolvent cocycle. We first establish the notation

$$R_s(\lambda) = (\lambda - (1+s^2+D^2))^{-1}.$$

Definition 2.1.27 ([CPRS06a, CPRS06b])**.** Let $(\mathcal{A},\mathcal{H},D,\gamma)$ be a smoothly summable spectral triple with spectral dimension p and suppose there exists $\mu > 0$ such that $D^2 \geq \mu^2$. For $a \in (0,\mu^2/2)$, let ℓ be the verical line $\ell = \{a+iv : v \in \mathbb{R}\}$. We define the resolvent cocycle $(\phi_m^r)_{m=0}^M$ for $\Re(r) > (1-m)/2$ as

$$\phi_m^r(a_0,\ldots,a_m) = \frac{\eta_m}{2\pi i}\int_0^\infty s^m \operatorname{Tr}\Big(\gamma\int_\ell \lambda^{-p/2-r} a_0 R_s(\lambda)[D,a_1]R_s(\lambda)\cdots[D,a_m]R_s(\lambda)\,\mathrm{d}\lambda\Big)\,\mathrm{d}s,$$

where

$$\eta_m = \left(-\sqrt{2i}\right)^{\bullet} 2^{m+1}\frac{\Gamma(m/2+1)}{\Gamma(m+1)}$$

with $\bullet = 0,1$ depending on whether the spectral triple is even or odd.

The integral over ℓ is well-defined by [CGRS14, Lemma 3.3]. We also note that while we require D to be invertible in order to write down the resolvent cocyle, invertibility of D is not required for the local index formula [CGRS14, Section 3.8].

The index formula is a pairing of a cocycle with an algebraic chain. If $e \in \mathcal{A}^\sim$ is a projection, we define $\mathrm{Ch}^0(e) = e$ and for $k \geq 1$,

$$\mathrm{Ch}^{2k}(e) = (-1)^k \frac{(2k)!}{k!} (e - 1/2) \otimes e \otimes \cdots \otimes e \in (\mathcal{A}^\sim)^{\otimes(2k+1)}.$$

If $u \in \mathcal{A}^\sim$ is a unitary, then we define for $k \geq 0$

$$\mathrm{Ch}^{2k+1}(u) = (-1)^k k!\, u^* \otimes u \otimes \cdots \otimes u^* \otimes u \in (\mathcal{A}^\sim)^{\otimes(2k+2)}.$$

We split up the theorem into odd and even cases.

Theorem 2.1.28 ([CM95, Ren04, CGRS14])**.** *Let $(\mathcal{A}, \mathcal{H}, D)$ be an odd smoothly summable spectral triple with spectral dimension p. Let $N = \lfloor \frac{q}{2} \rfloor + 1$, where $\lfloor \cdot \rfloor$ is the floor function, and let u be a unitary in the unitisation of $\mathcal{A}$. The index pairing can be computed with the resolvent cocycle*

$$\langle [u], [(\mathcal{A}, \mathcal{H}, D)] \rangle = \frac{-1}{\sqrt{2\pi i}} \operatorname*{res}_{r=(1-p)/2} \sum_{m=1,\mathrm{odd}}^{2N-1} \phi_m^r(\mathrm{Ch}^m(u))$$

and the sum $\sum_{m=1,\mathrm{odd}}^{2N-1} \phi_m^r(\mathrm{Ch}^m(u))$ *analytically continues to a deleted neighbourhood of $r = (1-p)/2$.*

If, moreover, the triple $(\mathcal{A}, \mathcal{H}, D)$ has isolated spectral dimension, then the index can be computed with the residue cocycle

$$\langle [u], [(\mathcal{A}, \mathcal{H}, D)] \rangle = \frac{-1}{\sqrt{2\pi i}} \sum_{m=1,\mathrm{odd}}^{2N-1} \phi_m(\mathrm{Ch}^m(u)).$$

We note that the minus sign in the formula for $\langle [u], [(\mathcal{A}, \mathcal{H}, D)] \rangle$ does not always appear in the literature. The minus sign is required so that the residue cocycle is homotopic to the Chern character and not its inverse. In particular, the sign ensures that the Gohberg-Krein Theorem

$$\left\langle [e^{2\pi i\theta}], \left[\left(C^\infty(S^1), L^2(S^1), \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}\theta} \right) \right] \right\rangle = -\mathrm{Wind}(e^{2\pi i\theta}) = -1$$

is reproduced, where $\mathrm{Wind}[f(\theta)]$ is the winding number of a continuous function f on the circle [GK60].

Theorem 2.1.29 ([CM95, Ren04, CGRS14])**.** *Let $(\mathcal{A}, \mathcal{H}, D, \gamma)$ be an even smoothly summable spectral triple with spectral dimension p. Let $N = \lfloor \frac{q+1}{2} \rfloor$ and $e \in \mathcal{A}^\sim$ be a self-adjoint projection. The index pairing can be computed by the resolvent cocycle*

$$\langle [e] - [1_e], [(\mathcal{A}, \mathcal{H}, D, \gamma)] \rangle = \operatorname*{res}_{r=(1-p)/2} \sum_{m=0,\mathrm{even}}^{2N} \phi_m^r(\mathrm{Ch}^m(e) - \mathrm{Ch}^m(1_e))$$

and the sum $\sum_{m=0,\text{even}}^{2N} \phi_m(\mathrm{Ch}^m(e))$ *analytically continues to a deleted neighbourhood of* $r = (1-p)/2$.

If $(\mathcal{A}, \mathcal{H}, D, \gamma)$ *has isolated spectral dimension, then the index can be computed with the residue cocycle*

$$\langle [e] - [1_e], [(\mathcal{A}, \mathcal{H}, D, \gamma)] \rangle = \sum_{m=0,\text{even}}^{2N} \phi_m(\mathrm{Ch}^m(e) - \mathrm{Ch}^m(1_e)).$$

An important remark is that while the cocycle formulas for the index look intimidating, in the examples we consider the expressions simplify substantially and we are left with a more tractable equation.

2.2 The unbounded Kasparov product

Associated to any spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is a Fredholm module $(A, \mathcal{H}, D(1+D^2)^{-1/2})$ representing a class in the K-homology of A, $K^j(A)$. This means that spectral triples represent K-homological data using more geometric or physical operators (which are typically unbounded). Of course, the K-homology of an algebra is a special case of the bivariant KK-groups as developed by Kasparov [Kas81] with $K^j(A) \cong KK^j(A, \mathbb{C})$. Kasparov's KK-theory is a far-reaching generalisation of the index theory studied thus far, the centrepiece of which is the intersection product

$$KK(A, B) \times KK(B, C) \to KK(A, C)$$

now often called the Kasparov product.

Our aim in this section is to give a brief exposition on unbounded KK-theory. Unbounded methods of studying KK-theory were first considered by [BJ83], who showed that such a viewpoint was possible (cf. Theorem 2.2.27). Of particular interest to us is the way in which unbounded theory may provide a more constructive approach to the Kasparov product. Baaj and Julg first considered the external product (a map $KK(A, B) \times KK(C, D) \to KK(A \hat{\otimes} B, C \hat{\otimes} D)$), where a natural formula in terms of unbounded classes can be given [BJ83]. The unbounded internal product was first studied by [Kuc97] and has more recently been developed by [BMv13, KL13, Mes14, MR15, FR15].

It is important to emphasise that unbounded Kasparov theory is still in development and there are examples that fall outside the theory as it presently stands. The obstacles to lifting the Kasparov product to the unbounded setting are highly technical and are outside the scope of this thesis. We will start with complex KK-theory, which is the most widely-studied, while Section 2.3 will cover the case of KK-theory for real C^*-algebras.

2.2.1 Preliminaries

Hilbert C^*-Modules

We begin with a brief review of Hilbert C^*-modules, which can be thought of as a noncommutative extension of a Hilbert space. Our reference unless otherwise stated is [RW98].

Definition 2.2.1. A right C^*-module over a C^*-algebra A is a linear space E together with a right action $E \times A \to E$ and an inner product $(\cdot\,|\,\cdot) : E \times E \to A$ such that for all $e, f, g \in E$, $\lambda, \rho \in \mathbb{C}$ and $a \in A$,

1. $(\lambda e) \cdot a = \lambda(e \cdot a) = e \cdot (\lambda a)$,
2. $(e|\lambda f + \rho g) = \lambda(e|f) + \rho(e|g)$,
3. $(e|f \cdot a) = (e|f) \cdot a$,
4. $(e|f) = (f|e)^*$ as an element of the C^*-algebra A,
5. $(e|e) \geq 0$ as an element of A,
6. $(e|e) = 0$ if and only if $e = 0$,
7. The space is complete in the norm $\|e\|_E^2 := \|(e|e)\|_A$, where $\|\cdot\|_A$ denotes the norm of A.

Remarks 2.2.2. 1. We will often use the notation E_A to denote a right-A C^*-module.

2. If we remove the completeness property, then the above definition will still make sense if, instead of a C^*-algebra A, we have a dense $*$-subalgebra $\mathcal{A}$ provided that the condition $(e|e) \geq 0$ means as an element of $A \supset \mathcal{A}$. If the $\mathcal{A}$-module is not complete, then it can be completed into an A-module [RW98, Lemma 2.16]. We will denote dense sub-modules over smooth subalgebras by the script lettering $\mathcal{E}_{\mathcal{A}}$.

3. There is still a notion of the Cauchy-Schwarz inequality for elements in a C^*-module. Given $e, f \in E_A$, one finds that $(e|f)^*(e|f) \leq \|(e|e)\|_A(f|f)$ [RW98, Lemma 2.5].

Example 2.2.3. Since $\mathbb{C}$ is a C^*-algebra, a rather trivial example is to take a complex Hilbert space and simply view it as a Hilbert $\mathbb{C}$-module.

Example 2.2.4. Let A be a C^*-algebra and consider A_A, the C^*-module of A over itself defined by the relations

$$a \cdot b = ab, \qquad\qquad (a|b) = a^*b$$

The only condition worth checking from the definition is that $(a|a) = 0$ if and only if $a = 0$. Using the C^*-norm condition,

$$(a|a) = 0 \Leftrightarrow a^*a = 0 \Leftrightarrow \|a^*a\| = 0 \Leftrightarrow \|a\|^2 = 0 \Leftrightarrow a = 0.$$

Example 2.2.5. Take $C_c(\mathbb{R}^n)$, the continuous functions of compact support on $\mathbb{R}^n$ and consider the module $C_c(\mathbb{R}^n)_{C_c(\mathbb{R}^n)}$ with action by right-multiplication and inner product $(f|g)(x) = \overline{f(x)}g(x)$. This is not a complete module but it can be completed in the way one would expect. The completion yields $C_0(\mathbb{R}^n)_{C_0(\mathbb{R}^n)}$ as in Example 2.2.4.

Example 2.2.6. Take $E \to X$ to be a complex vector bundle over a compact, Hausdorff space X. We pick a Hermitian form $(\cdot \mid \cdot)$ on E. Let $\Gamma(E)$ be the sections of E. Then using the Hermitian form as the inner product, $\Gamma(E)$ becomes a $C(X)$-C^*-module.

If X is only locally compact but still Hausdorff, then $\Gamma_0(E)$, the sections of E that vanish at infinity, is a $C_0(X)$-C^*-module.

The last example leads nicely into the Serre-Swan theorem, which links together C^*-modules and geometry.

Theorem 2.2.7 (Serre-Swan Theorem, [Swa62]). *Let X be a compact and Hausdorff space. Then a right module over $C(X)$ is finitely generated and projective if and only if $E \cong \Gamma(V)$ for some complex vector bundle $V \to X$.*

Recall that C^*-module is finitely generated if and only if there exist elements $e_1, \dots, e_n \in E$ such that for all $e \in E$, there exist $a_i \in A$ such that

$$e = \sum_{i=1}^{n} e_i a_i.$$

A C^*-module E is projective if and only if E can be written as a direct summand (as a module) of free modules. In the Serre-Swan case, this implies that

$$E \oplus F \cong C(X)^N$$

for some module F. Putting these two conditions together, if we have a $C(X)$-module $E = \Gamma(V)$ for some complex vector bundle $V \to X$, then it must have the form

$$E \cong pC(X)^N$$

for some $p \in M_n(C(X))$ with $p^2 = p$.

We typically deal with self-adjoint projections, $p^* = p$, but we have not chosen an inner product. If we do choose an inner product, then we can take $p^* = p$ as every idempotent is similar to a projection in a C^*-algebra [Bla98, Proposition 4.6.2].

The standard inner product on $pC(X)^N$ is

$$(e|f) = \sum_{i,j} e_i^* p_{ij} f_j.$$

If $V \to X$ is the restriction of another bundle $V^c \to X^c$ for some compactification X^c of X, then $\Gamma(V^c) = pC(X^c)^N$ for some $p \in M_n(C(X^c))$ and so

$$\Gamma_0(V^c) = \Gamma(V) = pC_0(X)^N.$$

More generally, if A is a unital C^*-algebra then a finitely generated projective C^*-module of A takes the form pA^N or some $p = p^* = p^2 \in M_N(A)$. The K-theory of an algebra A can be computed by considering the stable homotopy classes of finitely generated projective modules over A (see [GBVF01, Chapter 3]).

Much like the case of Hilbert spaces, we are interested in linear transformations between C^*-modules. Though there are many similarities between operators on C^*-modules and operators on Hibert spaces, an adjoint operator T^* for some operator T may not always be defined.

We will denote by $\mathrm{End}_A(E)$ the adjointable endomorphisms from the Hilbert C^*-module E_A to itself, and $\mathrm{Hom}_A(E, F)$ the adjointable linear maps from E_A to F_A.

Proposition 2.2.8 ([RW98], Lemma 2.18)**.** *If $T \in \mathrm{End}_A(E)$, then for all $e \in E$, $(Te|Te) \leq \|T\|^2(e|e)$.*

For $f, g \in E_A$, define the rank-1 endomorphism $\Theta_{f,g}h = f \cdot (g|h)$ for $h \in E_A$. We define $\mathrm{End}^{00}_A(E)$ to be the endomorphisms of finite rank and are given by the set

$$\mathrm{End}^{00}_A(E) = \mathrm{span}_{\mathbb{C}}\{\Theta_{e,f} \,:\, e, f \in E\}.$$

The compact operators $\mathrm{End}^0_A(E)$ are defined by

$$\mathrm{End}^0_A(E) = \overline{\mathrm{span}}_{\mathbb{C}}\{\Theta_{e,f} \,:\, e, f \in E\} = \overline{\mathrm{End}^{00}_A(E)},$$

where the closure is taken via the norm of A.

Example 2.2.9. Take A as C^*-module over itself. Then for $a, b, c \in A$, $\theta_{a,b}c = ab^*c$. Hence, if we take the closure of all linear combinations of $\theta_{a,b}$ for all $a, b \in A$, then we clearly get A back. Thus, $\mathrm{End}^0_A(A) = A$.

Example 2.2.10. (The Standard Module) Let $\mathcal{H}_A = \ell^2(\mathbb{N}) \otimes A$ (we can, of course, use any separable Hilbert space $\mathcal{H}$ instead of $\ell^2(\mathbb{N})$). We construct this module in the following steps:

1. Take the span of finite sums
$$\sum_{i=1}^{N} h_i \otimes a_i$$
for an orthonormal basis $\{h_i\}$ of $\ell^2(\mathbb{N})$ and $a_i \in A$.

2. Define the inner product

$$\left(\sum_i h_i \otimes a_i \middle| \sum_j h_j \otimes b_j\right) := \sum_{i,j} \langle h_i, h_j \rangle a_i^* b_j = \sum_i a_i^* b_i,$$

where $\langle \cdot\,, \cdot \rangle$ is the inner-product of $\ell^2(\mathbb{N})$,

3. Complete with respect to the norm $\|\xi\|^2_{\mathcal{H}_A} := \|(\xi|\xi)\|_A$.

If A is unital,

$$\mathrm{End}^0_A(\mathcal{H}_A) = \mathcal{K}(\mathcal{H}) \otimes A, \qquad \mathrm{End}_A(\mathcal{H}_A) = \mathcal{B}(\mathcal{H}) \otimes_\sigma A,$$

where $\otimes_\sigma$ denotes the spatial tensor product of C^*-algebras (we do not need this in the first equation as $\mathcal{K}(\mathcal{H})$ is nuclear so all tensor products are isomorphic). If A is σ-unital, then

$$\mathrm{End}^0_A(\mathcal{H}_A) = \mathcal{K}(\mathcal{H}) \otimes A, \qquad \mathrm{End}_A(\mathcal{H}_A) = \mathcal{M}(\mathcal{K}(\mathcal{H}) \otimes A),$$

where $\mathcal{M}(B)$ is the multiplier algebra of B [Bla98, §13].

Unbounded operators on C^*-modules

We give a brief review of unbounded operators on C^*-modules. The key reference for this section is [Lan95, Chpt 9].

Definition 2.2.11. Let E_A, F_A be right-A C^*-modules. An (unbounded) operator

$$D : \mathrm{Dom}(D) \subset E_A \to F_A$$

is a densely defined, A-linear map. An operator $\widetilde{D}$ is an extension of D if $\mathrm{Dom}(D) \subset \mathrm{Dom}(\widetilde{D})$ and $\widetilde{D}|_{\mathrm{Dom}(D)} = D$. In this case we write $D \subset \widetilde{D}$. Note that $D = \widetilde{D}$ if and only if $D \subset \widetilde{D}$ and $\widetilde{D} \subset D$.

Definition 2.2.12. Let D be a a densely defined, A-linear map.

$$\mathrm{Dom}(D^*) := \{f \in F_A : \exists e \in E_A \text{ such that } (Dh|f)_A = (h|e)_A \text{ for all } h \in \mathrm{Dom}(D)\}.$$

The adjoint of D is the A-linear map $D^* : \mathrm{Dom}(D^*) \to E_A$ defined by $D^* f = e$. Note that D^* need not be densely defined.

We say that D is symmetric if $D \subset D^*$ and D is self-adjoint if $D = D^*$.

Definition 2.2.13. Let D be a a densely defined, A-linear map. The graph $G(D) \subset E_A \oplus F_A$ is the submodule

$$G(D) = \{(e, De) : e \in \mathrm{Dom}(D)\}.$$

We say that D is closed if $G(D)$ is a closed submodule.

We define $v \in \mathrm{Hom}_A(E \oplus F, F \oplus E)$ by $v(e,f) = (f,-e)$. One can show that $G(D^*) = vG(D)^\perp$. This tells us that $G(D^*)$ is a closed submodule of $F \oplus E$ (i.e. D^* is closed). If E and F were Hilbert spaces we would have $E \oplus F = G(D) \oplus vG(D^*)$, but closed submodules of C^*-modules need not be complemented. In order to get such a decomposition in the C^*-module case we need to impose an additional condition on D.

Definition 2.2.14. An operator $D : \mathrm{Dom}(D) \subset E_A \to F_A$ is called regular if it is a closed operator such that D^* is densely defined and $(1 + D^*D)$ has dense range.

Theorem 2.2.15 ([Lan95], Theorem 9.3)**.** *Let $D : \mathrm{Dom}(D) \subset E_A \to F_A$ be regular. Let $v \in \mathrm{Hom}_A(E \oplus F, F \oplus E)$ be $v(e,f) = (f,-e)$. Then $G(D) \oplus vG(D^*) = E \oplus F$.*

Corollary 2.2.16. *If D is regular, then $(D^*)^* = D$.*

Proof. The graph $G(D)$ is complemented, so $G((D^*)^*) = (G(D)^\perp)^\perp = G(D)$. □

Proposition 2.2.17 ([Lan95], Proposition 9.9)**.** *If $D : \mathrm{Dom}(D) \subset E \to F$ is regular, then D^*D is self-adjoint and regular.*

Example 2.2.18. Let $\mathbb{T}^k$ denote the k-torus, and let $\sigma : \mathbb{T}^k \to \mathrm{Aut}(A)$ be a strongly continuous action on a C^*-algebra A with fixed point algebra A^σ. Define the map $\Phi : A \to A^\sigma$ by

$$\Phi(u) = \frac{1}{(2\pi)^k} \int_{\mathbb{T}^k} \sigma_{(z_1,\ldots,z_k)}(a)\, dt_1 \ldots dt_k,$$

where $z_j = e^{it_j}$. Complete $A \otimes \mathbb{C}^{2^{\lfloor k/2 \rfloor}}$ with respect to the norm coming from

$$((a_i)|(b_j))_{A^\sigma} := \sum_{i=1}^{2^{\lfloor k/2 \rfloor}} \Phi(a_i^* b_i),$$

and call this completion E_{A^σ}. Set

$$\mathrm{Dom}(D) = \left\{ e \in E : \lim_{t \to 0} \frac{U_t e - e}{|t|} \in E \right\}, \qquad t \in \mathbb{R}^k,$$

where U_t is the unitary implementation of σ on E. That is, if $(a_i) \in A \otimes \mathbb{C}^{2^{\lfloor k/2 \rfloor}} \subset E$, then

$$U_t(a_i) = (\sigma_t(a_i)).$$

On $\mathrm{Dom}(D)$, define

$$\partial_i e = \lim_{t \to 0} \frac{U_{t\delta_i} e - e}{t}, \qquad t \in \mathbb{R},$$

where $\delta_i = (0, \ldots, \underbrace{1}_{i^{\text{th}}}, \ldots, 0)$. So the ∂_i are like partial derivatives. Choose matrices $\gamma^i \in M_{2^{[k/2]}}(\mathbb{C})$ for $i = 1, \ldots, k$ such that

$$\gamma^i \gamma^j + \gamma^j \gamma^i = 2\delta_{i,j}.$$

The γ^i represent the generators of the complex Clifford algebra $\mathbb{C}\ell_k$, which we are irreducibly representing on $M_{2^{\lfloor k/2\rfloor}}(\mathbb{C})$. Defining $D : \mathrm{Dom}(D) \to E$ by

$$D = (-i)\sum_{j=1}^{k} \partial_j \otimes \gamma^j,$$

we see that D is densely defined.

There are projections Φ_n, $n \in \mathbb{Z}^k$ such that $\sum_{n\in\mathbb{Z}^k} \Phi_n$ converges strictly to 1_E, given by

$$\Phi_n(a) = \frac{1}{(2\pi)^k}\int_{\mathbb{T}^k} z^{-n}\sigma_z(a)\,dt,$$

where $z^n = z_1^{n_1}\cdots z_k^{n_k}$. Then

$$D = -i\sum_{n\in\mathbb{Z}^k} \Phi_n \otimes \gamma(n)$$

with $\gamma(n) = \sum_{j=1}^k \gamma^j n_j$ and $n = (n_1, \ldots, n_k)$. This holds because $U_t\Phi_n = z^n\Phi_n$ ($z_j = e^{it_j}$) and shows that D is A^σ-linear. A computation shows that D is symmetric and that

$$D^2 = \sum_{n\in\mathbb{Z}^k} n\cdot n\Phi_n.$$

To show that

$$1 + D^2 = 1 - \sum_i \partial_i^2$$

is surjective, for $e \in E$ write $e = \sum_{n\in\mathbb{Z}^k} e_n$ and define

$$f = \sum_{n\in\mathbb{Z}^k} (1 + n\cdot n)^{-1} e_n,$$

so that $(1 + D^2)f = e$. So D is regular and symmetric. Because we have expressed $D = \sum_n \Phi_n \otimes \gamma(n)$ in terms of its spectral decomposition, one finds that $\mathrm{Dom}(D^*) \subset \mathrm{Dom}(D)$ and so D is self-adjoint.

2.2.2 KK-theory, bounded and unbounded

Kasparov modules

Our primary references for the following material are [Kas81, Bla98].

Definition 2.2.19. A $\mathbb{Z}_2$-graded C^*-algebra A is a C^*-algebra $A = A^0 \oplus A^1$ such that $A^i \cdot A^j \subset A^{(i+j) \bmod 2}$.

A $\mathbb{Z}_2$-graded C^*-module E_A is a C^*-module $E_A = E_A^0 \oplus E_A^1$ such that $E_A^i \cdot A^j \subset E_A^{(i+j) \bmod 2}$.

We note that if A is non-trivially graded, there is in general *no* adjointable endomorphism defining the splitting.

Example 2.2.20. If A is trivially graded, then $A \otimes_{\mathbb{C}} \mathbb{C}\ell_n$ is $\mathbb{Z}_2$-graded, since $\mathbb{C}\ell_n$ is $\mathbb{Z}_2$-graded.

Given the $\mathbb{Z}_2$-graded algebras A and B, we can also define the graded tensor product $A\hat{\otimes}B$ by the relations on spanning elements

$$(a_1\hat{\otimes}b_1)(a_2\hat{\otimes}b_2) = (-1)^{|b_1|\,|a_2|}(a_1a_2\hat{\otimes}b_1b_2), \qquad (a\hat{\otimes}b)^* = (-1)^{|a|\,|b|}(a^*\hat{\otimes}b^*),$$

where $|a|$ denotes the degree of a (either 0 or 1). For A and B nuclear, all completions of $A\hat{\otimes}B$ are isomorphic. For the case of non-nuclear algebras, we take the completion in the spatial tensor product (see [Kas81, §2.6]).

Example 2.2.21. If E_A and F_B are $\mathbb{Z}_2$-graded C^*-modules and there is an adjointable left action of A on F, then we can define the $\mathbb{Z}_2$-graded module $(E\hat{\otimes}_AF)_B$ as follows. We first define the ungraded C^*-module $(E \otimes_A F)_B$ with right-action by B on F and the B-valued inner-product

$$(e_1 \otimes f_1 \mid e_2 \otimes f_2)_B = (f_1 \mid [(e_1 \mid e_2)_A] \cdot f_2)_B\,.$$

We divide out the zero-length vectors in this inner-product and complete. One then defines the grading $\deg(e\hat{\otimes}f) = \deg(e) + \deg(f)$ to obtain a $\mathbb{Z}_2$-graded C^*-module $(E\hat{\otimes}_AF)_B$.

We also note that the obvious map $\mathrm{End}_A(E) \ni T \mapsto T\hat{\otimes}1 \in \mathrm{End}_B(E\hat{\otimes}_AF)$ is a graded homomorphism. See [Bla98, §14] or [Kas81, §2] for more on $\mathbb{Z}_2$-graded modules and tensor products.

Definition 2.2.22. Given $\mathbb{Z}_2$-graded C^*-algebras A and B, a (bounded) *Kasparov A-B-module* $(A, {}_\phi E_B, F)$ is given by

- A $\mathbb{Z}_2$-graded, countably generated, right-B C^*-module E_B;
- A $\mathbb{Z}_2$-graded $*$-homomorphism $\phi\colon A \to \mathrm{End}_B(E)$;
- An odd operator (i.e. of degree 1) $F \in \mathrm{End}_B(E)$ such that

  $$\phi(a)(1 - F^2), \quad \phi(a)(F - F^*), \quad [F, \phi(a)]_\pm \in \mathrm{End}^0_B(E)$$

  for all $a \in A$, where $[\cdot,\cdot]_\pm$ denotes the graded commutator $[T,S]_\pm := TS - (-1)^{|T||S|}ST$.

Definition 2.2.23. Given $\mathbb{Z}_2$-graded C^*-algebras A and B, an *unbounded Kasparov A-B-module* $(\mathcal{A}, {}_\phi E_B, D)$ is given by

- A $\mathbb{Z}_2$-graded, countably generated, right-B C^*-module E_B;
- A $\mathbb{Z}_2$-graded $*$-homomorphism $\phi\colon A \to \mathrm{End}_B(E)$;

- A self-adjoint, regular, odd operator $D\colon \operatorname{Dom} D \subset E \to E$ such that $[D, \phi(a)]_\pm$ is an adjointable endomorphism and $\phi(a)(1+D^2)^{-1/2}$ is a compact endomorphism for all a in a dense subalgebra $\mathcal{A}$ of A.

We will write Kasparov modules as (A, E_B, F) and $(\mathcal{A}, E_B, D)$ when the representation ϕ is unambiguous.

Example 2.2.24. Recall Example 2.2.18 where we have an algebra A with torus action $\sigma : \mathbb{T}^k \to \operatorname{Aut}(A)$ and fixed point algebra A^σ. We have already constructed the module E_{A^σ} with self-adjoint regular operator

$$D = \sum_{n\in\mathbb{Z}^k} \Phi_n \otimes \gamma(n),$$

where Φ_n is a projection onto the subalgebra $A_n = \{a \in A : \sigma_z(a) = z^n a \text{ for all } z \in \mathbb{T}^k\}$, $\gamma(n) = \sum_{j=1}^k \gamma^j n_j$ for $n = (n_1, \dots, n_k)$ and γ^j are the generators of the irreducible Clifford representation of $\mathbb{C}\ell_k$ on $\mathbb{C}^{2^{\lfloor k/2\rfloor}}$. We say that the torus action σ satisfies the spectral subspaces condition if $\overline{A_n^* A_n}$ is a complemented ideal in A^σ for all $n \in \mathbb{Z}^k$.

We also have an adjointable action by A on E_{A^σ} by left-multiplication:

$$(ab_i|(c_j))_{A^\sigma} := \sum_{i=1}^{2^{\lfloor k/2\rfloor}} \Phi((ab_i)^*c_i) = \sum_{i=1}^{2^{\lfloor k/2\rfloor}} \Phi(b_i^* a^* c_i) = ((b)_i|a^*c_j)_{A^\sigma}.$$

Finally if k is even, we have a grading operator given by $\gamma = (-i)^{k/2}\gamma^1\cdots\gamma^k$. Provided that the action σ satisfies the spectral subspaces condition, $(\mathcal{A}, E_{A^\sigma}, D, \gamma)$ is an unbounded A-A^σ Kasparov module. See [CNNR11, Section 2.1] for a proof.

Example 2.2.25 (Spectral triples as unbounded Kasparov modules). Let $A = C(M)$ for a compact, oriented manifold M with Riemannian metric g. Let $B = \mathbb{C}$ and $\bigwedge^* T^*M = \bigoplus_k \bigwedge^k T^*M$.

1. Let $E_\mathbb{C} = L^2(\bigwedge^* T^*M \otimes \mathbb{C}, \mathrm{dvol}_g)$ and consider the operator $D := \gamma_L \circ \nabla^{\mathrm{LC}}$, where ∇^{LC} is the Levi-Civita connection on T^*M. We define γ_L by the map $\Gamma(T^*M \otimes \bigwedge^* T^*M \otimes \mathbb{C}) \to \Gamma(\bigwedge^* T^*M \otimes \mathbb{C})$ given by the Clifford multiplication $\gamma_L(\omega)\rho := \omega \wedge \rho - \iota(\omega)\rho$, where $\wedge$ denotes the exterior product and $\iota(\omega)$ is the contraction along ω (also called the interior product). Then $D = d + d^*$, where d is the exterior derivative.

2. Suppose M is a spin manifold with complex spinor bundle $S \to M$. Let $E_\mathbb{C} = L^2(S, \mathrm{dvol}_g)$ and consider $D := \gamma \circ \nabla^S$, where ∇^S is the lift of the Levi-Civita connection, and γ is the action of $\mathbb{C}\ell(T^*M, g)$ on S.

3. Let $E \to M$ be any complex vector bundle. Let $E_\mathbb{C} = L^2(\bigwedge^* T^*M \otimes E, \mathrm{dvol}_g)$ with $D_E = (\gamma_L \otimes 1) \circ (\nabla^{\mathrm{LC}} \otimes 1 + 1 \otimes \nabla^E)$ (from (1)), or $E_\mathbb{C} = L^2(S \otimes E, \mathrm{dvol}_g)$ with $D_E = (\gamma \otimes 1) \circ (\nabla^S \otimes 1 + 1 \otimes \nabla^E)$ (from (2)).

One can check that $\big(C^\infty(M), L^2(\bigwedge^* T^*M \otimes E, \mathrm{dvol}_g)_{\mathbb{C}}, D_E, \gamma_{\bigwedge^* T^*M}\big)$ and similarly $\big(C^\infty(M), L^2(S \otimes E, \mathrm{dvol}_g)_{\mathbb{C}}, D_E, \gamma_S\big)$ are unbounded $C^\infty(M)$-$\mathbb{C}$ Kasparov modules (i.e. spectral triples over $C^\infty(M)$).

Example 2.2.26 (Trivial module). Let A be a $\mathbb{Z}_2$-graded C^*-algebra and A_A the C^*-module with inner product $(a_1|a_2)_A = a_1^* a_2$ (cf. Example 2.2.4). There are left and right actions on A by left and right multiplication. Hence $(A, A_A, 0, \gamma_A)$ is a Kasparov module where γ_A is the grading on A.

Theorem 2.2.27 ([BJ83]). *An unbounded Kasparov module $(\mathcal{A}, E_B, D)$ defines a bounded Kasparov A-B-module $(A, E_B, F_D := D(1+D^2)^{-\frac{1}{2}})$.*

Proof. We immediately see that $a(1-F_D^2) = a(1+D^2)^{-1}$ is compact, and $F_D = F_D^*$ since $D = D^*$. The hard bit is to check that $[F_D, a]$ is compact.* We first argue that it suffices to prove that $[F_D, a]$ is compact for $a \in \mathcal{A}$, where $\mathcal{A}$ is the dense subalgebra of A such that $[D, a]$ is bounded for all $a \in \mathcal{A}$. Namely, for $a \in A$, we can choose a sequence $a_j \in \mathcal{A}$ such that $a_j \to a$, and then

$$\|[F_D, a_j - a_k]\| \leq \|F_D(a_j - a_k) - (a_j - a_k)F_D\| \leq 2\|a_j - a_k\| \to 0.$$

Second, recall the integral formula for fractional powers (see [BJ83]), which for any $0 < s < 1$ is given by

$$(1+D^2)^{-s} = \frac{\sin(s\pi)}{\pi} \int_0^\infty \lambda^{-s}(1+\lambda+D^2)^{-1} d\lambda.$$

Now, for $a, b \in \mathcal{A}$, write

$$[F_D, a]b = [D, a](1+D^2)^{-\frac{1}{2}}b + D[(1+D^2)^{-\frac{1}{2}}, a]b.$$

The first term is compact by assumption. The second term can be rewritten as

$$D[(1+D^2)^{-\frac{1}{2}}, a]b = D\frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}[(1+\lambda+D^2)^{-1}, a]b \,\mathrm{d}\lambda$$

Because $a \cdot (\mathrm{Dom}(D)) \subset \mathrm{Dom}(D)$ and $[D, a]$ is densely defined, we can use [CP98, Lemma 2.3] to obtain the equality,

$$\begin{aligned} D[(1+D^2)^{-\frac{1}{2}}, a]b = -\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\Big(&D(1+\lambda+D^2)^{-1}D[a, D](1+\lambda+D^2)^{-1} \\ &+ D(1+\lambda+D^2)^{-1}[a, D]D(1+\lambda+D^2)^{-1}\Big) b \,\mathrm{d}\lambda. \end{aligned}$$

Because of the norm estimates

$$\|D(x+D^2)^{-1}D\| \leq 1, \qquad \|D(x+D^2)^{-1}\| \leq \frac{1}{\sqrt{1+x}}, \qquad \|(x+D^2)^{-1}\| \leq \frac{1}{x}$$

*Note that the commutator $[\cdot, \cdot]$ is the graded commutator $[\cdot, \cdot]_\pm$.

for any $x > 0$, we find that

$$\left\| D[(1+D^2)^{-\frac{1}{2}}, a]b \right\| \leq \frac{2}{\pi} \int_0^\infty \lambda^{-\frac{1}{2}} \frac{1}{1+\lambda} \|[D,a]\| \, \|b\| \, \mathrm{d}\lambda < \infty.$$

Since the integrand is compact and converges in norm, we conclude that the endomorphism $D[(1+D^2)^{-\frac{1}{2}}, a]b$ is compact. An analogous argument will show that $a[F_D, b]$ is compact, whence $[F_D, ab]$ is compact. Because products are dense in A, $[F_D, a]$ is compact for all $a \in A$. $\square$

Equivalence relations and the KK-group

Let us now introduce the equivalence relations on Kasparov modules that are used to construct the KK-group.

Unitary equivalence Two A-B Kasparov modules $(A, {}_{\phi_1}E^1_B, F_1)$ and $(A, {}_{\phi_2}E^2_B, F_2)$ are unitarily equivalent if there exists an even unitary $U\colon E^1_B \to E^2_B$ such that $F_2 = UF_1U^*$ and $\phi_2(a) = U\phi_1(a)U^*$.

Operator homotopy Two A-B Kasparov modules $(A, {}_{\phi_1}E^1_B, F_1)$ and $(A, {}_{\phi_2}E^2_B, F_2)$ are operator homotopic if $E^1 = E^2 = E$, $\phi_1 = \phi_2 = \phi$, and there exists a norm-continuous path $(F_t)_{t\in[a,b]}$ such that $F_a = F_1$, $F_b = F_2$, and $(A, {}_\phi E, F_t)$ is a Kasparov module for all $t \in [a,b]$. (e.g. if $F_2 = F_1 + K$ for K compact, then $F_1 + tK$ is an operator homotopy for $t \in [0,1]$.)

Degenerate modules An A-B Kasparov module $(A, {}_\phi E, F)$ is called a degenerate module if $[F, \phi(a)] = \phi(a)(F - F^*) = \phi(a)(1 - F^2) = 0$.

Definition 2.2.28. We say that two A-B Kasparov modules (A, E^1_B, F_1) and (A, E^2_B, F_2) are equivalent if there is an operator homotopy from (A, E^1_B, F_1) to $(A, E^1_B, \tilde{F}_1)$, and $(A, E^1_B, \tilde{F}_1)$ is unitarily equivalent to $(A, E^2_B, F_2) \oplus D$, where D is a degenerate A-B Kasparov module.

Definition 2.2.29. We define $KK(A,B)$ to the set of Kasparov A-B modules modulo the equivalence relation generated by the relation from Definition 2.2.28.

Theorem 2.2.30 ([Kas81], §4, Theorem 1)**.** *The set $KK(A,B)$ forms a group, where the addition is given by the direct sum.*

Proof. First note that a degenerate module is by definition equivalent to the group identity. We leave it as a simple exercise to check that the direct sum of two Kasparov A-B modules is a Kasparov A-B module.

Given a Kasparov A-B module $(A, {}_\phi E_B, F)$, we claim that $-[(A, {}_\phi E_B, F)]$ is represented by $[(A, {}_{\tilde{\phi}}E^{\mathrm{op}}_B, -F)]$, where $(E^{\mathrm{op}})^0 = E^1$, $(E^{\mathrm{op}})^1 = E^0$, and $\tilde{\phi}(a^0 + a^1) := \phi(a^0) -$

$\phi(a^1)$. If there is a grading $\gamma \in \mathrm{End}_B(E)$, then $-[(A, E_B, F, \gamma)] = [(A, E_B^{\mathrm{op}}, -F, -\gamma)]$. To prove this claim, we need to show that the sum

$$-[(A, E_B, F)] + [(A, E_B^{\mathrm{op}}, -F)] := [(A, (E \oplus E^{\mathrm{op}})_B, F \oplus -F)]$$

is operator homotopic to a degenerate module.

Set $F_t := \begin{pmatrix} F\cos t & \mathrm{Id}\sin t \\ \mathrm{Id}\sin t & -F\cos t \end{pmatrix}$ for $t \in [0, \frac{\pi}{2}]$, where the identity map $\mathrm{Id}\colon E \to E^{\mathrm{op}}$ is an *odd* map. It is not too hard to check that F_t is an operator homotopy.

Since $\begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$ is self-adjoint and has square 1, the module will be degenerate if

$$\left[\begin{pmatrix} 0 & \mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}, \begin{pmatrix} \phi(a) & 0 \\ 0 & \tilde{\phi}(a) \end{pmatrix}\right] = 0,$$

This indeed follows:

$$\text{if } a \text{ even:} \quad \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \phi(a) & 0 \\ 0 & \tilde{\phi}(a) \end{pmatrix}\right] = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} - \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} = 0,$$

$$\text{if } a \text{ odd:} \quad \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \phi(a) & 0 \\ 0 & \tilde{\phi}(a) \end{pmatrix}\right]_+ = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = 0. \qquad \square$$

We briefly list the basic properties of the KK-groups.

Theorem 2.2.31 ([Kas81])**.** *The group $KK(\cdot, \cdot)$ is a bivariant functor from the category of separable and nuclear C^*-algebras to abelian groups that is contravariant in the 1st variable and covariant in the 2nd. This functor is homotopy invariant, stable and split-exact in both variables.*

It was shown by Higson [Hig87] that the KK-functor is the universal bivariant homology theory of C^*-algebras that is homotopy invariant, stable and split-exact.

Relation to K-theory

We first note a preliminary result.

Lemma 2.2.32 ([GBVF01], Corollary 3.10)**.** *Let E_A be a right-A C^*-module and $p \in \mathrm{End}^0_A(E)$ a projection. Then $p \in \mathrm{End}^{00}_A(E)$.*

Proposition 2.2.33 ([Kas81], §6, Theorem 3)**.** *If A is trivially graded, then $KK(\mathbb{C}, A) \cong K_0(A)$.*

Sketch proof. We assume A is unital (see [Kas81, §6] for the case that A is non-unital). Let the map $\varphi\colon K_0(A) \to KK(\mathbb{C}, A)$ be given by

$$[p] - [q] \overset{\varphi}{\mapsto} \left[\left(\mathbb{C}, E_A = (pA^N \oplus qA^M)_A, F = 0, \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\right],$$

for $p \in M_N(A)$ and $q \in M_M(A)$. Next we take a Kasparov module $(\mathbb{C}, E_A, F, \gamma)$. We can assume without loss of generality that $1_{\mathbb{C}}$ acts as 1_{E_A}; if not, $1_{\mathbb{C}}$ acts as a projection $p \in \mathrm{End}_A(E)$ and we compress the module to $(\mathbb{C}, pE_A, pFp, p\gamma)$, which lies in the same equivalence class in $KK(\mathbb{C}, A)$ [HR01, Lemma 8.3.8]. Suppose that F is regular in the sense of [GBVF01, Definition 4.3]. We can use the grading to represent $F = \begin{pmatrix} 0 & F_- \\ F_+ & 0 \end{pmatrix}$ and define the map $\psi \colon KK(\mathbb{C}, A) \to K_0(A)$

$$[(\mathbb{C}, E_A, F, \gamma)] \overset{\psi}{\mapsto} [P_{\mathrm{Ker}\, F_+}] - [P_{\mathrm{Ker}\, F_-}] = \mathrm{Index}\, F_+.$$

Since $(1 - F^2)$ is compact, the projections onto $\mathrm{Ker}\, F_\pm$ are compact and, by Lemma 2.2.32, finite-rank. We require F to be regular as this guarantees that the closed subspaces $\mathrm{Ker}(F_\pm)$ are complemented in E_A. If the operator F is not regular, we can amplify F to $\tilde{F} = \begin{pmatrix} F & 0 \\ (1-F^2)^{1/2} & 0 \end{pmatrix}$, which is regular [GBVF01, Lemma 4.10] and define $\mathrm{Index}\, F_+ = \mathrm{Index}\, \tilde{F}_+$.

We then check the isomorphisms by computing

$$(\psi \circ \varphi)([p] - [q]) = \psi([P_{pA^N}] - [P_{qA^M}]) = [p] - [q],$$

and

$$\begin{aligned}(\varphi \circ \psi)((\mathbb{C}, E_A, F, \gamma)) &= \varphi([P_{\mathrm{Ker}\, F_+}] - [P_{\mathrm{Ker} F_-}]) \\ &= \left(\mathbb{C}, (\mathrm{Ker}(F_+) \oplus \mathrm{Ker}(F_-))_A, 0, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right).\end{aligned}$$

Because F is regular, there exists a pseudoinverse $G \in \mathrm{End}_A(E)$ such that $1_{E_A} - GF$ is a compact endomorphism equal to $P_{\mathrm{Ker}(F)}$ (cf. [GBVF01, p146-147]). Hence as a class in $KK(\mathbb{C}, A)$, we can rewrite the Kasparov module

$$(\varphi \circ \psi)((\mathbb{C}, E_A, F, \gamma)) = \left[\left(\mathbb{C}, E_A, \begin{pmatrix} 0 & F_- \\ F_+ & 0 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)\right]. \quad \square$$

Higher KK-theory

Complex Clifford algebras are used to define the higher-order KK-groups, where

$$\mathbb{C}\ell_n = \mathrm{span}_{\mathbb{C}}\{e_1, \ldots, e_n \,:\, e_i^2 = 1,\, e_i^* = e_i\}.$$

There is a classification of Clifford algebras (see [LM89, Section I.4]), where

$$\mathbb{C}\ell_{2n} \cong M_{2^n}(\mathbb{C}), \qquad \mathbb{C}\ell_{2n+1} \cong M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C}).$$

Definition 2.2.34. Denote $KK^n(A, B) = KK(A \hat{\otimes} \mathbb{C}\ell_n, B)$.

Clifford algebras encode an algebraic periodicity of the KK-groups as, by stability,

$$KK^{2n}(A,B) = KK(A\hat{\otimes}\mathbb{C}\ell_{2n}, B) \cong KK(A\hat{\otimes}\,\mathrm{End}_{\mathbb{C}}(\mathbb{C}^{2^n}), B) \cong KK(A,B).$$

Therefore $KK^{2n+1}(A,B) \cong KK(A\hat{\otimes}\mathbb{C}\ell_1, B)$ and we only have two groups to consider.

Our next task is to characterise 'odd' Kasparov modules as ungraded modules. Let A and B be trivially graded C^*-algebras. Suppose that $(A\hat{\otimes}\mathbb{C}\ell_1, (E_+ \oplus E_-)_B, F, \gamma)$ is an $(A\hat{\otimes}\mathbb{C}\ell_1)$-$B$ Kasparov module. Because all algebras are trivially graded, without changing KK-classes we can assume the Kasparov module is of the form

$$\left(A\hat{\otimes}\mathbb{C}\ell_1, (E\oplus E)_B, F = \begin{pmatrix} 0 & F_- \\ F_+ & 0 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right),$$

where $E \cong E_+$ is trivially graded and the generator of $\mathbb{C}\ell_1$ acts as left-multiplication by $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which anticommutes with the grading γ [Con94, Proposition IV.A.13].

Calculating the compact commutators with F,

$$\begin{aligned} \left[F, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right] &= \begin{pmatrix} 0 & [F_-, a] \\ [F_+, a] & 0 \end{pmatrix}, \\ \left[F, \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}\right]_+ &= \begin{pmatrix} F_-b + bF_+ & 0 \\ 0 & F_+b + bF_- \end{pmatrix} \\ &= \begin{pmatrix} b(F_- + F_+) & 0 \\ 0 & b(F_- + F_+) \end{pmatrix} \text{ mod compacts,} \end{aligned}$$

we find that $[F_\pm, a]$ is compact and $bF_- = -bF_+$ modulo compacts. Since $a(F - F^*)$ is compact, we find that $aF_\pm = aF_\mp^*$ modulo compacts, and hence $aF_\pm = -aF_\pm^*$ modulo compacts. Therefore we can write $F = \begin{pmatrix} 0 & -i\tilde{F} \\ i\tilde{F} & 0 \end{pmatrix}$ modulo compacts, where $a\tilde{F} = a\tilde{F}^*$ modulo compacts. So the $(A\hat{\otimes}\mathbb{C}\ell_1)$-$B$ Kasparov module $(A\hat{\otimes}\mathbb{C}\ell_1, E_B, F)$ is in fact completely determined by the ungraded A-B Kasparov module $(A, E_B, \tilde{F})$.

Definition 2.2.35. We say a Kasparov module (A, E_B, F) is *odd* if there is no $\mathbb{Z}_2$-grading on E, and the C^*-algebras A and B are trivially graded.

Let (A, E_B, F) be an odd Kasparov module. Then one can construct the even $(A\hat{\otimes}\mathbb{C}\ell_1)$-$B$ Kasparov module

$$\left(A\hat{\otimes}\mathbb{C}\ell_1, \begin{pmatrix} E_B \\ E_B \end{pmatrix}, \begin{pmatrix} 0 & -iF \\ iF & 0 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right),$$

with representation such that $a \mapsto a\otimes 1_2$ and $\mathbb{C}\ell_1$ is generated by $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ [Con94, Proposition IV.A.13].

Example 2.2.36. Consider the unbounded Kasparov $C(S^1)$-$\mathbb{C}$ module (spectral triple) given by $(C^\infty(S^1), L^2(S^1)_\mathbb{C}, D = \frac{1}{i}\frac{d}{d\theta})$ and take the projection $P = \chi_{[0,\infty)}(D)$ onto $\overline{\operatorname{span}}\{z^n \mid n \geq 0\}$. Then the operator $F := 2P - 1$ is equal to $D(1+D^2)^{-\frac{1}{2}}$ modulo compacts.

The map $C^\infty(S^1) \ni a \mapsto PaP$ is *not* a $*$-homomorphism because $PaPPbP \neq PabP$. However, we do find

$$PaPPbP = PabP + Pa[P,b]P = PabP + Pa\left[\frac{F+1}{2}, b\right]P = PabP \text{ mod compacts.}$$

Let π denote the projection $\mathcal{B}[L^2(S^1)] \to \mathcal{B}[L^2(S^1)]/\mathcal{K}[L^2(S^1)] =: \mathcal{Q}$ onto the Calkin algebra $\mathcal{Q}$. Then $a \mapsto \pi(PaP)$ is a $*$-homomorphism $C(S^1) \to \mathcal{Q}$. We obtain a short exact sequence

$$0 \to \mathcal{K}[L^2(S^1)] \to C^*(PaP \mid a \in C(S^1), \mathcal{K}) \to C(S^1) \to 0$$

called the Toeplitz extension.

There is an equivalence between odd Kasparov modules and short-exact sequences. Given separable and nuclear algebras A and B, one can define $\operatorname{Ext}(A,B)$ as the Grothendieck group of equivalence classes of short exact sequences

$$0 \to B \to C \to A \to 0;$$

see [Bla98, §15] for more information.

Theorem 2.2.37 ([Kas81], §7)**.** *For separable and nuclear C^*-algebras A and B, $KK^1(A,B) \cong \operatorname{Ext}(A,B)$.*

We shall give a rough outline of the constructions underlying this theorem. Let (A, E_B, F) be an ungraded/odd Kasparov module. Suppose further that $F = F^*$ and $F^2 = 1$ (we can always do this while staying in the same equivalence class in $KK^1(A,B)$ [HR01, Lemma 8.3.5]). We set $P := \frac{1+F}{2}$ and for $a_1, a_2 \in A$ we find that

$$Pa_1PPa_2P = Pa_1a_2P + Pa_1[P,a_2]P = Pa_1a_2P \text{ mod compacts.}$$

We then get an extension, i.e. a short exact sequence

$$0 \longrightarrow \operatorname{End}^0_B(E) \longrightarrow C^*(PaP, \operatorname{End}^0_B(E)) \underset{\sigma}{\overset{\pi}{\rightleftarrows}} A \longrightarrow 0\,, \tag{2.2}$$

which is called the 'generalised Toeplitz extension'. The quotient of $C^*(PaP, \operatorname{End}^0_B(E))$ by $\operatorname{End}^0_B(E)$ gives the algebra A and not a quotient of A if the Busby invariant, the map $\varphi : A \to \mathcal{Q}(B) = \mathcal{M}(B)/B$ given by $\varphi(a) = \pi(PaP)$, is injective. Injectivity of the Busby invariant is equivalent to PaP being compact if and only if $a = 0$ [Bla98,

§15]. Such a condition can always be achieved by adding a degenerate module to our original odd Kasparov module

$$(A, E_B, 2P-1) \oplus (A, \mathcal{H} \otimes B_B, 1)\,,$$

where $\mathcal{H}$ is an infinite dimensional Hilbert space containing a faithful representation of A with $A \cap \mathcal{K}(\mathcal{H}) = \emptyset$. Adding on such a module does not change the original $KK^1(A, B)$ class. The extension from Equation (2.2) comes with a map $\sigma\colon A \to C^*(PaP)$ given by $\sigma(a) := PaP$. The map σ is not a $*$-homomorphism, but it is *completely positive* (i.e. positive on $M_n(A)$ for all n). The map σ also has the property that $\pi \circ \sigma = \mathrm{Id}_A$. In the case when σ is a $*$-homomorphism, there is no non-trivial splitting of Equation (2.2) and we say that the extension is trivial. We can carry out the same process with $1-P$ to obtain another extension.

Using both projections P and $1-P$, we obtain a short exact sequence

$$0 \longrightarrow \mathrm{End}^0_B(E) \longrightarrow C^*(PaP,\, (1-P)a(1-P),\, \mathrm{End}^0_B(E)) \underset{\sigma}{\overset{\pi}{\rightleftarrows}} A \longrightarrow 0\,,$$

where now the map $\sigma\colon A \to C^*(PaP, (1-P)a(1-P))$ is defined by

$$\sigma(a) := PaP + (1-P)a(1-P) + Pa(1-P) + (1-P)aP.$$

The map σ is now a $*$-homomorphism and so the extension is trivial.

Constructing a Kasparov module from an extension is more complicated. We start with a short exact sequence with positive splitting σ,

$$0 \longrightarrow B \longrightarrow C \underset{\sigma}{\overset{\pi}{\rightleftarrows}} A \longrightarrow 0\,.$$

Because B is an ideal in C, we have a left-action of C by multiplication on the module B_B. Hence we can think of $C \subset \mathrm{End}_B(B)$. The short exact sequence comes with the surjection $\pi\colon C \to A$ and completely positive map $\sigma\colon A \to \mathrm{End}_B(B)$ satisfying $\pi \circ \sigma = \mathrm{Id}_A$.

Theorem 2.2.38 (Kasparov-Stinespring dilation, [Kas81], §1.15)**.** *Let A, B be separable nuclear C^*-algebras and let $\sigma\colon A \to \mathrm{End}_B(B)$ be completely positive. Then there exists a C^*-module X_B and a $*$-homomorphism $\phi\colon A \to \mathrm{End}_B(B \oplus X)$ such that $P_B\phi(a)P_B = \sigma(a)$ for all $a \in A$, where $P_B\colon B \oplus X \to B$ is the projection onto B.*

By the Kasparov-Stinespring dilation theorem we have a module $(B \oplus X)_B$ with representation $\phi : A \to \mathrm{End}_B(B \oplus X)$ and projection $P_B \in \mathrm{End}_B(B \oplus X)$ such that $P_B\phi(a)P_B = \sigma(a)$. Kasparov shows $P_B\phi(a)(1-P_B)$ is compact and so we obtain an odd A-B Kasparov module $(A, {}_\phi(B \oplus X)_B, 2P_B - 1)$.

Because the passage from extension to Kasparov module requires an explicit positive splitting and uses the dilation theorem, it is in general quite difficult to compute Kasparov products of classes defined by extensions.

2.2.3 The product

We recall the central result of Kasparov theory.

Theorem 2.2.39 ([Kas81], §4, Theorem 4)**.** *Suppose the algebras A_1 and A_2 are separable and let B_1, B_2 and D have strictly positive elements. Then the intersection product*

$$KK(A_1, B_1\hat{\otimes}D) \times KK(D\hat{\otimes}A_2, B_2) \to KK(A_1\hat{\otimes}A_2, B_1\hat{\otimes}B_2)$$

is well-defined and associative.

Unfortunately, Theorem 2.2.39 is highly non-constructive and one only knows that such a map exists. It is here that unbounded Kasparov theory can be of use as it provides a geometric framework that may be used to compute the product explicitly.

Let $(\mathcal{A}, E^1_B, D_1)$ and $(\mathcal{B}, E^2_C, D_2)$ be even unbounded A-B and B-C Kasparov modules. Our goal is to construct an unbounded A-C module which, when we take the bounded transformation, represents the class of the product in $KK(A, C)$. Methods to give such a construction have been considered in [LRV12, Mes14, KL13, BMv13, MR15], though we will only cover the basic ideas.

Given $\mathbb{Z}_2$-graded modules E^1_B and ${}_BE^2_C$, we recall Example 2.2.21 which shows that $(E^1\hat{\otimes}_B E^2)_C$ is also $\mathbb{Z}_2$-graded module. Therefore there is an obvious choice for the right C-module representing the product. Similarly, one can check that if the representation of $\mathcal{A}$ on E^1_B is given by $\varphi : \mathcal{A} \to \mathrm{End}_B(E^1)$, then $\varphi\hat{\otimes}1$ gives a $\mathbb{Z}_2$-graded representation of $\mathcal{A}$ on the module $(E^1\hat{\otimes}_B E^2)_C$.

The only piece of information we are missing is the unbounded operator D. Unfortunately, the obvious choice $D = D_1\hat{\otimes}1 + 1\hat{\otimes}D_2$ does not work as the operator $1\hat{\otimes}D_2$ is not well-defined on the balanced tensor product $E^1\hat{\otimes}_B E^2$ (unless $B = \mathbb{C}$ in which case we have the external Kasparov product). It is in correcting this problem that most of the technical difficulties of the product arise.

We start by defining a creation operator. Given $e_1 \in E^1_B$ and a $*$-homomorphism $\psi : B \to \mathrm{End}_C(E^2)$, we let $T_{e_1} \in \mathrm{Hom}_B(E^2, E^1 \otimes_B E^2)$ be given by $T_{e_1}e_2 = e_1 \otimes e_2$. One can check that T_{e_1} is adjointable with $T^*_{e_1}(\tilde{e}_1 \otimes e_2) = \psi((e_1|\tilde{e}_1)_B)e_2$.

Theorem 2.2.40 (Kucerovsky's criterion [Kuc97], Theorem 13)**.** *Let $(\mathcal{A}, {}_{\phi_1}E^1_B, D_1)$ and $(\mathcal{B}, {}_{\phi_2}E^2_C, D_2)$ be unbounded Kasparov modules. Write $E := E^1\hat{\otimes}_B E^2$. Suppose that $(\mathcal{A}, {}_{\phi_1}E_C, D)$ is an unbounded Kasparov module such that*

Connection condition *For all e_1 in a dense subspace of $\phi_1(A)E^1$, the commutators*

$$\left[\begin{pmatrix} D & 0 \\ 0 & D_2 \end{pmatrix}, \begin{pmatrix} 0 & T_{e_1} \\ T^*_{e_1} & 0 \end{pmatrix}\right]$$

are bounded on $\mathrm{Dom}(D \oplus D_2) \subset E \oplus E^2$;

Domain condition $\mathrm{Dom}(D) \subset \mathrm{Dom}(D_1\hat{\otimes}1)$*;*

Positivity condition *For all* $e \in \mathrm{Dom}(D)$, $((D_1\hat{\otimes}1)e|De) + (De|(D_1\hat{\otimes}1)e) \geq K(e|e)$ *for some* $K \in \mathbb{R}$.

Then the class of $(\mathcal{A}, {}_{\phi_1}E_C, D)$ *in* $KK(A, C)$ *represents the Kasparov product.*

Kucerovsky's criterion gives us checkable conditions to see if an unbounded Kasparov module represents the product. What Kucerovsky's theorem does not provide is a way to construct the unbounded product module. A more constructive approach to taking the unbounded product is the subject of much of the current research in unbounded Kasparov theory.

Connections and the unbounded product

Here we briefly outline an approach to the Kasparov product via the unbounded picture. Central to this viewpoint are the so-called connections on a module $\mathcal{E}_{\mathcal{A}}$, defined to be a noncommutative analogue of the geometric notion of connection. Such operators do not always exist and we restrict to the case of smooth $*$-algebras (cf. Definition 2.1.4).

Definition 2.2.41. Let $m\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ denote the multiplication. Define

$$\Omega^1(\mathcal{A}) = \mathrm{Ker}(m) = \overline{\mathrm{span}}\{\sum_i a_i\, \delta b_i\},$$

where $\delta\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is the universal derivation defined by $\delta b := 1 \otimes b - b \otimes 1$, and satisfies $\delta(ab) = \delta(a)b + a\delta(b)$. This allows us to construct

$$\Omega^0(\mathcal{A}) := \mathcal{A}, \qquad \Omega^k(\mathcal{A}) := \Omega^0(\mathcal{A})^{\otimes_{\mathcal{A}} k}, \qquad \Omega^*(\mathcal{A}) := \bigoplus_{k=0}^{\infty} \Omega^k(\mathcal{A}).$$

For $\omega \in \Omega^{|\omega|}(\mathcal{A})$, $\rho \in \Omega^*(\mathcal{A})$, and $a_i \in \mathcal{A}$ we have

$$\delta(\omega\rho) = \delta(\omega)\rho + (-1)^{|\omega|}\omega\delta(\rho), \qquad \delta(a_0\delta a_1 \cdots \delta a_n) = \delta a_0 \delta a_1 \cdots \delta a_n, \qquad \delta^2 = 0.$$

We denote $\Omega^*(\mathcal{A})$ as the universal differential algebra over $\mathcal{A}$ with adjoint $\delta(a)^* = -\delta(a^*)$.

Definition 2.2.42. Given a right module $\mathcal{E}_{\mathcal{A}}$, a *connection* is a map

$$\nabla\colon \mathcal{E}_{\mathcal{A}} \to \mathcal{E}_{\mathcal{A}} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \quad \text{such that} \quad \nabla(ea) = (\nabla e)a + e \otimes \delta a.$$

A connection can be extended to

$$\begin{aligned}&\nabla\colon \mathcal{E}_{\mathcal{A}} \otimes_{\mathcal{A}} \Omega^*(\mathcal{A}) \to \mathcal{E}_{\mathcal{A}} \otimes_{\mathcal{A}} \Omega^{*+1}(\mathcal{A})\\ &\nabla(e \otimes \omega) = (-1)^{|\omega|}(\nabla e) \otimes \omega + e \otimes \delta\omega.\end{aligned}$$

One can show that ∇^2 is $\mathcal{A}$-linear and an $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$-valued 2-form.

Next, suppose that $(\mathcal{A}, E_B, D_1, \gamma_E)$ and $(\mathcal{B}, F_C, D_2, \gamma_F)$ are unbounded Kasparov modules, where $\mathcal{E}_{\mathcal{B}}$ is a dense submodule of E_B and $\overline{\mathcal{B} \cdot F_C} = F_C$. We can represent 1-forms on ${}_{\mathcal{B}}F_C$ by the operator D_2, where $\pi(b_0\delta(b_1))f = b_0[D_2, b_1]f$. Suppose $\mathcal{E}_{\mathcal{B}}$ has a connection ∇. Then we define the operator

$$1 \otimes_{\nabla} D_2 : \mathcal{E} \otimes_{\mathcal{B}} \mathrm{Dom}(D_2) \to E \otimes_B F,$$
$$(1 \otimes_{\nabla} D_2)(e \otimes f) = (e \otimes D_2 f) + (1 \otimes \pi) \circ (\nabla \otimes 1)(e \otimes f).$$

Theorem 2.2.43 ([LRV12, Mes14, KL13, MR15])**.** *The unbounded operator $D_1\hat{\otimes}1 + 1\hat{\otimes}_{\nabla}D_2$ acting on a dense subspace of $(E\hat{\otimes}_B F)_C$ satisfies the connection condition of Kucerovsky's criterion.*

The operator $D_1\hat{\otimes}1 + 1\hat{\otimes}_{\nabla}D_2$ gives us a candidate for the Dirac-type operator that represents the product. We do, however, emphasise that the tuple

$$\big(\mathcal{B}, (E\hat{\otimes}_B F)_C, D_1\hat{\otimes}1 + 1\hat{\otimes}_{\nabla}D_2, \gamma_E\hat{\otimes}\gamma_F\big) \tag{2.3}$$

may not be an unbounded Kasparov module, nor satisfy the positivity and domain conditions of Kucerovsky's criterion. For the products we take in this thesis, Equation (2.3) will be an unbounded Kasparov module representing the product, which we check using Kucerovsky's criterion. For newer developments on the constructive approach to the Kasparov product, the reader may consult [BMv13, KL13, Mes14, MR15, FR15].

2.3 Kasparov theory for real algebras

Our work so far has so far been concerned with complex KK-theory, which was constructed as a unifying approach to K-theory and K-homology, theories that arise from studying topological properties of complex vector bundles and elliptic operators on manifolds. Of course, Atiyah, Singer and others also studied finer invariants than those which solely related to complex vector bundles. This lead to, amongst others, KO-theory and KR-theory, which deal with real bundles or complex bundles with a "real" involution (see for example [LM89, Ati66]).

One of the novel aspects of KK-theory is that these finer invariants can also be fitted into Kasparov's framework by dealing with real or Real C^*-algebras (note that the capitalisation makes a difference). A key difference between complex KK-theory and KKO and KKR-theory is that the latter two theories posses an 8-fold periodicity. The finer nature of the invariants that appear in the real/Real theories means that torsion groups also play a prominent role in this setting. This is of particular interest to us if we are interested in properly understanding, say, the $\mathbb{Z}_2$-invariant of the quantum spin-Hall effect. Indeed, the reason we are introducing this more complicated version of KK-theory is that we will find it necessary to pass to the real setting in order to properly understand the invariants that arise in the so-called periodic table of topological insulators.

This thesis will focus on the case of real Kasparov theory. While the basic results presented in this section can be expressed in both real and Real KK-theory, Real Kasparov theory is not well adapted to studying systems with complex anti-linear group actions. Such group actions arise in many examples of topological insulator systems (see Chapter 5). While KKR-theory can still be used in particular examples of topological states of matter, we leave this investigation to another place.

2.3.1 KKO-theory

We shall give a brief introduction to KK-theory for real C^*-algebras.

Definition 2.3.1. A real C^*-algebra A is a real Banach $*$-algebra such that $\|a^*a\| = \|a\|^2$ and $a^*a + 1 \geq 0$ for all $a \in A$.

Remark 2.3.2 (A note on real vs Real C^*-algebras)*.* The real Gelfand-Naimark theorem says that commutative real C^*-algebras are isomorphic to algebras of the form

$$C_0(X)^\tau = \{f \in C_0(X, \mathbb{C}) \,:\, f(x^\tau) = \overline{f(x)} \text{ for all } x \in X\},$$

where (X, τ) is a locally compact Hausdorff space with involution τ, see [AK48, Ros15].

More generally, given a Real C^*-algebra (A, τ) (a complex C^*-algebra A with anti-linear involution τ that preserves multiplication), the subalgebra of elements in A invariant under τ, $A^\tau = \{a \in A \,:\, a^\tau = a\}$, is a real C^*-algebra. There is an equivalence of the category of Real C^*-algebras with the category of real C^*-algebras (see [LS10] for more detail on the relation between real and Real algebras).

Definition 2.3.3. A real Hilbert A-module is a linear space E over $\mathbb{R}$ with right action by a real C^*-algebra A and A-valued inner product $(\cdot \mid \cdot)_A$ such that the conditions of Definition 2.2.1 hold.

Many of the examples we considered in Section 2.2.1 on complex Hilbert C^*-modules have natural real analogues.

Example 2.3.4. Take $E \to X$ to be a real vector bundle over a locally compact Hausdorff space X. Provided that there exists a positive real-valued form $(\cdot \mid \cdot)$ on E, then we can define the real C^*-module $\Gamma_0(E)_{C(X,\mathbb{R})}$ with right-action by multiplication and inner-product via $(\cdot \mid \cdot)$.

One can check that the key definitions concerning operators on complex C^*-modules in Section 2.2.1 (e.g. adjointable, finite-rank, compact, regular) can be easily translated to the real setting.

Definition 2.3.5. A real unbounded Kasparov module $(\mathcal{A}, {}_\phi E_B, D, \gamma)$ is $\mathbb{Z}_2$-graded real C^*-module E_B with graded real endomorphism $\phi : \mathcal{A} \to \mathrm{End}_B(E)$ and unbounded regular operator D such that for all $a \in \mathcal{A}$,

1. $[D, \phi(a)]_\pm \in \mathrm{End}_B(E)$,
2. $\phi(a)(1 + D^2)^{-1/2} \in \mathrm{End}^0_B(E)$.

The results of Baaj and Julg [BJ83] continue to hold for real Kasparov modules. Therefore we may apply the bounded tranformation of an unbounded module $(\mathcal{A}, E_B, D)$ to obtain the real Kasparov module $(A, E_B, D(1+D^2)^{-1/2})$, where A is the C^*-closure of the dense subalgebra† $\mathcal{A}$.

One can define notions of unitary equivalence, homotopy and degenerate modules from Section 2.2.2 in the real setting. Hence we can define the group $KKO(A, B)$ as the equivalence classes of real (bounded) Kasparov modules modulo these relations.

The generality of the constructions and proofs in [Kas81] mean that all the central results in complex KK-theory carry over into the real (and Real) setting. In particular, the intersection product

$$KKO(A, B) \times KKO(B, C) \to KKO(A, C)$$

is still a well-defined map and other important properties such as stability

$$KKO(A\hat{\otimes}\mathcal{K}(\mathcal{H}), B) \cong KKO(A, B)$$

continue to hold, where $\mathcal{K}(\mathcal{H})$ is the real compact operators on a separable real Hilbert space.

If we wish to consider the unbounded picture and the product, one finds that Kucerovsky's criterion (Theorem 2.2.40) can also be used in KKO-setting [Kuc97, Theorem 13]. Such a result relies on technical considerations of real C^*-algebras that are implicit in Kasparov's work. The modules and products we consider in this thesis are simple enough that these technicalities will not play a role and all computations are explicit.

Higher-order groups

Clifford algebras are used to define higher KKO-groups and encode periodicity. In the real setting, we define

$$C\ell_{p,q} = \mathrm{span}_{\mathbb{R}}\big\{\, \gamma^1, \ldots, \gamma^p,\, \rho^1, \ldots, \rho^q \,\big|\, (\gamma^i)^2 = 1,\, (\gamma^i)^* = \gamma^i,\, (\rho^i)^2 = -1,\, (\rho^i)^* = -\rho^i \,\big\}\,.$$

†Smooth subalgebras have a different meaning for real algebras as stability under the holomorphic functional calculus may not be a well-defined concept in the real category. A general approach to this issue is not available, but for the simple examples arising in this thesis it can be seen directly that every K-theory class can be represented by an element of the dense subalgebra we use.

Example 2.3.6. Consider the real space $\mathbb{R}^{p,q}$ with basis $\{e_1, \ldots, e_p, \epsilon_1, \ldots, \epsilon_q\}$ from which we construct the exterior algebra $\bigwedge^* \mathbb{R}^{p,q}$. We can define an action of $C\ell_{p,q}$ on $\bigwedge^* \mathbb{R}^{p,q}$ by Clifford multiplication. We define $\eta^j(\omega) = e_j \wedge \omega + \iota(e_j)\omega$ for $1 \leq j \leq p$ and $\nu^j(\omega) = \epsilon_j \wedge \omega + \iota(\epsilon_j)\omega$ for $1 \leq j \leq q$, where $\iota(v)$ denotes the contraction of a form along v. One readily checks that the η^j and ν^j satisfy the requirements to be a Clifford generators.

Similarly, given $\mathbb{R}^d$ we can construct $\bigwedge^* \mathbb{R}^d$ and define a representation of $C\ell_{d,0}$ or $C\ell_{0,d}$ with the generators

$$\gamma^j(\omega) = e_j \wedge \omega + \iota(e_j)\omega, \qquad \rho^j(\omega) = e_j \wedge \omega - \iota(e_j)\omega$$

respectively. We note the sign change that ensures $(\rho^j)^2 = -1$.

We define higher-order KKO groups by tensoring with real Clifford algebras. Kasparov defines

$$K_{p,q}K^{r,s}O(A, B) := KKO(A\hat{\otimes}C\ell_{p,q}, B\hat{\otimes}C\ell_{r,s}). \tag{2.4}$$

The definition from Equation (2.4) simplifies immediately with the following result.

Theorem 2.3.7 (§5, Theorem 4 of [Kas81])**.** *Given real algebras A and B, then for a fixed difference $(p-q)-(r-s)$ the groups $K_{p,q}K^{r,s}O(A, B)$ are canonically isomorphic.*

Proof. We note that $C\ell_{n,n} \cong \mathrm{End}_{\mathbb{R}}(\bigwedge^* \mathbb{R}^n)$. Therefore $C\ell_{n,n} \cong \mathcal{K}(\mathcal{H})$ for $\mathcal{H} = \bigwedge^* \mathbb{R}^n$ and by stability $KKO(A\hat{\otimes}C\ell_{r,s}, B) \cong KK(A\hat{\otimes}C\ell_{r+1,s+1}, B)$. Hence

$$KKO(A\hat{\otimes}C\ell_{p,q}, B\hat{\otimes}C\ell_{r,s}) \cong KKO(A\hat{\otimes}C\ell_{p,q}\hat{\otimes}C\ell_{s,r}, B) \cong KK(A\hat{\otimes}C\ell_{p+s,q+r}, B).$$

Up to stable isomorphism, the algebra $C\ell_{p+s,q+r}$ depends solely on $(p+s)-(q+r) = (p-q)-(r-s)$. □

Remark 2.3.8. Theorem 2.3.7 implies that it is sufficient to define higher KKO-groups by tensoring by real Clifford algebras of the form $C\ell_{n,0}$ or $C\ell_{0,n}$ (though cases like $C\ell_{r,s}$ may still arise in examples).

By Theorem 2.3.7, we find that

$$KKO(A\hat{\otimes}C\ell_{n,0}, B) \cong KKO(A, B\hat{\otimes}C\ell_{0,n}).$$

It is clear that in the real picture that the placement of a Clifford algebra on the left or right in the bivariant group $KKO(\cdot, \cdot)$ is important. Furthermore, there is a difference between the algebra $C\ell_{n,0}$ and $C\ell_{0,n}$ that does not occur in the complex theory.

We now clarify the relation between real KK-groups and real K-theory.

Proposition 2.3.9 ([Kas81], §6 Theorem 3)**.** *If A is trivially graded and σ-unital, then $KKO(C\ell_{n,0}, A) \cong KO_n(A)$.*

Proposition 2.3.9 implies that if $A \cong C(X)$ for some compact Hausdorff space X, then $KKO(C\ell_{n,0}, C(X)) \cong KO^{-n}(X)$ (note the sign change that one can ignore in the complex setting) and so we are back in the setting of Atiyah's KO-theory for spaces (see, for example, [LM89] for more on topological KO-theory). The reader may also consult [BL15] for a useful characterisation of $KO_n(A)$ in terms of unitaries and involutions.

Like the complex case, there is an equivalence between short exact sequences of real C^*-algebras and real Kasparov modules, where

$$\mathrm{Ext}_{\mathbb{R}}(A, B) \cong KKO(A\hat{\otimes}C\ell_{0,1}, B) \cong KKO(A, B\hat{\otimes}C\ell_{1,0})$$

for real, nuclear and separable algebras A and B [Kas81, §7].

We also briefly consider Bott periodicity. Because KK-groups are stable and Clifford algebras encode an algebraic periodicity with $C\ell_{0,8} \cong C\ell_{8,0} \cong M_{16}(\mathbb{R})$, it follows that $KKO(A\hat{\otimes}C\ell_{8,0}, B) \cong KKO(A, B\hat{\otimes}C\ell_{8,0}) \cong KKO(A, B)$. We would like to relate the algebraic periodicity of the KK-groups to a topological periodicity. Kasparov defines the suspension of an algebra A by $\Sigma A = C_0(\mathbb{R}, A)$. A complicated argument (involving the product) shows that

$$KKO(\Sigma^n A\hat{\otimes}C\ell_{n,0}, B) \cong KKO(A, B) \cong KKO(A\hat{\otimes}C\ell_{0,n}, \Sigma^n B),$$

which relates algebraic periodicity to the more familiar topological periodicity (see [Kas81, §5] for a proof).

Chapter 3

The quantum Hall effect and Chern numbers

3.1 Introduction

This chapter studies the links between condensed matter systems and index theory using the machinery of spectral triples. The quantum Hall effect is our motivating example. As briefly outlined in the introduction, there have been many explanations for the quantisation of the Hall conductance in the physics literature. The most widely accepted interpretation is that the Kubo formula for conductance can be expressed in terms of the integral of the curvature of a particular connection on the Brillouin zone (momentum space) of the sample [TKNdN82]. Hence the Hall conductance is proportional to a pairing of a Chern class with a homology class, which takes integer values. Such a viewpoint helps us directly understand the link between the Hall conductance and topology, but can not account for the case of irrational magnetic field. It is also difficult to introduce disorder and impurities into the purely geometric models, so the robust nature of the Hall conductance is not fully explained.

The solution to the problem of irrational magnetic field strength came from Bellissard, who used C^*-algebras and techniques from Alain Connes' noncommutative geometry to perform a noncommutative analogue of the Thouless et al. argument. In particular, the noncommutative method was able to account for irrational magnetic field strength and disorder could be added into the system without changing the fundamental result. Bellissard wrote many papers on the noncommutative approach to solid state physics and the quantum Hall effect, which are summarised (and expanded upon) in [BvS94]. Bellissard and co-authors were able to prove the quantisation of the Hall conductance by linking the Kubo formula to a Fredholm index, including the case when disorder is present.

The paper [BvS94] contains many results that apply to both the discrete and continuous models, but some of the more technical details were only proved in the discrete setting. Morally speaking, there should not be a difference between discrete or continuous models in terms of the result that one gets. However, because a continuous model acts on $L^2(\mathbb{R}^2)$, which has a non-compact Brillouin zone (momentum space), the technical difficulties that one needs to work around can be much greater.

Recent results on non-unital spectral triples and index theory as outlined in Chapter Section 2.1 and [CGRS14] mean that tools and constructions now exist that allow us to consider non-unital or non-compact index problems. We find that while there are extra details that need to be checked, the essence of the discrete quantum Hall picture also holds in the continuous setting.

In this chapter, we consider a continuous d-dimensional system subject to a uniform magnetic field normal to the sample. The 2-dimensional Landau Hamiltonian used to model the quantum Hall effect is an important example. We then construct the 'noncommutative Brillouin zone' and a spectral triple encoding its geometry. By applying the local index formula from Chapter 2.1.4, we obtain tractable expressions for the pairing of unitaries and projections in our algebra with the K-homology class represented by the spectral triple. These expressions are the non-unital analogue of the 'higher-dimensional Chern numbers' studied in [PLB13, PS14, Pro15]. In the case $d = 2$, we recover the Kubo formula for the Hall conductance as derived by Bellissard. The formulas we derive can also be applied to the so-called strong topological phases of complex classes of topological insulators in arbitrary dimension, see Chapter 5.

Some of the material in Section 3.3 was investigated under the guidance of Prof. Hermann Schulz-Baldes during a visit to Friedrich-Alexander Universität Erlangen-Nürnberg in October-November 2014.

It should also be noted that the techniques we use in this chapter to derive computable expressions for the index pairing can *not* be used in the case of topological insulators with torsion invariants. This is because one of the key tools we use to derive the Chern numbers is the local index formula, which involves expressing the index pairing of K-theory and K-homology as a pairing of cyclic homology and cyclic cohomology. Such pairings are the same in the case of non-torsion invariants, but a pairing of cyclic homology and cohomology is unable to detect invariants arising from torsion groups. We will study the problem of torsion index pairings in Chapter 5.

3.2 The noncommutative Brillouin zone

We model a particle in $\mathbb{R}^d$ subject to a uniform magnetic field perpendicular to the sample. There is a choice of magnetic potential A, where $B = \mathrm{d}A + A \wedge A$ is the magnetic field. In general we take $A = (A_1, \ldots, A_d)$ such that $A_j \in L^2_{\text{loc.}}(\mathbb{R}^d)$ and differentiable with

$$\frac{\partial}{\partial x_j}A_k - \frac{\partial}{\partial x_k}A_j = B_{j,k} = \text{const.}$$

for all $j, k \in \{1, \dots, d\}$. The Schrödinger operator is given by

$$H_0 = \frac{1}{2m^*}\sum_{j=1}^{d}\left(-i\hbar\frac{\partial}{\partial x_j} - eA_j\right)^2,$$

where m^* is the effective mass of the particle. We choose units such that $m^* = \frac{\hbar}{2}$ and introduce the operators

$$K_j = -i\frac{\partial}{\partial x_j} - \frac{e}{\hbar}A_j, \qquad j = 1, \dots, d.$$

The operators K_j are acting as an analogue of the wave-vector, usually given by $-i\frac{\partial}{\partial x_j}$ (K_j reduces to this case when there is no magnetic field). We choose the symmetric gauge and define $A_j = -\frac{1}{2}\sum_{k=1}^{d} B_{j,k}x_k$ for $j = 1, \dots, d$, where $B_{j,k}$ is antisymmetric and real. We introduce the parameter $\theta_{j,k}$ so that we can rewrite

$$K_j = -i\frac{\partial}{\partial x_j} - \sum_{k=1}^{d}\theta_{j,k}X_k \quad \text{and} \quad \sum_{j=1}^{d} K_j^2 = \frac{2m^*}{\hbar^2}H_0 = H_0.$$

Example 3.2.1 (Quantum Hall Hamiltonian). In the case where $d = 2$, our Hamitonian is given in the symmetric gauge as

$$H_0 = \left(-i\frac{\partial}{\partial x_1} + \theta X_2\right)^2 + \left(-i\frac{\partial}{\partial x_2} - \theta X_1\right)^2,$$

where $\theta \in \mathbb{R}$ represents the magnetic flux through a unit cell. We recognise this Hamiltonian as the 2-dimensional Landau Hamiltonian used to model the quantum Hall effect. We shall return to this example repeatedly.

The presence of the magnetic field means that H_0 does not commute with ordinary translation operators S_a, where $(S_a\psi)(x) = \psi(x-a)$ for $\psi \in L^2(\mathbb{R}^d)$ and $x, a \in \mathbb{R}^d$. However, we may define the so-called *magnetic translations* U_a such that in the symmetric gauge $(U_a\psi)(x) = e^{-i\theta(x\wedge a)}(x-a)$ for $\psi \in L^2(\mathbb{R}^d)$, where $\theta(x\wedge a) = \sum_{j,k=1}^{d}\theta_{j,k}x_ja_k$. We note that $\theta(x \wedge x) = 0$ and $\theta(x\wedge y) = -\theta(y \wedge x)$. One checks that $[U_a, K_j] = 0$ on $\mathrm{Dom}(K_j)$ for any $a \in \mathbb{R}^d$ and $j \in \{1, \dots, d\}$ (see [Zak64] for more details on magnetic translations for general gauge choices).

Remark 3.2.2. We choose the symmetric gauge as it is particularly amenable to computations, though all results of interest in this chapter do not depend on our gauge choice (provided B is constant and normal to the sample).

If we consider a physical system with edge or boundary, the presence of an edge will affect our choice of magnetic potential. We will return to this issue in in Chapter 4 (see Remark 4.2.7).

Definition 3.2.3. Let $\mathcal{H}$ be a separable Hilbert space and G a locally compact group. The map $G \ni g \mapsto U_g \in \mathcal{U}(\mathcal{H})$ is a projective unitary representation if

1. $U_{g_1}U_{g_2} = \sigma(g_1, g_2)U_{g_1g_2}$ for all $g_1, g_2 \in G$ with σ a 2-cocycle of G. That is, a continous map $G \times G \to \mathbb{T}$ such that

$$\sigma(g, e) = \sigma(e, g) = 1, \qquad \sigma(g_1, g_2g_3)\sigma(g_2, g_3) = \sigma(g_1g_2, g_3)\sigma(g_1, g_2);$$

2. The map $g \mapsto U_g$ is continuous in the strong operator topology.

A simple check proves the following proposition.

Proposition 3.2.4. *In the symmetric gauge, the operators* $\{U_a : a \in \mathbb{R}^d\}$ *on* $L^2(\mathbb{R}^d)$ *are a projective representation of* $\mathbb{R}^d$ *with* $\sigma(a, b) = e^{i\theta(a\wedge b)}$.

3.2.1 Homogeneous operators

Many of the results outlined below (until Section 3.3) come from the articles [Bel92] and [BvS94, Section 3.5, 3.6]. We will unpack what Bellissard means by homogeneous Schrödinger operators and some of the more important results.

Remark 3.2.5. This section is one where the differences between the discrete and continuous case are very plain. The main reason for this difference is that H_{disc} acting on $\ell^2(\mathbb{Z}^d)$ is bounded, whereas the magnetic Schrödinger Hamiltonian on $L^2(\mathbb{R}^d)$ is unbounded. From the perspective of operator algebras, we consider the resolvent of the Hamiltonian in the continuous case, while in the discrete setting we deal directly with the Hamiltonian. Generally the proofs for continuous systems are more technical than their discrete counterparts.

We begin by introducing a potential into our system. For our Hamiltonian on $L^2(\mathbb{R}^d)$, we now have

$$H = \sum_{j=1}^{d} K_j^2 + V = H_0 + V, \tag{3.1}$$

where V is an essentially bounded, real-valued and measurable function on $\mathbb{R}^d$. By [Iwa90, Theorem 1.1], H is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$ and so has a unique self-adjoint extension.

Definition 3.2.6. Let $\mathcal{H}$ be a separable Hilbert space and $\{U_g : g \in G\}$ a projective unitary representation of a locally compact group G on $\mathcal{H}$. A self-adjoint operator H acting on $\mathcal{H}$ is homogeneous with respect to G if for each $z \in \rho(H)$, the resolvent set of H, the family

$$\Omega(z) = \overline{\{U_g(z - H)^{-1}U_g^{-1} : g \in G\}} \tag{3.2}$$

is compact, with closure in the strong operator topology.

We will find (cf. Corollary 3.2.10) that the Hamiltonian $H = \sum_j K_j^2 + V$ is homogeneous with respect the representation of $\mathbb{R}^d$ by magnetic translations U_a, $a \in \mathbb{R}^d$.

Lemma 3.2.7. *Let H be a homogeneous operator with respect to G. Suppose $z, z' \in \rho(H)$ with $z \neq z'$. Then the spaces $\Omega(z)$ and $\Omega(z')$ are homeomorphic.*

Proof. Using the resolvent equation, for any $g \in G$

$$\begin{aligned} U_g(z' - H)^{-1}U_g^{-1} &= U_g\left((z' - H)^{-1} - (z - H)^{-1} + (z - H)^{-1}\right)U_g^{-1} \\ &= \left(U_g(z' - H)^{-1}U_g^{-1}(z - z') + 1\right)U_g(z - H)^{-1}U_g^{-1}. \end{aligned}$$

Therefore the sequence $(U_{g_j}(z - H)^{-1}U_{g_j}^{-1})_{j\geq 0}$ will converge strongly to an operator T if and only if $(U_{g_j}(z' - H)^{-1}U_{g_j}^{-1})_{j\geq 0}$ converges strongly to an operator T'. The map $\Omega(z) \ni T \mapsto T' \in \Omega(z')$ gives a homeomorphism. ☐

Because $\Omega(z) \cong \Omega(z')$ for all $z, z' \in \rho(H)$, we can consider Ω to be an abstract compact space with an action by the group G.

Definition 3.2.8. Let H be a homogeneous operator with respect to a locally compact group G. The hull of H is the dynamical system (Ω, G, T), where Ω is the compact space $\Omega(z)$ from Equation (3.2) for any $z \in \rho(H)$ and G acts on Ω through T.

We will denote the action of G on Ω by $T_g\omega$ for $g \in G$ and $\omega \in \Omega$. An alternative method can be used to define the hull by considering the translations of a bounded potential V. In the case that $H = H_0 + V$, these constructions are equivalent (see Corollary 3.2.11 below), though it requires a little work.

Theorem 3.2.9 ([NB90], Appendix)**.** *Let H be as in Equation (3.1). Denote by $L_w^\infty(\mathbb{R}^d)$ the space of measurable essentially bounded functions over $\mathbb{R}^d$ with the weak topology of $L^1(\mathbb{R}^d)$ and $\mathcal{B}_s[L^2(\mathbb{R}^d)]$ the space of bounded operators on $L^2(\mathbb{R}^d)$ with the strong topology. Then the map*

$$L_w^\infty(\mathbb{R}^d) \ni V \mapsto (z - H_0 - V)^{-1} \in \mathcal{B}_s[L^2(\mathbb{R}^d)]$$

is continuous for any $z \in \mathbb{C}$ with $\Im(z) \neq 0$.

We remark that Theorem 3.2.9 is proved in [NB90] for the case $d = 2$, though there is a natural extension to arbitrary dimension. Theorem 3.2.9 has two important consequences.

Corollary 3.2.10. *The Hamiltonian given by Equation (3.1) is homogenous with respect to the magnetic translations.*

Proof. We have already checked that the magnetic translations give a projective representation of $\mathbb{R}^d$, so we just need to determine that $\{U_a(z-H)^{-1}U_{-a} : a \in \mathbb{R}^d\}$ has compact strong closure. Any ball $\{f \in L^\infty_w(\mathbb{R}^d) : \|f\| \leq R\}$ is pre-compact in the weak $L^1(\mathbb{R}^d)$-topology. We also have that $[U_a V U_{-a}]\psi(x) = V(x-a)\psi(x)$ almost surely by a simple computation. This observation implies that $V_a(x) = V(x-a)$ belongs to the ball $\{V' \in L^\infty_w(\mathbb{R}^d) : \|V'\| \leq \|V\|\}$. Hence the weak closure of $\{V_a : a \in \mathbb{R}^d\}$ is a closed subspace of a compact space and therefore compact. By Theorem 3.2.9, the family $\{V_a : a \in \mathbb{R}^n\}$ maps continuously to $\{(z-H_0-V_a)^{-1} : a \in \mathbb{R}^d\}$. As $[H_0, U_a] = 0$, $(z-H_0-V_a)^{-1} = U_a(z-H_0-V)^{-1}U_{-a}$ and $\{U_a(z-H)^{-1}U_{-a} : a \in \mathbb{R}^d\}$ is the image of $\{V_a : a \in \mathbb{R}^n\}$, a compact set, by the continuous function from Theorem 3.2.9. Therefore $\{U_a(z-H)^{-1}U_{-a} : a \in \mathbb{R}^d\}$ is compact in the strong operator topology. □

The next corollary gives us a convenient way of looking at the hull.

Corollary 3.2.11. *Let H be as in Equation (3.1). The hull of H is homeomorphic to the hull of V, i.e. the weak closure of $\Omega = \{U_a V U_{-a} : a \in \mathbb{R}^d\}$. Moreover, if we denote by V_ω the bounded function representing the point $\omega \in \Omega$, then there is a Borel function v on Ω such that $V_\omega(x) = v(T_{-x}\omega)$ for almost all $x \in \mathbb{R}^d$ and all $\omega \in \Omega$. If in addition V is uniformly continuous and bounded, v is continuous.*

Proof. By Theorem 3.2.9, the map

$$\Omega_V \ni V_a \mapsto (z-H_0-V_a)^{-1} = U_a(z-H_0-V)U_{-a} \in \Omega_H$$

is continuous. Similarly the inverse map

$$\Omega_H \ni U_a(z-H_0-V)^{-1}U_{-a} = (z-H_0-U_a V U_{-a})^{-1} \mapsto U_a V U_{-a} \in \Omega_V$$

is continuous. Hence we have that $\Omega_V \cong \Omega_H \cong \Omega$.

We let ρ_k be a sequence of non-negative bump functions acting as an approximate unit for the convolution product. That is, for any $\delta > 0$

$$\lim_{k\to\infty} \int_{|x|>\delta} \rho_k(x)\,\mathrm{d}x = 0, \qquad \lim_{k\to\infty} \int_{\mathbb{R}^d} \rho_k(x-a)F(x)\,\mathrm{d}x = F(a) \tag{3.3}$$

for any $F \in L^1(\mathbb{R}^d)$. Now, for all $\omega \in \Omega$, $V_\omega \in L^\infty_w(\mathbb{R}^d)$. We define functions v_k by $v_k(\omega) = \int_{\mathbb{R}^d} V_\omega(x)\rho_k(x)\,\mathrm{d}x$. Because $\omega \mapsto V_\omega$ is continuous as a map $\Omega \to L^\infty_w(\mathbb{R}^d)$, $(v_k)_{k\geq 0}$ is a sequence of continuous functions on Ω. We set $v(\omega) = \lim_{k\to\infty} v_k(\omega)$ if it exists. If v exists, then it is a Borel function because for any closed interval $[a,b] \subset \mathbb{R}$,

$$v^{-1}([a,b]) = \bigcap_{n\geq 1}\bigcup_{k\geq 1}\bigcap_{p\geq k} \left\{\omega \in \Omega : v_p(\omega) \in \left[a - \frac{1}{n}, b + \frac{1}{n}\right]\right\}$$

and the set on the right hand side is Borel (as each v_k is continuous). We now take $F \in L^1(\mathbb{R}^d)$ and compute

$$\begin{aligned}\int_{\mathbb{R}^d} v(T_{-x}\omega)F(x)\,\mathrm{d}x &= \lim_{k\to\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V_{T_{-x}\omega}(y)\rho_k(y)F(x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \lim_{k\to\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V_\omega(y+x)\rho_k(y)F(x)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

We make the substitution $u = y + x$, $v = y$ and find that

$$\begin{aligned}\int_{\mathbb{R}^d} v(T_{-x}\omega)F(x)\,\mathrm{d}x &= \lim_{k\to\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V_\omega(u)\rho_k(v)F(u-v)\,\mathrm{d}v\,\mathrm{d}u \\ &= \lim_{k\to\infty}\int_{\mathbb{R}^d} V_\omega(u)(F*\rho_k)(u)\,\mathrm{d}u \\ &= \int_{\mathbb{R}^d} V_\omega(u)F(u)\,\mathrm{d}u \qquad (3.4)\end{aligned}$$

as ρ_k is an approximate identity for convolution product in $L^1(\mathbb{R})$. Equation (3.4) holds for any $F \in L^1(\mathbb{R}^d)$, so we may say $v(T_{-x}\omega) = V_\omega(x)$ for all $\omega \in \Omega$ and almost all $x \in \mathbb{R}^d$.

We now assume V_ω to be uniformly continuous and bounded on $\mathbb{R}^d$, so $V_\omega \in L^\infty_w(\mathbb{R}^d)$ for all ω. For v_k defined as above, we claim that the sequence $(v_k)_{k\geq 0}$ is Cauchy in the uniform topology. Recall from the definition of v_k,

$$v_k(\omega) = \int_{\mathbb{R}^d} V_\omega(x)\rho_k(x)\,\mathrm{d}x = \lim_{j\to\infty}\int_{\mathbb{R}^d} V(x-a_j)\rho_k(x)\,\mathrm{d}x$$

for some sequence $(a_j) \in \mathbb{R}^d$. Now, because $\int_{\mathbb{R}^d}\rho_k = 1$,

$$\begin{aligned}|v_k(\omega) - v_{k'}(\omega)| &= \left|\int_{\mathbb{R}^d} V_\omega(x)\rho_k(x)\,\mathrm{d}x - \int_{\mathbb{R}^d} V_\omega(y)\rho_{k'}(y)\,\mathrm{d}y\right| \\ &= \lim_{j\to\infty}\left|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V(x-a_j)\rho_k(x)\rho_{k'}(y)\,\mathrm{d}y\,\mathrm{d}x\right. \\ &\qquad \left. - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} V(y-a_j)\rho_{k'}(y)\rho_k(x)\,\mathrm{d}x\,\mathrm{d}y\right| \\ &\leq \lim_{j\to\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|V(x-a_j) - V(y-a_j)|\,\rho_k(x)\rho_{k'}(y)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

We take an $\epsilon > 0$. As V is uniformly continuous, we can always find a $\delta > 0$ such that $|x-y| < \delta$ implies that $|V(x-a_j) - V(y-a_j)| < \epsilon/2$. We now split up our integral into two parts,

$$\begin{aligned}|v_k(\omega) - v_{k'}(\omega)| &\leq \lim_{j\to\infty}\int_{|x-y|<\delta}|V(x-a_j) - V(y-a_j)|\,\rho_k(x)\rho_{k'}(y)\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad + \int_{|x-y|\geq\delta}|V(x-a_j) - V(y-a_j)|\,\rho_k(x)\rho_{k'}(y)\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \frac{\epsilon}{2} + 2\|V\|_\infty\left(\int_{|x-y|>\delta}\rho_k(x)\rho_{k'}(y)\,\mathrm{d}x\,\mathrm{d}y\right).\end{aligned}$$

We note that $|x-y| < \delta$ implies that $|x| > \delta/2$ or $|y| > \delta/2$. Therefore we decompose the last integral into parts and estimate

$$\int_{|x-y|>\delta} \rho_k(x)\rho_{k'}(y)\,\mathrm{d}x\,\mathrm{d}y \leq \int_{|x|>\delta/2} \rho_k(x)\,\mathrm{d}x \int_{|y|>\delta/2} \rho_{k'}(y)\,\mathrm{d}y.$$

Using Equation (3.3), we take k and k' sufficiently large so that each integral is bounded by is bounded by $\frac{1}{2}\sqrt{\frac{\epsilon}{\|V\|_\infty}}$. Putting these results together

$$|v_k(\omega) - v_{k'}(\omega)| \leq \frac{\epsilon}{2} + 2\|V\|_\infty \left(\frac{\epsilon}{4\|V\|_\infty}\right) = \epsilon.$$

Because this inequality is independent of ω, it remains true when we take the supremum over all $\omega \in \Omega$. Therefore, $(v_k)_{k\geq 0}$ is a Cauchy sequence in the uniform topology and so converges to a continous function v. □

We remind the reader that analogues of Theorem 3.2.9 and Corollaries 3.2.10 and 3.2.11 exist in the case of a discrete Hamiltonian acting on $\ell^2(\mathbb{Z}^d)$ (also called the tight-binding approximation).

3.2.2 The algebra, representations and twisted crossed products

We shall now construct our algebra of observables. Such an observable algebra needs to satisfy several properties in order to adequately model the quantum Hall effect or a higher-dimensional system. First the algebra must be large enough to contain the observables of interest such as the Hamiltonian and current operators (or their resolvents). The algebra also needs to be sufficiently small so that the topological data does not disappear (e.g. we can not use the von Neumann algebra $\mathcal{N} = \{U_a : a \in \mathbb{R}^d\}'$ since $K_0(\mathcal{N}) = 0$).

The algebra of interest is a reduced twisted crossed-product C^*-algebra, $C(\Omega)\rtimes_\theta \mathbb{R}^d$. We start with the compact space Ω introduced in Definition 3.2.6 and 3.2.8 with a strongly continuous $\mathbb{R}^d$-action $\omega \mapsto T_a\omega$, $a \in \mathbb{R}^d$. We consider the continuous functions of $\Omega \times \mathbb{R}^d$ with compact support, $C_c(\Omega\times\mathbb{R}^d)$. We can make $C_c(\Omega\times\mathbb{R}^d)$ into a $*$-algebra with the twisted convolution and involution

$$\begin{aligned}(fg)(\omega, x) &= \int_{\mathbb{R}^d} e^{i\theta(x\wedge y)} f(\omega, y) g(T_{-y}\omega, x-y)\,\mathrm{d}y \\ f^*(\omega, x) &= \overline{f(T_{-x}\omega, -x)}\end{aligned}$$

for all $f, g \in C_c(\Omega\times\mathbb{R}^d)$. One checks (cf. [Bel92, Section 2.5]) that $C_c(\Omega\times\mathbb{R}^d)$ with the convolution product and adjoint forms a $*$-algebra. For a fixed $\omega \in \Omega$, we can represent this algebra on $L^2(\mathbb{R}^d)$ by the map π_ω, where

$$[\pi_\omega(f)\psi](x) = \int_{\mathbb{R}^d} e^{-i\theta(x\wedge y)} f(T_{-x}\omega, y-x)\psi(y)\,\mathrm{d}y$$

for all $\psi \in L^2(\mathbb{R}^d)$. A computation will check that π_ω respects the $*$-algebra structure on $C_c(\Omega \times \mathbb{R}^d)$ (see [PR89] or [Wil07, Section 7.4] for more on twisted convolution algebras).

The family of representations $\{\pi_\omega : \omega \in \Omega\}$ can be compared by the action of magnetic translations on Ω. Specifically, we have the following.

Proposition 3.2.12. *The representations π_ω satisfy the covariance condition,*

$$U_a \pi_\omega(f) U_{-a} = \pi_{T_a\omega}(f)$$

for all $\omega \in \Omega$ and $f \in C_c(\Omega \times \mathbb{R}^d)$.

Proof. We take $\psi \in L^2(\mathbb{R}^d)$ and start with the left hand side,

$$\begin{aligned}
(U_a \pi_\omega(f) U_{-a}\psi)(x) &= e^{-i\theta(x\wedge a)}(\pi_\omega(f)U_{-a}\psi)(x-a) \\
&= e^{-i\theta(x\wedge a)} \int_{\mathbb{R}^d} e^{-i\theta[(x-a)\wedge y]} f(T_{-(x-a)}\omega, y-(x-a))(U_{-a}\psi)(y)\,\mathrm{d}y \\
&= e^{-i\theta(x\wedge a)} \int_{\mathbb{R}^d} e^{-i\theta[(x-a)\wedge y]} f(T_{-(x-a)}\omega, y+a-x) e^{i\theta(y\wedge a)}\psi(y+a)\,\mathrm{d}y \\
&= e^{-i\theta(x\wedge a)} \int_{\mathbb{R}^d} e^{-i\theta[(x-a)\wedge(y-a)]} f(T_{-(x-a)}\omega, y-x) e^{i\theta[(y-a)\wedge a]}\psi(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d} e^{-i\theta[(x-a)\wedge(y-a)+x\wedge a-y\wedge a]} f(T_{-x}(T_a\omega), y-x)\psi(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d} e^{-i\theta(x\wedge y)} f(T_{-x}(T_a\omega), y-x)\psi(y)\,\mathrm{d}y \\
&= [\pi_{T_a\omega}(f)\psi](x),
\end{aligned}$$

where we have made the substitution $y \mapsto y+a$. ☐

Because of the covariance condition, we can define a norm on $C_c(\Omega \times \mathbb{R}^d)$ using the operator norm on $\mathcal{B}[L^2(\mathbb{R}^d)]$,

$$\|f\| = \sup_{\omega\in\Omega} \|\pi_\omega(f)\|_{\mathcal{B}[L^2(\mathbb{R}^d)]}.$$

We define our algebra of observables A to be the C^*-completion of $C_c(\Omega\times\mathbb{R}^d)$ with the convolution product subject to the covariance condition. For convenience, we denote by $\mathcal{A}$ the dense $*$-subalgebra of continuous compactly supported functions on $\Omega \times \mathbb{R}^d$. Because this algebra is a twisted crossed-product, we may also denote it by the standard notation $C(\Omega) \rtimes_\theta \mathbb{R}^d$, where θ represents the $\mathbb{R}^d$-action twisted by the magnetic field. As $\mathbb{R}^d$ is amenable, we can be sloppy about the distinction between full and reduced crossed-product algebras.

Theorem 3.2.13 ([Bel92], Theorem 6)**.** *Take $H = \sum_j K_j^2 + V_\omega$ acting on $L^2(\mathbb{R}^d)$ with hull Ω. For each $z \in \rho(H)$ and $x \in \mathbb{R}^d$ there is an element $R(z;x) \in A$ such that for all $\omega \in \Omega$, $\pi_\omega[R(z;x)] = (z - H_{T_{-x}\omega})^{-1}$.*

Proof. We first consider the operator e^{-tH_0} for $t > 0$. We claim that there is a function $f_t(x)$ that is smooth and fast-decreasing in x such that

$$\big(e^{-tH_0}\psi\big)(x) = \int_{\mathbb{R}^d} e^{-i\theta(x\wedge y)} f_t(x-y)\psi(y)\,\mathrm{d}y \tag{3.5}$$

for all $t > 0$. The proof of this claim is quite cumbersome and can be found in [Bel92, Theorem 6]; we will present the case $d = 2$ with the quantum Hall Hamiltonian

$$H_0 = K_1^2 + K_2^2 = \left(-i\frac{\partial}{\partial x_1} + \theta X_2\right)^2 + \left(-i\frac{\partial}{\partial x_2} - \theta X_1\right)^2, \quad \theta \in \mathbb{R}.$$

The operator $e^{-tH_0}\psi$ is the solution to the heat-like equation $(\partial_t + H_0)\phi = 0$. We seek a function $k_t^0(x, y)$ such that

$$\big(e^{-tH_0}\psi\big)(x) = \int_{\mathbb{R}^2} k_t^0(x, y)\psi(y)\,\mathrm{d}y.$$

We use the ansatz

$$k_t^0(x, y) = \exp\Big(\frac{a_t}{2}|x-y|^2 + b_t(x\wedge y) + c_t\Big),$$

where in the 2-dimensional setting, $x \wedge y = x_1y_2 - x_2y_1$. Because $(\partial_t + H_0)e^{-tH_0}\psi = 0$ for all $\psi \in L^2(\mathbb{R}^2)$, $(\partial_t + H_0)k_t^0(x, y) = 0$ (this is a slightly formal expression as we apply H_0 to the first variable of k_t^0). We find that

$$\begin{aligned}
(\partial_t k_t^0)(x, y) &= \left[\frac{\dot a_t}{2}|x-y|^2 + \dot b_t(x\wedge y) + \dot c_t\right] k_t(x, y) \\
(K_1^2 k_t^0)(x, y) &= \big[-a_t - (a_t(x_1-y_1) + b_ty_2)^2 - 2i\theta x_2(a_t(x_1-y_1) + b_ty_2) + \theta^2x_2^2\big]k_t^0(x, y) \\
(K_2^2 k_t^0)(x, y) &= \big[-a_t - (a_t(x_2-y_2) - b_ty_1)^2 + 2i\theta x_1(a_t(x_2-y_2) - b_ty_1) + \theta^2x_1^2\big]k_t^0(x, y).
\end{aligned}$$

Setting $(\partial_t + H_0)k_t^0 = (\partial_t + K_1^2 + K_2^2)k_t^0 = 0$ and dividing through by k_t^0, we obtain the following system of differential equations

$$\frac{\dot a_t}{2} = a_t^2 - \theta^2 = a_t^2 + b_t^2 = a_t^2 - i\theta b_t, \qquad \dot b_t = 2a_tb_t + 2i\theta a_t, \qquad \dot c_t = 2a_t.$$

This system has the solution

$$a_t = -\theta\tanh(2\theta t + C_1), \qquad b_t = -i\theta, \qquad c_t = -\log\left[\cosh(2\theta t + C_1)\right] + C_2,$$

where C_1 and C_2 are constants. Putting everything together, we can write

$$k_t^0(x, y) = \frac{C}{\cosh(2\theta t + C_1)}\exp\left[-\frac{\theta}{2}\tanh(2\theta t + C_1)|x-y|^2 - i\theta(x\wedge y)\right]$$

for constants C and C_1. By the functional calculus, we know that $\lim_{t\to 0} e^{-tH_0}\psi = \psi$, so we require that $\lim_{t\to 0} k_t^0(x, y) = \delta(x-y)$, the Dirac delta distribution. We can apply this condition to find explicitly what C and C_1 are. One finds that $C_1 = \frac{i\pi}{2}$. In the context of this proof the value of C is inessential so we will skip it. We can rewrite the kernel as

$$\big(e^{-tH_0}\psi\big)(x) = \int_{\mathbb{R}^2} e^{-i\theta(x\wedge y)} f_t(x-y)\psi(y)\,\mathrm{d}y,$$

where the function f_t is defined by

$$\begin{aligned} f_t(x) &= \frac{C}{\cosh(2\theta t + \frac{i\pi}{2})}\exp\left[-\frac{\theta}{2}\tanh\left(2\theta t + \frac{i\pi}{2}\right)|x|^2\right] \\ &= \frac{-iC}{\sinh(2\theta t)}\exp\left(-\frac{\theta}{2}\coth\left(2\theta t\right)|x|^2\right) \end{aligned} \tag{3.6}$$

and f_t is a Schwartz function for all $t > 0$. Therefore as $|x-y| \to \infty$, $f_t(x-y) \to 0$ rapidly. We remark that in the d-dimensional setting, f_t is of an similar form to Equation (3.6); we direct the reader to [Bel92, p569] for the details.

Let us return to the general d-dimensional setting and define the function $F_t \in C(\Omega) \rtimes_\theta \mathbb{R}^d$ by $F_t(\omega, x) = f_t(-x)$ for all ω. By the definition of π_ω and Equation (3.5), $e^{-tH_0} = \pi_\omega(F_t)$ for all $\omega \in \Omega$ and $t > 0$.

We now consider $e^{-t(H_0+V_\omega)}$. By a Dyson expansion $e^{-t(H_0+V_\omega)} = e^{-tH_0} + D_t(H_\omega)$, where $D_t(H_\omega)$ is given by the sum

$$\sum_{n=0}^{\infty}(-i)^n\int_0^t \mathrm{d}s_1\int_0^{s_1}\mathrm{d}s_2\cdots\int_0^{s_{n-1}}\mathrm{d}s_n\, e^{-(t-s_1)H_0}V_\omega e^{-(s_1-s_2)H_0}V_\omega\cdots V_\omega e^{-s_nH_0}.$$

Because $V_\omega \in L^\infty_w(\mathbb{R}^d)$ and e^{-sH_0} are bounded for any $\omega \in \Omega$ and $s > 0$, $D_t(H_\omega)$ satisfies the hypothesis of [RS75, Theorem X.70] and therefore converges uniformly in the strong operator topology for $t < \infty$. We want to show that $e^{-tH_0}V_\omega e^{-sH_0}$ is of the form $\pi_\omega(G_{t,s})$ for some $G_{t,s} \in C(\Omega) \rtimes_\theta \mathbb{R}^d$. By Equation (3.5),

$$\begin{aligned} \big(e^{-tH_0}V_\omega e^{-sH_0}\psi\big)(x) &= \int_{\mathbb{R}^d} e^{-i\theta(x\wedge y)} f_t(x-y)V_\omega(y)\int_{\mathbb{R}^d} e^{-i\theta(y\wedge u)} f_s(y-u)\psi(u)\,\mathrm{d}u\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-i\theta x\wedge(x-v)+(x-v)\wedge u)} f_t(v)V_\omega(x-v)f_s(x-v-u)\psi(u)\,\mathrm{d}v\,\mathrm{d}u \\ &= \int_{\mathbb{R}^d} e^{-i\theta(x\wedge u)}\left(\int_{\mathbb{R}^d} e^{i\theta(x\wedge v+v\wedge u)} f_t(v)V_{T_{-x}\omega}(-v)f_s(x-u-v)\,\mathrm{d}v\right)\psi(u)\,\mathrm{d}u \\ &= \int_{\mathbb{R}^d} e^{-i\theta(x\wedge u)}\left(\int_{\mathbb{R}^d} e^{-i\theta(u-x)\wedge v} f_t(v)V_{T_{-x}\omega}(-v)f_s(-(u-x)-v)\,\mathrm{d}v\right)\psi(u)\,\mathrm{d}u \\ &= \int_{\mathbb{R}^d} e^{-i\theta(x\wedge u)} G_{t,s}(T_{-x}\omega, u-x)\psi(u)\,\mathrm{d}u, \end{aligned}$$

where we have made the substitution $v = x - y$ in the second line and the regularity of f_t allows us to use Fubini's theorem. We recall the definition of V_ω as a weak limit of a sequence of the form $V(\cdot - a_j)$ for $(a_j)_{j\geq 1}$ a sequence in $\mathbb{R}^d$. Therefore we write

$$G_{t,s}(\omega, x) = \lim_{j\to\infty}\int_{\mathbb{R}^d} e^{-i\theta(x\wedge y)} f_t(y)V(-y-a_j)f_s(-x-y)\,\mathrm{d}y.$$

Because f_t and f_s are smooth and fast decreasing and

$$\|G_{s,t}\| \leq \|V\|_\infty \|F_t\| \|F_s\|$$

uniformly in s and t, it follows that $G_{t,s} \in C(\Omega) \rtimes_\theta \mathbb{R}^d$ for all $s, t > 0$ and so $e^{-tH_0} V_\omega e^{-sH_0} = \pi_\omega(G_{s,t})$. Because the Dyson expansion converges in $C(\Omega) \rtimes_\theta \mathbb{R}^d$ and $\|G_{s,t}\|$ is bounded, $e^{-t(H_0+V_\omega)} = \pi_\omega(g_t)$ for some $g_t \in C(\Omega) \rtimes_\theta \mathbb{R}^d$.

Now, suppose that $V_\omega(x) \geq V'$ for all ω and almost all x. Because H_0 is positive, $H_0 + V_\omega$ is bounded from below by V'. From this estimate, we have that

$$(z - H_\omega)^{-1} = \int_0^\infty e^{-t(H_0+V_\omega - z)} \, \mathrm{d}t$$

is well-defined for $\Re(z) < V'$. For other values of $\tilde{z} \in \rho(H)$, we use the resolvent formula

$$(\tilde{z} - H_\omega)^{-1} = (z - H_\omega)^{-1} + (z - \tilde{z})(\tilde{z} - H_\omega)^{-1}(z - H_\omega)^{-1}$$

for $\Re(z) < V'$ and $(z - \tilde{z})$ small enough so that $(z - H_\omega)^{-1} + (z - \tilde{z})(\tilde{z} - H_\omega)^{-1}(z - H_\omega)^{-1}$ is contained in the C^*-closure of $\pi_\omega(A)$. This process can be iterated to obtain the remaining $\tilde{z} \in \rho(H)$. □

Corollary 3.2.14. *Let $H = H_0 + V_\omega$. Then $f(H) \in \pi_\omega(A)$ for every function $f \in C_0(\mathbb{R})$.*

Proof. Theorem 3.2.13 tells us that the resolvent of H is in $\pi_\omega(A)$. Polynomials of the resolvent $(z - H)^{-1}$ are dense in $C_0(H) = \{f(H) : f \in C_0(\mathbb{R})\}$. Hence $C_0(H)$ is contained in the C^*-closure. □

3.2.3 The noncommutative calculus

Theorem 3.2.13 ensures that the resolvent of the disordered Hamiltonian is contained in our observable algebra. If this were our only requirement for the observable algebra, then we could have simply taken the C^*-algebra generated by the resolvent of the (disordered) Hamiltonian. Because the quantum Hall effect involves a disordered Hamiltonian, current operators and the geometry of the momentum space, we require the larger crossed-product algebra. The algebra $C(\Omega) \rtimes_\theta \mathbb{R}^d$ is also required to determine the topological properties of higher-dimensional systems.

One of the strengths of Bellissard's noncommutative Brillouin zone is that there is enough structure on A and the dense subalgebra $\mathcal{A} \cong C_c(\Omega \times \mathbb{R}^d)$ to define a calculus of sorts. This extra structure is of interest to us as we would like to consider the current operators $J_k = i[H, X_k]$, where X_k is the position operator on $L^2(\mathbb{R}^d)$ for $k \in \{1, \ldots, d\}$. In the quantum Hall example, such operators give the Hall current and come from a 'noncommutative derivative' of the Hamiltonian. The noncommutative calculus of $\mathcal{A} \subset A$ allows us to make sense of these derivatives. Furthermore, by constructing an 'integration theory' on the algebra of observables, we can also consider the measurements of such current operators.

We start by defining a measure-theory on our algebra, which we do via a trace. We will consider two traces: an abstract trace defined on the algebra $\mathcal{A}$ and another coming from measurements in translation invariant systems. Under suitable hypotheses, we will show that these traces coincide.

Definition 3.2.15. Suppose the dynamical system $(\Omega, \mathbb{R}^d, T)$ has an invariant Borel probability measure $\mathbf{P}$. For $f \in \mathcal{A}$ and $f \geq 0$, we define

$$\mathcal{T}(f) = \int_\Omega f(\omega, 0)\, \mathrm{d}\mathbf{P}(\omega).$$

Lemma 3.2.16. *The functional $\mathcal{T}$ is a semifinite norm lower-semicontinous trace with $\mathcal{A} \subset \mathrm{Dom}(\mathcal{T})$. If the support of $\mathbf{P}$ is Ω, then the trace $\mathcal{T}$ is faithful.*

Proof. We first check that

$$\begin{aligned}
\mathcal{T}(f^*f) &= \int_\Omega (f^*f)(\omega, 0)\, \mathrm{d}\mathbf{P}(\omega) \\
&= \int_\Omega \int_{\mathbb{R}^d} e^{i\theta(0\wedge y)} f^*(\omega, y) f(T_{-y}\omega, 0 - y)\, \mathrm{d}y\, \mathrm{d}\mathbf{P}(\omega) \\
&= \int_\Omega \int_{\mathbb{R}^d} \overline{f(T_{-y}\omega, -y)} f(T_{-y}\omega, -y)\, \mathrm{d}y\, \mathrm{d}\mathbf{P}(\omega) \\
&= \int_\Omega \int_{\mathbb{R}^d} |f(T_y\omega, y)|^2\, \mathrm{d}y\, \mathrm{d}\mathbf{P}(\omega),
\end{aligned}$$

which is finite and non-negative for $f \in C_c(\Omega \times \mathbb{R}^d)$. Hence $\mathcal{T}$ is well-defined for any positive $f \in \mathcal{A}$.

It is a simple check that $\mathcal{T}$ satisfies the linearity properties required for a trace. We then compute

$$\begin{aligned}
\mathcal{T}(ff^*) &= \int_\Omega \int_{\mathbb{R}^d} f(\omega, y) f^*(T_{-y}\omega, -y)\, \mathrm{d}y\, \mathrm{d}\mathbf{P}(\omega) = \int_\Omega \int_{\mathbb{R}^d} f(\omega, y)\overline{f(T_yT_{-y}\omega, y)}\, \mathrm{d}y\, \mathrm{d}\mathbf{P}(\omega) \\
&= \int_\Omega \int_{\mathbb{R}^d} |f(\omega, y)|^2\, \mathrm{d}y\, \mathrm{d}\mathbf{P}(\omega), \qquad (3.7)
\end{aligned}$$

which is the same as $\mathcal{T}(f^*f)$ as $\mathbf{P}$ is invariant under the action of T. Therefore the functional $\mathcal{T} : \mathcal{A}_+ \to [0, \infty]$ satisfies the conditions required to be a trace, where $\mathcal{A}_+$ is the positive cone of $\mathcal{A}$.

The trace is semifinite as it is well-defined on $\mathcal{A}$, which is norm-dense in A. Next, suppose $g_n \to g$ in norm. As $\|g\| = \sup_{\omega\in\Omega} \|\pi_\omega(g)\|_{\mathcal{B}[L^2(\mathbb{R}^d)]}$, we see that if $g_n \to g$ in norm, then by the definition of $\pi_\omega(g_n)$, $g_n(\omega, x)$ will converge pointwise to $g(\omega, x)$ almost everywhere. As $g_n, g \geq 0$, we can suppose $g_n = f_n^* f_n$ and $g = f^* f$. We then compute

$$\mathcal{T}(f^*f) = \mathcal{T}\Big(\lim_{n\to\infty} f_n^* f_n\Big) = \int_\Omega \int_{\mathbb{R}^d} \Big|\lim_{n\to\infty} f_n(T_y\omega, y)\Big|^2 \,\mathrm{d}y\,\mathrm{d}\mathbf{P}(\omega)$$
$$\leq \liminf_{n\to\infty} \int_\Omega \int_{\mathbb{R}^d} |f_n(T_y\omega, y)|^2 \,\mathrm{d}y\,\mathrm{d}\mathbf{P}(\omega) = \liminf_{n\to\infty} \mathcal{T}(f_n^* f_n),$$

where we have used Fatou's Lemma on the product measure defined by the Lebesgue measure on $\mathbb{R}^d$ and $\mathbf{P}$ on Ω.

Finally, if $\operatorname{supp}(\mathbf{P}) = \Omega$ and $\mathcal{T}(f^*f) = \mathcal{T}(ff^*) = 0$, then Equation (3.7) implies that $f(\omega, x) = 0$ for all ω and x (as f^*f is continuous). □

Definition 3.2.17. For $\Lambda \subset \mathbb{R}^d$ open and convex, define $\mathrm{Tr}_\Lambda(T) = \mathrm{Tr}(Q_\Lambda T Q_\Lambda)$ where $Q_\Lambda : L^2(\mathbb{R}^d) \to L^2(\Lambda)$ is the projection. Then taking an increasing sequence Λ_j with $\bigcup_j \Lambda_j = \mathbb{R}^d$, the *trace per unit area* on $\mathcal{B}[L^2(\mathbb{R}^d)]$ is given by

$$\mathrm{Tr}_{\mathrm{ar}}(T) = \lim_{j\to\infty} \frac{1}{|\Lambda_j|} \mathrm{Tr}_{\Lambda_j}(T), \qquad T \geq 0,$$

where $|\Lambda_j|$ denotes the Lebesgue measure of Λ_j.

Proposition 3.2.18. *Let $f \in \mathcal{A}_+$. If $\mathbf{P}$ is an ergodic measure (that is, the only functions in $L^2(\Omega, \mathbf{P})$ such that $v(T_x\omega) = v(\omega)$ are constant functions), then for almost all $\omega \in \Omega$,*

$$\mathcal{T}(f) = \mathrm{Tr}_{\mathrm{ar}}[\pi_\omega(f)].$$

Proof. Given $g \in \mathcal{A}$, we know that

$$[\pi_\omega(g)\psi](x) = \int_{\mathbb{R}^d} e^{-i\theta(x\wedge y)} g(T_{-x}\omega, y - x)\psi(y)\,\mathrm{d}y$$

so $\pi_\omega(g)$ is an integral operator with kernel $k_\omega(x,y) = e^{-i\theta(x\wedge y)} g(T_{-x}\omega, y-x)$. Because Λ is bounded and $k_\omega(x,y)$ is continuous, $\pi_\omega(g)$ is Hilbert-Schmidt on $L^2(\Lambda)$ by [RS72, Theorem VI.23] for any $g \in C_c(\Omega\times\mathbb{R}^d)$. Therefore we can say that the product $\pi_\omega(g^*g)$ is trace-class by [RS72, Theorem VI.22, part (h)] for $g \in C_c(\Omega \times \mathbb{R}^d)$. We can take the trace Tr_Λ by integrating along the diagonal [Sim05, Theorem 3.9]. Computing the trace for $f = g^*g$,

$$\mathrm{Tr}_\Lambda[\pi_\omega(f)] = \int_\Lambda k_\omega(x,x)\,\mathrm{d}x = \int_\Lambda e^{-i\theta(x\wedge x)} f(T_{-x}\omega, x - x)\,\mathrm{d}x$$
$$= \int_\Lambda f(T_{-x}\omega, 0)\,\mathrm{d}x.$$

As the action of $\mathbb{R}^d$ by T on Ω is $\mathbf{P}$-measure preserving, a continuous version of Birkhoff's Ergodic Theorem in higher dimensions [NZ79, Section 4] gives that

$$\mathrm{Tr}_{\mathrm{ar}}[\pi_\omega(f)] = \lim_{j\to\infty} \frac{1}{|\Lambda_j|} \mathrm{Tr}_\Lambda[\pi_\omega(f)] = \lim_{j\to\infty} \frac{1}{|\Lambda_j|} \int_{\Lambda_j} f(T_{-x}\omega, 0)\,\mathrm{d}x$$
$$= \int_\Omega f(\omega, 0)\,\mathrm{d}\mathbf{P}(\omega) = \mathcal{T}(f)$$

for almost all ω. □

Remark 3.2.19. We shall assume from now on that the probability measure $\mathbf{P}$ is ergodic under the action of $\mathbb{R}^d$ on Ω with $\mathrm{supp}(\mathbf{P}) = \Omega$. In an abuse of notation, we will also denote the trace per unit volume by $\mathcal{T}$, where $\mathcal{T}(f) = \mathcal{T}(\pi_\omega(f))$ almost surely.

Definition 3.2.20. For $p \geq 1$, denote by $L^p(A, \mathcal{T})$ the completion of $\mathcal{A}$ in the norm

$$\|f\|_p = [\mathcal{T}(|f|^p)]^{1/p}.$$

In particular, $L^2(A, \mathcal{T})$ is a Hilbert space with inner product $\langle f_1, f_2 \rangle = \mathcal{T}(f_1^* f_2)$. The space $L^2(A, \mathcal{T})$ comes with a canonical representation $\pi_{GNS} : A \to \mathcal{B}[L^2(A, \mathcal{T})]$ given by left multiplication.

Now that we have a measure theory on our algebra, we construct a differential structure. In the noncommutative framework, derivations on an algebra take the place of derivatives of functions.

Lemma 3.2.21. *For all $j \in \{1, \ldots, d\}$, the mapping $(\partial_j f)(\omega, x) = ix_j f(\omega, x)$ is a $*$-derivation on $C_c(\Omega \times \mathbb{R}^d)$.*

Proof. The only claims that are not clear are the Leibniz rule $\partial_j(fg) = \partial_j(f)g + f\partial_j(g)$ and that $\partial_j(f^*) = [\partial_j(f)]^*$. By direct calculation,

$$\begin{aligned}
[\partial_j(f)g) + f\partial_j(g)](\omega, x) &= \int_{\mathbb{R}^d} e^{i\theta(x\wedge y)} (\partial_j f)(\omega, y) g(T_{-y}\omega, x - y)\, \mathrm{d}y \\
&\quad + \int_{\mathbb{R}^d} e^{i\theta(x\wedge y)} f(\omega, y)(\partial_j g)(T_{-y}\omega, x - y)\, \mathrm{d}y \\
&= \int_{\mathbb{R}^d} e^{i\theta(x\wedge y)} iy_j f(\omega, y) g(T_{-y}\omega, x - y)\, \mathrm{d}y \\
&\quad + \int_{\mathbb{R}^d} e^{i\theta(x\wedge y)} f(\omega, y) i(x_j - y_j) g(T_{-y}\omega, x - y)\, \mathrm{d}y \\
&= \int_{\mathbb{R}^d} e^{i\theta(x\wedge y)} i(y_j + x_j - y_j) f(\omega, y) g(T_{-y}\omega, x - y)\, \mathrm{d}y \\
&= ix_j \int_{\mathbb{R}^d} e^{i\theta(x\wedge y)} f(\omega, y) g(T_{-y}\omega, x - y)\, \mathrm{d}y \\
&= ix_j (fg)(\omega, x) = [\partial_j(fg)](\omega, x).
\end{aligned}$$

Also

$$[\partial_j(f^*)](\omega, x) = ix_j \overline{f(T_{-x}\omega, -x)} = \overline{i(-x_j) f(T_{-x}\omega, -x)} = (ix_j f)^*(\omega, x) = [\partial_j(f)]^*(\omega, x)$$

as required. ☐

Because the derivations $\{\partial_j\}_{j=1}^d$ commute on $C_c(\Omega \times \mathbb{R}^d)$, we may exponentiate to obtain a d-parameter group of $*$-automorphsims on A given by

$$[\rho_k(f)](\omega, x) = e^{ik\cdot x} f(\omega, x)$$

for $k = (k_1, \ldots, k_d) \in \mathbb{R}^d$ and $f \in \mathcal{A}$. This action then extends to all of A by continuity.

Lemma 3.2.22. *Let $X = (X_1, \ldots, X_d)$ be the position operator on $L^2(\mathbb{R}^d)$ and $f \in \mathcal{A}$. Then for all $k \in \mathbb{R}^d$ and $j \in \{1, \ldots, d\}$,*

$$\pi_\omega[\rho_k(f)] = e^{-ik\cdot X}\pi_\omega(f)e^{ik\cdot X}, \qquad \pi_\omega(\partial_j f) = -i[X_j, \pi_\omega(f)].$$

Proof. We check for any $\psi \in L^2(\mathbb{R}^d)$,

$$\begin{aligned}
[\pi_\omega(\rho_k(f))\psi](x) &= \int_{\mathbb{R}^d} e^{-i\theta(x\wedge y)}(\rho_k f)(T_{-x}\omega, y-x)\psi(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d} e^{-i\theta(x\wedge y)}e^{ik\cdot(y-x)}f(T_{-x}\omega, y-x)\psi(y)\,\mathrm{d}y \\
&= e^{-ik\cdot X}\int_{\mathbb{R}^d} e^{-i\theta(x\wedge y)}f(T_{-x}\omega, y-x)e^{ik\cdot y}\psi(y)\,\mathrm{d}y \\
&= \left[e^{-ik\cdot X}\pi_\omega(f)e^{ik\cdot X}\psi\right](x).
\end{aligned}$$

We also find

$$\begin{aligned}
[\pi_\omega(\partial_j f)\psi](x) &= \int_{\mathbb{R}^d} e^{-i\theta(x\wedge y)}(\partial_j f)(T_{-x}\omega, y-x)\psi(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d} e^{-i\theta(x\wedge y)}i(y_j - x_j)f(T_{-x}\omega, y-x)\psi(y)\,\mathrm{d}y \\
&= -i\left[x_j\int_{\mathbb{R}^d} e^{-i\theta(x\wedge y)}f(T_{-x}\omega, y-x)\psi(y)\,\mathrm{d}y\right. \\
&\qquad \left. - \int_{\mathbb{R}^d} e^{-i\theta(x\wedge y)}f(T_{-x}\omega, y-x)y_j\psi(y)\,\mathrm{d}y\right] \\
&= -i\left(X_j\pi_\omega(f)\psi - \pi_\omega(f)X_j\psi\right)(x) \\
&= (-i[X_j, \pi_\omega(f)]\psi)(x)
\end{aligned}$$

□

Because of the result $\pi_\omega(\partial_j f) = -i[X_j, \pi_\omega(f)]$, we will also denote $\partial_j(a) = -i[X_j, a]$ for a a bounded operator on $L^2(\mathbb{R}^d)$ with $a \cdot \mathrm{Dom}(X_j) \subset \mathrm{Dom}(X_j)$ and $j \in \{1, \ldots, d\}$.

An immediate consequence of Lemma 3.2.22 is that if the resolvent of a Hamiltonian $\pi_\omega(f) = (\lambda - H)^{-1}$ is in the domain $\mathrm{Dom}(\partial_j)$ (as is the case for $H_0 = \sum_j K_j^2$), then

$$\partial_j[\pi_\omega(f)] = i[(\lambda - H)^{-1}, X_j] = (\lambda - H)^{-1}J_j(\lambda - H)^{-1}.$$

Hence the differential structure on the algebra of observables allows us to detect information about the current operators.

3.3 Topology and the index pairing

Now that we have constructed the noncommutative Brillouin zone and a notion of calculus on this space, we represent this geometric data as a spectral triple.

3.3.1 The spectral triple

We let $S \to \mathbb{R}^d$ be the complex spinor bundle over $\mathbb{R}^d$. The spinor bundle has an irreducible representation of the Clifford algebra $\mathbb{C}\ell_d$ with the generators $\{\gamma^j\}_{j=1}^d$ satisfying

$$\gamma^i\gamma^j + \gamma^j\gamma^i = 2\delta_{i,j}$$

with $\delta_{i,j}$ the Kronecker delta. Using these generators (which can be represented as matrices acting on $\mathbb{C}^\nu$, where $\nu = 2^{\lfloor \frac{d+1}{2} \rfloor}$ is the rank of the bundle), there is a natural unbounded representative of the Fredholm modules studied in [PLB13, PS14].

Proposition 3.3.1. *Define the algebra of products $\pi_\omega(\mathcal{A})^2 = \{\pi(fg) : f, g \in \mathcal{A}\}$. Then the tuple*

$$\left(\pi_\omega(\mathcal{A})^2, L^2(\mathbb{R}^d) \otimes \mathbb{C}^\nu, D = \sum_{j=1}^d X_j \otimes \gamma^j\right)$$

is a finitely summable spectral triple with spectral dimension d for all $\omega \in \Omega$.

Proof. By the representation $f \mapsto \pi_\omega(f) \otimes 1_\nu$ for $f \in C_c(\Omega \times \mathbb{R}^d)$ and Lemma 3.2.22, we have that

$$[D, \pi_\omega(f) \otimes 1_\nu] = \sum_{j=1}^d [X_j, \pi_\omega(f)] \otimes \gamma^j = i \sum_{j=1}^d \pi_\omega(\partial_j f) \otimes \gamma^j.$$

Hence $[D, \pi_\omega(f) \otimes 1_\nu]$ is bounded. Next we consider $(\pi_\omega(fg) \otimes 1_\nu)(1 + D^2)^{-s/2}$ for $f, g \in C_c(\Omega \times \mathbb{R}^d)$. To prove the this operator is trace-class, we will first show that $(1 + D^2)^{-s/4}\pi_\omega(g^*)$ is Hilbert-Schmidt for any $g \in C_c(\Omega \times \mathbb{R}^d)$ and $s > d$, where

$$\begin{aligned}[(1 + D^2)^{-s/4}\pi_\omega(g^*)\psi](x) &= \int_{\mathbb{R}^d} e^{-i\theta(x\wedge y)}(1 + |x|^2)^{-s/4} g^*(T_{-x}\omega, y - x)\psi(y)\,\mathrm{d}y \otimes 1_\nu \\ &= \int_{\mathbb{R}^d} e^{-i\theta(x\wedge y)}(1 + |x|^2)^{-s/4} \overline{g(T_{-y}\omega, x - y)}\psi(y)\,\mathrm{d}y \otimes 1_\nu\end{aligned}$$

The operator $(1+|X|^2)^{-s/4}\pi_\omega(g^*)$ has an integral kernel on $L^2(\mathbb{R}^d)$ given by $k_\omega(x, y) = e^{-i\theta(x\wedge y)}(1 + |x|^2)^{-s/4}\overline{g(T_{-y}\omega, x - y)}$. We use an argument that will be employed repeatedly for kernels of this kind. Because g has compact spatial support, we can write

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |k_\omega(x, y)|^2\,\mathrm{d}x\,\mathrm{d}y &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (1 + |x|^2)^{-s/2}\,|g(T_{-y}\omega, x - y)|^2\,\mathrm{d}x\,\mathrm{d}y \\ &\le C_1 \int_{|x-y|<N} (1 + |x|^2)^{-s/2}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

We make the substitution $u = x$, $v = x - y$ and estimate

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |k_\omega(x, y)|^2\,\mathrm{d}x\,\mathrm{d}y &\le C_1 \int_{\mathbb{R}^d}\int_{|v|<N} \left(1 + |u|^2\right)^{-s/2}\mathrm{d}v\,\mathrm{d}u \\ &\le C_2 \int_{\mathbb{R}^d} (1 + |u|^2)^{-s/2}\,\mathrm{d}u. \qquad (3.8)\end{aligned}$$

The final integral will converge if $s > d$ and therefore $(1+|X|^2)^{-s/4}\pi_\omega(g^*)$ is Hilbert-Schmidt for any $g \in \mathcal{A}$ and $s > d$ by [RS72, Theorem VI.23].

We note that if $s \to 2$, then $(\pi_\omega(g) \otimes 1_\nu)(1 + D^2)^{-s/4} \to (\pi_\omega(g) \otimes 1_\nu)(1 + D^2)^{-1/2}$ in norm. Therefore $(\pi_\omega(g) \otimes 1_\nu)(1 + D^2)^{-1/2}$ is a norm-limit of compact operators and so is compact for all $g \in \mathcal{A}$.

Finally, we need to show that

$$\begin{aligned}(\pi_\omega(f)\pi_\omega(g) \otimes 1_\nu)(1 + D^2)^{-s/2} &= (\pi_\omega(f) \otimes 1_\nu)(1 + D^2)^{-s/2}(\pi_\omega(g) \otimes 1_\nu) \\ &\quad + (\pi_\omega(f) \otimes 1_\nu)[\pi_\omega(g) \otimes 1_\nu, (1 + D^2)^{-s/2}] \qquad (3.9)\end{aligned}$$

is trace-class for $s > d$. The first term on the right hand side of Equation (3.9) is a product of Hilbert-Schmidt operators and is trace-class by [RS72, Theorem VI.22, part (h)] for $s > d$.

For the term $(\pi_\omega(f) \otimes 1_\nu)[\pi_\omega(g) \otimes 1_\nu, (1 + D^2)^{-s/2}]$, we use an argument similar to [CGP+15, Lemma 3.6]. It suffices to assume that $(1+D^2)^{-s/2} = (1+|X|^2)^{-rd/2} \otimes 1_\nu$ with $1 < r < 2$. We use the Leibniz rule to express the commutator

$$[\pi_\omega(g), (1 + |X|^2)^{-rd/2}] = \sum_{j+k=d-1} C_{j,k}(1 + |X|^2)^{-\frac{jr}{2}}[\pi_\omega(g), (1 + |X|^2)^{-\frac{r}{2}}](1 + |X|^2)^{-\frac{kr}{2}}.$$

Using the integral formula for fractional powers (see [CP98, p701]), we can express

$$\begin{aligned}[\pi_\omega(g) \otimes 1_\nu, (1 + D^2)^{-r/2}] &= C_r \int_0^\infty t^{-r/2}[\pi_\omega(g) \otimes 1_\nu, (1 + t + D^2)^{-1}]\, \mathrm{d}t \\ &= C_r \int_0^\infty t^{-r/2}(1 + t + D^2)^{-1}[D^2, \pi_\omega(g) \otimes 1_\nu](1 + t + D^2)^{-1}\, \mathrm{d}t \\ &= iC_r \sum_{l=1}^{d} \Big(\int_0^\infty t^{-r/2}X_l(1 + t + |X|^2)^{-1}\pi_\omega(\partial_l g)(1 + t + |X|^2)^{-1}\, \mathrm{d}t \\ &\qquad + \int_0^\infty t^{-r/2}(1 + t + |X|^2)^{-1}\pi_\omega(\partial_l g)X_l(1 + t + |X|^2)^{-1}\, \mathrm{d}t \Big) \otimes 1_\nu\end{aligned}$$

where $C_r = \frac{\sin(r\pi/2)}{\pi}$. Therefore we find that each term in the Leibniz expansion of $\pi_\omega(f)[\pi_\omega(g), (1 + |X|^2)^{-rd/2}] \otimes 1_\nu$ is of the form

$$\begin{aligned}iC_r \sum_{l=1}^{d} \int_0^\infty t^{-r/2}\pi_\omega(f)(1 + |X|^2)^{-\frac{jr}{2}}(1 + t + |X|^2)^{-1}\,(\pi_\omega(\partial_l g)X_l \\ +X_l\pi_\omega(\partial_l g))\,(1 + t + |X|^2)^{-1}(1 + |X|^2)^{-\frac{kr}{2}}\, \mathrm{d}t \otimes 1_\nu.\end{aligned}$$

Our aim is to factorise the integrand

$$\pi_\omega(f)(1 + |X|^2)^{-\frac{jr}{2}}(1 + t + |X|^2)^{-1}\pi_\omega(\partial_l g)X_l(1 + t + |X|^2)^{-1}(1 + |X|^2)^{-\frac{kr}{2}}$$

as a product of operators in the Schatten ideals $\mathcal{L}^{2u}[L^2(\mathbb{R}^d)]$ and $\mathcal{L}^{2v}[L^2(\mathbb{R}^d)]$ with $u, v \in \mathbb{Z}$ and such that $(2u)^{-1} + (2v)^{-1} > 1$ for $r > 1$ (the other term in the integrand will follow by an analogous argument). We first consider

$$\pi_\omega(f)(1+|X|^2)^{-\frac{jr}{2}}(1+t+|X|^2)^{-1} = \pi_\omega(f)(1+|X|^2)^{-\frac{jr}{2}}(1+t+|X|^2)^{-1/2}(1+t+|X|^2)^{-1/2}.$$

An operator T is in the space $\mathcal{L}^{2u}[L^2(\mathbb{R}^d)]$ if T^u has a square-summable integral kernel. The operator $\left[\pi_\omega(f)(1+|X|^2)^{-\frac{jr}{2}}(1+t+|X|^2)^{-1/2}\right]^u$ has integral kernel given by

$$\begin{aligned}
k_\omega^u(z_0, z_u) &= \int_{z_1}\cdots\int_{z_{u-1}} k_\omega(z_0, z_1)\cdots k_\omega(z_{u-1}, z_u)\,\mathrm{d}z_{u-1}\cdots\mathrm{d}z_1 \\
k_\omega(z_i, z_{i+1}) &= e^{-i\theta(z_i\wedge z_{i+1})} f(T_{-z_i}\omega, z_{i+1}-z_i)(1+|z_{i+1}|^2)^{-\frac{jr}{2}}(1+t+|z_{i+1}|^2)^{-1/2}.
\end{aligned}$$

We use the compact support of f to estimate

$$\begin{aligned}
\int_{z_0}\int_{z_u} |k_\omega^u(z_0, z_u)|^2\,\mathrm{d}z_0\,\mathrm{d}z_u &\leq \int_{z_0}\int_{z_1}\cdots\int_{z_u} |k_\omega(z_0, z_1)\cdots k_\omega(z_{u-1}, z_u)|^2\,\mathrm{d}z_u\,\mathrm{d}z_{u-1}\cdots\mathrm{d}z_0 \\
&\leq C_1 \int_{|z_1-z_0|<N,\ldots,|z_u-z_{u-1}|<N} (1+t+|z_1|^2)^{-jr-1}\cdots(1+t+|z_u|^2)^{-jr-1}\,\mathrm{d}z_u\cdots\mathrm{d}z_0.
\end{aligned}$$

Next we make the substitution $w_0 = z_0$ and $w_i = z_i - z_{i-1}$ for $i \in \{1, \ldots, u\}$. We rewrite

$$\begin{aligned}
\int_{z_0}\int_{z_u} |k_\omega^u(z_0, z_u)|^2\,\mathrm{d}z_0\,\mathrm{d}z_u \leq \int_{w_0}\int_{|w_1|<N}\cdots\int_{|w_u|<N} &(1+t+|w_0+w_1|^2)^{-jr-1} \\
&\times (1+t+|w_0+w_1+w_2|^2)^{-jr-1}\cdots(1+|w_0+\cdots+w_u|^2)^{-jr-1}\mathrm{d}z_u\cdots\mathrm{d}z_0.
\end{aligned}$$

One then notes that for any $i \in \{1, \ldots, u\}$,

$$\begin{aligned}
(1+t+|w_0+\cdots+w_i|^2)^{-jr-1} &\leq C_{j,r}(1+t+|w_0+\cdots+w_i|)^{-2(jr+1)} \\
&\leq \tilde{C}_{j,r}(1+t+|w_0|)^{-2(jr+1)}(1+t+|w_1+\cdots+w_i|)^{2(jr+1)}.
\end{aligned}$$

where we have used the inequality $(1+t+|x+y|)^s \leq C_s(1+t+|x|)^s(1+t+|y|)^{|s|}$ for any $t \geq 0$ from [Gil95, Lemma 1.1.8]. Because $(1+t+|w_1+\cdots+w_i|)^{2(jr+1)}$ is continuous and the domain of integration over $(w_1, \ldots, w_i)$ is compact, we can say that

$$\begin{aligned}
\int_{z_0}\int_{z_u} |k_\omega^u(z_0, z_u)|^2\,\mathrm{d}z_0\,\mathrm{d}z_u &\leq C_{j,r}\int_{w_0}(1+t+|w_0|)^{-2u(jr+1)}\,\mathrm{d}w_0 \\
&\quad\times \int_{|w_1|<N}\cdots\int_{|w_u|<N}\prod_{i=1}^u\left(1+\left|\sum_{l=1}^i w_l\right|\right)^{2(jr+1)}\mathrm{d}w_u\cdots\mathrm{d}w_1 \\
&\leq \tilde{C}_{j,r}\int_{w_0}(1+t+|w_0|)^{-2u(jr+1)}\,\mathrm{d}w_0.
\end{aligned}$$

The final integral will converge if $u \geq \lceil \frac{d}{2(1+jr)} \rceil$, where $\lceil\cdot\rceil$ is the ceiling function (as $jr+1 \notin \mathbb{Z}$). We take the minimum such u and observe that under the Schatten norm for all $t \geq 0$,

$$
\begin{aligned}
&\left\|\pi_\omega(f)(1+|X|^2)^{-\frac{jr}{2}}(1+t+|X|^2)^{-1}\right\|_{2u} \\
&\qquad = \left\|\pi_\omega(f)(1+|X|^2)^{-\frac{jr}{2}}(1+t+|X|^2)^{-1/2}\right\|_{2u}\left\|(1+t+|X|^2)^{-1/2}\right\|_{\mathrm{op}} \\
&\qquad \leq \tilde{C}_u \frac{1}{\sqrt{1+t}}. \qquad (3.10)
\end{aligned}
$$

Next we consider $\pi_\omega(\partial_l g)X_l(1+t+|X|^2)^{-1}(1+|X|^2)^{-\frac{kr}{2}}$. We take the v-th power, which has integral kernel

$$
\begin{aligned}
k^v_\omega(z_0,z_v) &= \int_{z_1}\cdots\int_{z_{v-1}} k_\omega(z_0,z_1)\cdots k_\omega(z_{v-1},z_v)\,\mathrm{d}z_1\cdots\mathrm{d}z_{v-1} \\
k_\omega(z_i,z_{i+1}) &= c_{z_i,z_{i+1}}(\partial_l g)(T_{-z_i}\omega, z_{i+1}-z_i)(z_{i+1})_l(1+|z_{i+1}|^2)^{-\frac{kr}{2}}(1+t+|z_{i+1}|^2)^{-1}
\end{aligned}
$$

with $c_{z_i,z_{i+1}} = e^{-i\theta(z_i\wedge z_{i+1})}$. Because $\partial_l g \in \mathcal{A}$ and has compact spatial support, we can use the same argument as the case of $\mathcal{L}^{2u}$ to estimate the L^2-norm of the kernel, where

$$
\begin{aligned}
&\int_{z_0}\int_{z_v}|k^v_\omega(z_0,z_v)|^2\,\mathrm{d}z_0\,\mathrm{d}z_v \\
&\quad \leq C_1 \int_{\substack{|z_1-z_0|<M,\\ \cdots \\ |z_v-z_{v-1}|<M}} (z_1)_l^2(1+t+|z_1|^2)^{-kr-2}\cdots(z_v)_l^2(1+t+|z_v|^2)^{-kr-2}\,\mathrm{d}z_v\cdots\mathrm{d}z_0.
\end{aligned}
$$

We make the substitution $w_0 = z_0$, $w_i = z_i - z_{i-1}$ for $i \in \{1,\ldots,v\}$ and reduce

$$
\begin{aligned}
\int_{z_0}\int_{z_v}|k^v_\omega(z_0,z_v)|^2\,\mathrm{d}z_0\,\mathrm{d}z_v &\leq C_1\int_{w_0}\int_{\substack{|w_1|<M,\\ \cdots \\ |w_v|<M}} (w_0+w_1)_l^2(1+t+|w_0+w_1|^2)^{-kr-2}\times\cdots \\
&\quad \cdots\times(w_0+\cdots+w_v)_l^2(1+t+|w_0+\cdots+w_v|^2)^{-kr-2}\,\mathrm{d}w_v\cdots\mathrm{d}w_0.
\end{aligned}
$$

We have previously estimated

$$
(1+t+|w_0+\cdots+w_i|^2)^{-kr-2} \leq C_{k,r}(1+t+|w_0|)^{-2(kr+2)}(1+t+|w_1+\cdots+w_i|)^{2(kr+2)}
$$

for any $i \in \{1,\ldots,v\}$. Similarly, one finds

$$
\begin{aligned}
(w_0+\ldots+w_i)_l^2 &\leq (1+|(w_0)_l+\cdots+(w_i)_l|)^2 \\
&\leq C(1+|(w_0)_l|)^2(1+|(w_1)_l+\cdots+(w_i)_l|)^2
\end{aligned}
$$

and so

$$
\begin{aligned}
\int_{z_0}\int_{z_v}|k^v_\omega(z_0,z_v)|^2\,\mathrm{d}z_0\,\mathrm{d}z_v &\leq C_{k,r}\int_{w_0}(1+|(w_0)_l|)^{2v}(1+t+|w_0|)^{-2v(kr+2)}\,\mathrm{d}w_0 \\
&\quad\times\int_{|w_1|<M}\cdots\int_{|w_v|<M}\prod_{i=1}^{v}\left(1+\left|\sum_{m=1}^{i}(w_m)_l\right|\right)^2\left(1+\left|\sum_{n=1}^{i}w_n\right|\right)^{2(kr+2)}\mathrm{d}w_1\cdots\mathrm{d}w_u \\
&\leq \tilde{C}_{k,r}\int_{w_0}(1+|(w_0)_l|)^{2v}(1+t+|w_0|)^{-2v(kr+2)}\,\mathrm{d}w_0.
\end{aligned}
$$

We take $v = \lceil \frac{d}{2(1+kr)} \rceil$ so that the integral converges and therefore we can say that $\pi_\omega(\partial_l g)X_l(1+t+|X|^2)^{-1}(1+|X|^2)^{-\frac{kr}{2}} \in \mathcal{L}^{2v}[L^2(\mathbb{R}^d)]$ for any $l \in \{1,\dots,d\}$.

Using our values of u and v, we define $\frac{1}{w} = \frac{1}{2u} + \frac{1}{2v}$ and find that

$$\frac{1}{w} = \frac{1}{2\lceil \frac{d}{2(1+jr)} \rceil} + \frac{1}{2\lceil \frac{d}{2(1+kr)} \rceil} \leq \frac{1}{2\frac{d}{2(1+jr)}} + \frac{1}{2\frac{d}{2(1+kr)}} = \frac{1+jr+1+kr}{d} = \frac{(d-1)r+2}{d}$$

as $j+k = d-1$. By assumption $1 < r < 2$, so $1 < \frac{1}{w} < 2$, which implies that $w < 1$ and so the product is trace-class for $1 < r < 2$ by the Hölder inequality on Schatten norms,

$$\|ST\|_w \leq \|S\|_{2u}\|T\|_{2v}, \qquad \frac{1}{w} = \frac{1}{2u} + \frac{1}{2v},$$

see [Sim05, Theorem 2.8]. Furthermore,

$$\begin{aligned} C_r \int_0^\infty t^{-r/2} \Big\| \pi_\omega(f)(1+|X|^2)^{-\frac{jr}{2}}(1+t+|X|^2)^{-1}\left(\pi_\omega(\partial_l g)X_l \right. \\ \left. + X_l\pi_\omega(\partial_l g)\right)(1+t+|X|^2)^{-1}(1+|X|^2)^{-\frac{kr}{2}} \Big\|_1 \, \mathrm{d}t \\ \leq \tilde{C}_r \int_0^\infty t^{-r/2} \frac{1}{\sqrt{1+t}} \, \mathrm{d}t, \end{aligned}$$

where we have used Equation (3.10). The function $t \mapsto t^{-r/2}(1+t)^{-1/2}$ is summable for $1 < r < 2$ and so the integral limit converges in the $\|\cdot\|_1$-norm.

We conclude that $(\pi_\omega(f) \otimes 1_\nu)[\pi_\omega(g) \otimes 1_\nu, (1+D^2)^{-dr/2}]$ is trace-class for $1 < r < 2$, which completes the proof. $\square$

Remark 3.3.2. We note that when d is even, the spectral triple of Proposition 3.3.1 is even via the grading $\gamma = (-i)^{d/2}\gamma^1 \cdots \gamma^d$. One readily checks

$$\gamma^2 = (-1)^{d/2}(\gamma^1 \cdots \gamma^d)^2 = (-1)^d = 1, \qquad \gamma\gamma^j = -\gamma^j\gamma, \qquad [\gamma, \pi_\omega(f) \otimes 1_\nu] = 0.$$

Remark 3.3.3 (Summability and the product algebra). We have used the algebra of products $\pi_\omega(\mathcal{A})^2 \subset \pi_\omega(\mathcal{A})$ in the proof of Proposition 3.3.1 to obtain the summability properties of the spectral triple. If we ignore summability, then $\big(\pi_\omega(\mathcal{A}), L^2(\mathbb{R}^d) \otimes \mathbb{C}^\nu, D\big)$ is a spectral triple. The use of the product algebra is a technicality that emerges in the non-unital setting (see [CGP+15, Section 3] for a similar example). Clearly if $\mathcal{A}$ were unital, then $\pi_\omega(\mathcal{A})^2 \cong \pi_\omega(\mathcal{A})$. While we conjecture that the results in this section can be extended to the full algebra $\pi_\omega(\mathcal{A})$, we leave this issue as an open problem.

By [CGRS14, Proposition 2.14], the spectral triple of Proposition 3.3.1 can be used to define a $(d+1)$-summable Fredholm module $\big(\pi_\omega(\mathcal{A})^2, L^2(\mathbb{R}^d) \otimes \mathbb{C}^\nu, D(1+D^2)^{-1/2}\big)$.

Proposition 3.3.4. *The spectral triple* $\Big(\pi_\omega(\mathcal{A})^2, L^2(\mathbb{R}^d) \otimes \mathbb{C}^\nu, \sum_j X_j \otimes \gamma^j\Big)$ *is smoothly summable.*

Proof. Using the notation from Chapter 2.1.2, our spectral triple will be smoothly summable with spectral dimension d if we can show $\pi_\omega(\mathcal{A})^2 \cup [D, \pi_\omega(\mathcal{A})^2] \subset \mathcal{B}_1^\infty(D, d)$. We can characterise $\mathcal{B}_1^\infty(D, d)$ by the operator L, where $L(T) = (1 + D^2)^{-1/2}[D^2, T]$ for $T \in \mathrm{Dom}(L) \subset \mathcal{B}(\mathcal{H})$ and, by [CGRS14, Lemma 1.29],

$$\mathcal{B}_1^\infty(D, d) \cong \Big\{ T \in \mathcal{B}(\mathcal{H}) : \text{ for all } k \in \mathbb{N},\ L^k(T) \in \mathcal{B}_1(D, d) \Big\}.$$

The expression $L^k(T)$ involves the k-th iterated commutator of T with D^2, denoted $T^{(k)}$. Given $f \in \mathcal{A}$, we note that D^2 and $\pi_\omega(f) \otimes 1_\nu$ act diagonally on $L^2(\mathbb{R}^d) \otimes \mathbb{C}^\nu$ and so

$$(\pi_\omega(f) \otimes 1_\nu)^{(k)} = \big([|X|^2, [|X|^2, \ldots, [|X|^2, \pi_\omega(f)] \ldots]]\big) \otimes 1_\nu,$$

which we write as $\pi_\omega(f)^{(k)_X} \otimes 1_\nu$. Provided the iterated commutator is well-defined, one checks that

$$[|X|^2, \pi_\omega(f)]\psi(x) = \int_{\mathbb{R}^d} e^{-i\theta(x \wedge y)}(|x|^2 - |y|^2) f(T_{-x}\omega, y - x)\psi(y)\, \mathrm{d}y.$$

Then supposing that

$$\pi_\omega(f)^{(k)_X}\psi(x) = \int_{\mathbb{R}^d} e^{-i\theta(x \wedge y)}(|x|^2 - |y|^2)^k f(T_{-x}\omega, y - x)\psi(y)\, \mathrm{d}y, \tag{3.11}$$

one computes

$$\begin{aligned}
[|X|^2, \pi_\omega(f)^{(k)_X}]\psi(x) &= \int_{\mathbb{R}^d} e^{-i\theta(x \wedge y)}|x|^2(|x|^2 - |y|^2)^k f(T_{-x}\omega, y - x)\psi(y)\, \mathrm{d}y \\
&\quad - \int_{\mathbb{R}^d} e^{-i\theta(x \wedge y)}(|x|^2 - |y|^2)^k f(T_{-x}\omega, y - x)|y|^2\psi(y)\, \mathrm{d}y \\
&= \int_{\mathbb{R}^d} e^{-i\theta(x \wedge y)}(|x|^2 - |y|^2)^{k+1} f(T_{-x}\omega, y - x)\psi(y)\, \mathrm{d}y.
\end{aligned}$$

Provided the iterated commutator is well-defined, Equation (3.11) is true by induction. Therefore by direct calculation

$$\begin{aligned}
&\Big[L^k(\pi_\omega(f) \otimes 1_\nu)\psi\Big](x) \\
&\quad = (1 + |x|^2)^{-k/2} \int_{\mathbb{R}^d} e^{-i\theta(x \wedge y)}(|x|^2 - |y|^2)^k f(T_{-x}\omega, y - x)\psi(y)\, \mathrm{d}y \otimes 1_\nu.
\end{aligned}$$

To estimate this operator, we first note that $\sum_{j=1}^d \partial_j^{2k} f(\omega, x) = i^{2k}|x|^{2k} f(\omega, x)$ using the notation $|x|^{2k} = \sum_{j=1}^d x_j^{2k}$ and so

$$\pi_\omega\left(\sum_{j=1}^d \partial_j^{2k} f\right)\psi(x) = (-1)^k \int_{\mathbb{R}^d} e^{-i\theta(x \wedge y)}|y - x|^{2k} f(T_{-x}\omega, y - x)\psi(y)\, \mathrm{d}y.$$

Estimating the integral kernel of $L^k(\pi_\omega(f))$ on $L^2(\mathbb{R}^d)$, we let $\langle x, y\rangle$ be the standard inner product in $\mathbb{R}^d$ and compute

$$\begin{aligned}
|k_{L^k(\pi_\omega(f))}(x,y)| &= (1+|x|^2)^{-k/2}\left|(|x|^2-|y|^2)^k f(T_{-x}\omega, y-x)\right| \\
&= (1+|x|^2)^{-k/2}\left|(\langle x-y, x+y\rangle)^k f(T_{-x}\omega, y-x)\right| \\
&\leq (1+|x|^2)^{-k/2}|x+y|^k|x-y|^k|f(T_{-x}\omega, y-x)| \\
&\leq (1+|x|^2)^{-k/2}|1+|2x+y-x|\,|^k|x-y|^k|f(T_{-x}\omega, y-x)| \\
&\leq (1+|x|^2)^{-k/2}(1+|2x|)^k(1+|y-x|)^k|y-x|^k|f(T_{-x}\omega, y-x)| \\
&\leq C_k(1+|x|)^{-k}(1+|x|)^k(1+|y-x|)^k|y-x|^k|f(T_{-x}\omega, y-x)| \\
&= C_k\sum_{j=0}^k \binom{k}{j}|y-x|^{j+k}|f(T_{-x}\omega, y-x)| \\
&= C_k\sum_{j=0}^k \binom{k}{j}\left|\sum_{l=1}^d(\partial_l^{j+k}f)(T_{-x}\omega, y-x)\right| \\
&= C_k|\tilde{f}(T_{-x}\omega, y-x)|,
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality in the third line and the inequality $(1+|x+y|)^s \leq (1+|x|)^s(1+|y|)^{|s|}$ from [Gil95, Lemma 1.1.8] in the fifth line. Because $\sum_j \partial_j^{j+k} f \in C_c(\Omega\times\mathbb{R}^d)$, we see that $\tilde{f}\in C_c(\Omega\times\mathbb{R}^d)$ and so for any $k\in\mathbb{N}$ the integral kernel of $L^k(\pi_\omega(f))$ is bounded by the integral kernel of elements in $\pi_\omega(\mathcal{A})$. Therefore we can estimate the Hilbert-Schmidt norm of $L^k(\pi_\omega(f))(1+|X|^2)^{-s/4}$ by the same argument as was employed in Equation (3.8). Namely, for any $k\in\mathbb{N}$

$$\begin{aligned}
\left\|L^k(\pi_\omega(f))(1+|X|^2)^{-s/4}\right\|_2^2 &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|k_{L^k(\pi_\omega(f))}(x,y)|^2(1+|y|^2)^{-s/2}\,\mathrm{d}x\,\mathrm{d}y \\
&\leq C_k\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\tilde{f}(T_{-x}\omega, y-x)|^2(1+|y|^2)^{-s/2}\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \tilde{C}_k\int_{\mathbb{R}^d}(1+|y|^2)^{-s/2}\,\mathrm{d}y,
\end{aligned}$$

which is finite for any $s>d$. Therefore $L^k(\pi_\omega(f))(1+|X|^2)^{-s/4}$ is Hilbert-Schmidt by [RS72, Theorem VI.23] and given any $f,g\in C_c(\Omega\times\mathbb{R}^d)$,

$$(1+|X|^2)^{-s/4}L^{k_1}(\pi_\omega(f^*))L^{k_2}(\pi_\omega(g))(1+|X|^2)^{-s/4}\in\mathcal{L}^1[L^2(\mathbb{R}^d)]$$

for any $k_1,k_2\in\mathbb{N}$ and $s>d$ [RS72, Theorem VI.22]. Under the notation of Chapter 2.1.2, we obtain that $L^k(\pi_\omega(f))\otimes 1_\nu\in\mathrm{Dom}(\varphi_s)^{1/2}$ for $s>d$, which then implies that $L^k[\pi_\omega(f)]\otimes 1_\nu\in\mathcal{B}_2(D,d)$ for any $k\in\mathbb{N}$ and $f\in C_c(\Omega\times\mathbb{R}^d)$. We may then say that products $L^{k_1}[\pi_\omega(f)]L^{k_2}[\pi_\omega(g)]\otimes 1_\nu\in\mathcal{B}_1(D,d)$ for any $k_1,k_2\in\mathbb{N}$ and $f,g\in\mathcal{A}$. Hence we obtain that $\pi_\omega(\mathcal{A})^2\otimes 1_\nu\subset\mathcal{B}_1^\infty(D,d)$.

Next we consider $L^k([D,\, \pi_\omega(fg)\otimes 1_\nu])$ and note that

$$[D,\, \pi_\omega(fg)\otimes 1_\nu] = \sum_{j=1}^{d}[X_j,\, \pi_\omega(fg)]\otimes\gamma^j = i\sum_{j=1}^{d}\partial_j(fg)\otimes\gamma^j$$

by Lemma 3.2.22. Because $\partial_j(fg)\in\pi_\omega(\mathcal{A})^2$, $[D,\, \pi_\omega(\mathcal{A})^2]\subset\mathcal{B}_1^\infty(D,d)$ by the same argument as $\pi_\omega(\mathcal{A})^2$ and we are done. □

Lemma 3.3.5. *The K-homology class of the spectral triple from Proposition 3.3.1 is almost surely independent of the choice of $\omega\in\Omega$ coming from the representation π_ω of $\mathcal{A}$.*

Proof. We compare different representations of the disorder parameter by the covariance relation (Proposition 3.2.12), which gives the unitarily equivalent spectral triple

$$\widehat{S}^a\lambda_\omega\widehat{S}^{-a} = \left(\pi_{T_a\omega}(\mathcal{A})^2,\, L^2(\mathbb{R}^d)\otimes\mathbb{C}^\nu,\, \sum_{j=1}^{d}(X_j-a_j)\otimes\gamma^j,\, \gamma\right)$$

as $\widehat{S}^aX_j\widehat{S}^{-a} = X_j - a_j$. The straight line homotopy $D_t = \sum_j(X_j - t\alpha_j)\otimes\gamma^j$ for $t\in[0,1]$ shows that $[\lambda_\omega] = [\lambda_{T_\alpha\omega}]$ at the level of K-homology classes. As the action of $\mathbb{R}^d$ on Ω is taken to be ergodic, the Kasparov class almost surely independent of the choice of $\omega\in\Omega$. □

Example 3.3.6 (Quantum Hall spectral triple). Let us again consider the case $d=2$ and the quantum Hall system. We choose explicit Clifford generators so that

$$\left(\pi_\omega(\mathcal{A})^2, L^2(\mathbb{R}^2)\oplus L^2(\mathbb{R}^2), D = \begin{pmatrix} 0 & X_1 - iX_2 \\ X_1 + iX_2 & 0\end{pmatrix}, \gamma = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)$$

is a smoothly summable spectral triple with spectral dimension 2. We recognise this spectral triple as the unbounded version of the Fredholm module studied in [BvS94].

Because the spectral triple $(\pi_\omega(\mathcal{A})^2, L^2(\mathbb{R}^d)\otimes\mathbb{C}^\nu, D)$ is smoothly summable, the algebra $\pi_\omega(\mathcal{A})^2$ has a completion $\mathcal{C} = \pi_\omega(\mathcal{A})^2_{\delta,\varphi}\subset\mathcal{B}_1^\infty(D,d)$ in the δ-φ topology (cf. Equation (2.1)). The tuple $\big(\mathcal{C}, L^2(\mathbb{R}^d)\otimes\mathbb{C}^\nu, D\big)$ is a smoothly summable spectral triple with spectral dimension d by Proposition 2.1.19 with $\mathcal{C}$ Fréchet and stable under the holomorphic functional calculus. Furthermore, and any index formulas on $\pi_\omega(\mathcal{A})^2$ will extend to the completion $\mathcal{C}$.

To consider the index pairing of the class represented by the spectral triple of Proposition 3.3.1, we employ the double construction introduced in Definition 2.1.21. Because the spectral triple of Proposition 3.3.1 is smoothly summable, we can apply the (non-unital) local index formula from Theorem 2.1.28 and 2.1.29 to compute the index pairing of unitaries and projections in $\pi_\omega(\mathcal{A}^2)^\sim$ with the spectral triple. It is our goal to relate the index formula to the higher-dimensional Chern numbers studied by [PLB13, PS14], with Bellissard's cocycle formula for the Hall conductance a special case. We first simplify the residue-trace terms that appear in the local index formula.

Lemma 3.3.7. *Let $\mathcal{T}$ be the trace from Definition 3.2.15. Then*

$$\mathcal{T}(fg) = \frac{1}{\mathrm{Vol}_{d-1}(S^{d-1})} \operatorname*{res}_{s=d} \mathrm{Tr}\Big(\pi_\omega(fg)(1+|X|^2)^{-s/2}\Big)$$

for any $f, g \in (C_c(\Omega \times \mathbb{R}^d))$ and almost all $\omega \in \Omega$.

Proof. We recall that the algebraic trace is given by

$$\mathcal{T}(fg) = \int_\Omega (fg)(\omega, 0)\,\mathrm{d}\mathbf{P}(\omega).$$

Because the spectral triple $(\pi_\omega(\mathcal{A})^2, \mathcal{H}, D)$ has spectral dimension d, we can take the trace of $\pi_\omega(fg)(1+|X|^2)^{-s/2}$ by integrating along the diagonal of the integral kernel for $s > d$, where

$$\mathrm{Tr}\Big(\pi_\omega(fg)(1+|X|^2)^{-s/2}\Big) = \int_{\mathbb{R}^d} (fg)(T_{-x}\omega, 0)(1+|x|^2)^{-s/2}\,\mathrm{d}x.$$

We denote by $G(\omega, s) = \mathrm{Tr}\big(\pi_\omega(fg)(1+|X|^2)^{-s/2}\big)$ for $\Re(s) > d$. For any $a \in \mathbb{R}^d$, we compute that

$$\begin{aligned}
G(T_a\omega, s) &= \int_{\mathbb{R}^d} (fg)(T_{a-x}\omega, 0)(1+|x|^2)^{-s/2}\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} (fg)(T_{-u}\omega, 0)(1+|a+u|^2)^{-s/2}\,\mathrm{d}u \\
&= \int_{\mathbb{R}^d} (fg)(T_{-u}\omega, 0)(1+|u|^2)^{-s/2}\,\mathrm{d}u \\
&\quad + \int_{\mathbb{R}^d} (fg)(T_{-u}\omega, 0)\Big((1+|a+u|^2)^{-s/2} - (1+|u|^2)^{-s/2}\Big)\,\mathrm{d}u.
\end{aligned}$$

We use the Laplace transform to rewrite

$$\begin{aligned}
G(T_a\omega, s) - G(\omega, s) &= \int_{\mathbb{R}^d} (fg)(T_{-u}\omega, 0)\Big((1+|a+u|^2)^{-s/2} - (1+|u|^2)^{-s/2}\Big)\,\mathrm{d}u \\
&= \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_{\mathbb{R}^d} (fg)(T_{-u}\omega, 0) \int_0^\infty t^{s/2-1} \left(e^{-t(1+|a+u|^2)} - e^{-t(1+|u|^2)}\right) \mathrm{d}t\,\mathrm{d}u \\
&= \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_{\mathbb{R}^d} (fg)(T_{-u}\omega, 0) \int_0^\infty t^{s/2-1} \int_0^a \nabla_b \left(e^{-t(1+|b+u|^2)}\right) \mathrm{d}b\,\mathrm{d}t\,\mathrm{d}u.
\end{aligned}$$

Taking the derivative in b we find that (using multi-index notation)

$$\begin{aligned}
&G(T_a\omega, s) - G(\omega, s)\\
&\qquad = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_{\mathbb{R}^d} (fg)(T_{-u}\omega, 0) \int_0^\infty t^{s/2-1} \int_0^a (-2t|b+u|) e^{-t(1+|b+u|^2)} \, \mathrm{d}b \, \mathrm{d}t \, \mathrm{d}u\\
&\qquad = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_{\mathbb{R}^d} (fg)(T_{-u}\omega, 0) \int_0^\infty t^{s/2} \int_0^a (-2|b+u|) e^{-t(1+|b+u|^2)} \, \mathrm{d}b \, \mathrm{d}t \, \mathrm{d}u\\
&\qquad = \frac{\Gamma\left(\frac{s}{2}+1\right)}{\Gamma\left(\frac{s}{2}\right)} \int_{\mathbb{R}^d} (fg)(T_{-u}\omega, 0) \int_0^a (-2|b+u|)(1+|b+u|^2)^{-s/2-1} \, \mathrm{d}b \, \mathrm{d}u\\
&\qquad = -s \int_{\mathbb{R}^d} (fg)(T_{-u}\omega, 0) \int_0^a |b+u|(1+|b+u|^2)^{-s/2-1} \, \mathrm{d}b \, \mathrm{d}u.
\end{aligned}$$

We note that the last integral will coverge for $\Re(s) > d-1$. The difference $G(T_a\omega, s) - G(\omega, s)$ is holomorphic for $\Re(s) > d-1$. To prove this claim, we first compute

$$\begin{aligned}
&\frac{1}{h}\left(G(T_a\omega, s+h) - G(\omega, s+h) - G(T_a\omega, s) + G(\omega, s)\right)\\
&= -\int_{\mathbb{R}^d} (fg)(T_{-u}\omega, 0) \int_0^a |b+u| \left(\frac{(s+h)(1+|b+u|^2)^{-\frac{h}{2}} - s}{h} \right) (1+|b+u|^2)^{-\frac{s}{2}-1} \, \mathrm{d}b \, \mathrm{d}u
\end{aligned}$$

and compare to the formal derivative

$$-\int_{\mathbb{R}^d} (fg)(T_{-u}\omega, 0) \int_0^a |b+u| \left(1 - \frac{1}{2}\ln(1+|b+u|^2)\right)(1+|b+u|^2)^{-\frac{s}{2}-1} \, \mathrm{d}b \, \mathrm{d}u$$

whose integral will also converge for $\Re(s) > d-1$. We then check that

$$\begin{aligned}
\lim_{h\to 0} \frac{(s+h)(1+|b+u|^2)^{-\frac{h}{2}} - s}{h} &= \lim_{h\to 0} \frac{(s+h)\exp\left(-\frac{h}{2}\ln(1+|b+u|^2)\right) - s}{h}\\
&= \lim_{h\to 0} \frac{(s+h)\left(1 - \frac{h}{2}\ln(1+|b+u|^2) + \mathcal{O}(h^2)\right) - s}{h}\\
&= 1 - \frac{1}{2}\ln(1+|b+u|^2).
\end{aligned}$$

Therefore $G(T_a\omega, s) - G(\omega, s)$ has a well-defined complex derivative for $\Re(s) > d-1$.

Next we fix $\omega_0 \in \Omega$ and consider the function $\omega \mapsto G(\omega, s) - G(\omega_0, s)$. Integrating yields

$$\int_\Omega (G(\omega, s) - G(\omega_0, s)) \, \mathrm{d}\mathbf{P}(\omega) = \int_\Omega G(\omega, s) \, \mathrm{d}\mathbf{P}(\Omega) - G(\omega_0, s)$$

as $\mathbf{P}(\Omega) = 1$. For $\Re(s) > d$,

$$\begin{aligned}
\int_\Omega G(\omega, s) \, \mathrm{d}\mathbf{P}(\Omega) &= \int_\Omega \int_{\mathbb{R}^d} (fg)(T_{-x}\omega, 0)(1+|x|^2)^{-s/2} \, \mathrm{d}x \, \mathrm{d}\mathbf{P}(\omega)\\
&= \int_\Omega \int_{\mathbb{R}^d} (fg)(\omega, 0)(1+|x|^2)^{-s/2} \, \mathrm{d}x \, \mathrm{d}\mathbf{P}(\omega)
\end{aligned}$$

where we have used the invariance of the action of T on Ω to make a substitution. By switching to polar coordinates, we can explicitly compute that

$$\begin{aligned}\int_\Omega G(\omega,s)\,\mathrm{d}\mathbf{P}(\Omega) &= \mathrm{Vol}_{d-1}(S^{d-1})\int_\Omega (fg)(\omega,0)\,\mathrm{d}\mathbf{P}(\omega)\int_0^\infty (1+r^2)^{-s/2}r^{d-1}\,\mathrm{d}r\\ &= \mathcal{T}(fg)\,\mathrm{Vol}_{d-1}(S^{d-1})\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{s-d}{2}\right)}{2\Gamma\left(\frac{s}{2}\right)}.\end{aligned}$$

As $G(\omega,s)-G(\omega_0,s)$ is holomorphic in a neighbourhood of $s=d$, we can say that

$$\mathcal{T}(fg)\,\mathrm{Vol}_{d-1}(S^{d-1})\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{s-d}{2}\right)}{2\Gamma\left(\frac{s}{2}\right)} = G(s,\omega_0)+g(s) \tag{3.12}$$

with $g(s)$ holomorphic in a neighbourhood of $s=d$. By the functional equation for the Γ-function, the left hand side of Equation (3.12) has an analytic continuation to the complex plane with a simple pole at $s=d$. Because g analytically extends to a neighbourhood of $s=d$, we conclude that $G(\omega_0,s)$ analytically extends to a neighbourhood of $s=d$ such that $(s-d)G(\omega_0,s)$ is holomorphic at $s=d$ for all $\omega_0\in\Omega$. Computing the residue of $G(\omega_0,s)$,

$$\begin{aligned}\operatorname*{res}_{s=d}\mathrm{Tr}\Big(\pi_{\omega_0}(fg)(1+|X|^2)^{-s/2}\Big) &= \operatorname*{res}_{s=d}\mathcal{T}(fg)\,\mathrm{Vol}_{d-1}(S^{d-1})\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{s-d}{2}\right)}{2\Gamma\left(\frac{s}{2}\right)}\\ &= \mathcal{T}(fg)\,\mathrm{Vol}_{d-1}(S^{d-1})\end{aligned}$$

as required. Because the $G(T_a\omega,s)-G(\omega,s)$ is holomorphic at $s=d$ for any $a\in\mathbb{R}^d$, the residue trace is almost-surely independent of the choice of $\omega\in\Omega$. □

Corollary 3.3.8. *The trace per unit volume of $\pi_\omega(fg)$ can be computed with the residue trace*

$$\frac{1}{\mathrm{Vol}_{d-1}(S^{d-1})}\operatorname*{res}_{s=d}\mathrm{Tr}\Big(\pi_{\omega'}(fg)(1+|X|^2)^{-s/2}\Big)$$

for almost any choice of ω, $\omega'\in\Omega$.

Proof. Apply Proposition 3.2.18 to the results in Lemma 3.3.7. □

Our smoothly summable spectral triple allows us to use the local index formula and Lemma 3.3.7 means we can relate the result back to physical quantities. To compute the index pairing we separate into the cases where d is odd or even.

3.3.2 The odd Chern numbers

We assume that $d=2n+1$ for some $n\in\mathbb{N}$. Our aim is to use the local index formula to derive computable expressions for the index pairing. We state the main result.

Theorem 3.3.9. *Let u_ω be a unitary in $M_q[\pi_\omega(\mathcal{A}^2)^\sim]$ and $[X_{\mathrm{odd}}]$ the K-homology class represented by the spectral triple of Proposition 3.3.1 in odd dimensions. Then the index pairing is given by the formula*

$$\langle [u_\omega], [X_{\mathrm{odd}}]\rangle = \lambda_d \sum_{\sigma\in S_d} (-1)^\sigma \left(\mathrm{Tr}_{\mathbb{C}^q}\otimes\mathcal{T}\right)\left(\prod_{i=1}^d u_\omega^* \partial_{\sigma(i)}(u_\omega)\right),$$

where $\lambda_d = -\frac{2^{(d+1)/2}\pi^{(d-1)/2}((d-1)/2)!}{i^{(d+1)/2}d!}$, $\mathrm{Tr}_{\mathbb{C}^q}$ is the matrix trace on $\mathbb{C}^q$, $\mathcal{T}$ is the trace per unit volume on $L^2(\mathbb{R}^d)$ and S_d is the permutation group on d letters. The result is almost surely independent of the choice of $\omega\in\Omega$.

We focus on the case $q=1$ and can extend to matrices by the method outlined in Chapter 2.1.3. Because the spectral triple of Proposition 3.3.1 is smoothly summable with spectral dimension d, the odd local index formula (Theorem 2.1.28) gives that

$$\langle [u_\omega], [(\pi_\omega(\mathcal{A}^2), \mathcal{H}, D)]\rangle = \frac{-1}{\sqrt{2\pi i}} \operatorname*{res}_{r=(1-d)/2} \sum_{m=1,\mathrm{odd}}^{2N-1} \phi_m^r(\mathrm{Ch}^m(u_\omega)),$$

where u_ω is a unitary in the unitisation of $\pi_\omega(\mathcal{A}^2)$, $N=\lfloor d/2\rfloor+1$ and

$$\mathrm{Ch}^{2n+1}(u) = (-1)^n n!\, u^*\otimes u\otimes u^*\otimes\cdots\otimes u \qquad (2n+2 \text{ entries}).$$

The functional ϕ_m^r is the resolvent cocycle from Definition 2.1.27. To compute the index pairing we make the following important observation.

Lemma 3.3.10 ([BCP+06], §11.1)**.** *The only term in the sum $\sum_{m=1,\mathrm{odd}}^{2N-1}\phi_m^r(\mathrm{Ch}^m(u_\omega))$ that contributes to the index pairing is the term with $m=d$.*

Proof. We first note that the spinor trace on Clifford generators is given by

$$\mathrm{Tr}_{\mathbb{C}^\nu}(i^d\gamma^1\cdots\gamma^d) = (-i)^{\lfloor (d+1)/2\rfloor}2^{\lfloor (d-1)/2\rfloor} \tag{3.13}$$

and will vanish on any product of k Clifford generators with $0<k<d$. The resolvent cocycle involves the spinor trace of terms

$$a_0R_s(\lambda)[D,a_1]R_s(\lambda)\cdots[D,a_m]R_s(\lambda), \qquad R_s(\lambda) = (\lambda-(1+s^2+D^2))^{-1},$$

for $a_0,\ldots,a_m\in\pi_\omega(\mathcal{A})^2$. We note that $[D,a_l] = i\sum_{j=1}^d \partial_j(a_l)\otimes\gamma^j$ and $R_s(\lambda) = (\lambda-(1+s^2+|X|^2))^{-1}\otimes 1_\nu$ in the spinor representation. Therefore the product $a_0R_s(\lambda)[D,a_1]R_s(\lambda)\cdots[D,a_m]R_s(\lambda)$ will be in the span of m Clifford generators acting on $L^2(\mathbb{R}^d)\otimes\mathbb{C}^\nu$ for $0<m<d$. Furthermore, our trace estimates ensure that each spinor component

$$\int_\ell \lambda^{-d/2-r}a_0(\lambda-(1+s^2+|X|^2))^{-1}\partial_{j_1}(a_1)\cdots\partial_{j_m}(a_m)(\lambda-(1+s^2+|X|^2))^{-1}\,\mathrm{d}\lambda$$

is trace-class for $a_0, \ldots, a_m \in \pi_\omega(\mathcal{A})^2$ and $\Re(r)$ sufficiently large. Hence for $0 < m < d$, the spinor trace will vanish for $\Re(r)$ large. Similar to the proof of Lemma 3.3.7, we can analytically extend $\phi_m^r(\mathrm{Ch}^m(u_\omega))$ as a function holomorphic in a neigbourhood of $r = (1-d)/2$ for $0 < m < d$. Thus $\phi_m^r(\mathrm{Ch}^m(u_\omega))$ does not contribute to the index pairing for $0 < m < d$. □

Proof of Theorem 3.3.9. Lemma 3.3.10 simplifies what we have to do substantially. The index is given by the expression

$$\langle [u_\omega], [(\pi_\omega(\mathcal{A})^2, \mathcal{H}, D)] \rangle = \frac{-1}{\sqrt{2\pi i}} \operatorname*{res}_{r=(1-d)/2} \phi_d^r(\mathrm{Ch}^d(u_\omega))$$

Therefore we need to compute the residue at $r = (d-1)/2$ of

$$\frac{(-1)^{n+1} n! \eta_d}{(2\pi i)^{3/2}} \int_0^\infty s^m \operatorname{Tr}\Big(\int_\ell \lambda^{-p/2-r} u_\omega^* R_s(\lambda)[D, u_\omega] R_s(\lambda)[D, u_\omega^*] \cdots [D, u_\omega] R_s(\lambda)\, \mathrm{d}\lambda \Big) \mathrm{d}s,$$

where $d = 2n+1$. To compute this residue we move all terms $R_s(\lambda)$ to the right, which can be done up to a function holomorphic at $r = (1-d)/2$ by an argument analogous to the proof of Lemma 3.3.7. This allows us to take the Cauchy integral. We then observe that $\underbrace{[D, u_\omega][D, u_\omega^*] \cdots [D, u_\omega]}_{d \text{ terms}} \in \pi_\omega(\mathcal{A}^2)^\sim \otimes 1_\nu$, so Lemma 3.3.7 implies that the zeta function

$$\operatorname{Tr}\Big(u_\omega^* [D, u_\omega][D, u_\omega^*] \cdots [D, u_\omega](1 + D^2)^{-z/2} \Big)$$

has at worst a simple pole at $\Re(z) = d$. Therefore we can explicitly compute

$$\begin{aligned} &\frac{-1}{\sqrt{2\pi i}} \operatorname*{res}_{r=(1-d)/2} \phi_d^r(\mathrm{Ch}^d(u_\omega)) \\ &\qquad = (-1)^{n+1} n! \frac{1}{d!} \tilde{\sigma}_{n,0} \operatorname*{res}_{z=d} \operatorname{Tr}\Big(u_\omega^* [D, u_\omega][D, u_\omega^*] \cdots [D, u_\omega](1 + D^2)^{-z/2} \Big). \end{aligned}$$

Recall that under the notation from Chapter 2.1.4, the numbers $\tilde{\sigma}_{n,j}$ are defined by the formula

$$\prod_{j=0}^{n-1} (z + j + 1/2) = \sum_{j=0}^{n} z^j \tilde{\sigma}_{n,j}.$$

Hence the number $\tilde{\sigma}_{n,0}$ is the coefficient of 1 in the product $\prod_{l=0}^{n-1}(z + l + 1/2)$. This is the product of all the non-z terms, which can be written as

$$(1/2)(3/2) \cdots (n - 1/2) = \frac{1}{\sqrt{\pi}} \Gamma(d/2).$$

Putting this back together, our index pairing can be written as

$$\mathrm{Index}(P \hat{u}_\omega P) = (-1)^{n+1} \frac{n! \Gamma(d/2)}{d! \sqrt{\pi}} \operatorname*{res}_{z=d} \operatorname{Tr}\Big(u_\omega^* [D, u_\omega][D, u_\omega^*] \cdots [D, u_\omega](1 + D^2)^{-z/2} \Big).$$

We make use of the identity $[D, u_\omega^*] = -u_\omega^*[D, u_\omega]u_\omega^*$, which allows us to rewrite

$$\begin{aligned} u_\omega^* \underbrace{[D, u_\omega][D, u_\omega^*] \cdots [D, u_\omega]}_{d=2n+1 \text{ terms}} &= (-1)^n u_\omega^*[D, u_\omega]u_\omega^*[D, u_\omega]u_\omega^* \cdots u_\omega^*[D, u_\omega] \\ &= (-1)^n \left(u_\omega^*[D, u_\omega]\right)^d . \end{aligned}$$

Recall that $[D, u_\omega] = \sum_{j=1}^d [X_j, u_\omega] \otimes \gamma^j = i \sum_{j=1}^d \partial_j(u_\omega) \otimes \gamma^j$ so we have the relation $u_\omega^*[D, u_\omega] = i \sum_{j=1}^d u_\omega^* \partial_j(u_\omega) \otimes \gamma^j$. Taking the d-th power

$$\left(u_\omega^*[D, u_\omega]\right)^d = i^d \sum_{J=(j_1,\ldots,j_d)} u_\omega^* \partial_{j_1}(u_\omega) \cdots u_\omega^* \partial_{j_d}(u_\omega) \otimes \gamma^{j_1} \cdots \gamma^{j_d}$$

where the sum is extended over all multi-indices J. Note that every term in the sum is a multiple of the volume form and so has a non-zero spinor trace. Writing this product in terms of permutations,

$$\left(u_\omega^*[D, u_\omega]\right)^d = i^d \sum_{\sigma \in S_d} (-1)^\sigma \prod_{i=1}^d u_\omega^* \partial_{\sigma(i)}(u_\omega) \otimes \gamma^{\sigma(i)},$$

where S_d is the permutation group of d letters.

Let's put what we have back together.

$$\begin{aligned} \mathrm{Index}(P\hat{u}_\omega P) &= (-1)^{n+1} \frac{n!\Gamma(d/2)}{d!\sqrt{\pi}} \operatorname*{res}_{z=d} \mathrm{Tr}\Big(u_\omega^*[D, u_\omega][D, u_\omega^*] \cdots [D, u_\omega](1 + D^2)^{-z/2}\Big) \\ &= -\frac{n!\Gamma(d/2)}{d!\sqrt{\pi}} \operatorname*{res}_{z=d} \mathrm{Tr}\left[i^d \left(\sum_{\sigma \in S_d} (-1)^\sigma \prod_{i=1}^d u_\omega^* \partial_{\sigma(i)}(u_\omega) \otimes \gamma^{\sigma(i)} \right) (1 + D^2)^{-z/2} \right] \\ &= -\frac{n!\Gamma(d/2) 2^{\lfloor (d-1)/2 \rfloor}}{i^{\lfloor (d+1)/2 \rfloor} d! \sqrt{\pi}} \operatorname*{res}_{z=d} \mathrm{Tr}_{L^2(\mathbb{R}^d)} \left(\sum_{\sigma \in S_d} (-1)^\sigma \prod_{i=1}^d u_\omega^* \partial_{\sigma(i)}(u_\omega)(1 + |X|^2)^{-z/2} \right), \end{aligned}$$

where we have used Equation (3.13). Because $\prod_{i=1}^d u_\omega^* \partial_{\sigma(i)}(u_\omega) \in \pi_\omega(\mathcal{A})^2$ for any $d \geq 1$, we can apply Lemma 3.3.7 and Corollary 3.3.8 to reduce the formula to

$$\mathrm{Index}(P\hat{u}_\omega P) = -\frac{n!\Gamma(d/2)\mathrm{Vol}_{d-1}(S^{d-1}) 2^{\lfloor (d-1)/2 \rfloor}}{i^{\lfloor (d+1)/2 \rfloor} d! \sqrt{\pi}} \sum_{\sigma \in S_d} (-1)^\sigma \mathcal{T}\left(\prod_{i=1}^d u_\omega^* \partial_{\sigma(i)}(u_\omega) \right),$$

Corollary 3.3.8 also ensures that $\mathrm{Index}(P\hat{u}_\omega P)$ is almost surely independent of ω. We use the equation $\mathrm{Vol}_{d-1}(S^{d-1}) = \frac{d\pi^{d/2}}{\Gamma(d/2+1)}$ to simplify

$$\frac{n!\Gamma(d/2)\mathrm{Vol}_{d-1}(S^{d-1}) 2^{\lfloor (d-1)/2 \rfloor}}{i^{\lfloor (d+1)/2 \rfloor} d! \sqrt{\pi}} = \frac{2(2\pi)^n n!}{i^{n+1}(2n+1)!},$$

for $d = 2n + 1$, and therefore

$$\mathrm{Index}(P\hat{u}_\omega P) = \lambda_d \sum_{\sigma \in S_d} (-1)^\sigma \mathcal{T}\left(\prod_{i=1}^d u_\omega^* \partial_{\sigma(i)}(u_\omega) \right)$$

with $\lambda_{2n+1} = \frac{-2(2\pi)^n n!}{i^{n+1}(2n+1)!}$. □

We think of Theorem 3.3.9 as a continuous and non-unital version of the higher-order (odd) Chern numbers as studied by Prodan and Schulz-Baldes [PS14].

Remark 3.3.11. The reduction of the index pairing to the residue of $\phi_d^r(\mathrm{Ch}^d(u_\omega))$ gives an expression for the index in terms of the functional ϕ_d appearing in the residue cocycle (Definition 2.1.26). We suspect that the spectral triple $\big(\pi_\omega(\mathcal{A})^2, L^2(\mathbb{R}^d)\otimes\mathbb{C}^\nu, D\big)$ has isolated spectral dimension, which would allow us to use the residue cocycle directly. Because the final result does not require isolated spectral dimension, we have not pursued this question further.

3.3.3 The even Chern numbers

We now consider the case $d = 2n$. We will find that many of the simplifications we made in the odd case also occur in the even-dimensional setting. It suffices to take the pairing with a projection $p_\omega \in M_q[\pi_\omega(\mathcal{A})^\sim] \supset M_q[\pi_\omega(\mathcal{A}^2)^\sim]$ as $p_\omega = p_\omega^2 \in M_q[\pi_\omega(\mathcal{A}^2)^\sim]$. We recall the even local index formula (Theorem 2.1.29),

$$\langle [p_\omega] - [1_{p_\omega}], [(\pi_\omega(\mathcal{A})^2, \mathcal{H}, D)]\rangle = \operatorname*{res}_{r=(1-d)/2} \sum_{m=0,\mathrm{even}}^{d} \phi_m^r(\mathrm{Ch}^m(p_\omega) - \mathrm{Ch}^m(1_{p_\omega})),$$

$$\mathrm{Ch}^{2n}(p_\omega) = (-1)^n \frac{(2n)!}{2(n!)}(2p_\omega - 1)\otimes p_\omega^{\otimes 2n}, \qquad \mathrm{Ch}^0(p_\omega) = p_\omega,$$

where ϕ_m^r is the resolvent cocycle and $1_{p_\omega} = \pi^q(p_\omega)$ for $\pi^q(b) : M_q(\pi_\omega(\mathcal{A})^\sim) \to M_q(\mathbb{C})$ the quotient map coming from the minimal unitisation of $C_c(\Omega\times\mathbb{R}^d)$.

Theorem 3.3.12. *Let $p_\omega \in M_q(\pi_\omega(\mathcal{A})^\sim)$ be a projection and $[X_{\mathrm{even}}]$ the even K-homology class represented by the spectral triple of Proposition 3.3.1 in even dimensions. Then the index pairing can be expressed by the formula*

$$\langle [p_\omega] - [1_{p_\omega}], [X_{\mathrm{even}}]\rangle = \frac{(2\pi i)^{d/2}}{(d/2)!}\sum_{\sigma\in S_d}(-1)^\sigma\,(\mathrm{Tr}_{\mathbb{C}^q}\otimes\mathcal{T})\left(p_\omega \prod_{i=1}^{d}\partial_{\sigma(i)}(p_\omega)\right),$$

where S_d is the permuation group of d letters. The result is almost surely independent of the choice of $\omega\in\Omega$.

Like the setting with d odd, our computation can be substantially simplified with some preliminary results. We again focus on the case $q = 1$, and refer to Chapter 2.1.3 for the extension to matrices $p_\omega \in M_q(\pi_\omega(\mathcal{A})^\sim)$.

Lemma 3.3.13. *The index pairing reduces to the computation* $\operatorname*{res}_{r=(d-1)/2} \phi_d^r(\mathrm{Ch}^d(p_\omega))$.

Proof. We first note that for $m > 0$, $\phi_m^r(\mathrm{Ch}(1_{p_\omega})) = 0$ as these terms involve the commutators $[D, 1_{p_\omega}] = 0$. The proof used in Lemma 3.3.10 also holds here to show that $\phi^r_m(\mathrm{Ch}^m(p_\omega))$ does not contribute to the index pairing for $0 < m < d$. The $m = 0$ term is of the form

$$\phi^r_0(p_\omega - 1_{p_\omega}) = 2\int_0^\infty \mathrm{Tr}\Big(\gamma(p_\omega - 1_{p_\omega})(1 + s^2 + D^2)^{-d/2-r}\Big)\,\mathrm{d}s,$$

Because there is a symmetry of the operator $(p_\omega - 1_{p_\omega})(1 + s^2 + D^2)^{-d/2-r}$ between the ± 1 eigenspaces of $\gamma = (-i)^{d/2}\gamma^1\gamma^2\cdots\gamma^d$, the graded trace will vanish provided $\Re(r)$ is sufficiently large. Therefore $\phi^r_0(p_\omega - 1_{p_\omega})$ analytically continues as a function holomorphic in a neighbourhood of $r = (1-d)/2$, hence the residue will vanish. □

As a side-remark, one finds that $\phi^r_0(p_\omega)$ is in general not well-defined for non-unital algebras. Taking the pairing with $\phi^r_0(p_\omega - 1_{p_\omega})$ is an important change we need to make in this setting.

Proof of Theorem 3.3.12. Lemma 3.3.13 implies our index computation is reduced to

$$\big\langle [p_\omega] - [1_{p_\omega}], \big[(\pi_\omega(\mathcal{A})^2, \mathcal{H}, D)\big]\big\rangle = \operatorname*{res}_{r=(1-d)/2} \phi^r_d(\mathrm{Ch}^d(p_\omega)),$$

which is a residue at $r = (1-d)/2$ of the term

$$\frac{(-1)^{d/2} d!\eta_d}{(d/2)!2\pi i}\int_0^\infty s^m \mathrm{Tr}\bigg(\gamma\int_\ell \lambda^{-p/2-r}(2p_\omega - 1)R_s(\lambda)[D,p_\omega]R_s(\lambda)\cdots[D,p_\omega]R_s(\lambda)\,\mathrm{d}\lambda\bigg)\mathrm{d}s.$$

Like the case of d odd, we can move the resolvent terms to the right up to a holomorphic error in order to take the Cauchy integral. Lemma 3.3.7 also implies that $\mathrm{Tr}\big(\gamma(2p_\omega - 1)([D,p_\omega])^d(1 + D^2)^{-s/2}\big)$ has at worst a simple pole at $s = d$. Computing the residue explicitly,

$$\operatorname*{res}_{r=(1-d)/2} \phi^r_d(\mathrm{Ch}^d(p_\omega)) = \frac{(-1)^{d/2}}{2((d/2)!)}\sigma_{d/2,1} \operatorname*{res}_{z=d} \mathrm{Tr}\Big(\gamma(2p_\omega - 1)([D,p_\omega])^d(1 + D^2)^{-z/2}\Big),$$

where $\sigma_{d/2,1}$ is the coefficient of z in $\prod_{j=0}^{d/2-1}(z+j)$. One finds that $\sigma_{d/2,1} = ((d/2)-1)!$. Putting these results back together,

$$\mathrm{Index}(\hat{p}_\omega D_+ \hat{p}_\omega) = (-1)^{d/2}\frac{1}{d}\operatorname*{res}_{z=d}\mathrm{Tr}\Big(\gamma(2p_\omega - 1)([D,p_\omega])^d(1 + D^2)^{-z/2}\Big).$$

Next we claim that $\mathrm{Tr}\big(\gamma([D,p_\omega])^d(1 + D^2)^{-z/2}\big) = 0$ for $\Re(z) > d$. To see this, we compute

$$\begin{aligned}[D,p_\omega]^d &= \sum_{J=(j_1,\ldots,j_d)} [X_{j_1},p]\cdots[X_{j_d},p_\omega] \otimes \gamma^{j_1}\cdots\gamma^{j_d} \\ &= C\sum_{J=(j_1,\ldots,j_d)} [X_{j_1},p_\omega]\cdots[X_{j_d},p_\omega] \otimes 1_\nu.\end{aligned}$$

Because $\sum_{J=(j_1,\ldots,j_d)}[X_{j_1},p]\cdots[X_{j_d},p_\omega]$ is symmetric with respect to the ±1 eigenspaces of γ, the spinor trace $\mathrm{Tr}(\gamma[D,p_\omega]^d(1+D^2)^{-z/2})$ will vanish for $\Re(z)>d$. The zeta function $\mathrm{Tr}(\gamma[D,p_\omega]^d(1+D^2)^{-z/2})$ analytically continues as a function holomorphic in a neighbourhood of $z=d$ and its residue does not contribute to the index.

We know that $[D,p_\omega]=\sum_{j=1}^d[X_j,p_\omega]\otimes\gamma^j=i\sum_{j=1}^d\partial_j(p_\omega)\otimes\gamma^j$ and so

$$p_\omega([D,p])^d=i^dp_\omega\sum_{J=(j_1,\ldots,j_d)}\partial_{j_1}(p_\omega)\cdots\partial_{j_d}(p_\omega)\otimes\gamma^{j_1}\cdots\gamma^{j_d}$$

for the multi-index J. We express the product in terms of permutation groups as

$$p_\omega([D,p_\omega])^d=(-1)^{d/2}p_\omega\sum_{\sigma\in S_d}(-1)^\sigma\prod_{i=1}^d\partial_{\sigma(i)}(p_\omega)\otimes\gamma^{\sigma(i)}.$$

Therefore, using the relation $\mathrm{Tr}_{\mathbb{C}^\nu}(\gamma\gamma^1\cdots\gamma^d)=i^{d/2}2^{d/2-1}$,

$$\begin{aligned}\mathrm{Index}(\hat p_\omega D_+\hat p_\omega)&=(-1)^{d/2}\frac{1}{d}\operatorname*{res}_{z=d}\mathrm{Tr}\Big(\gamma\,2p_\omega([D,p_\omega])^d(1+D^2)^{-z/2}\Big)\\ &=(-1)^{d/2}(-1)^{d/2}\frac{2i^{d/2}2^{d/2-1}}{d}\operatorname*{res}_{z=d}\mathrm{Tr}_{L^2(\mathbb{R}^d)}\left(p_\omega\sum_{\sigma\in S_d}(-1)^\sigma\prod_{i=1}^d\partial_{\sigma(i)}(p_\omega)(1+|X|^2)^{-z/2}\right)\\ &=\frac{(2i)^{d/2}\mathrm{Vol}_{d-1}(S^{d-1})}{d}\,\mathcal{T}\left(\sum_{\sigma\in S_d}(-1)^\sigma\prod_{i=1}^d\partial_{\sigma(i)}(p_\omega)\right)\end{aligned}$$

by Corollary 3.3.8. The results of Corollary 3.3.8 also imply that the index is almost surely independent of the choice of $\omega\in\Omega$. We use the equation $\mathrm{Vol}_{d-1}(S^{d-1})=\frac{d\pi^{d/2}}{(d/2)!}$ for d even to simplify

$$\mathrm{Index}(\hat p_\omega D_+\hat p_\omega)=\frac{(2\pi i)^{d/2}}{(d/2)!}\sum_{\sigma\in S_d}(-1)^\sigma\,\mathcal{T}\left(p_\omega\prod_{i=1}^d\partial_{\sigma(i)}(p_\omega)\right). \tag{3.14}$$

□

Comparing the index formula in Equation (3.14) to [PLB13, Equation (4)], we have reproduced the expression for the higher-dimensional even Chern numbers in the continuous (non-unital) setting. In particular, if we take $d=2$, then

$$\mathrm{Index}(\hat p_\omega D_+\hat p_\omega)=2\pi i\,\mathcal{T}(p_\omega[\partial_1p_\omega,\partial_2p_\omega])$$

and we recover the Kubo formula for the Hall conductance as derived in [BvS94, Section 4]. Theorem 3.3.12 for $d=2$ gives an alternate proof of the quantisation of the Hall conductance to [BvS94], which uses Fredholm modules, and [ASS94a], which uses a geometric integral identity.

Remark 3.3.14. The author was recently made aware of the work [And14], which includes results quite similar to the central results of this chapter, Theorem 3.3.9 and 3.3.12. Andersson uses so-called Rieffel deformations of an algebra whereas we work with twisted crossed products. Under suitable hypotheses, twisted crossed-product algebras are stably isomorphic to Rieffel-deformed algebras and therefore represent the same topological data at the level of K-theory and K-homology. We see our work in this section as a complement to Andersson's work in the area.

Remark 3.3.15. As was considered in [BvS94, Section 5] and [PLB13, PS14] for the discrete setting, we would like to use localisation to extend the class of operators for which the Chern number is well-defined. What we have proved so far is that the continuous higher-dimensional Chern numbers are well-defined for unitaries and projections in $M_q[\pi_\omega(\mathcal{A}^2)^\sim]$, which is quite restrictive. We leave a proper investigation on the extension of this pairing to another place, though make some preliminary observations.

As previously remarked, all index formulae over $\pi_\omega(\mathcal{A})^2$ extend to the δ-φ completion, $\mathcal{C}$, though we can extend our Chern numbers further. Suppose d is even and take a projection $p \in M_q[\mathcal{B}(L^2(\mathbb{R}^d))]$. Because the local index formula reduces to a single term (Lemma 3.3.10), then it is not necessary that $p \in M_q[\pi_\omega(\mathcal{A})^\sim]$. Instead all that is required is that $(2p-1)[D,p]^d$ is an element of $M_q(\mathcal{C}^\sim)$ for the higher-dimensional Chern number to be well-defined and, more importantly, to represent a Fredholm index.

In the case that d is odd, the same argument implies that $(u^*[D,u])^d \in M_q(\mathcal{C}^\sim)$ is sufficient for a unitary $u \in M_q[\mathcal{B}(L^2(\mathbb{R}^d))]$ to have a well-defined Chern number that represents an index pairing. Hence the higher-dimensional Chern numbers can be extended to a broader class of operators, which we can think of as like a noncommutative Sobolev space.

A task for future work is to relate the operators for which the Chern numbers extend to localised states and disorder.

Chapter 4

The bulk-edge correspondence of the discrete quantum Hall effect

4.1 Introduction

4.1.1 Boundaries and the bulk-edge correspondence

Chapter 3 outlined how techniques from noncommutative geometry can be used to extract topological information from a quantum Hall system (and higher-dimensional analogues). Our method involved studying the index pairing between particular K-theory and K-homology classes of the algebra of observables.

All systems that were considered in Chapter 3 were defined on the space $L^2(\mathbb{R}^d)$ for some d. However, we would also like to consider a system with boundary, e.g. $L^2(\mathbb{R}^{d-1} \times [0, \infty))$ or $\ell^2(\mathbb{Z}^{d-1} \times \mathbb{N})$ in the tight-binding picture. In such a system, the Lorentz drift of electrons that characterises the 'bulk' (boundary-free) Hall current is interrupted by the presence of an edge. Therefore one expects a net current along the boundary that is related to the original Hall current.

Given that the quantum Hall effect is an experimentally verified phenomena, our mathematical models should be able to account for boundary effects in the quantum Hall system. Furthermore, because the bulk Hall conductance is topological, we expect the conductance of the edge current to be topological in nature.

The relationship between topological invariants that come from our bulk and edge picture is the so-called bulk-edge correspondence. The bulk-edge correspondence for the quantum Hall effect says the two invariants are equal. Because the bulk Hall current is what emerged in our description of the quantum Hall effect in Chapter 3, it is associated with a system without boundary. The edge invariant (also called the edge conductance) comes from studying observables concentrated at the boundary of a sample. Linking these two quantities together is quite a non-trivial task, both on the level of the algebra of observables of our systems of interest as well as their topological properties. The first problem was solved by Kellendonk et al. [SBKR02, KSB04b, KR08], who linked the bulk and boundary algebra of observables by a short exact sequence.

The relation between topological properties of bulk and edge systems requires quite powerful machinery. It is here that studying the K-theory, K-homology and the index pairing of our observable algebras is not quite enough. Instead we need to generalise to the full bivariant KK-machinery outlined in Chapter 2 to put all the pieces together. In particular, the KK-setting and Kasparov product give us expressions for the index pairing without using the local index formula, which measures pairings of cyclic homology and cohomology and cannot detect torsion invariants. This observation is important when we pass to topological insulator systems in Chapter 5.

4.1.2 Overview of this chapter

We revisit the bulk-edge correspondence in the discrete (or tight-binding) version of the quantum Hall effect as previously studied in [EG02, EGS05, SBKR00, SBKR02, KSB04a, KSB04b]. In these papers, the motivation is to incorporate the presence of a boundary or edge into Bellissard's initial explanation of the quantum Hall effect [BvS94]. This is done by introducing an 'edge conductance', σ_e, which is then shown to be the same as Bellissard's initial expression for the (quantised) Hall conductance, σ_H. Our motivation comes from the more K-theoretic arguments used in [SBKR02, KSB04b].

We propose a new method based on explicit representations of extension classes as Kasparov modules. Given a short exact sequence of C^*-algebras,

$$0 \to B \to C \to A \to 0,$$

we know by results of Kasparov [Kas81] (Theorem 2.2.37) that this sequence gives rise to a class in $\mathrm{Ext}(A, B)$, which is the same as $KK^1(A, B)$ for the algebras we study. By representing our short exact sequence as an unbounded Kasparov module, we can use the methods developed in [BMv13, KL13, Mes14, MR15] to take the Kasparov product of our module with spectral triples representing elements in $K^j(B) \cong KK^j(B, \mathbb{C})$ to give elements in $K^{j+1}(A, \mathbb{C})$.

In this chapter we focus on a simple model so as not to obscure the main idea with technical details. Thus we consider the short exact sequence representing the Toeplitz extension of the rotation algebra, A_ϕ. An unbounded Kasparov module can be built from this extension by considering the circle action on the rotation algebra A_ϕ, as in [CNNR11].

We outline an alternative method for constructing a Kasparov module representing an extension class (generalised in [RRS15]) via a singular functional. We introduce this method with a view towards more complicated examples, where the circle-action picture breaks down. Such examples include the following.

1. For the case of a finite group G with $K \triangleleft G$, the short exact sequence

$$0 \to B \rtimes K \to C \rtimes G \to A \rtimes G/K \to 0$$

can no longer be represented by circle actions. Such crossed products may emerge by considering the symmetry group of topological insulator systems, for example.

2. For models with internal degrees of freedom (such as a honeycomb lattice), we would no longer be working with the Cuntz-Pimsner algebra of a self Morita-equivalence bimodule (as defined in [RRS15, Section 2]) and so the singular functional method is necessary.

See [RRS15] for more examples of extensions requiring this viewpoint. The flexibility of our approach to representing extensions as Kasparov modules (with which products can be taken) will allow many more systems-with-edge to be investigated.

4.1.3 Statement of the main result

We begin with a Toeplitz-like extension of the rotation algebra A_ϕ, and show how to construct an unbounded Kasparov module $\beta = \left(\mathcal{A}_\phi, Z_{C^*(\widehat{U})}, N\right)$ for a smooth subalgebra $\mathcal{A}_\phi \subset A_\phi$ representing the extension in KK-theory. Here $Z_{C^*(\widehat{U})}$ is a Hilbert C^*-module coming from the extension, $\widehat{U}$ is the shift operator on $\ell^2(\mathbb{Z})$ along the boundary $\mathbb{Z}$ and the unbounded operator N is a number operator (defined later).

We also introduce a 'boundary spectral triple' $\Delta = \left(\mathcal{B}, \ell^2(\mathbb{Z}), M\right)$, where $\mathcal{B} \subset C^*(\widehat{U})$ is a dense $*$-algebra acting on the boundary. We think of the spectral triple Δ capturing K-homological data of observables concentrated at the boundary of our sample. Our main result of the chapter, Theorem 4.3.3, is as follows.

Theorem. *The internal Kasparov product $\beta \hat{\otimes}_{\mathcal{B}} \Delta$ is unitarily equivalent to the negative of the spectral triple modelling the boundary-free quantum Hall effect.*

We note that the unitary equivalence of the Kasparov modules in the theorem is at the unbounded level, a stronger equivalence than in the bounded setting.

Recall from the work of Bellissard [BvS94] and Chapter 3 that the quantised Hall conductance in the discrete setting without boundary comes from the pairing of the Fermi projection with an element in $K^0(A_\phi)$. Our main result says that this K-homology class can be 'factorised' into a product of a K-homology class representing the boundary and a KK^1-class representing the short exact sequence linking the boundary and boundary-free systems. We can then use the associativity of the Kasparov product to immediately obtain an edge conductance, and the equality of the bulk and edge conductances.

It is in this point that our work differs from, but complements, the boundary picture developed in [SBKR02, KSB04b], where the authors had to define a separate edge conductance and then show equality with the usual Hall conductance. Instead, our method derives the bulk-edge correspondence as a direct consequence of the factorisation of the boundary-free K-homology class. Indeed, our work demonstrates how we can obtain the bulk-edge correspondence of [SBKR02] without passing to cyclic theory. This allows our method to be applied to more complicated situations with torsion invariants, topological insulators being an important example (see Chapter 5).

We also note that by working in the unbounded Kasparov picture, all computations are explicit. As Kasparov theory can also be extended to accommodate group actions, real/Real algebras etc. this means our method has potential applications to a much wider array of physical models.

This chapter is organised into two major sections. Section 4.2 contains the construction of the Kasparov module that is needed in Section 4.3 where the main theorem is proved.

4.2 A Kasparov module representing the Toeplitz extension

4.2.1 The discrete quantum Hall system

Our model of interest will be the discrete or tight-binding quantum Hall system as considered in [MC96]. This model allows our constructions and computations to be as transparent as possible. In the case without boundary, where $\mathcal{H} = \ell^2(\mathbb{Z}^2)$, we have magnetic translations $\widehat{U}$ and $\widehat{V}$ acting as unitary operators on $\ell^2(\mathbb{Z}^2)$. These operators commute with the unitaries U and V that generate the Hamiltonian $H = U + U^* + V + V^*$. We choose the Landau gauge so that

$$\begin{aligned}
(\widehat{U}\lambda)(m,n) &= \lambda(m-1,n), & (\widehat{V}\lambda)(m,n) &= e^{-2\pi i\phi m}\lambda(m,n-1),\\
(U\lambda)(m,n) &= e^{-2\pi i\phi n}\lambda(m-1,n), & (V\lambda)(m,n) &= \lambda(m,n-1),
\end{aligned}$$

where ϕ has the interpretation as the magnetic flux through a unit cell and $\lambda \in \ell^2(\mathbb{Z}^2)$. We note that $\widehat{U}\widehat{V} = e^{2\pi i\phi}\widehat{V}\widehat{U}$ and $UV = e^{-2\pi i\phi}VU$, so $C^*(\widehat{U},\widehat{V}) \cong A_\phi$, the rotation algebra, and $C^*(U,V) \cong A_{-\phi}$. We can also interpret $A_{-\phi} \cong A_\phi^{\mathrm{op}}$, where A_ϕ^{op} is the opposite algebra with multiplication $(ab)^{\mathrm{op}} = b^{\mathrm{op}}a^{\mathrm{op}}$. To see this identification we compute,

$$\widehat{U}^{\mathrm{op}}\widehat{V}^{\mathrm{op}} = \left(\widehat{V}\widehat{U}\right)^{\mathrm{op}} = e^{-2\pi i\phi}\left(\widehat{U}\widehat{V}\right)^{\mathrm{op}} = e^{-2\pi i\phi}\widehat{V}^{\mathrm{op}}\widehat{U}^{\mathrm{op}}.$$

Our choice of gauge also means that $C^*(\widehat{U},\widehat{V}) \cong C^*(\widehat{U}) \rtimes_\eta \mathbb{Z}$, where $\widehat{V}$ is implementing the crossed-product structure via the automorphism $\eta(\widehat{U}^m) = \widehat{V}^*\widehat{U}^m\widehat{V}$. Such a decomposition of the algebra $C^*(\widehat{U},\widehat{V})$ will be useful for when we consider a bulk-edge system (see Section 4.2.2 and Remark 4.2.7).

Next, we impose a boundary on the system and consider the Hamiltonian acting on the half-plane $\ell^2(\mathbb{Z}\times\mathbb{N})$. The Hamiltonian now takes the form $H_{\mathrm{hp}} = U_{\mathrm{hp}} + U_{\mathrm{hp}}^* + V_{\mathrm{hp}} + V_{\mathrm{hp}}^*$, where

$$(U_{\mathrm{hp}}\lambda)(m,n) = e^{-2\pi i\phi n}\lambda(m-1,n), \qquad (V_{\mathrm{hp}}\lambda)(m,n) = \chi(n-1)\lambda(m,n-1),$$

and

$$\chi(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}.$$

Our gauge choice comes with the corresponding magnetic translations

$$(\widehat{U}_{\mathrm{hp}}\lambda)(m,n) = \lambda(m-1,n), \qquad (\widehat{V}_{\mathrm{hp}}\lambda)(m,n) = \chi(n-1)e^{-2\pi i\phi m}\lambda(m,n-1).$$

Lemma 4.2.1. *The operators U_{hp} and V_{hp} commute with $\widehat{U}_{hp}$ and $\widehat{V}_{hp}$.*

Proof. We shall consider the case $[U_{\mathrm{hp}}, \widehat{V}_{\mathrm{hp}}]$. One checks that

$$\begin{aligned}(U_{\mathrm{hp}}\widehat{V}_{\mathrm{hp}}\lambda)(m,n) &= e^{-2\pi i\phi n}(\widehat{V}_{\mathrm{hp}}\lambda)(m-1,n) \\ &= \chi(n-1)e^{-2\pi i\phi n}e^{-2\pi i\phi(m-1)}\lambda(m-1,n-1), \\ (\widehat{V}_{\mathrm{hp}}U_{\mathrm{hp}}\lambda)(m,n) &= \chi(n-1)e^{-2\pi i\phi m}(U_{\mathrm{hp}}\lambda)(m,n-1) \\ &= \chi(n-1)e^{-2\pi i\phi m}e^{-2\pi i\phi(n-1)}\lambda(m-1,n-1)\end{aligned}$$

and so $[U_{\mathrm{hp}}, \widehat{V}_{\mathrm{hp}}] = 0$. The other cases follow similar arguments. □

We find that in the presence of a boundary there are still unitaries U_{hp} and $\widehat{U}_{\mathrm{hp}}$, but now V_{hp} and $\widehat{V}_{\mathrm{hp}}$ are partial isometries, where

$$V_{\mathrm{hp}}^*V_{\mathrm{hp}} = \widehat{V}_{\mathrm{hp}}^*\widehat{V}_{\mathrm{hp}} = 1, \qquad V_{\mathrm{hp}}V_{\mathrm{hp}}^* = \widehat{V}_{\mathrm{hp}}\widehat{V}_{\mathrm{hp}}^* = 1 - P_{n=0}.$$

Remark 4.2.2 (A note on boundary conditions)*.* Our choice of translations are encoding Dirichlet-style boundary conditions at $n = 0$. We note that changing the boundary conditions in the discrete/tight-binding picture will change the operators V_{hp} and $\widehat{V}_{\mathrm{hp}}$ by a finite-rank operator. Hence the difference is a compact perturbation.

One of our reasons for choosing 'Dirichlet translations' is that such a choice has a natural link to the representation theory of semigroups (say $\mathbb{Z}\times\mathbb{N}$ or $\mathbb{R}\times[0,\infty)$). Define $W^k = U_{\mathrm{hp}}^{k_1}V_{\mathrm{hp}}^{k_2}$ and $\widehat{W}^k = \widehat{U}_{\mathrm{hp}}^{k_1}\widehat{V}_{\mathrm{hp}}^{k_2}$ for $k \in \mathbb{Z}\times\mathbb{N}$ and $\sigma(k,k') = e^{2\pi i\phi k_1' k_2}$ for $k,k' \in \mathbb{Z}\times\mathbb{N}$. It is a simple check that σ is a semigroup 2-cocycle for $\mathbb{Z}\times\mathbb{N}$.

Lemma 4.2.3. *The operator W generates a σ-representation of $\mathbb{Z}\times\mathbb{N}$. The operator $\widehat{W}$ generates a $\overline{\sigma}$-representation of $\mathbb{Z}\times\mathbb{N}$ that commutes with W.*

Proof. We first compute that

$$\begin{aligned}(W^k\lambda)(m,n) = (U_{\mathrm{hp}}^{k_1}V_{\mathrm{hp}}^{k_2}\lambda)(m,n) &= e^{-2\pi i\phi k_1 n}(V_{\mathrm{hp}}\lambda)(m-k_1,n)\\ &= \chi(n-k_2)e^{-2\pi i\phi k_1 n}\lambda(m-k_1,n-k_2).\end{aligned}$$

We need to show that $W^kW^{k'} = \sigma(k,k')W^{k+k'}$. We compute,

$$\begin{aligned}(W^kW^{k'}\lambda)(m,n) &= \chi(n-k_2)e^{-2\pi i\phi k_1 n}(W^{k'}\lambda)(m-k_1,n-k_2)\\ &= \chi(n-k_2)e^{-2\pi i\phi k_1 n}\chi(n-k_2-k_2')e^{-2\pi i\phi k_1'(n-k_2)}\lambda(m-k_1-k_1',n-k_2-k_2')\\ &= e^{2\pi i\phi k_1' k_2}\chi(n-k_2-k_2')e^{-2\pi i\phi(k_1+k_1')n}\lambda(m-k_1-k_1',n-k_2-k_2')\\ &= \sigma(k,k')(W^{k+k'}\lambda)(m,n),\end{aligned}$$

where we have used that $\chi(n-k_2)\chi(n-k_2-k_2') = \chi(n-k_2-k_2')$ for all $k_2,k_2' \in \mathbb{N}$. This gives the result for W^k.

Next we find $(\widehat{W}^k\lambda)(m,n) = (\widehat{U}_{\mathrm{hp}}^{k_1}\widehat{V}_{\mathrm{hp}}^{k_2}\lambda)(x) = \chi(n-k_2)e^{-2\pi i\phi(m-k_1)}\lambda(m-k_1,n-k_2)$. Then,

$$\begin{aligned}(\widehat{W}^k\widehat{W}^{k'}\lambda)(m,n) &= \chi(n-k_2)e^{-2\pi i\phi(m-k_1)}\chi(n-k_2-k_2')e^{-2\pi i\phi k_2'(m-k_1-k_1')}\\ &\qquad\times\lambda(m-k_1-k_1',n-k_2-k_2')\\ &= e^{-2\pi i\phi k_2 k_1'}\chi(n-k_2-k_2')e^{-2\pi i\phi(k_2+k_2')(m-k_1-k_1')}\lambda(m-k_1-k_1',n-k_2-k_2')\\ &= \overline{\sigma(k,k')}(\widehat{W}^{k+k'}\lambda)(m,n)\end{aligned}$$

as required. Finally $[W,\widehat{W}] = 0$ by Lemma 4.2.1. $\square$

We may use analogous arguments from the proof of Lemma 4.2.3 to obtain a similar result for the adjoint operators W^* and $\widehat{W}^*$. We omit the details for brevity.

Lemma 4.2.4. *Let $\sigma^*(k,k') = e^{-2\pi i\phi k_1 k_2'}$ for $k,k' \in \mathbb{Z}\times\mathbb{N}$. The operators W^* (resp. $\widehat{W}^*$) generate a σ^*-representation (resp. $\overline{\sigma^*}$-representation) of $\mathbb{Z}\times\mathbb{N}$. Furthermore, the two representations commute.*

Our notation of $\sigma^*(k,k') = e^{-2\pi i\phi k_1 k_2'}$ is reasonable as $\sigma^*(k,k') = \overline{\sigma(k',k)}$. To recapitulate, the map $k \mapsto W^k$ is a right σ-representation of the semigroup $\mathbb{Z}\times\mathbb{N}$, with $k \mapsto \widehat{W}^k$ the corresponding left $\overline{\sigma}$-representation that commutes with W^k. An analogous result holds for $k \mapsto (W^*)^k$ with σ^* and $k \mapsto (\widehat{W}^*)^k$.

One can also consider the Hamiltonian $H_{\mathrm{hp}} = U_{\mathrm{hp}}+U_{\mathrm{hp}}^*+V_{\mathrm{hp}}+V_{\mathrm{hp}}^*+g(m,n)$, where $g(m,n)$ is a bounded periodic potential, that is $g \in \ell^\infty(\mathbb{Z}\times\mathbb{N})$ and $g(m+k_1,n+k_2) = g(m,n)$ for any $k \in \mathbb{Z}\times\mathbb{N}$. Such a Hamiltonian will still commute with $\widehat{W}^k$, which encodes the magnetic translations.

Considering algebras with shift operators on the half-plane has put us in the domain of Toeplitz algebras and, in particular, Toeplitz extensions. It was observed in [SBKR00, SBKR02] that we can link a system without boundary with an edge system via such an extension.

4.2.2 The bulk-edge short exact sequence

We outline an idea loosely based on that of Kellendonk et al. [SBKR02, KSB04b], who employed constructions from Pimsner and Voiculescu [PV80]. The essence of the idea is to relate the bulk and edge algebras via a Toeplitz-like extension.

Proposition 4.2.5 (§2 of [PV80])**.** *Let S be the usual shift operator on $\ell^2(\mathbb{N})$ with $S^*S = 1$, $SS^* = 1 - P_{n=0}$. There is a short exact sequence,*

$$0 \to C^*(\widehat{U}) \otimes \mathcal{K}[\ell^2(\mathbb{N})] \xrightarrow{\psi} C^*(\widehat{U} \otimes 1, \widehat{V} \otimes S) \to C^*(\widehat{U}) \rtimes_\eta \mathbb{Z} \to 0.$$

The map ψ given in Proposition 4.2.5 is such that

$$\psi(\widehat{U}^m \otimes e_{jk}) = (\widehat{V}^*)^j \widehat{U}^m \widehat{V}^k \otimes S^j P_{n=0} (S^*)^k$$

for matrix units e_{jk} in $\mathcal{K}[\ell^2(\mathbb{N})]$ and is then extended to the full algebra by linearity. One checks that ψ is an injective map into the ideal of $C^*(\widehat{U} \otimes 1, \widehat{V} \otimes S)$ generated by $1 \otimes P_{n=0}$ [PV80]. We also have the isomorphism $C(S^1) \rtimes_\eta \mathbb{Z} \cong C^*(\widehat{U} \otimes 1, \widehat{V} \otimes V) \cong C^*(\widehat{U}, \widehat{V})$, where V is the image of S under the map to the Calkin algebra.

Remark 4.2.6. The quotient A_ϕ represents our ‘bulk algebra’ as it can be derived from a magnetic Hamiltonian on $\ell^2(\mathbb{Z}^2)$ as in [MC96]. The ideal $C^*(\widehat{U}) \otimes \mathcal{K}[\ell^2(\mathbb{N})]$ is interpreted as representing the ‘boundary algebra’. To see this we note that $C^*(\widehat{U})$ acts on the edge $\ell^2(\mathbb{Z} \times \{0\})$, (this action being describable in terms of the bilateral shift operator). Tensoring $C^*(\widehat{U})$ by the compacts in the direction perpendicular to the boundary has a physical interpretation as observables acting on $\ell^2(\mathbb{Z} \times \mathbb{N})$ that act near the boundary and decay sufficiently fast so that the operator is compact normal to the boundary. Because we expect the Hall current to be concentrated at the edge of a sample with a fast decay into the interior, our bulk-edge model lines up with this picture.

Remark 4.2.7 (The Pimsner-Voiculescu sequence and the choice of gauge). An important aspect of setting up a bulk-edge system using a short-exact sequence is that the bulk algebra A can be related to the edge algebra B by a crossed-product, $A \cong B \rtimes \mathbb{Z}$ with the action on B given by (twisted) translation operators normal to the boundary. Our choice of the Landau gauge ensures that this decomposition can be naturally observed, with $A_\phi \cong C^*(\widehat{U}) \rtimes_{\mathrm{Ad}\widehat{V}} \mathbb{Z}$ and $C^*(\widehat{U})$ interpreted as a boundary algebra.

While different gauge choices can be considered for a system with boundary (see [KR08, Section 4], which uses ideas from [PR89]), for transparency we choose the Landau gauge, where translations are untwisted along the boundary.

Our aim is to associate an odd complex Kasparov module to the Pimsner-Voiculescu short exact sequence. The reader may consult Chapter 2.2 for a basic overview of KK-theory.

4.2.3 Constructing the Kasparov module

In the last section, we introduced the short exact sequence

$$0 \to C^*(\widehat{U}) \otimes \mathcal{K} \xrightarrow{\psi} \mathcal{T} \to \mathcal{A}_\phi \to 0. \tag{4.1}$$

We know that this sequence gives rise to a class in KK-theory using Ext groups, but in order to compute the Kasparov product, it is desirable to have an explicit Kasparov module that represents a class in $KK^1(A_\phi, C^*(\widehat{U}) \otimes \mathcal{K}) \cong KK^1(A_\phi, C^*(\widehat{U}))$.

To do this, we introduce our main technical innovation, a singular functional Ψ on the subalgebra $C^*(S)$ of $\mathcal{T}$, which is given by

$$\Psi(T) = \operatorname*{res}_{s=1} \sum_{k=0}^{\infty} \langle e_k, T e_k \rangle (1+k^2)^{-s/2}$$

and $\{e_k\}$ is any basis of $\ell^2(\mathbb{N})$. A generalisation of this construction appears in [RRS15].

Proposition 4.2.8. *The functional Ψ is a well-defined trace on $C^*(S)$ such that $\Psi\left(S^{l_2}(S^*)^{l_1} S^{n_1}(S^*)^{n_2}\right) = \delta_{l_1-l_2, n_1-n_2}$, where $\delta_{a,b}$ is the Kronecker delta. Moreover, $\Psi(T) = 0$ for any compact T.*

Proof. Because Ψ is constructed from the usual trace on $\ell^2(\mathbb{N})$, it is a straightforward check that Ψ is a trace by properties of the trace on $\ell^2(\mathbb{N})$ and complex residues. Thus, for $S^\alpha (S^*)^\beta \in C^*(S)$, we see that

$$\langle e_k, S^\alpha (S^*)^\beta e_k \rangle = \delta_{\alpha,\beta} \langle (S^*)^\alpha e_k, (S^*)^\alpha e_k \rangle = \delta_{\alpha,\beta}\, \chi_{[k,\infty)}(\alpha),$$

where $\chi_{[k,\infty)}$ is the indicator function. Hence

$$\begin{aligned}\Psi\left[S^\alpha (S^*)^\beta\right] &= \operatorname*{res}_{s=1} \sum_{k=0}^{\infty} \delta_{\alpha,\beta} \chi_{[k,\infty)}(\alpha)(1+k^2)^{-s/2} \\ &= \operatorname*{res}_{s=1} \sum_{k=\alpha}^{\infty} \delta_{\alpha,\beta}(1+k^2)^{-s/2} = \delta_{\alpha,\beta}.\end{aligned}$$

Similarly $\Psi\big((S^*)^\alpha S^\beta\big) = \delta_{\alpha,\beta}$. From this we have that, for $l_1 \geq n_1$,

$$\Psi\left(S^{l_2}(S^*)^{l_1} S^{n_1}(S^*)^{n_2}\right) = \Psi\left(S^{l_2}(S^*)^{l_1-n_1+n_2}\right) = \delta_{l_2, l_1-n_1+n_2} = \delta_{l_1-l_2, n_1-n_2};$$

or, for $l_1 \leq n_1$,

$$\Psi\left(S^{l_2}(S^*)^{l_1} S^{n_1}(S^*)^{n_2}\right) = \Psi\left(S^{l_2-l_1+n_1}(S^*)^{n_2}\right) = \delta_{l_2-l_1+n_1, n_2} = \delta_{l_1-l_2, n_1-n_2}.$$

Since $(S^*)^\alpha S^\alpha = 1_{C^*(S)}$, one now readily checks that

$$\Psi(T) \leq \|T\|\, \Psi\big(1_{C^*(S)}\big) = \|T\| \tag{4.2}$$

for all $T \in C^*(S)$ and so Ψ extends by continuity to $C^*(S)$. For any finite-rank operator, $F \in C^*(S)$, $\langle e_k, F e_k \rangle \neq 0$ for finitely many k. This tells us that $\sum_k \langle e_k, F e_k \rangle (1+k^2)^{-s/2}$ is holomorphic at $s = 1$, whence $\Psi(F) = 0$. By Equation (4.2), Ψ vanishes on all the compacts operators on $\ell^2(\mathbb{N})$. $\square$

In order to simplify computations, we realise $\mathcal{T}$ as the norm closure of the linear span of the operators

$$(\widehat{V} \otimes S)^{n_1}[(\widehat{V} \otimes S)^*]^{n_2}(\widehat{U} \otimes 1)^m = \widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}$$

for $m \in \mathbb{Z}$ and $n_1, n_2 \in \mathbb{N}$. We put the $\widehat{U}$ on the right as we are going to construct a right $C^*(\widehat{U})$-module using this presentation.

The first step is the inner product: $(\cdot \mid \cdot) : \mathcal{T} \times \mathcal{T} \to C^*(\widehat{U})$ given by

$$\begin{aligned}\Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1} \otimes S^{l_1}(S^*)^{l_2}\Big|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2} \otimes S^{n_1}(S^*)^{n_2}\Big)&\\ := \Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\Big)^* \widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\, \Psi\Big[\Big(S^{l_1}(S^*)^{l_2}\Big)^* S^{n_1}(S^*)^{n_2}\Big].&\end{aligned}$$

To show the inner product actually takes values in $C^*(\widehat{U})$, we use Proposition 4.2.8 to compute that

$$\begin{aligned}\Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1} \otimes S^{l_1}(S^*)^{l_2}\Big|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2} \otimes S^{n_1}(S^*)^{n_2}\Big)&\\ = \widehat{U}^{-m_1}\widehat{V}^{l_2-l_1}\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\delta_{l_1-l_2,n_1-n_2} = \widehat{U}^{m_2-m_1}\delta_{l_1-l_2,n_1-n_2},&\end{aligned}$$

which is in $C^*(\widehat{U})$. With this in mind we construct a right $C^*(\widehat{U})$ module.

Proposition 4.2.9. *The map $(\cdot \mid \cdot) : \mathcal{T} \times \mathcal{T} \to C^*(\widehat{U})$ together with an action by right multiplication makes $\mathcal{T}$ a right $C^*(\widehat{U})$-inner product module. Quotienting by vectors of zero length and completing yields a right $C^*(\widehat{U})$-module.*

Proof. Using the equation

$$\Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1} \otimes S^{l_1}(S^*)^{l_2}\Big|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2} \otimes S^{n_1}(S^*)^{n_2}\Big) = \widehat{U}^{m_2-m_1}\delta_{l_1-l_2,n_1-n_2}$$

most of the requirements for $(\cdot \mid \cdot)$ to be a $C^*(\widehat{U})$-valued inner-product follow in a straightforward way. We will check compatibility with multiplication on the right by elements of $C^*(\widehat{U})$. We compute that

$$\begin{aligned}\Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1} \otimes S^{l_1}(S^*)^{l_2}\Big|\,\Big(\widehat{V}^{n_1-n_2}\widehat{U}^{m_2} \otimes S^{n_1}(S^*)^{n_2}\Big)\cdot(\widehat{U}^{\alpha} \otimes 1)\Big)&\\ = \Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1} \otimes S^{l_1}(S^*)^{l_2}\Big|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2+\alpha} \otimes S^{n_1}(S^*)^{n_2}\Big)&\\ = \widehat{U}^{m_2-m_1+\alpha}\delta_{l_1-l_2,n_1-n_2}&\\ = \Big(\widehat{U}^{m_2-m_1}\delta_{l_1-l_2,n_1-n_2}\Big)\widehat{U}^{\alpha}&\\ = \Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1} \otimes S^{l_1}(S^*)^{l_2}\Big|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2} \otimes S^{n_1}(S^*)^{n_2}\Big)\cdot\widehat{U}^{\alpha}&\end{aligned}$$

for $\alpha \in \mathbb{Z}$. Obtaining the result for arbitrary elements in $C^*(\widehat{U})$ is a simple extension of this. $\square$

We denote our C^*-module by $Z_{C^*(\widehat{U})}$ and inner-product by $(\cdot \mid \cdot)_{C^*(\widehat{U})}$. The point of the singular trace Ψ becomes apparent in the next proposition where we construct a left action of A_ϕ on $Z_{C^*(\widehat{U})}$.

Proposition 4.2.10. *There is an adjointable representation if A_ϕ on $Z_{C^*(\widehat{U})}$.*

Proof. Clearly we can multiply elements of $Z_{C^*(\widehat{U})}$ by $\mathcal{T}$ on the left, but by Proposition 4.2.8, we know that $(\widehat{U}^j\widehat{V}^k\otimes k)\cdot Z_{C^*(\widehat{U})} = 0$ if $k \in \mathcal{K}[\ell^2(\mathbb{N})]$. Therefore the representation of $\mathcal{T}$ descends to a representation of $\mathcal{T}/\psi[C(S^1)\otimes\mathcal{K}] \cong A_\phi$. This gives us the explicit left-action generated by

$$\begin{aligned}(\widehat{U}^\alpha\widehat{V}^\beta)\cdot\Big(\widehat{V}^{n_1-n_2}\widehat{U}^m\otimes S^{n_1}(S^*)^{n_2}\Big) &= (\widehat{U}^\alpha\widehat{V}^\beta\widehat{V}^{n_1-n_2}\widehat{U}^m)\otimes S^{n_1+\beta}(S^*)^{n_2} \\ &= e^{2\pi i\phi\alpha(n_1-n_2+\beta)}\widehat{V}^{\beta+n_1-n_2}\widehat{U}^{m+\alpha}\otimes S^{\beta+n_1}(S^*)^{n_2}\end{aligned}$$

for $\alpha,\beta\in\mathbb{Z}$ with $\beta\geq 0$ and

$$(\widehat{U}^\alpha\widehat{V}^\beta)\cdot\Big(\widehat{V}^{n_1-n_2}\widehat{U}^m\otimes S^{n_1}(S^*)^{n_2}\Big) = e^{2\pi i\phi\alpha(n_1-n_2+\beta)}\widehat{V}^{\beta+n_1-n_2}\widehat{U}^{m+\alpha}\otimes S^{\beta}(S^*)^{n_2+|\beta|}$$

for $\beta<0$. It follows that, as operators on $Z_{C^*(\widehat{U})}$, $\widehat{U}\widehat{V} = e^{2\pi i\phi}\widehat{V}\widehat{U}$. Next we just need to verify that the action is adjointable as a module over $C^*(\widehat{U})$. We compute that

$$\begin{aligned}&\Big(\widehat{U}\cdot\Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\otimes S^{l_1}(S^*)^{l_2}\Big)\Big|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2}\Big)_{C^*(\widehat{U})} \\ &\quad= \Big(e^{2\pi i\phi(l_1-l_2)}\widehat{V}^{l_1-l_2}\widehat{U}^{m_1+1}\otimes S^{l_1}(S^*)^{l_2}\,\Big|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2}\Big)_{C^*(\widehat{U})} \\ &\quad= e^{-2\pi i\phi(l_1-l_2)}\widehat{U}^{m_2-1-m_1}\delta_{l_1-l_2,n_1-n_2} \\ &\quad= \Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\otimes S^{l_1}(S^*)^{l_2}\,\Big|\,e^{-2\pi i\phi(n_1-n_2)}\widehat{V}^{n_1-n_2}\widehat{U}^{m_2-1}\otimes S^{n_1}(S^*)^{n_2}\Big)_{C^*(\widehat{U})} \\ &\quad= \Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\otimes S^{l_1}(S^*)^{l_2}\,\Big|\,\widehat{U}^{-1}\cdot\Big(\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2}\Big)\Big)_{C^*(\widehat{U})}\end{aligned}$$

and then

$$\begin{aligned}&\Big(\widehat{V}\cdot\Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\otimes S^{l_1}(S^*)^{l_2}\Big)\Big|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2}\Big)_{C^*(\widehat{U})} \\ &\quad= \Big(\widehat{V}^{l_1-l_2+1}\widehat{U}^{m_1}\otimes S^{l_1+1}(S^*)^{l_2}\,\Big|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2}\Big)_{C^*(\widehat{U})} \\ &\quad= \widehat{U}^{m_2-m_1}\delta_{l_1-l_2+1,n_1-n_2} \\ &\quad= \widehat{U}^{m_2-m_1}\delta_{l_1-l_2,n_1-n_2-1} \\ &\quad= \Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\otimes S^{l_1}(S^*)^{l_2}\,\Big|\,\widehat{V}^{n_1-n_2-1}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2+1}\Big)_{C^*(\widehat{U})} \\ &\quad= \Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\otimes S^{l_1}(S^*)^{l_2}\,\Big|\,\widehat{V}^{-1}\cdot\Big(\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2}\Big)\Big)_{C^*(\widehat{U})}.\end{aligned}$$

Therefore our generating elements are adjointable and unitary on the dense span of monomials in $Z_{C^*(\widehat{U})}$. Thus if $\widehat{U}$, $\widehat{V}$ are bounded, they will generate an adjointable representation of A_ϕ. To consider the boundedness of $\widehat{U}$ and $\widehat{V}$, we first note that the inner-product in $Z_{C^*(\widehat{U})}$ is defined from multiplication in $\mathcal{T}$ and the functional Ψ, which has the property $\Psi(T) \leq \|T\|$, by Equation (4.2). These observations imply that

$$\|a\|_{\mathrm{End}(Z)} = \sup_{\substack{z\in Z\\ \|z\|=1}} (a\cdot z \mid a\cdot z)_{C^*(\widehat{U})} \leq \sup_{\substack{z\in Z\\ \|z\|=1}} \|aa^*\| \, (z \mid z)_{C^*(\widehat{U})} = \|aa^*\|.$$

Therefore the action of A_ϕ is bounded, and so extends to an adjointable action on $Z_{C^*(\widehat{U})}$. □

In Section 4.2.4, we show that by considering a left module ${}_{C^*(\widehat{U})}Z$, we may also obtain an adjointable representation of A_ϕ^{op}. Before we finish building our Kasparov module, we need some further results arising from properties of the singular trace Ψ.

Proposition 4.2.11. *Let $l_1 - l_2 = n_1 - n_2$. Then $\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2} = \widehat{V}^{l_1-l_2}\widehat{U}^m \otimes S^{l_1}(S^*)^{l_2}$ as elements in $Z_{C^*(\widehat{U})}$.*

Proof. We can assume without loss of generality that $l_1 = n_1 + k$ and $l_2 = n_2 + k$ for some $k \in \mathbb{Z}$. As a preliminary, we compute $S^{n_1}(S^*)^{n_2} - S^{n_1+k}(S^*)^{n_2+k}$ under the norm induced by Ψ. Firstly we expand

$$\begin{aligned}
\Big(S^{n_1}(S^*)^{n_2} - S^{n_1+k}(S^*)^{n_2+k}\Big)^* & \Big(S^{n_1}(S^*)^{n_2} - S^{n_1+k}(S^*)^{n_2+k}\Big) \\
&= S^{n_2}(S^*)^{n_1}S^{n_1}(S^*)^{n_2} - S^{n_2}(S^*)^{n_1}S^{n_1+k}(S^*)^{n_2+k} \\
&\quad - S^{n_2+k}(S^*)^{n_1+k}S^{n_1}(S^*)^{n_2} + S^{n_2+k}(S^*)^{n_1+k}S^{n_1+k}(S^*)^{n_2+k} \\
&= S^{n_2}(S^*)^{n_2} - S^{n_2+k}(S^*)^{n_2+k} - S^{n_2+k}(S^*)^{n_2+k} + S^{n_2+k}(S^*)^{n_2+k} \\
&= S^{n_2}(S^*)^{n_2} - S^{n_2+k}(S^*)^{n_2+k}.
\end{aligned}$$

We now recall that $\Psi(S^\alpha(S^*)^\beta) = \delta_{\alpha,\beta}$, so that

$$\begin{aligned}
\Psi\Big[(S^{n_1}(S^*)^{n_2} - S^{n_1+k}(S^*)^{n_2+k})^*(S^{n_1}(S^*)^{n_2} - S^{n_1+k}(S^*)^{n_2+k})\Big]& \\
= \Psi(S^{n_2}(S^*)^{n_2}) - \Psi(S^{n_2+k}(S^*)^{n_2+k}) = 0.&
\end{aligned}$$

From this point, it is a simple task to show that $\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}$ is equal to $\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1+k}(S^*)^{n_2+k}$ in the norm induced by $(\cdot \mid \cdot)_{C^*(\widehat{U})}$. □

Lemma 4.2.12. *Let $z_{n_1,n_2,m}$ denote the element $\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2} \in Z_{C^*(\widehat{U})}$. Then for all $k \in \mathbb{Z}$*

$$\Theta_{z_{l_1,l_2,k}, z_{l_1,l_2,k}}(z_{n_1,n_2,m}) = \delta_{l_1-l_2, n_1-n_2}\, z_{n_1,n_2,m},$$

where $\Theta_{e,f}(g) = e\cdot(f|g)_{C^*(\widehat{U})}$ are the rank-1 endomorphisms that generate $\mathrm{End}^0_{C^*(\widehat{U})}(Z)$.

Proof. We check that

$$\begin{aligned}\Theta_{z_{l_1,l_2,k},z_{l_1,l_2,k}}(z_{n_1,n_2,m}) &= \widehat{V}^{l_1-l_2}\widehat{U}^k \otimes S^{l_1}(S^*)^{l_2} \\ &\quad\times\left(\widehat{V}^{l_1-l_2}\widehat{U}^k \otimes S^{l_1}(S^*)^{l_2}\,\middle|\,\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}\right)_{C^*(\widehat{U})} \\ &= \left(\widehat{V}^{l_1-l_2}\widehat{U}^k \otimes S^{l_1}(S^*)^{l_2}\right)\cdot \widehat{U}^{m-k}\delta_{l_1-l_2,n_1-n_2} \\ &= \widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}\delta_{l_1-l_2,n_1-n_2},\end{aligned}$$

where we have used Proposition 4.2.11. □

With these preliminary results out the way, we now state the main result of this subsection.

Proposition 4.2.13. *Define the operator* $N : \mathrm{span}\left\{\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}\right\} \subset Z \to Z$ *such that* $N\left(\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}\right) = (n_1-n_2)\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}$*. Then* $\left(\mathcal{A}_\phi, Z_{C^*(\widehat{U})}, N\right)$ *is an unbounded, odd Kasparov module.*

Proof. Lemma 4.2.12 shows that for any n_1, n_2 with $n_1 - n_2 = k$, the operator $\Phi_k = \Theta_{z_{n_1,n_2,0},z_{n_1,n_2,0}}$ is an adjointable projection. These projections form an orthogonal family

$$\Phi_l\Phi_k = \delta_{l,k}\Phi_k$$

by Lemma 4.2.12, and it is straightforward to show that $\sum_{k\in\mathbb{Z}}\Phi_k$ is the identity of Z (convergence in the strict topology). The arguments used in [PR06] show that given $z \in Z$ and defining $\Phi_k z = z_k$, we have that

$$z = \sum_{k\in\mathbb{Z}} z_k.$$

This allows us to define a number operator on the finite span

$$Nz = \sum_k k z_k,$$

whose closure has domain $\mathrm{Dom}(N) = \left\{\sum_k z_k : \sum_k k^2 (z_k|z_k)_{C^*(\widehat{U})} < \infty\right\}$. As N is given in its spectral representation, standard proofs show that N is self-adjoint (again, see [PR06] for an explicit proof).

To show that N is regular, we observe that

$$N^2\left(\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}\right) = (n_1-n_2)^2\,\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}$$

and so N^2 has the spanning set of $\mathcal{T}$ as eigenvectors. Therefore $(1+N^2)$ has dense range and so N is regular.

To check that we have an unbounded Kasparov module, we need to show that $[N, a]$ is a bounded endomorphism for a in a smooth dense subalgebra $\mathcal{A}_\phi \subset A_\phi$ and that $(1+N^2)^{-1/2} \in \mathrm{End}^0_{C^*(\widehat{U})}(Z)$. We have that, for $\beta \geq 0$

$$\begin{aligned} N(\widehat{U}^\alpha \widehat{V}^\beta)\left(\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}\right) \\ &= N\left(e^{2\pi i\phi\alpha(n_1-n_2+\beta)}\widehat{V}^{n_1-n_2+\beta}\widehat{U}^{m+\alpha} \otimes S^{n_1+\beta}(S^*)^{n_2}\right) \\ &= (n_1-n_2+\beta)e^{2\pi i\phi\alpha(n_1-n_2+\beta)}\widehat{V}^{n_1-n_2+\beta}\widehat{U}^{m+\alpha} \otimes S^{n_1+\beta}(S^*)^{n_2} \end{aligned}$$

and

$$\begin{aligned} (\widehat{U}^\alpha \widehat{V}^\beta)N\left(\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}\right) \\ &= (n_1-n_2)e^{2\pi i\phi\alpha(n_1-n_2+\beta)}\widehat{V}^{n_1-n_2+\beta}\widehat{U}^{m+\alpha} \otimes S^{n_1+\beta}(S^*)^{n_2}, \end{aligned}$$

which implies that $[N, \widehat{U}^\alpha\widehat{V}^\beta] = \beta\widehat{U}^\alpha\widehat{V}^\beta$ since the span of $\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}$ is dense in the domain of N in the graph norm. If we take elements $a \in \mathcal{A}_\phi$ to be $a = \sum_{\alpha,\beta} a_{\alpha,\beta}\widehat{U}^\alpha\widehat{V}^\beta$ with $(a_{\alpha,\beta}) \in \mathcal{S}(\mathbb{Z}^2)$, the Schwartz class sequences, then $\mathcal{A}_\phi$ is dense and we have that

$$[N, a] = \sum_{\alpha,\beta} \beta a_{\alpha,\beta}\widehat{U}^\alpha\widehat{V}^\beta$$

is in $\mathcal{A}_\phi$ as $\beta a_{\alpha,\beta} \in \mathcal{S}(\mathbb{Z}^2)$ and therefore is bounded. An entirely analogous argument also works for $\beta < 0$.

Finally, we recall that N^2 has a set of eigenvectors given by the spanning functions $\left\{\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2} \,:\, n_1, n_2 \in \mathbb{N},\, m \in \mathbb{Z}\right\}$. This means that we can write

$$N^2 = \bigoplus_{k\in\mathbb{Z}} k^2 \Phi_k$$

where Φ_k is the projection onto $\mathrm{span}\{z_{n_1,n_2,m} \in Z_{C^*(\widehat{U})} : n_1 - n_2 = k,\, m \in \mathbb{Z}\}$. As the projections Φ_k can be written as a rank one operator $\Theta_{z_{n_1,n_2,0}, z_{n_1,n_2,0}} \in \mathrm{End}^{00}_{C^*(\widehat{U})}(Z)$, we have that

$$(1+N^2)^{-1/2} = \bigoplus_{k\in\mathbb{Z}} \left(1+k^2\right)^{-1/2} \Phi_k$$

is a norm-convergent sum of elements in $\mathrm{End}^{00}_{C^*(\widehat{U})}(Z)$ and is therefore in $\mathrm{End}^0_{C^*(\widehat{U})}(Z)$. $\square$

4.2.4 A left module with A_ϕ^{op}-action

The module $Z_{C^*(\widehat{U})}$ has more structure. It is in fact a left C^*-module over $C^*(\widehat{U})$ where we define an inner-product by

$$\begin{aligned}
&{}_{C^*(\widehat{U})}\left(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\otimes S^{l_1}(S^*)^{l_2}\,\middle|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2}\right)\\
&\qquad = \widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\left(\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\right)^*\Psi\left[S^{l_1}(S^*)^{l_2}\left(S^{n_1}(S^*)^{n_2}\right)^*\right]\\
&\qquad = \widehat{V}^{l_1-l_2}\widehat{U}^{m_1-m_2}\widehat{V}^{n_2-n_1}\delta_{l_1-l_2,n_1-n_2}\\
&\qquad = \eta^{-1}_{n_1-n_2}(\widehat{U}^{m_1-m_2})\delta_{l_1-l_2,n_1-n_2},
\end{aligned}$$

recalling that $\eta_n(\widehat{U}^m) = \widehat{V}^{-n}\widehat{U}^m\widehat{V}^n$ is the automorphism defining the crossed-product structure. We check compatibility of ${}_{C^*(\widehat{U})}(\,\cdot\mid\cdot\,)$ with left-multiplication by $C^*(\widehat{U})$, where

$$\begin{aligned}
&{}_{C^*(\widehat{U})}\left(\widehat{U}\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\otimes S^{l_1}(S^*)^{l_2}\,\middle|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2}\right)\\
&\qquad = \widehat{U}\widehat{V}^{l_1-l_2}\widehat{U}^{m_1-m_2}\widehat{V}^{n_1-n_1}\delta_{l_1-l_2,n_1-n_2}\\
&\qquad = \widehat{U}\cdot{}_{C^*(\widehat{U})}\left(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\otimes S^{l_1}(S^*)^{l_2}\,\middle|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2}\right).
\end{aligned}$$

The other axioms for a left $C^*(\widehat{U})$-valued inner-product are straightforward. We complete in the induced norm and denote our left-module by ${}_{C^*(\widehat{U})}Z$.

Proposition 4.2.14. *There is an adjointable representation of $A_{-\phi}\cong A_\phi^{\mathrm{op}}$ on ${}_{C^*(\widehat{U})}Z$.*

Proof. We construct an action by $C^*(U,V)\cong A_\phi^{\mathrm{op}}$ by defining

$$\begin{aligned}
U\cdot\left(\widehat{V}^{n_1-n_2}\widehat{U}^{m}\otimes S^{n_1}(S^*)^{n_2}\right) &= \left(\widehat{V}^{n_1-n_2}\widehat{U}^{m}\otimes S^{n_1}(S^*)^{n_2}\right)\cdot\widehat{U}\\
&= \widehat{V}^{n_1-n_2}\widehat{U}^{m+1}\otimes S^{n_1}(S^*)^{n_2},\\
V\cdot\left(\widehat{V}^{n_1-n_2}\widehat{U}^{m}\otimes S^{n_1}(S^*)^{n_2}\right) &= \left(\widehat{V}^{n_1-n_2}\widehat{U}^{m}\otimes S^{n_1}(S^*)^{n_2}\right)\cdot\widehat{V}\\
&= e^{2\pi i\phi m}\widehat{V}^{n_1-n_2+1}\widehat{U}^{m}\otimes S^{n_1+1}(S^*)^{n_2}.
\end{aligned}$$

One finds that $UV = e^{-2\pi i\phi}VU$ as operators on ${}_{C^*(\widehat{U})}Z$. As previously, we check adjointability on generating elements, where

$$\begin{aligned}
&{}_{C^*(\widehat{U})}\left(U\cdot\left(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\otimes S^{l_1}(S^*)^{l_2}\right)\,\middle|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2}\right)\\
&\qquad = \eta^{-1}_{n_1-n_2}(\widehat{U}^{m_1+1-m_2})\delta_{n_1-n_2,l_1-l_2}\\
&\qquad = {}_{C^*(\widehat{U})}\left(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\otimes S^{l_1}(S^*)^{l_2}\,\middle|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2-1}\otimes S^{n_1}(S^*)^{n_2}\right)\\
&\qquad = {}_{C^*(\widehat{U})}\left(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\otimes S^{l_1}(S^*)^{l_2}\,\middle|\,U^{-1}\cdot\left(\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2}\right)\right)
\end{aligned}$$

as expected. For V, we find that

$$
\begin{aligned}
&{}_{C^*(\widehat{U})}\Big(V\cdot\Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\otimes S^{l_1}(S^*)^{l_2}\Big)\Big|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2}\Big)\\
&\quad={}_{C^*(\widehat{U})}\Big(e^{2\pi i\phi m_1}\widehat{V}^{l_1-l_2+1}\widehat{U}^{m_1}\otimes S^{l_1+1}(S^*)^{l_2}\,\Big|\,\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2}\Big)\\
&\quad=e^{2\pi i\phi m_1}\widehat{V}^{l_1-l_2+1}\widehat{U}^{m_1-m_2}\widehat{V}^{n_2-n_1}\delta_{l_1-l_2+1,n_1-n_2}\\
&\quad=e^{2\pi i\phi m_1}e^{-2\pi i\phi(m_1-m_2)}\widehat{V}^{l_1-l_2}\widehat{U}^{m_1-m_2}\widehat{V}^{n_2-n_1+1}\delta_{l_1-l_2,n_1-n_2-1}\\
&\quad={}_{C^*(\widehat{U})}\Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\otimes S^{l_1}(S^*)^{l_2}\,\Big|\,e^{-2\pi i\phi m_2}\widehat{V}^{n_1-n_2-1}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2+1}\Big)\\
&\quad={}_{C^*(\widehat{U})}\Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\otimes S^{l_1}(S^*)^{l_2}\,\Big|\,(\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2})\widehat{V}^{-1}\Big)\\
&\quad={}_{C^*(\widehat{U})}\Big(\widehat{V}^{l_1-l_2}\widehat{U}^{m_1}\otimes S^{l_1}(S^*)^{l_2}\,\Big|\,V^{-1}\cdot\Big(\widehat{V}^{n_1-n_2}\widehat{U}^{m_2}\otimes S^{n_1}(S^*)^{n_2}\Big)\Big)
\end{aligned}
$$

and so our generating elements are adjointable and unitary on the dense span of monomials in ${}_{C^*(\widehat{U})}Z$. Thus if U, V are bounded, they will generate an adjointable representation of A_ϕ^{op}. To consider the boundedness of U and V, we first note that the inner-product in ${}_{C^*(\widehat{U})}Z$ is defined from multiplication in $\mathcal{T}$ and the functional Ψ, which has the property $\Psi(T)\leq\|T\|$, by Equation (4.2). These observations imply that

$$
\|a^{\mathrm{op}}\|_{\mathrm{End}(Z)}=\sup_{\substack{z\in Z\\ \|z\|=1}}{}_{C^*(\widehat{U})}(a^{\mathrm{op}}\cdot z\mid a^{\mathrm{op}}\cdot z)\leq\sup_{\substack{z\in Z\\ \|z\|=1}}\|a^{\mathrm{op}}(a^{\mathrm{op}})^*\|\,{}_{C^*(\widehat{U})}(z\mid z)=\|a^{\mathrm{op}}(a^{\mathrm{op}})^*\|.
$$

Therefore the action of A_ϕ^{op} is bounded, and so extends to an adjointable action on ${}_{C^*(\widehat{U})}Z$. ☐

Remark 4.2.15. Our construction of ${}_{C^*(\widehat{U})}Z$ shows that Z can be equipped with a bimodule structure over $C^*(\widehat{U})$. Proposition 4.2.10 and 4.2.14 show that the right (resp. left) module comes with an adjointable representation of A_ϕ (resp. A_ϕ^{op}). However, we emphasise that the representation of A_ϕ (resp. A_ϕ^{op}) is *not* adjointable on the left (resp. right) module.

Another point to note is that the actions of A_ϕ and A_ϕ^{op} on Z commute. To see this, we compute that

$$
\widehat{U}V\Big(\widehat{V}^{n_1-n_2}\widehat{U}^{m}\otimes V^{n_1}(V^*)^{n_2}\Big)=e^{2\pi i\phi(n_1-n_2+1)}e^{2\pi i\phi m}\widehat{V}^{n_1-n_2+1}\widehat{U}^{m+1}\otimes V^{n_1}(V^*)^{n_2}
$$

and

$$
V\widehat{U}\Big(\widehat{V}^{n_1-n_2}\widehat{U}^{m}\otimes V^{n_1}(V^*)^{n_2}\Big)=e^{2\pi i\phi(m+1)}e^{2\pi i\phi(n_1-n_2)}\widehat{V}^{n_1-n_2+1}\widehat{U}^{m+1}\otimes V^{n_1}(V^*)^{n_2}.
$$

Hence we see that $[\widehat{U},V]=0$ (and similarly for other generators). Once again, we reiterate that these actions cannot be considered as simultaneous representations on the level of right or left $C^*(\widehat{U})$-modules.

All the technical results in Section 4.2.3 about the singular trace Ψ still hold in the left-module setting. In particular, a completely analogous argument to the proof of Proposition 4.2.13 gives us the following.

Proposition 4.2.16. *The tuple* $\left(\mathcal{A}_\phi^{\mathrm{op}}, {}_{C^*(\widehat{U})}Z, N\right)$ *is an odd, unbounded* $\mathcal{A}_\phi^{\mathrm{op}}$-$C^*(\widehat{U})^{\mathrm{op}}$ *Kasparov module.*

4.2.5 Relating the module to the extension class

Now we put the pieces together. By [Kas81, Section 7], the extension class associated to $\left(\mathcal{A}_\phi, Z_{C^*(\widehat{U})}, N\right)$ comes from the short exact sequence

$$0 \to \mathrm{End}^0_{C^*(\widehat{U})}(PZ) \to C^*(PA_\phi P, \mathrm{End}^0_{C^*(\widehat{U})}(PZ)) \to A_\phi \to 0, \tag{4.3}$$

where $P = \chi_{[0,\infty)}(N)$ is the non-negative spectral projection and we have added a degenerate module if necessary to ensure the Busby map is injective (see the discussion following Theorem 2.2.37).

We have that the map $Q : Z \to \ell^2(\mathbb{Z}) \otimes C^*(\widehat{U})$ given by

$$Q\left(\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2}\right) = e_{n_1-n_2} \otimes \widehat{U}^m$$

is an adjointable unitary isomorphism with adjoint

$$Q^*(e_n \otimes \widehat{U}^m) \mapsto \widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2},$$

where n_1, n_2 are *any* natural numbers such that $n = n_1 - n_2$ (cf. Proposition 4.2.11). Conjugation by Q gives an explicit isomorphism $\mathrm{End}^0_{C^*(\widehat{U})}(PZ) \cong \mathcal{K}[\ell^2(\mathbb{N})] \otimes C^*(\widehat{U})$. This isomorphism is compatible with the sequence in Equation (4.3) in that the commutators $[P, S^k]$ and $[P, (S^*)^k]$ generate $\mathcal{K}[\ell^2(\mathbb{N})]$. With a suitable identification, the map

$$\mathrm{End}^0_{C^*(\widehat{U})}(PZ) \overset{\iota}{\hookrightarrow} C^*(PA_\phi P, \mathrm{End}^0_{C^*(\widehat{U})}(PZ))$$

is just inclusion.

Now define the isomorphism $\zeta : C^*(PA_\phi P, \mathrm{End}^0_{C^*(\widehat{U})}(PZ)) \to \mathcal{T}$ by

$$\zeta(P\widehat{V}^n P) = (\widehat{V} \otimes S)^n, \qquad \zeta(P\widehat{V}^{-n} P) = [(\widehat{V} \otimes S)^*]^n$$

for $n \geq 0$ and

$$\zeta(P\widehat{U}^m P) = \widehat{U}^m \otimes 1, \quad \zeta(S^j(1 - SS^*)(S^*)^k) = (\widehat{V}^* \otimes S)^j(1 \otimes 1 - SS^*)(\widehat{V} \otimes S^*)^k$$

and extend accordingly. Then we have that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \otimes C^*(\widehat{U}) \arrow[r] & C^*(\widehat{U} \otimes 1, \widehat{V} \otimes S) \arrow[r] & A_\phi \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{End}^0_{C^*(\widehat{U})}(P\mathcal{T}) \arrow[r] \arrow[u, "\cong"', "\mathrm{Ad}Q"] & C^*(PA_\phi P, \mathrm{End}^0_{C^*(\widehat{U})}(PZ)) \arrow[r] \arrow[u, "\cong"', "\zeta"] & A_\phi \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

commutes, and so these extensions are unitarily equivalent. We summarise this section by the following.

Proposition 4.2.17. *The extension class representing the short exact sequence of Equation (4.1) is the same as the class represented by the unbounded Kasparov module* $\left(\mathcal{A}_\phi, Z_{C^*(\widehat{U})}, N\right)$ *in* $KK^1(A_\phi, C^*(\widehat{U}))$.

4.3 The bulk-edge correspondence and Kasparov theory

4.3.1 Overview of the main result

Once again recall the short exact sequence

$$0 \to C^*(\widehat{U}) \otimes \mathcal{K}[\ell^2(\mathbb{N})] \xrightarrow{\psi} \mathcal{T} \to A_\phi \to 0.$$

The ideal is considered as our boundary data, as we can consider it acting on $\ell^2(\mathbb{Z}\times\mathbb{N})$ but with compact operators acting in the direction perpendicular to the boundary. The quotient A_ϕ describes a quantum Hall system in the absence of the boundary.

There is a spectral triple coming from the discrete quantum Hall effect without disorder or boundaries related to the results in Chapter 3 (in turn based off [BvS94]). We use the notation $\left(\mathcal{A}_{-\phi}, \ell^2(\mathbb{Z}^2)\oplus\ell^2(\mathbb{Z}^2), X, \gamma\right)$ for this triple, where X is the matrix $\begin{pmatrix} 0 & X_1 - iX_2 \\ X_1 + iX_2 & 0 \end{pmatrix}$ and X_j are the position (or, equivalently, number) operators on $\ell^2(\mathbb{Z}^2)$ for $j = 1, 2$. The quantum Hall spectral triple represents a class in $KK^0(A_{-\phi}, \mathbb{C})$,

We also have the natural spectral triple for $\mathcal{B}$ a dense $*$-subalgebra of $C^*(\widehat{U})$ that gives us a class $\left[(\mathcal{B}, \ell^2(\mathbb{Z})_\mathbb{C}, M)\right] \in KK^1(C(S^1),\mathbb{C}) \cong KK^1(C^*(\widehat{U})\otimes\mathcal{K},\mathbb{C})$ for M the position/number operator on $\ell^2(\mathbb{Z})$. Our idea is to use the Kasparov module that represents the Toeplitz extension to relate the bulk and boundary spectral triples via the internal Kasparov product. Namely, we claim that, under the map

$$KK^1(A_\phi, C^*(\widehat{U})) \times KK^1(C^*(\widehat{U}),\mathbb{C}) \to KK^0(A_\phi,\mathbb{C}),$$

we have that

$$\left[(\mathcal{A}_\phi, Z_{C^*(\widehat{U})}, N)\right] \hat{\otimes}_{C^*(\widehat{U})} \left[(\mathcal{B}, \ell^2(\mathbb{Z})_\mathbb{C}, M)\right] = -\left[(\mathcal{A}_\phi, \ell^2(\mathbb{Z}^2)_\mathbb{C}, X, \Gamma)\right].$$

Of course, our original boundary-free spectral triple is in $K^0(A_{-\phi})$, not $K^0(A_\phi)$. By using the extra structure coming from the left-module $\left(\mathcal{A}_\phi^{\mathrm{op}}, {}_{C^*(\widehat{U})}\mathcal{T}, N\right)$, we are able to resolve this discrepancy and obtain Bellissard's spectral triple from the product module up to an explicit unitary equivalence.

4.3.2 The details

The boundary spectral triple and the product

We have our module $\beta = \left(\mathcal{A}_\phi, Z_{C^*(\widehat{U})}, N\right)$ giving rise to a class in $KK^1(A_\phi, C^*(\widehat{U}))$. We now obtain our 'boundary module' by considering the space $\ell^2(\mathbb{Z})$ with action of

$C^*(\widehat{U})$ by translations; i.e, $(\widehat{U}\lambda)(m) = \lambda(m-1)$. We have a natural spectral triple in this setting denoted by $\Delta = \big(\mathcal{B}, \ell^2(\mathbb{Z}), M\big)$, where $\mathcal{B}$ is a dense $*$-subalgebra of $C^*(\widehat{U})$ and $M : \mathrm{Dom}(M) \to \ell^2(\mathbb{Z})$ is given by $M\lambda(m) = m\lambda(m)$. It is a simple exercise to show that $\big(\mathcal{B}, \ell^2(\mathbb{Z}), M\big)$ is indeed a spectral triple and therefore an odd, unbounded $\mathcal{B}$-$\mathbb{C}$ Kasparov module. This is also what we would expect for a boundary system as the operator M becomes the Dirac operator on the circle if we switch to momentum space by the Fourier transform. Our goal is to take the internal Kasparov product over $\mathcal{B} \subset C^*(\widehat{U})$ and obtain a class in $KK^0(A_\phi, \mathbb{C})$, which we then link to Bellisard's spectral triple modelling a boundaryless quantum Hall system.

We take the product $\beta \hat{\otimes}_{C^*(\widehat{U})} \Delta$ in the unbounded setting (see Chapter 2.2.3 for an overview of the unbounded product).

Lemma 4.3.1. *The Kasparov product of the unbounded modules* $\beta = \Big(\mathcal{A}_\phi, Z_{C^*(\widehat{U})}, N\Big)$ *and* $\Delta = \big(\mathcal{B}, \ell^2(\mathbb{Z}), M\big)$ *is given by*

$$\beta \hat{\otimes}_{C^*(\widehat{U})} \Delta = -\left[\left(\mathcal{A}_\phi, \begin{pmatrix} Z \otimes_{C^*(\widehat{U})} \ell^2(\mathbb{Z}) \\ Z \otimes_{C^*(\widehat{U})} \ell^2(\mathbb{Z}) \end{pmatrix}_{\mathbb{C}}, \begin{pmatrix} 0 & 1 \otimes_\nabla M - iN \otimes 1 \\ 1 \otimes_\nabla M + iN \otimes 1 & 0 \end{pmatrix}\right)\right],$$

where $\mathcal{A}_\phi$ *acts diagonally and* $\nabla : \mathcal{Z} \to \mathcal{Z} \otimes_{\mathrm{poly}(\widehat{U})} \Omega^1(\mathrm{poly}(\widehat{U}))$ *is a connection on a smooth submodule* $\mathcal{Z}$ *of* Z*. The overall minus sign means the negative of this class in* $KK(A_\phi, \mathbb{C})$.

Proof. It is proved in [KL13, Theorem 7.5] that the KK-class of the product

$$\Big[\Big(\mathcal{A}_\phi, Z_{C^*(\widehat{U})}, N\Big)\Big] \hat{\otimes}_{\mathcal{B}} \big[\big(\mathcal{B}, \ell^2(\mathbb{Z}), M\big)\big]$$

is represented by

$$\left(\mathcal{A}_\phi, \begin{pmatrix} Z \otimes_{C^*(\widehat{U})} \ell^2(\mathbb{Z}) \\ Z \otimes_{C^*(\widehat{U})} \ell^2(\mathbb{Z}) \end{pmatrix}_{\mathbb{C}}, \begin{pmatrix} 0 & N \otimes 1 - i 1 \otimes_\nabla M \\ N \otimes 1 + i 1 \otimes_\nabla M & 0 \end{pmatrix}\right).$$

There are several conditions to check in order to apply [KL13, Theorem 7.5], but the product we are taking turns out to be of the simplest kind, and we omit these simple checks. Here A_ϕ acts diagonally on column vectors, and the grading is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. To define $1 \otimes_\nabla M$, we let $\mathcal{Z}_{C^*(\widehat{U})}$ be the submodule of Z given by finite sums of elements $z_{n_1,n_2,m} = \widehat{V}^{n_1-n_2} \widehat{U}^m \otimes S^{n_1} (S^*)^{n_2}$ and take the connection

$$\nabla : \mathcal{Z} \to \mathcal{Z} \otimes_{\mathrm{poly}(\widehat{U})} \Omega^1(\mathrm{poly}(\widehat{U}))$$

given by

$$\nabla\left(\sum_{n_1,n_2,m} z_{n_1,n_2,m}\right) = \sum_{n_1,n_2} z_{n_1,n_2,0} \otimes \delta(\widehat{U}^m),$$

where δ is the universal derivation, and we represent 1-forms on $\ell^2(\mathbb{Z})$ via

$$\tilde{\pi}\left(a_0\delta(a_1)\right)\lambda = a_0[M,a_1]\lambda$$

for $\lambda \in \ell^2(\mathbb{Z})$. We define

$$(1\otimes_\nabla M)(z\otimes\lambda) := (z\otimes M\lambda) + (1\otimes\tilde{\pi})\circ(\nabla\otimes 1)(z\otimes\lambda).$$

The need to use to a connection to correct the naive formula $1\otimes M$ is because $1\otimes M$ is not well-defined on the balanced tensor product. Computing yields that

$$\begin{aligned}(1\otimes_\nabla M)\left(\sum_{n_1,n_2,\beta} z_{n_1,n_2,\beta}\otimes\lambda\right) &= \sum_{n_1,n_2} z_{n_1,n_2,0}\otimes\widehat{U}^\beta M\lambda + \sum_{n_1,n_2} z_{n_1,n_2}\otimes[M,\widehat{U}^\beta]\lambda \\ &= \sum_{n_1,n_2} z_{n_1,n_2}\otimes M\widehat{U}^\beta\lambda.\end{aligned}$$

Now conjugating the representation, operator and grading by $\begin{pmatrix}0 & i\\ 1 & 0\end{pmatrix}$ yields the unitarily equivalent spectral triple

$$\left(\mathcal{A}_\phi, \begin{pmatrix} Z\otimes_{C^*(\widehat{U})}\ell^2(\mathbb{Z}) \\ Z\otimes_{C^*(\widehat{U})}\ell^2(\mathbb{Z})\end{pmatrix}_{\mathbb{C}}, \begin{pmatrix} 0 & -(1\otimes_\nabla M - iN\otimes 1) \\ -(1\otimes_\nabla M + iN\otimes 1) & 0\end{pmatrix}\right)$$

with grading $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$. In turn, the KK-class of this spectral triple is given by

$$-\left[\left(\mathcal{A}_\phi, \begin{pmatrix} Z\otimes_{C^*(\widehat{U})}\ell^2(\mathbb{Z}) \\ Z\otimes_{C^*(\widehat{U})}\ell^2(\mathbb{Z})\end{pmatrix}_{\mathbb{C}}, \begin{pmatrix} 0 & 1\otimes_\nabla M - iN\otimes 1 \\ 1\otimes_\nabla M + iN\otimes 1 & 0\end{pmatrix}\right)\right]$$

with grading $\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. $\square$

Our task now is to relate the product Kasparov module to the boundary-free quantum Hall system.

Equivalence of the product triple and boundary-free triple

Recall once again [BvS94, MC96] our 'bulk' spectral triple

$$\left(\mathcal{A}_{-\phi}, \begin{pmatrix}\ell^2(\mathbb{Z}^2)\\ \ell^2(\mathbb{Z}^2)\end{pmatrix}_{\mathbb{C}}, \begin{pmatrix} 0 & X_1 - iX_2 \\ X_1 + iX_2 & 0\end{pmatrix}, \gamma = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right),$$

where $(X_1 \pm iX_2)\lambda(m,n) = (m\pm in)\lambda(m,n)$ for $\lambda\in\mathrm{Dom}(M\pm iN)\subset\ell^2(\mathbb{Z}^2)$ and $\mathcal{A}_{-\phi}\cong C^*(U,V)$ has the representation generated by

$$(U\lambda)(m,n) = e^{-2\pi i\phi n}\lambda(m-1,n), \qquad (V\lambda)(m,n) = \lambda(m,n-1),$$

with $H = U + U^* + V + V^*$ and $\lambda \in \ell^2(\mathbb{Z}^2)$.

Analogous arguments as in Lemma 4.2.3 and Lemma 4.2.4 tell us that $C^*(U,V)$ gives a right σ-representation of $\mathbb{Z}^2$ and there is a corresponding left $\overline{\sigma}$-representation of $\mathbb{Z}^2$ by $C^*(\widehat{U},\widehat{V})$ commuting with the right representation, where $\sigma(k,k') = e^{2\pi i\phi k_1' k_2}$ (cf. [MC96]). Because $C^*(U,V) \cong A_{-\phi} \cong A_\phi^{\mathrm{op}}$, we obtain the following.

Proposition 4.3.2. *The tuple*

$$\left(\mathcal{A}_\phi \otimes \mathcal{A}_\phi^{\mathrm{op}}, \ell^2(\mathbb{Z}^2) \oplus \ell^2(\mathbb{Z}^2), \begin{pmatrix} 0 & X_1 - iX_2 \\ X_1 + iX_2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

defines an even spectral triple.

Proof. The only thing we need to check is that our Dirac-type operator has bounded commutators with a smooth subalgebra of $C^*(\widehat{U},\widehat{V})$. Simple computations show that

$$[X_1,\widehat{U}] = \widehat{U}, \qquad [X_2,\widehat{U}] = 0$$
$$[X_1,\widehat{V}] = 0, \qquad [X_2,\widehat{V}] = \widehat{V}.$$

Hence these commutators will be bounded for finite polynomials of $\widehat{U}$ and $\widehat{V}$, which are dense in $C^*(\widehat{U},\widehat{V})$. □

Our aim is to reproduce this spectral triple via an explicit unitary equivalence with the module we have constructed via the Kasparov product. We state our central result.

Theorem 4.3.3. *Let $\varrho : Z_{C^*(\widehat{U})} \otimes_{C^*(\widehat{U})} \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}^2)$ be the map*

$$\varrho\left(\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2} \otimes_{C^*(\widehat{U})} e_j\right) = e^{-2\pi i\phi(j+m)(n_1-n_2)} e_{j+m,n_1-n_2},$$

where e_j and $e_{j,k}$ are the standard basis elements of $\ell^2(\mathbb{Z})$ and $\ell^2(\mathbb{Z}^2)$ respectively. Then there is a representation of $A_\phi \otimes A_\phi^{\mathrm{op}}$ on $Z \otimes_{C^*(\widehat{U})} \ell^2(\mathbb{Z})$ such that ϱ gives a unitary equivalence between the spectral triple

$$\left(\mathcal{A}_\phi \otimes \mathcal{A}_\phi^{\mathrm{op}}, \begin{pmatrix} Z \otimes_{C^*(\widehat{U})} \ell^2(\mathbb{Z}) \\ Z \otimes_{C^*(\widehat{U})} \ell^2(\mathbb{Z}) \end{pmatrix}, \begin{pmatrix} 0 & 1\otimes_\nabla M - iN\otimes 1 \\ 1\otimes_\nabla M + iN\otimes 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$$

arising from the product triple of Lemma 4.3.1 and the bulk quantum Hall triple in Proposition 4.3.2.

Proof. We first check that, by moving elements of $C^*(\widehat{U})$ across the balanced tensor product,

$$\begin{aligned}
\widehat{V}^{n_1-n_2}\widehat{U}^m \otimes S^{n_1}(S^*)^{n_2} \otimes_{C^*(\widehat{U})} e_j &= (\widehat{V}^{n_1-n_2} \otimes S^{n_1}(S^*)^{n_2}) \cdot \widehat{U}^m \otimes_{C^*(\widehat{U})} e_j \\
&= \widehat{V}^{n_1-n_2} \otimes S^{n_1}(S^*)^{n_2} \otimes_{C^*(\widehat{U})} \widehat{U}^m \cdot e_j \\
&= \widehat{V}^{n_1-n_2} \otimes S^{n_1}(S^*)^{n_2} \otimes_{C^*(\widehat{U})} e_{j+m},
\end{aligned}$$

we see that the map ϱ respects the inner-products on $Z\hat{\otimes}_{C^*(\widehat{U})}\ell^2(\mathbb{Z})$ and on $\ell^2(\mathbb{Z}^2)$. Hence ϱ is an isometric isomorphism between Hilbert spaces.

Next we need to define a commuting representation of A_ϕ^{op} on our product module. We can do this by pulling back the representation of $C^*(U,V)$ on $\ell^2(\mathbb{Z}^2)$ via the isomorphism ϱ. Alternatively, the same representation comes from the left action of A_ϕ^{op} on ${}_{C^*(\widehat{U})}Z$, the module we constructed in Section 4.2.4. We first note that generating elements of $Z_{C^*(\widehat{U})}\otimes_{C^*(\widehat{U})}\ell^2(\mathbb{Z})$ can be written as $\widehat{V}^{n_1-n_2}\otimes S^{n_1}(S^*)^{n_2}\otimes e_j$ for some $j\in\mathbb{Z}$ and $n_1,n_2\in\mathbb{N}$. Then

$$U^\alpha V^\beta\cdot\Big(\widehat{V}^{n_1-n_2}\otimes S^{n_1}(S^*)^{n_2}\otimes e_j\Big)=e^{2\pi i\phi\beta j}\widehat{V}^{n_1-n_2+\beta}\otimes S^{n_1+\beta}(S^*)^{n_2}\otimes e_{j+\alpha}$$

for $\beta\geq 0$. A similar formula but replacing $S^{n_1+\beta}(S^*)^{n_2}$ with $S^{n_1}(S^*)^{n_2+|\beta|}$ gives the action for $\beta<0$. This left-action of A_ϕ^{op} is compatibile with the isomorphism, that is,

$$\varrho\Big[U^\alpha V^\beta\cdot\Big(\widehat{V}^{n_1-n_2}\widehat{U}^m\otimes S^{n_1}(S^*)^{n_2}\otimes e_j\Big)\Big]=U^\alpha V^\beta\cdot\varrho\Big(\widehat{V}^{n_1-n_2}\widehat{U}^m\otimes S^{n_1}(S^*)^{n_2}\otimes e_j\Big)$$

and this relation extends appropriately.

What remains to check is that the map ϱ is compatible with the representation of A_ϕ and the Dirac-type operator. That is, we need to show that

$$\varrho\Big[\widehat{U}^\alpha\widehat{V}^\beta\cdot\Big(\widehat{V}^{n_1-n_2}\otimes S^{n_1}(S^*)^{n_2}\otimes e_j\Big)\Big]=\widehat{U}^\alpha\widehat{V}^\beta\cdot\varrho\Big(\widehat{V}^{n_1-n_2}\otimes S^{n_1}(S^*)^{n_2}\otimes e_j\Big)$$

and

$$\begin{aligned}\varrho\Big[(1\otimes_\nabla M\pm iN\otimes 1)\Big(\widehat{V}^{n_1-n_2}\otimes S^{n_1}(S^*)^{n_2}\otimes e_j\Big)\Big]&\\ =(X_1\pm iX_2)\cdot\varrho\Big(\widehat{V}^{n_1-n_2}\otimes S^{n_1}(S^*)^{n_2}\otimes e_j\Big)&.\end{aligned}$$

For the first claim, more computations give that, for $\beta\geq 0$,

$$\begin{aligned}\varrho\Big[\widehat{U}^\alpha\widehat{V}^\beta\cdot\Big(\widehat{V}^{n_1-n_2}\otimes S^{n_1}(S^*)^{n_2}\otimes e_j\Big)\Big]&\\ &=\varrho\Big(e^{2\pi i\phi\alpha(\beta+n_1-n_2)}\widehat{V}^{n_1-n_2+\beta}\otimes S^{n_1+\beta}(S^*)^{n_2}\otimes e_{j+\alpha}\Big)\\ &=e^{2\pi i\phi\alpha(\beta+n_1-n_2)}e^{-2\pi i\phi(j+\alpha)(\beta+n_1-n_2)}e_{j+\alpha,n_1-n_2+\beta}\\ &=e^{-2\pi i\phi j(\beta+n_1-n_2)}e_{j+\alpha,n_1-n_2+\beta}\end{aligned}$$

and

$$\begin{aligned}\widehat{U}^\alpha\widehat{V}^\beta\cdot\varrho\Big(\widehat{V}^{n_1-n_2}\otimes S^{n_1}(S^*)^{n_2}\otimes e_j\Big)&=\widehat{U}^\alpha\widehat{V}^\beta e^{-2\pi i\phi j(n_1-n_2)}e_{j,n_1-n_2}\\ &=e^{-2\pi i\phi j\beta}e^{-2\pi i\phi j(n_1-n_2)}e_{j+\alpha,n_1-n_2+\beta}.\end{aligned}$$

Again, the case for $\beta<0$ is basically identical. Because the result holds on generating elements, it extends to the whole algebra and space. For the second claim, we once more check the result on spanning elements. We recall the construction of $1\otimes_\nabla M$, where

$$\begin{aligned}(1\otimes_\nabla M)\left(\widehat{V}^{n_1-n_2}\otimes S^{n_1}(S^*)^{n_2}\otimes e_j\right) &= \widehat{V}^{n_1-n_2}\otimes S^{n_1}(S^*)^{n_2}\otimes M\widehat{U}^0 e_j \\ &= j\left(\widehat{V}^{n_1-n_2}\otimes S^{n_1}(S^*)^{n_2}\otimes e_j\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\varrho\Big[(1\otimes_\nabla M \pm iN\otimes 1)\Big(\widehat{V}^{n_1-n_2}\otimes S^{n_1}(S^*)^{n_2}\otimes e_j\Big)\Big] \\ &\qquad = (j\pm i(n_1-n_2))\varrho\Big(\widehat{V}^{n_1-n_2}\otimes S^{n_1}(S^*)^{n_2}\otimes e_j\Big) \\ &\qquad = (j\pm i(n_1-n_2))e^{-2\pi i\phi j(n_1-n_2)}e_{j,n_1-n_2} \\ &\qquad = (X_1\pm iX_2)\,\varrho\Big(\widehat{V}^{n_1-n_2}\otimes S^{n_1}(S^*)^{n_2}\otimes e_m\Big)\end{aligned}$$

and the main result follows by extending in the standard way. □

Remark 4.3.4 (Factorisation and Poincaré duality). In the proof of Theorem 4.3.3, the bimodule structure of Z can be used to obtain the left-action of A_ϕ^{op} on the product module. An important observation is that we can either take the Kasparov product of $\left(\mathcal{A}_\phi, Z_{C^*(\widehat{U})}, N\right)$ or $\left(\mathcal{A}_\phi^{\text{op}}, {}_{C^*(\widehat{U})}Z, N\right)$ with our boundary module *and the resulting module is the same.* Hence we pick up an extra representation on our product module, which is necessary in order to completely link up the product module to the bulk spectral triple. The deeper meaning behind this extra structure is related to Poincaré duality for A_ϕ: see [Con96] for more information.

By setting up a unitary equivalence of spectral triples, we can conclude that the K-homological data encoded in Bellissard's spectral triple is the same as that presented by the product module we have constructed. The unitary equivalence is of course much stronger than just stable homotopy equivalence on the level of K-homology.

4.3.3 Pairings with K-Theory and the edge conductance

We know abstractly that the KK^1 class defined by the Kasparov module $(\mathcal{A}_\phi, Z_{C^*(\widehat{U})}, N)$ represents the boundary map in K-homology [Kas81, §7]. We examine this more closely by considering the pairings related to the quantisation of the Hall conductance.

We recall that the bulk spectral triple $(\mathcal{A}_\phi, \ell^2(\mathbb{Z}^2)\oplus\ell^2(\mathbb{Z}^2), X, \gamma)$ pairs with elements in $K_0(A_\phi)\cong\mathbb{Z}[1]\oplus\mathbb{Z}[p_\phi]$, where p_ϕ is the Powers-Rieffel projection. For simplicity, we denote the corresponding K-homology class of our spectral triple by $[X]$, where we know that $[X]=[\beta]\hat{\otimes}_{C^*(\widehat{U})}[\Delta]$. Now, $[X]$ pairs non-trivially with $[P_\mu]$, the Fermi projection, to give the Hall conductance up to a factor of e^2/h. Hence we have that

$$\sigma_H = \frac{e^2}{h}\left([P_\mu]\hat{\otimes}_{A_\phi}[X]\right) = -\frac{e^2}{h}\left([P_\mu]\hat{\otimes}_{A_\phi}\left([\beta]\hat{\otimes}_{C^*(\widehat{U})}[\Delta]\right)\right),$$

where the minus sign arises from Lemma 4.3.1. We can now use the associativity of the Kasparov product to rewrite this equation as

$$[P_\mu]\hat{\otimes}_{A_\phi}\Big([\beta]\hat{\otimes}_{C^*(\widehat{U})}[\Delta]\Big) = \big([P_\mu]\hat{\otimes}_{A_\phi}[\beta]\big)\,\hat{\otimes}_{C^*(\widehat{U})}[\Delta].$$

We see that this new product $[P_\mu]\hat{\otimes}_{A_\phi}[\beta]$ is in $KK^1(\mathbb{C}, C^*(\widehat{U})) \cong K_1(C^*(\widehat{U})) \cong \mathbb{Z}$, where the last group has generator $[\widehat{U}]$. So $[P_\mu]\hat{\otimes}_{A_\phi}[\beta]$ is represented by $\widehat{U}^m \in C^*(\widehat{U})$ for some $m \in \mathbb{Z}$ and we are now taking an odd index pairing.

Next we note that the map

$$K_1(C^*(\widehat{U})) \times K^1(C^*(\widehat{U})) \to \mathbb{Z} \quad \text{where } \big([P_\mu]\hat{\otimes}_{A_\phi}[\beta]\big) \times [\Delta] \mapsto \big([P_\mu]\hat{\otimes}_{A_\phi}[\beta]\big)\,\hat{\otimes}_{C^*(\widehat{U})}[\Delta]$$

depends only on our boundary data, so this pairing is the mathematical formulation of the so-called edge conductance which, as we have seen, is the same as our bulk Hall conductance up to sign.

Our definition of the edge conductance is purely mathematical, but one can see that the unitaries and spectral triples being used come quite naturally from considering the algebra $C^*(\widehat{U})$ acting on $\ell^2(\mathbb{Z})$, which is exactly what we would consider as a 'boundary system' in the discrete picture. Hence our approach to the edge conductance is physically reasonable. Furthermore, the computation of the edge conductance boils down to computing $\mathrm{Index}\Big(\Pi\widehat{U}^m\Pi\Big) = -m$ for $\Pi : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N})$, which is a much easier computation than $[P_\mu]\hat{\otimes}_{A_\phi}[X]$.

Kellendonk, Richter and Schulz-Baldes justify our use of the term 'edge conductance' by linking the edge index pairing to the conductance of an edge current [SBKR02]. We will return to this issue in the case of topological insulator systems and torsion invariants (see Chapter 5.3.2).

Chapter 5

Topological insulators

5.1 A brief review

Symmetries and invariants

Topological insulators can be loosely described as physical systems possessing certain symmetries which give rise to invariants topologically protected by these symmetries. The symmetries of most interest to physicists are time-reversal symmetry, particle-hole symmetry (also called charge-conjugation symmetry) and chiral symmetry (also called sub-lattice symmetry). We do, however, note that other symmetries such as spatial inversion symmetry may be considered though they will not play a central role here.

The first (non)-example of a topological inuslator system is the quantum Hall effect. The quantum Hall effect is topological because the Hall conductance can be expressed in terms of a pairing of homology classes of certain bundles over the Brillouin zone (momentum space) of our sample. Of course, in order to properly understand the meaning of a bundle over the Brillouin zone in the case of irrational magnetic field, one has to pass to the noncommutative picture as outlined in Chapter 3. Because the quantised Hall conductance is a topological property, it is stable under small perturbations and the addition of impurities into the system. Indeed, disorder plays an important role in the localisation of electrons and stable nature of the Hall conductance in between jumps [BvS94, Section 5].

We think of the quantum Hall effect as a non-example of a topological insulator as while the Hall conductance is linked to topological information, the Hamiltonian of the system (a single particle in a 2-dimensional system with a perpendicular magnetic field) does not obey any of the symmetry properties of interest.

Much more recently, the prediction of a new topological state of matter came from Kane and Mele [KM05], who consider a Haldane system (that is, a single-particle Hamiltonian acting on a honeycomb lattice) and impose time-reversal symmetry on their model. By conducting a similar analysis to early explanations of the quantum Hall effect, namely that of Thouless et al. [TKNdN82], the authors associate a $\mathbb{Z}_2$-number to their system. This number is 'topologically protected' because one cannot pass from one number to the other unless time-reversal symmetry is broken. In particular, if the spin-orbit coupling of the model is sufficiently large, then the $\{0, 1\}$-invariant is 1, which is interpreted as the existence of a 'spin current' flowing along the edge of the sample. That is, the spin-up and spin-down electrons separate and give currents travelling in opposite directions. The net current is zero, but each spin component has a non-trivial conductance that Kane-Mele link to topological invariants of bundles over the Brillouin zone. Otherwise, the $\{0, 1\}$-invariant is 0 and we have a 'trivial insulator'. This effect is called the quantum spin-Hall effect and was the first example of a topological insulator to use the internal symmetries of the Hamiltonian to obtain invariants.

The quantum spin-Hall effect was initially predicted to occur in graphene, but this is hard to work with experimentally. The effect was later predicted to be found in HgTe [BHZ06], a compound much more amenable to laboratory work, and subsequently the effect was experimentally confirmed in [KWB$^+$07].

The Kane-Mele invariant opened up a new avenue of theoretical research to see if similar invariants of a finer type could be found in other models and systems. This included higher-dimensional time-reversal invariant insulators, experimentally found in [HQW$^+$08]. Particle-hole symmetric systems were also considered, which drew a link to superconductors, whose current can be considered as the scattering of an electron by a hole (see for example [QHZ08, SRFL08]).

Lots of possible models were quickly discovered and the question began to turn towards how to properly classify such systems from their symmetry data. This involved showing how the 'topological numbers' derived in the various systems could be connected to algebraic topology, specifically classifying spaces and homotopy groups of symmetry compatible Hamiltonians. While there are many papers on this topic, one of the most influential came from Kitaev [Kit09], who outlined how symmetry data can be linked to Clifford algebras and, in particular, K-theory. Specifically, if one considers a system with time-reversal, particle-hole or chiral symmetry, then then one finds ten different outcomes depending on the nature of the symmetry (see [Kit09, RSFL10] for more on this). Kitaev argued that these different outcomes correspond precisely to the 10 different K-theory groups (8 real groups and 2 complex groups), where the K-theory is again coming from bundles over the Brillouin zone. The paper also showed how the dimension of the system affects the kind of invariant that may arise.

The work of Kitaev and Ryu et al. has been expanded and developed in newer papers by Stone et al. [SCR11] and Kennedy and Zirnbauer [KZ14], which were recently brought to the author's attention. To briefly summarise, Stone et al. and Kennedy-Zirnbauer are able to link the symmetries of interest to stable homotopy groups and Clifford algebras in a way that is more physically concrete than Kitaev's original outline.

In particular, Kennedy and Zirnbauer show how the Bott periodicity of complex and real K-theory can be understood in terms of the symmetries of the system [KZ14].

The bulk-edge correspondence

So far our discussion has been focused on how single-particle Hamiltonians with some additional symmetry data give rise to certain topological invariants, but the way in which these properties are physically realised is a key aspect of insulator materials. Namely, the observables that are measured in experiment are said to be carried on the edge or boundary of a sample. So on the one hand, we have a Hamiltonian acting on the whole space, often assumed to be translation invariant, which gives topological properties of the material via the Bloch bundles over the Brillouin zone (or a non-commutative analogue of this). On the other hand, the topological invariants are also related to the 'current' that is concentrated at the edge of a sample. Loosely speaking, the relationship between these two properties is the bulk-edge correspondence of topological insulator materials.

For example, the Kane-Mele model [KM05] can be reduced to a two-band model, where the Hamiltonian acting on the boundary-free (bulk) space $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^N$ has a spectral gap at 0 with spectral bands above and below. However, when a boundary is introduced, say $\ell^2(\mathbb{Z} \times \mathbb{N}) \otimes \mathbb{C}^N$, there is now spectrum that crosses this gap and connects the two bands. One says that this new spectrum corresponds to the edge states that are carrying the spin current. By linking these edge states with the topological properties of the original Hamiltonian, we can say that the edge states are also topologically protected. Like the invariants of the Hamiltonian, topological properties of edge states do not change under small perturbations or the addition of impurities into the system [HK10]. It is from such an interpretation that one can begin to understand topological insulator systems as behaving like an insulator in the interior (which on a local scale is translation invariant and boundary-free) but with a topologically stable current on the edge of the sample.

Unfortunately, much of the physics literature can be unclear as to how one concretely compares the bulk invariant of the Hamiltonian and its symmetries with the topological properties of edge currents. To resolve this issue we turn to the more mathematical arguments developed for the bulk-edge correspondence of the quantum Hall effect in Chapter 4. It is still a work in progress to properly establish how the bulk-edge correspondence fits together in the case that there are additional symmetries of the Hamiltonian to consider and take into account. This chapter hopes to resolve some of these issues.

Contributions in the mathematical literature

While there have been thousands of articles in the physics literature about topological insulators and their properties, there are comparatively few mathematical physics papers on the subject (that is, papers with a mathematical focus, but with physical applications in mind). Despite this, there have been some important contributions from mathematical physicists in understanding the mechanics of insulator systems, particularly with regards to making Kitaev's K-theoretic classification of matter more explicit. We briefly review the work that has been done on these problems, focusing on the more K-theoretic papers as this is closest to our viewpoint. We do not claim that our review is comprehensive or complete, but serves to highlight what is understood and what remains open.

Most of the literature that has so far arisen deals with the bulk theory only, so boundaries and edges are not considered (with a few important exceptions considered at the end).

Almost commuting matrices and insulator systems (Loring et al.)

Some of the first mathematical attempts to understand the topological insulator problem came from Loring in collaboration with Hastings and Sørensen [HL10, LH10, HL11, LS10, LS13, LS14, Lor15].

Very roughly speaking, these papers start with the model of a finite lattice on a torus or sphere. There are translation operators U_i between atom sites that commute. However, when these operators are compressed by the Fermi projection $P_\mu U_i P_\mu$, they may no longer commute. The observation of the authors is that the act of approximating the matrices $P_\mu U_i P_\mu$ with commuting matrices can be viewed as a lifting problem in C^*-algebras. The authors then argue that the obstructions to approximating almost-commuting matrices with commuting matrices lead to K-theory invariants (both complex and real). These obstructions can then be related back to the various symmetries that arise in insulator systems.

The papers of Loring, Hastings and Sørensen are able to mathematically establish a link between insulator systems and K-theory of operator algebras, though the physical models used (a finite lattice on a d-torus or d-sphere) do not line up easily with the models that are usually considered. Another drawback is that the methods Loring et al. use are quite different to any other treatment of such systems (including the various explanations of the quantum Hall effect) and so are difficult to adapt to the physical interpretations of such systems.

By considering the topological insulator problem and it's link to real/Real K-theory, there have also been some useful mathematical papers explaining KKR and KO-theory [BLR12, BL15]. In particular, the paper [BL15] provides a helpful characterisation of all 8 KO-groups in terms of unitary matrices and involutions.

Bloch bundles and K-theory (De Nittis-Gomi)

An alternate viewpoint comes from the papers of De Nittis and Gomi [NG14, DG14, DG15b, DG15a], who are developing a more explicitly geometric interpretation of the insulator invariants that arise. This is done by constructing a theory of Real or quaternionic or chiral vector bundles, and showing how the topological properties of insulator systems can be interpreted as geometric invariants of these bundles over the Brillouin zone. This work serves to correct some inconsistencies in the physics literature, where many of the bundles considered are trivial and symmetry structures implemented globally. De Nittis and Gomi show that when only local trivialisations are considered, much more care needs to be taken to properly construct and work with the invariants of interest.

The Bloch bundle picture is advantageous as it links much more clearly to the geometric explanations of the quantum Hall effect by [TKNdN82] and others, explicitly relating physical quantities to homology theories and pairings. The limitation of such a viewpoint is that it cannot fully take into account the situation with a magnetic field present, which may include systems with particle-hole or chiral symmetry. In such a picture, one would need to perform an analysis similar to Bellissard for the quantum Hall effect and construct Real/quaternionic/chiral bundles over the noncommutative Brillouin zone. It is also quite difficult to work disorder into the Bloch bundle viewpoint as much of the geometric framework no longer holds. This is an advantage of the noncommutative method as Bellissard and others have been able to demonstrate.

Chern numbers, spin-Chern numbers and disorder (Prodan, Schulz-Baldes)

A concerted attempt to adapt the ideas and constructions of Bellissard's noncommutative Brillouin zone and Chern numbers into the general insulator picture has been made by Prodan and Schulz-Baldes in several papers [Pro10, Pro11, SB13, Sch13, Pro14].

Part of this process involves showing how Bellissard's cocycle formula for the Hall conductance has natural generalisations to higher dimensions [PLB13, PS14]. We have already considered this problem in Chapter 3.3.

Another important aspect of Prodan and Schulz-Baldes' work has been defining the so-called spin-Chern numbers. Roughly speaking, a system with additional symmetries (usually time-reversal is considered) can be split into the ± 1 eigenspaces of a Pauli matrix representing spin, usually σ_3. One can then restrict to the $+1$ or -1 eigenspace and consider a Chern-like number on this subspace, denoted the spin-Chern number. In the case of time-reversal symmetry, the two separate spin-Chern numbers will add up to zero but the individual spin-Chern numbers may be non-zero. Hence one can interpret these invariants as capturing the conductance of the spin-up and spin-down currents of the quantum spin-Hall effect. The use of noncommutative methods also means that the models considered by Schulz-Baldes and Prodan are among the few that allow disorder to be included in the system (see [SB13] for more details).

The spin-Chern number picture shows how the noncommutative explanation of the quantum Hall effect can be applied to other insulator systems, but it is an incomplete picture so far. One of the main reasons for this is that the Chern number comes from the pairing of the periodic cyclic homology and cohomology of an algebra and takes integer values. Early results of Connes show that this cyclic pairing is the same as the index pairing of K-theory with K-homology in the case of finitely summable Fredholm modules over complex algebras [Con85]. However, such a relation breaks down in the case of torsion invariants, which are common in the K-groups of real/Real algebras. Cyclic cohomology can not detect torsion invariants in insulator systems, which means its use in such examples is limited. The absence of a connection to the computationally tractable cyclic theory is one reason why linking spin-Chern numbers to a bona-fide pairing in KR or KO-theory is a very hard problem and has not yet been resolved. We do however note that the Chern numbers of systems of arbitrary dimension considered by [PLB13, PS14] and in Chapter 3.3 can be applied to insulator systems where only chiral symmetry is considered.

The way one works around the problem of pairings in cyclic cohomology and homology is by dealing with the K-theory and K-homology groups directly for complex, and Real/real algebras, which can to detect torsion invariants. Indeed, this is the picture that Schulz-Baldes and co-authors adopt in later papers [DNSB14, GS15]. We also adopt this viewpoint, but from the perspective of KK-theory, which is necessary to consider the bulk-edge problem.

Symmetry groups and equivariant K-theory (Freed-Moore, Thiang)

So far our various symmetries have been considered on a case by case basis with no unifying theory linking systems together as Kitaev outlined. Such a theory in the commutative setting was developed by Freed and Moore [FM13], and then generalised to possibly noncommutative algebras by Thiang [Thi15].

The paper by Freed and Moore is very long and detailed so we will only give the most basic of summaries. The symmetries of interest to us (time-reversal, particle-hole and chiral) are put together in a symmetry group G. Then, symmetry compatible Hamiltonians correspond to projective unitary/anti-unitary representations of G (or a subgroup of thereof). Using the Bloch-bundle viewpoint to derive topological invariants of the system under consideration, the quantities of interest can be derived by looking at the equivariant K-theory of subgroups of G. In certain cases, lattice symmetries and the crystallographic group of the lattice of the sample can also be incorporated, giving rise to possibly twisted equivariant K-theory classes and invariants.

The work of Thiang showed how Freed-Moore's constructions can be carried out in the noncommutative setting. In particular, Thiang links symmetry data to Clifford algebras and constructs a homology theory similar to Karoubi's $K^{p,q}$-theory (see [Kar08, Chapter III]) that encodes these symmetries. Such a construction means that the Kitaev's classification (also called the 10-fold way) can be described in a unified framework.

Freed-Moore and Thiang's work allows all the symmetry data to be considered on an equal footing and gives a rigorous proof of Kitaev's classification. The work of Thiang in particular opens the door to further research as it provides a concrete framework to consider disordered systems and impurities. The main limitation is that the theory deals solely with a bulk system and K-theory. The use of K-homology or a system with edge is not considered.

KR-Theory and pairings (Grossmann-Schulz-Baldes)

A recurring characteristic of the literature on topological insulators, both physical and mathematical, is that the links to topology are solely discussed via K-theory. However, as we saw in Chapter 3, the expression for the Hall conductance is not just a K-theory construction, but a pairing (i.e. Kasparov product) between a K-theory class and a K-homology class coming from a particular spectral triple or Fredholm module. Most literature on topological insulators does not consider this extra K-homological information, though an exception are the papers of Schulz-Baldes and co-authors [DNSB14, GS15].

De Nittis, Grossmann and Schulz-Baldes show that a discrete condensed matter system with additional symmetries naturally gives rise to a Real spectral triple in the sense of Connes [Con95, GBVF01] and represents a KR-homology class. De Nittis and Schulz-Baldes consider the 2-dimensional case [DNSB14] and Grossmann-Schulz-Baldes generalise this to arbitrary dimension [GS15]. In particular, [DNSB14, GS15] show that the Fermi projection of a symmetry compatible Hamiltonian pairs with the Real spectral triple via an index and it is this pairing that gives the various classification groups of Kitaev, Freed-Moore and Thiang.

De-Nittis, Grossmann and Schulz-Baldes's work provides a useful picture of the bulk-theory of insulators. Working the bulk-edge correspondence into such a framework remains to be done. It would also be advantageous to highlight how the work of Thiang and Grossmann-Schulz-Baldes are related under the broader framework of KK-theory.

The bulk-edge correspondence (Graf-Porta, Schulz-Baldes, Mathai-Thiang)

While a mathematical understanding of the bulk-edge correspondence is still in development, there have been a few important contributions. Firstly there was the work of [ASBVB13, SB13], who consider 2-dimensional time-reversal invariant systems and prove a bulk-edge correspondence using the spin-Chern perspective and an argument using transfer matrices. A 2-dimensional bulk edge correspondence for systems with time-reversal symmetry is also considered in [GP13]. By using more elementary functional analytic techniques, Graf and Porta reproduce the result of [ASBVB13, SB13] for a broader class of possible Hamiltonians.

These are both useful results and important contributions to the literature, though the link between the bulk-edge picture described in these papers and the K-theoretic classification picture is very difficult to establish, though the two should be compatible.

Section 7 of [Lor15] considers the bulk-edge correspondence in arbitrary dimension. The drawback of this result is that, because the viewpoint is quite detached from the more widely studied K-theoretic picture, the link between Loring's argument and the work of Grossmann-Schulz-Baldes and Thiang is not transparent.

Finally we mention the papers by Mathai and Thiang [MT15b, MT15c], which emerged as this thesis was nearing completion. These papers use a short-exact sequence to link bulk and edge systems as considered by [KR08, SBKR02, KSB04b] in the case of the quantum Hall effect. One can then check that the invariants of interest (including torsion invariants for time-reversal symmetric systems) pass from bulk to edge in the Pimsner-Voiculescu sequence in complex or real K-theory. Mathai and Thiang also use real and complex T-duality to show in a variety of examples that when the boundary map in K-theory is T-dualised, the map can be expressed as a conceptually simpler restriction map. In the real case, the Kane-Mele $\mathbb{Z}_2$ invariant is also identified with the 2nd Stiefel-Whitney class under T-duality.

One of the goals of this chapter is to prove a bulk-edge correspondence of insulator systems using Kasparov theory. A KK-theoretic bulk-edge correspondence links the bulk and edge duality to the associativity of the Kasparov product, as was demonstrated in Chapter 4 for the quantum Hall effect. Our main result shows that an analogous statement of [MT15b, MT15c] is true for spectral triples and K-homology, which allows for arbitrary symmetry types to be considered.

Overview of this chapter

Our work in this chapter is split up into two main components.

1. A derivation of Kitaev's classification of topological states of matter. In doing so, we show how the work of Thiang and Grossmann-Schulz-Baldes can be understood in terms of Kasparov theory.

2. A KK-theoretic proof of the bulk-edge correspondence of discrete insulators of any symmetry type in arbitrary dimension.

The first part serves to bring together the already substantial contributions made in this area and to clarify how Kitaev's classification can be naturally cast into the language of KK-theory, complex and real. While a derivation of the periodic table is not exactly new, both in the physics and mathematical literature, we are of the opinion that an understanding of how we can apply Kasparov's powerful machinery to the insulator problem can potentially allow for much more sophisticated models and systems to be considered. Systems with disorder and impurities are possible examples.

To our knowledge, a rigorous bulk-edge correspondence of topological insulators using Kasparov theory has not yet appeared in the literature. The use of Kasparov theory and the intersection product to study systems with boundary can potentially be extended further than what is considered in this thesis. We will consider some possible future directions for work in this problem at the end of the chapter.

We also show how our general method applies to some of the examples of interest in the physics and mathematics literature. This includes the well-known Kane-Mele model of the quantum spin-Hall effect as well as 3-dimensional insulator systems.

5.2 Bulk theory

5.2.1 Symmetry types and representations

In our basic setup, we consider a self-adjoint single-particle Hamiltonian H acting on a complex Hilbert space $\mathcal{H}$. We work under the tight-binding model so $\mathcal{H}$ will usually take the form $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$, where d captures the dimension we are considering and N encodes any internal degrees of freedom coming from properties such as spin or the structure of our lattice. Our outline of the basic symmetries is quite similar to that discussed in, amongst others, [DNSB14, GS15].

The Hamiltonian is a one-particle representation of a system of independent fermions, and so we may ask what symmetries are compatible with H. The symmetries of interest to us are time-reversal symmetry, particle-hole symmetry (also called charge-conjugation symmetry) and chiral symmetry (also called sublattice symmetry). The time-reversal, particle-hole and chiral involutions interact and form the PT-symmetry group $\{1, T, P, PT\} = \mathbb{Z}_2 \times \mathbb{Z}_2$, where $C = TP = PT$.

Definition 5.2.1. A Hamiltonian H acting on a complex Hilbert space $\mathcal{H}$ respects time-reversal and/or particle-hole and/or chiral symmetry if there are complex anti-linear operators R_T and/or R_P and/or a complex-linear operator R_C acting on $\mathcal{H}$ such that $R_T^2, R_P^2, R_C^2 \in \{\pm 1_{\mathcal{H}}\}$ and

$$R_T H R_T^* = H, \qquad R_P H R_P^* = -H, \qquad R_C H R_C^* = -H \tag{5.1}$$

In the case of R_T and R_P, our Hamiltonian is said to have even (resp. odd) symmetry if $R^2 = 1$ (resp. $R^2 = -1$).

Because R_C is complex-unitary, the sign of its square is irrelevant (in the same way that $\mathbb{C}\ell_1$ may have a generator that squares to $+1$ or -1). We note that a Hamiltonian may only respect a single symmetry. However, if H is compatible with two symmetries, then by the underlying group structure it is compatible with the third symmetry. We will examine the link between symmetry compatible Hamiltonians and group representations in Section 5.2.2.

There is no general form that the symmetry operators R_T, R_P and R_C need take apart from the properties outlined in Definition 5.2.1, and instead are determined by the example under consideration. A useful characterisation of the conjugate-linear operators R_T and R_P is as operators acting on a complex Hilbert space that anti-commute with the Real involution given by complex conjugation.

Example 5.2.2 (Anti-linear symmetries via complex conjugation). We consider the Hilbert space $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{2N}$ and define the operator

$$R = \begin{pmatrix} 0 & \mathcal{C} \\ \eta\mathcal{C} & 0 \end{pmatrix},$$

where $\mathcal{C}$ is complex conjugation and $\eta \in \{\pm 1\}$. At this stage we are not restricting whether R represents time-reversal or particle-hole symmetry (though we are considering a single symmetry only). We note that $R^2 = \eta 1_{2N}$ so R can represent an even or odd symmetry depending on the sign of η. Given an operator $a \in \mathcal{B}[\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N]$ we define the operator $\overline{a} = \mathcal{C}a\mathcal{C}$. One computes that for $a, b, c, d \in \mathcal{B}[\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N]$

$$R \begin{pmatrix} a & b \\ c & d \end{pmatrix} R^* = \begin{pmatrix} \overline{d} & \eta\overline{c} \\ \eta\overline{b} & \overline{a} \end{pmatrix}. \tag{5.2}$$

Consider the case that R is implementing a time-reversal involution. By Equation (5.2), a matrix acting on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{2N}$ will be time-reversal symmetric if it takes the form $A = \begin{pmatrix} a & b \\ \eta\overline{b} & \overline{a} \end{pmatrix}$. If we wish to consider time-reversal invariant Hamiltonians, then we require the additional property that the operator A is self-adjoint.

We may also want to consider the case that R is representing particle-hole symmetry. An operator A is particle-hole symmetric if $RAR^* = -A$, so Equation (5.2) tells us that A must be of the form $\begin{pmatrix} a & b \\ -\eta\overline{b} & -\overline{a} \end{pmatrix}$.

Let's now consider the Dirac-type operator that appears in the quantum Hall effect, namely $X = \begin{pmatrix} 0 & X_1 - iX_2 \\ X_1 + iX_2 & 0 \end{pmatrix}$ acting on $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2$. We see that X is even time-reversal symmetric (that is $RXR^* = X$ with $\eta = 1$) or has odd particle-hole symmetry ($RXR^* = -X$ with $\eta = -1$) depending on the symmetry involution R is representing.

Example 5.2.3 (Symmetries via spatial involution). We start with the space $\ell^2(\mathbb{Z}^d)$ and define the anti-linear operator J such that $(J\lambda)(x) = \overline{\lambda(-x)}$ for $\lambda \in \ell^2(\mathbb{Z}^d)$. We define on $\mathcal{H} = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{2N}$ the operator

$$R = \begin{pmatrix} 0 & J \\ \eta J & 0 \end{pmatrix}$$

with $R^2 = \eta 1_{2N}$ as before. As a transformation on matrices acting on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{2N}$, one computes that

$$R \begin{pmatrix} a & b \\ c & d \end{pmatrix} R^* = \begin{pmatrix} JdJ & \eta JcJ \\ \eta JbJ & JaJ \end{pmatrix}.$$

We again consider the case that R models the time-reversal or partial-hole involution. Operators that are time-reversal invariant under conjugation by $R = R_T$ have the general characterisation $\begin{pmatrix} a & b \\ \eta JbJ & JaJ \end{pmatrix}$, whereas particle-hole symmetric operators under $R = R_P$ are of the form $\begin{pmatrix} a & b \\ -\eta JbJ & -JaJ \end{pmatrix}$.

Considering again the quantum Hall Dirac-type operator, we first note that $JX_kJ = -X_k$ and $J(\pm iX_k)J = \pm iX_k$ for the position operators X_k, $k = 1, 2$. Therefore we have that

$$R \begin{pmatrix} 0 & X_1 - iX_2 \\ X_1 + iX_2 & 0 \end{pmatrix} R^* = \begin{pmatrix} 0 & \eta(-X_1 + iX_2) \\ \eta(-X_1 - iX_2) & 0 \end{pmatrix},$$

which implies that X now has odd time reversal symmetry and even particle-hole symmetry.

Remark 5.2.4 (Time-reversal and particle-hole as 0-dimensional phenomena). The example of the quantum Hall Dirac-type operator shows that changing how we represent the involutions R_T and R_P may change whether an operator has a particular symmetry type. This is an important observation and indicates that the spatial involution is bringing extra data into our system (namely, that we have a d-dimensional sample with $d > 0$). In comparison, time-reversal and particle-hole involutions can exist in 0-dimensional samples and do not need the extra information that spatial involution does. We emphasise that systems with anti-linear symmetries defined using spatial involution are topologically inequivalent to systems with anti-linear symmetries defined from complex conjugation (see [MT15a] for more detail on the inequivalence of symmetry types).

Example 5.2.5 (Chiral symmetry). In most examples in the literature, the chiral symmetry involution is represented by the matrix $R_C = \begin{pmatrix} 1_N & 0 \\ 0 & -1_N \end{pmatrix}$ on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^{2N}$, so a self-adjoint Hamiltonian H is chiral symmetric if $H = \begin{pmatrix} 0 & h \\ h^* & 0 \end{pmatrix}$.

An important remark is that if H obeys a particular symmetry and the Fermi level μ is in a gap of the spectrum of H (we can assume without loss of generality that $\mu = 0$), then the 'spectrally flattened' Hamiltonian $\mathrm{sgn}(H) = H|H|^{-1}$ also obeys this symmetry.

5.2.2 Symmetries, group actions and Clifford algebras

We have briefly explained the symmetries that arise in our insulator systems but we would like a more structural understanding of how these symmetries fit into a unifying picture. Here the recent work of Thiang as developed in [Thi15, Thi14, MT15a], which develops ideas from [FM13], is of great use. One of the key insights in [FM13, Thi15] is to see that a symmetry compatible Hamiltonian with a spectral gap H can be expressed as a graded projective unitary/anti-unitary representation of the finite symmetry group $G \subset \{1, T, P, C\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. We explain this link below.

Definition 5.2.6. Let G be a finite group and $\mathcal{H}$ a complex Hilbert space. For each $g \in G$, let θ_g be a real-linear operator on $\mathcal{H}$ and suppose $\phi : G \to \{\pm 1\}$ is a continuous homomorphism. The triple (G, ϕ, σ) is a projective unitary/anti-unitary (PUA) representation if θ_g is unitary (resp. anti-unitary) if $\phi(g) = 1$ (resp. -1) and $\theta_{g_1}\theta_{g_2} = \sigma(g_1, g_2)\theta_{g_1 g_2}$ with $\sigma : G \times G \to \mathbb{T}$ a generalised 2-cocycle satisfying

$$\sigma(g_1, g_2)\sigma(g_1 g_2, g_3) = \sigma(g_2, g_3)^{g_1}\sigma(g_1, g_2 g_3), \quad g_1, g_2, g_3 \in G,$$

where for $z \in \mathbb{T}$, $z^g = z$ if $\phi(g) = 1$ and $z^g = \overline{z}$ if $\phi(g) = -1$.

We can now re-formulate the definition of a symmetry compatible Hamiltonian in terms of group representations.

Definition 5.2.7. Given a projective unitary/anti-unitary representation (G, ϕ, σ) and a gapped self-adjoint Hamiltonian H acting on a complex Hilbert space $\mathcal{H}$, we say that H is compatible with G if there is a continuous homomorphism $c : G \to \{\pm 1\}$ such that

$$\theta_g H = c(g) H \theta_g \quad \text{for all } g \in G. \tag{5.3}$$

Remark 5.2.8. Because $0 \notin \sigma(H)$, we can deform a symmetry-compatible H to its phase $H|H|^{-1} = \mathrm{sgn}(H)$ without changing Equation (5.3). This means that $\Gamma = \mathrm{sgn}(H)$ is acting like a grading of our PUA representation. Therefore, we say that a symmetry compatible Hamiltonian on $\mathcal{H}$ is precisely realised as a graded PUA representation (G, c, ϕ, σ) on $\mathcal{H}$ with grading $\Gamma = \mathrm{sgn}(H)$. The map ϕ determines if the symmetry involution θ_g is represented unitarily or anti-unitarily and the map c determines if the involution has even or odd grading. We emphasise that the grading of a symmetry involution θ_g as even or odd is different from whether the symmetry is denoted even or odd, which comes from whether $\theta_g^2 = 1$ or -1 respectively.

Our definition of a symmetry compatible Hamiltonian may apply to any finite group G, though we are interested in a subgroup G of $\{1, P, T, PT\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Equation (5.1) and the surrounding discussion tells us that a symmetry compatible Hamiltonian H can be expressed as a PUA representation of a subgroup G of $\{1, P, T, C\}$ on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$ with $\theta_g = R_g$, $\Gamma = \mathrm{sgn}(H)$ and

$$(\phi, c)(T) = (-1, 1), \qquad (\phi, c)(P) = (-1, -1), \qquad (\phi, c)(C) = (1, -1).$$

The values of these maps fix the cocycle σ coming from the projective representation and, hence, determine the commutation/anti-commutation relations of the elements $\{R_g : g \in G\}$.

Representations of G are in 1-1 correspondence with representations of the real or complex group C^*-algebra $C^*(G)$. From the perspective of Kasparov theory we would like to link the algebra $C^*(G)$ with real or complex Clifford algebras as such algebras play a fundamental role in KK-theory.

Proposition 5.2.9 ([FM13], Appendix B; [Thi15], Section 6)**.** *Let G be a subgroup of the symmetry group $\{1, T, P, PT\}$ with $G \neq \{1, C\}$. If a Hamiltonian H acting on $\mathcal{H}$ is compatible with G, then there is a graded representation of the real Clifford algebra $C\ell_{r,s}$ on $\mathcal{H}$ with the generators coming from the PUA representation of G. If $G = \{1, C\}$, then there is a graded representation of $\mathbb{C}\ell_1$ on $\mathcal{H}$. The representations are summarised in Table 5.1 up to stable isomorphism.*

The natural grading of Clifford algebras imply that all generators have odd degree. Therefore all generators of a Clifford representation must be odd with respect to the grading $\Gamma = \mathrm{sgn}(H)$.

Proof of Proposition 5.2.9. We do the proof on a case by case basis. We first use [Thi15, Proposition 6.2] to 'normalise' the twist σ of the PUA representation so that the operators R_P and R_T commute and $R_P R_T = R_{PT}$. For the full symmetry group $G = \{1, P, T, PT\}$, we use the operators R_g for $g \in G$ to consider the real algebra generated by $\{R_P, iR_P, iR_P R_T\}$. One checks that these generators are odd with respect to the grading Γ, mutually anti-commute and are self-adjoint (resp. skew-adoint) if they square to $+1$ (resp. -1). Therefore the real algebra generated by $\{R_P, iR_P, iR_{PT}\}$ is precisely a graded representation of a particular real Clifford algebra $C\ell_{r,s}$ with grading $\Gamma = \mathrm{sgn}(H)$.

Next we consider the subgroup $\{1, P\}$, to which we assign the real algebra generated by $\{R_P, iR_P\}$ and graded by $\mathrm{sgn}(H)$.

Representations of the subgroup $\{1, C\}$ give rise to a representation generated by R_C with grading $\mathrm{sgn}(H)$. Because R_C acts complex-linearly, we may consider the complex span of R_C as acting on $\mathcal{H}$. Hence the representation generated by R_C is a graded representation of $\mathbb{C}\ell_1$.

Symmetry generators	R_P^2	R_T^2	Graded Clifford representation (up to stable isomorphism)
T		$+1$	$C\ell_{0,0}$
P,T	$+1$	$+1$	$C\ell_{1,0}$
P	$+1$		$C\ell_{2,0}$
P,T	$+1$	-1	$C\ell_{3,0}$
T		-1	$C\ell_{4,0}$
P,T	-1	-1	$C\ell_{5,0}$
P	-1		$C\ell_{6,0}$
P,T	-1	$+1$	$C\ell_{7,0}$
N/A			$\mathbb{C}\ell_0$
C	$R_C^2 = 1$		$\mathbb{C}\ell_1$

Table 5.1: Symmetry types and their corresponding graded Clifford representations [Thi15, Table 1].

The case of the subgroup $\{1, T\}$ is a little different as R_T commutes with $\mathrm{sgn}(H)$. For the case that $R_T^2 = 1$, R_T defines a Real structure on the Hilbert space and gives no additional Clifford generators. If $R_T^2 = -1$, then R_T defines a quaternionic structure on $\mathcal{H}$ under the identification $\{i, j, k\} \sim \{i, R_T, iR_T\}$. There is an equivalence between a graded quaternionic vector space and a graded action of $C\ell_{4,0}$ on $\mathcal{H}$. Specifically, we take $\mathcal{H} \oplus \mathcal{H}$ and the real span of the Clifford generators

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -R_T \\ R_T & 0 \end{pmatrix}, \begin{pmatrix} 0 & -iR_T \\ iR_T & 0 \end{pmatrix} \right\}, \quad \Gamma = \begin{pmatrix} \mathrm{sgn}(H) & 0 \\ 0 & -\mathrm{sgn}(H) \end{pmatrix}.$$

Therefore, the subgroup $\{1, T\}$ gives rise to a graded representation of $C\ell_{0,0}$ or $C\ell_{4,0}$. □

Remark 5.2.10 (The 10-fold way). A graded PUA representation of $\{1, T, P, PT\}$ gives rise to the real Clifford generators $\{R_P, iR_P, iR_{PT}\}$. These generators represent four different Clifford algebras determined by the sign of R_P^2 and R_T^2. Similarly, the representations of the subgroup $\{1, P\}$ give representations for two real Clifford algebras generated by $\{R_P, iR_P\}$ and vary depending on whether $R_P^2 = \pm 1$. Graded representations of $\{1, C\}$ correspond to the Clifford algebra $\mathrm{span}_{\mathbb{C}}\{R_C\} \cong \mathbb{C}\ell_1$, which is the same whether $R_C^2 = \pm 1$ (again, these representations come with the grading $\Gamma = \mathrm{sgn}(H)$). A Hamiltonian compatible with the symmetry group $\{1, T\}$ gives rise to two real Clifford algebras depending on whether $R_T^2 = \pm 1$. In total, we have nine possible representations of symmetry subgroups as *distinct* Clifford algebras and a lack of any symmetry gives us one more possibility. This is the well-known '10-fold way' that arises when we consider symmetries of this kind (see for example [SRFL08]).

Because we are interested in the link between Clifford representations and KK-theory, we may choose representations up to stable isomorphism, where $C\ell_{r+1,s+1} \cong C\ell_{r,s}\hat{\otimes}M_2(\mathbb{R})$ for real Clifford algebras and $\mathbb{C}\ell_{n+2} \cong \mathbb{C}\ell_n\hat{\otimes}M_2(\mathbb{C})$ for complex algebras. We summarise the results in Table 5.1.

We note that in Table 5.1, each symmetry type gives rise to a distinct graded Clifford representation. Therefore (up to stable isomorphism), the process is reversible. That is, given a graded representation of $C\ell_{n,0}$ or $\mathbb{C}\ell_n$, we may think of this representation as encoding internal the symmetries of a subgroup of the PT-group, where the symmetries are compatible with a gapped Hamiltonian H such that $\Gamma = \mathrm{sgn}(H)$.

5.2.3 Internal symmetries and KK-classes

In the previous section, we outlined how symmetry-compatible gapped self-adjoint Hamiltonians give rise to a graded $*$-representation of $C\ell_{n,0}$ or $\mathbb{C}\ell_n$ with the number n determined (up to stable isomorphism) by the symmetries present and whether they are even or odd. Our next task is to relate this characterisation to the K-theory of our observable algebra.

Before we specify our observable algebra, we must first specify the class of of bulk Hamiltonians our method can be adapted to. As observed in the quantum Hall example (Chapter 3), in order to study the geometry and topology of the Brillouin zone, we require an algebra of observables larger than the algebra generated by the Hamiltonian (or its resolvent).

Assumption 5.2.11. Unless otherwise stated, we will assume the Hamiltonians we consider act on $\ell^2(\mathbb{Z}^d)\otimes\mathbb{C}^N$ have a spectral gap containing the Fermi level. Furthermore, we assume the Hamiltonians are represented by matrices whose entries are either finite polynomials of (possibly twisted) shift operators or infinite polynomials with Schwartz-class coefficients.

If H is compatible with the symmetry group G, a subgroup of $\{1, T, P, PT\}$, then we also assume that the symmetry action $H \mapsto R_g H R_g^*$ extends to an action on the algebra generated by the (twisted) shift operators that generate H.

We note that essentially all tight-binding (discrete) model Hamiltonians without disorder satisfy our criterion. We consider the algebra generated by the shift operators that give rise to H and act on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$ (as matrices if necessary). Specifically, this is the (possibly twisted) group C^*-algebra $C^*_\phi(\mathbb{Z}^d)$, where ϕ represents a twist coming from, say, an external magnetic field (of course ϕ may be 0). Therefore $H \in C^*_\phi(\mathbb{Z}^d)$ and, unless otherwise stated, we shall take our observable algebra $A := C^*_\phi(\mathbb{Z}^d)$. This will be a *real* C^*-algebra in most cases (cf. Chapter 2.3), though may be complexified in systems with either no symmetries or chiral symmetry only. We denote by $A_{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C}$ the complexification. The twisted group algebra $C^*_\phi(\mathbb{Z}^d)$ can also be represented on the real Hilbert space $\ell^2(\mathbb{Z}^d) \otimes \mathbb{R}^M$, which will be important when we construct real spectral triples.

We require the symmetry action on H to extend to the observable algebra (Assumption 5.2.11) in order to determine symmetry properties of the whole Brillouin zone. Such an assumption is required in the case of abstract representations of the symmetry group $G \subset \{1, T, P, PT\}$, though is easily satisfied in the common representations that arise in examples (e.g. symmetry involutions defined by complex conjugation or spatial involution).

Using the action of G on $A = C^*_\phi(\mathbb{Z}^d)$ we can take the crossed product $A \rtimes G$. This is one of the key reasons we require A to be a real algebra. In the case $g = P$ or T, the automorphism $\alpha_g(a) = R_g a R_g^*$ is complex anti-linear and so one can not take the crossed product of this automorphism if A is a complex algebra. We can realise this crossed product concretely as

$$A \rtimes G \cong \overline{\mathrm{span}}_{\mathbb{R}} \left\{ \sum_{g \in G} a_g R_g \; : \; a_g \in A \right\} \subset \mathrm{End}_{\mathbb{R}}(\mathcal{H}).$$

We can take the expectation of the action on the crossed-product, $\Phi : A \rtimes G \to A$. As G is a finite group, this takes the form

$$\Phi\left(\sum_{g \in G} a_g R_g \right) = a_e \in A.$$

Proposition 5.2.12. *Let G be a subgroup of the symmetry group $\{1, T, P, PT\}$ acting on $A = C^*_\phi(\mathbb{Z}^d)$ a real C^*-algebra. Then there is a real Hilbert A-module E_A defined as the completion of $A \rtimes G$ under the inner product $(e_1|e_2)_A = \Phi(e_1^* e_2)$ and with right-action given by right-multiplication. If $G = \{1, C\}$ then the algebras and modules can be complexified to give a complex Hilbert $A_{\mathbb{C}}$-module.*

Proof. The proof that $\Phi : A \rtimes G \to A$ gives an A-valued inner product is a simple check that we will omit for brevity. We check that right-multiplication is compatible with the inner product where, for $c \in A$,

$$\begin{aligned}
\left(\sum_{g \in G} a_g R_g \,\middle|\, \sum_{h \in G} b_h R_h c \right)_A &= \Phi\left(\sum_{g,h \in G} R_g^* a_g^* b_h R_h c \right) \\
&= \Phi\left(\sum_{g,h \in G} R_g^* a_g^* b_h \alpha_h(c) R_h \right) \\
&= \sum_{g,h \in G} \delta_{g,h} \alpha_g^{-1}(a_g^* b_h \alpha_h(c)) R_g^* R_h
\end{aligned}$$

as Φ evaluates at the identity. We then simplify

$$\left(\sum_{g\in G} a_g R_g \middle| \sum_{h\in G} b_h R_h c\right)_A = \sum_{g\in G} \alpha_g^{-1}(a_g^* b_g)c = \left(\sum_{g\in G} a_g R_g \middle| \sum_{h\in G} b_h R_h\right)_A c.$$

We can complete $A \rtimes G$ in the norm defined from this inner product to obtain the real module E_A. In the case that $G = \{1, C\}$, the action by α_C is complex-linear and so all algebras and modules can be complexified without affecting linearity. □

We note that left-multiplication by an element in the crossed product $A \rtimes G$ is adjointable on E_A by the simple relation

$$(e_1 e_2 | e_3)_A = \Phi(e_2^* e_1^* e_3) = (e_2 | e_1^* e_3)_A \tag{5.4}$$

for any $e_j \in A \rtimes G$. In particular, this means that a left-action by multiplication by the real C^*-algebra $C^*(G) \subset A \rtimes G$ is adjointable. In the spirit of Proposition 5.2.9, we obtain the following.

Proposition 5.2.13. *Let H be a Hamiltonian satisfying Assumption 5.2.11 that is compatible with a subgroup G of the symmetry group $\{1, T, P, PT\}$. Then there is a real Kasparov module $\left(C\ell_{n,0}, E_A^N, 0, \Gamma\right)$, where Γ is a matrix of the operator $\operatorname{sgn}(H)$ and $N \in \{1, 2, 4\}$ is determined by the symmetries present. If $G = \{1, C\}$ then the module can be complexified to a complex Kasparov module. The number n is determined up to stable isomorphism by Table 5.1.*

Proof. We first note that left-multiplication by R_g is adjointable for any $g \in G$ by Equation (5.4). The same argument applies to show that the grading $\operatorname{sgn}(H) \in A$ is an adjointable operator.

From this point our proof is quite similar to the proof of Proposition 5.2.9 and is done on a case by case basis. We can once again use [Thi15, Proposition 6.2] to normalise our symmetry involutions so R_T commutes with R_P and $R_T R_P = R_{PT}$.

We start with the full group $G = \{1, T, P, PT\}$ and define a left-action on $E_A \oplus E_A$ given by left-multiplication by the real algebra generated by the elements

$$\left\{ \begin{pmatrix} R_P & 0 \\ 0 & -R_P \end{pmatrix}, \begin{pmatrix} 0 & R_P \\ R_P & 0 \end{pmatrix}, \begin{pmatrix} 0 & -R_{PT} \\ R_{PT} & 0 \end{pmatrix} \right\}, \quad \Gamma = \begin{pmatrix} \operatorname{sgn}(H) & 0 \\ 0 & \operatorname{sgn}(H) \end{pmatrix}.$$

One readily checks as in Proposition 5.2.9 that the generating elements have odd grading and mutually anti-commute. The left-action generated by these elements gives rise to four distinct Clifford algebras depending on whether $R_T^2 = \pm 1$ and $R_P^2 = \pm 1$. Because our Dirac-type operator is 0, the tuple $\left(C\ell_{n,0}, E_A^{\oplus 2}, 0, \operatorname{sgn}(H) \otimes 1_2\right)$ satisfies the remaining requirements to be a real Kasparov module.

Similarly for the case $G = \{1, P\}$ we take a left-action generated by

$$\left\{ \begin{pmatrix} R_P & 0 \\ 0 & -R_P \end{pmatrix}, \begin{pmatrix} 0 & R_P \\ R_P & 0 \end{pmatrix} \right\}, \quad \Gamma = \begin{pmatrix} \mathrm{sgn}(H) & 0 \\ 0 & \mathrm{sgn}(H) \end{pmatrix}.$$

We obtain an adjointable left-action of either $C\ell_{2,0}$ or $C\ell_{0,2}$ depending on whether $R_P^2 = \pm 1$.

If $G = \{1, C\}$ then we take the (complex) left-action generated by R_C on the complex module $(E \otimes_{\mathbb{R}} \mathbb{C})_{A_{\mathbb{C}}}$ with grading $\mathrm{sgn}(H)$. Hence the left-action is a graded representation of $\mathbb{C}\ell_1$.

Once again the case of $G = \{1, T\}$ is slightly more complicated as R_T is evenly graded. If $R_T^2 = 1$, then R_T implements a Real involution on the module E_A and gives no additional Clifford representation. If $R_T^2 = -1$, then R_T encodes a quaternionic structure on E_A. There is an equivalence between graded quaternionic modules and graded real modules with a left $C\ell_{4,0}$-action. Specifically, we take $E_A \oplus E_A$ and consider the real action generated by

$$\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & -R_T \\ R_T & 0 \end{pmatrix}, \begin{pmatrix} 0 & -iR_T \\ iR_T & 0 \end{pmatrix} \right\}, \quad \Gamma = \begin{pmatrix} \mathrm{sgn}(H) & 0 \\ 0 & -\mathrm{sgn}(H) \end{pmatrix}.$$

We may also replace i with $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and consider the action on $E_A^{\oplus 4}$. In either case we obtain a graded adjointable representation of $C\ell_{4,0}$. □

The Clifford representations that we construct in Proposition 5.2.13 are analogous to the representations in Proposition 5.2.9 and therefore are distinct up to stable isomorphism by Table 5.1. Hence, like the Hilbert space picture, there is a 1-1 correspondence between symmetry compatible Hamiltonians and graded Clifford representations on the C^*-module E_A^N (again, up to stable isomorphism).

We shall denote the class of the Kasparov module of Proposition 5.2.13 by $[H^G]$, an element in $KKO(C\ell_{n,0}, A)$ (or in the complex case $KK(\mathbb{C}\ell_n, A_{\mathbb{C}})$). We think of the class $[H^G]$ as encoding the internal symmetries of the Hamiltonian.

For trivially graded algebras A, the class $[H^G]$ can be associated to a class in either $KO_n(A)$ or $K_n(A_{\mathbb{C}})$ (cf. Proposition 2.3.9). Indeed for $A = C^*_\phi(\mathbb{Z}^d)$ and trivially graded, we have that

$$KKO(C\ell_{n,0}, C^*_\phi(\mathbb{Z}^d)) \cong KKO(C\ell_{n,0}, C^*(\mathbb{Z}^d)\hat{\otimes}\mathcal{K}) \cong KO_n(C^*(\mathbb{Z}^d)) \cong KO^{-n}(\mathbb{T}^d),$$

where we have used Proposition 2.3.9 and the Packer-Raeburn stabilisation trick to 'untwist' the group C^*-algebra up to stable isomorphism [PR89]. Hence we recover the real K-theory of the discrete Brillouin zone, though we note that the noncommutative method allows for more complicated algebras and spaces to be considered.

Remark 5.2.14 (Anti-linear symmetries and Real C^*-algebras). We have shown how the symmetries coming from the group $\{1, T, P, PT\}$ can be linked to real C^*-algebras and KKO-theory. One may ask whether we can also study this question from the perspective of Real C^*-algebras and KKR-theory. The construction of the crossed product $A \rtimes G$ where $\alpha_g(a) = R_g a R_g^*$ will *not* hold in the Real category if $G = \{1, T, P, PT\}$ as this will involve two anti-linear automorphisms α_P and α_T. However, if we consider the subgroups $\{1, T\}$ or $\{1, P\}$ with R_T or R_P defining a Real structure on the (complex) Hilbert space $\mathcal{H}$, then the action $\alpha_T(a) = R_T a R_T^*$ defines a Real involution on the complex algebra $C^*_\phi(\mathbb{Z}^d) \otimes_{\mathbb{R}} \mathbb{C}$ with $a^\tau = \alpha_g(a)$ for $g = T$ or P.

We expect a similar result to Proposition 5.2.13 to hold in the Real picture provided $G = \{1, T\}$ or $\{1, P\}$. In the interest of brevity, we will leave a proper investigation of the wider links between insulator systems and KKR-theory to another place.

5.2.4 Spectral triples and pairings

Our discussion up to this point has centred mostly on K-theory, but this is not the end of the story. Recall from Chapter 3 that if we are interested in the conductance of a physical system, then for gapped Hamiltonians this can be represented as the index pairing of the Fermi projection with a particular K-homology class.

For complex discrete systems without disorder, a Hamiltonian H that satisfies Assumption 5.2.11 is contained in $C^*_\phi(\mathbb{Z}^d) \otimes_{\mathbb{R}} \mathbb{C}$. We can consider a dense $*$-subalgebra $\mathcal{A}_{\mathbb{C}} \subset C^*_\phi(\mathbb{Z}^d) \otimes_{\mathbb{R}} \mathbb{C}$ of finite polynomials of shift operators and construct the complex spectral triple

$$\left(\mathcal{A}_{\mathbb{C}},\, \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N \otimes \mathbb{C}^\nu,\, \sum_{j=1}^{d} X_j \otimes 1_N \otimes \gamma^j,\, \gamma = (-i)^{d/2} \gamma^1 \cdots \gamma^d \right), \tag{5.5}$$

where the matrices γ^j have the relation $\gamma^i\gamma^j + \gamma^j\gamma^i = 2\delta_{i,j}$. As we saw in Chapter 3.3, we obtain 'higher order Chern numbers' by taking the index pairing of this spectral triple with the Fermi projection or some unitary $u \in \mathcal{A}$ (see also [PLB13, PS14]). Putting this in the language of Kasparov theory, for d even

$$\begin{aligned} &KK(\mathbb{C}, A) \times KK(A, \mathbb{C}) \to KK(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z} \\ &C_d = [P_\mu] \hat{\otimes}_A \left[\left(\mathcal{A}_{\mathbb{C}}, \ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^N \otimes \mathbb{C}^\nu, X = \sum_{j=1}^{d} X_j \otimes 1_N \otimes \gamma^j, \gamma \right) \right] \\ &\quad = \mathrm{Index}(P_\mu X_+ P_\mu), \end{aligned}$$

where $X = \begin{pmatrix} 0 & X_- \\ X_+ & 0 \end{pmatrix}$ is decomposed by the grading γ. The case of d odd has an analogous formula but we are taking a product of $KK(\mathbb{C}\ell_1, A)$ with $KK(A, \mathbb{C}\ell_1)$.

Our goal is to refine the complex index pairing to the real picture when one considers time-reversal and particle-hole symmetry.

The bulk spectral triple

Real spectral triples require representations on real Hilbert spaces. Hence, we take $A = C^*_\phi(\mathbb{Z}^d)$ acting on the bulk Hilbert space $\mathcal{H}_b$, which can be $\ell^2(\mathbb{Z}^d)\otimes\mathbb{R}^N$ or $\ell^2(\mathbb{Z}^d)\otimes\mathbb{C}^M$, where $\mathbb{C}\cong\mathbb{R}\oplus i\mathbb{R}$ is considered as a real space. For the case of a uniform magnetic field present, we take $\ell^2(\mathbb{Z}^d)\otimes\mathbb{C}^M$ to more easily line up with the magnetic field picture developed in Chapters 3 and 4. The twisted shift operators $\widehat{S}_j$ that generate H may be represented by

$$(\widehat{S}^\alpha\psi)(x) = e^{i\frac{e}{\hbar}A(x)\cdot\alpha}\psi(x-\alpha),$$

where $\widehat{S}^\alpha = \widehat{S}_1^{\alpha_1}\cdots\widehat{S}_d^{\alpha_d}$ for $\alpha = (\alpha_1,\ldots,\alpha_d)\in\mathbb{Z}^d$ and $A(x)$ is the magnetic potential. Such translation operators give a projective representation of $\mathbb{Z}^d$ with 2-cocyle given by $\sigma(a,b) = e^{i\frac{e}{\hbar}A(a)\cdot b}$. In the setting of real algebras, additional restrictions are placed on the twist σ; we will largely avoid these issues and refer to [Kel15] for more details. The twisted shift operators also commute with the the magnetic translations, which generate a $\overline{\sigma}$-representation of $\mathbb{Z}^d$.

Our task is to construct a Kasparov module that is capturing the geometry of the (possibly noncommutative) Brillouin zone. If a Hamiltonian H satisfies Assumption 5.2.11, then we take $\mathcal{A}$ to be the $*$-algebra of finite polynomials of (twisted) shift operators (or infinite polynomials with Schwartz-class coefficients) over $\mathbb{R}$. Such an algebra $\mathcal{A}$ is dense in $C^*_\phi(\mathbb{Z}^d)$ (and similarly $\mathcal{A}\otimes_\mathbb{R}\mathbb{C}$ is dense in $A_\mathbb{C}$). We require a dense subalgebra in order to deal with spectral triples and unbounded modules over $C^*_\phi(\mathbb{Z}^d)$. Similar to [Kas88, LRV12], we have the following result.

Proposition 5.2.15. *If a Hamiltonian H satisfies Assumption 5.2.11 with $\mathcal{A}\subset C^*_\phi(\mathbb{Z}^d)$, then*

$$\lambda = \left(\mathcal{A}\hat{\otimes}C\ell_{0,d},\, \mathcal{H}_b\otimes{\textstyle\bigwedge}^*\mathbb{R}^d,\, \sum_{j=1}^d X_j\otimes\gamma^j,\, \gamma_{\bigwedge^*\mathbb{R}^d}\right)$$

is a real spectral triple, where X_j is the position operator on $\ell^2(\mathbb{Z}^d)$ and acts diagonally on $\mathcal{H}_b$. The left-action of $C\ell_{0,d}$ is generated by the operators $\{\rho^j\}_{j=1}^d$ and the operators $\{\gamma^j\}_{j=1}^d$ generate $C\ell_{d,0}$. The Clifford algebras $C\ell_{0,d}$ and $C\ell_{d,0}$ are represented as left and right actions on $\bigwedge^*\mathbb{R}^d$ respectively by the formulae

$$\rho^j(\omega) = e_j\wedge\omega - \iota(e_j)\omega, \qquad\qquad \gamma^j(\omega) = e_j\wedge\omega + \iota(e_j)\omega, \tag{5.6}$$

with $\omega\in\bigwedge^*\mathbb{R}^d$, $\{e_j\}_{j=1}^d$ the standard basis of $\mathbb{R}^d$ and $\iota(v)\omega$ the contraction of ω along v. The grading $\gamma_{\bigwedge^*\mathbb{R}^d}$ is given in terms of the isomorphism $C\ell_{0,d}\hat{\otimes}C\ell_{d,0}\cong\mathrm{End}_\mathbb{R}(\bigwedge^*\mathbb{R}^d)$, where $\gamma_{\bigwedge^*\mathbb{R}^d} = (-1)^d\rho^1\cdots\rho^d\hat{\otimes}\gamma^d\cdots\gamma^1$.

One can check that ρ^j and γ^k anti-commute (i.e. they graded-commute). We note that, despite a right-action by $C\ell_{d,0}$ on $\bigwedge^* \mathbb{R}^d$, we do *not* get an $A\hat{\otimes}C\ell_{0,d}$-$C\ell_{d,0}$ Kasparov module as the graded-commutator of $1 \otimes \gamma^k$ with $\sum_j X_j \otimes \gamma^j$ is not bounded (see [LRV12, Section 4.3] for a more detailed discussion on these Clifford actions and the link to Kasparov's fundamental class).

Proof of Proposition 5.2.15. Because $[\rho^j, \gamma^k]_+ = 0$, we obtain that ρ^j graded-commutes with $\sum_k X_k \otimes \gamma^k$. Therefore we just need to check $[X_j \otimes 1_N, a]$ is bounded for all j and $a(1+D^2)^{-1/2}$ is compact for $a \in \mathcal{A}$. We let $\widehat{S}^\alpha = \widehat{S}_1^{\alpha_1} \cdots \widehat{S}_d^{\alpha_d}$ for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{Z}^d$. Then, for $\psi \in \mathrm{Dom}(X_j) \subset \ell^2(\mathbb{Z}^d)$,

$$\begin{aligned}[X_j, \widehat{S}^\alpha]\psi(x) &= x_j c_{\alpha,x}\psi(x-\alpha) - c_{\alpha,x}(x_j - \alpha_j)\psi(x-\alpha) \\ &= \alpha_j(\widehat{S}^\alpha\psi)(x),\end{aligned}$$

where the scalar $c_{\alpha,x}$ comes from that $\widehat{S}^\alpha$ is possibly twisted by a magnetic field. Therefore $[X_j, a]$ extends to a bounded operator for a any finite polynomial of $\widehat{S}^\alpha$ on $\ell^2(\mathbb{Z}^d)$. Because such elements generate $\mathcal{A}$, $[X_j, a]$ is bounded for any $a \in \mathcal{A}$.

Next we note that $(1+D^2)^{-1/2} = (1+|X|^2)^{-1/2} \otimes 1_N \otimes 1_{\bigwedge^* \mathbb{R}^d}$ as an operator on on $\mathcal{H}_b = \ell^2(\mathbb{Z}^d) \otimes \mathbb{F}^N \otimes \bigwedge^* \mathbb{R}^d$ for $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. On $\ell^2(\mathbb{Z}^d)$,

$$(1+|X|^2)^{-1/2} = \bigoplus_{k\in\mathbb{Z}^d} (1+|k|^2)^{-1/2} P_k,$$

where P_k is the projection onto the span of $e_{(k_1,\ldots,k_d)}$ with $\{e_k\}_{k\in\mathbb{Z}^d}$ the standard basis of $\ell^2(\mathbb{Z}^d)$. Hence $(1+|X|^2)^{-1/2}$ is a norm-convergent sum of finite-rank operators and so is compact. From this we conclude that $(1+D^2)^{-1/2}$ is compact on $\mathcal{H}_b$. □

For the case of complex algebras, the spectral triple of interest is given in Equation (5.5). Such a spectral triple can be considered as the discrete analogue of the spectral triple from Chapter 3, Proposition 3.3.1.

We think of the real spectral triple of Proposition 5.2.15 as encoding geometric information of the (possibly noncommutative) Brillouin torus, including dimension. The Kasparov module represented by $[H^G]$ on the other hand captures information about the internal symmetries of the Hamiltonian. By taking the pairing/product of the $[H^G]$ with the spectral triple, we obtain all the topological information of interest in the system.

Remark 5.2.16 (Pairings and the periodic table). Our unbounded module gives a class $[\lambda] \in KKO(A\hat{\otimes}C\ell_{0,d}, \mathbb{R})$ [BJ83]. We would like to consider an analogous notion of the Chern numbers in the real category. However, because we are dealing with representatives of KKO-classes, we need to generalise the complex pairing to the internal product of $[\lambda]$ with the class $[H^G]$ from Proposition 5.2.13 that represents the symmetries of

Symmetry generators	R_P^2	R_T^2	Graded Representation	$[H^G]\hat{\otimes}[\lambda] \in KO_{n-d}(\mathbb{R})$ or $K_{n-d}(\mathbb{C})$ $d=0$	$d=1$	$d=2$	$d=3$
T		$+1$	$C\ell_{0,0}$	$\mathbb{Z}$	0	0	0
P,T	$+1$	$+1$	$C\ell_{1,0}$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
P	$+1$		$C\ell_{2,0}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
P,T	$+1$	-1	$C\ell_{3,0}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
T		-1	$C\ell_{4,0}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
P,T	-1	-1	$C\ell_{5,0}$	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
P	-1		$C\ell_{6,0}$	0	0	$\mathbb{Z}$	0
P,T	-1	$+1$	$C\ell_{7,0}$	0	0	0	$\mathbb{Z}$
N/A			$\mathbb{C}\ell_0$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
C	$R_C^2 = 1$		$\mathbb{C}\ell_1$	0	$\mathbb{Z}$	0	$\mathbb{Z}$

Table 5.2: Symmetry types, their corresponding graded Clifford representation and the pairing of the Fermi projection with the d-dimensional spectral triple (shown for $d \leq 3$).

the Hamiltonian.

$$\begin{aligned} &KKO(C\ell_{n,0}, A) \times KKO(A\hat{\otimes}C\ell_{0,d}, \mathbb{R}) \to KKO(C\ell_{n,0}\hat{\otimes}C\ell_{0,d}, \mathbb{R}) \\ &C_{n,d} = [H^G]\hat{\otimes}_A[\lambda] \end{aligned}$$

We represent the index pairing as a Kasparov product rather than a pairing of a projection with a cyclic cocyle as the latter involves a map to periodic cyclic cohomology, which is unable to detect torsion invariants. We note that the class $[H^G]\hat{\otimes}_A[\lambda]$ takes values in $KKO(C\ell_{n,0}\hat{\otimes}C\ell_{0,d}, \mathbb{R}) \cong KO_{n-d}(\mathbb{R})$ [Kas81, §6]. Therefore, by considering the various symmetry subgroups of $\{1, T, P, TP\}$ that give rise to graded Clifford representations of $C\ell_{n,0}$ for different n outlined in Table 5.1, we are able to derive the celebrated periodic table of Kitaev, which is given in Table 5.2.

We summarise our work in this section.

Proposition 5.2.17. *The periodic table of topological insulators can be realised as the index pairing (Kasparov product) of the real/complex Kasparov module of Proposition 5.2.13 with the bulk spectral triple of Proposition 5.2.15 or Equation* (5.5).

5.2.5 The Kasparov product and the Clifford index

So far we have identified the invariants of interest in topological insulator systems as a Kasparov product, $[H^G]\hat{\otimes}_A[\lambda]$, of KK-classes (complex or real) capturing internal symmetries and geometric information. In the case of complex algebras and modules, this abstract pairing can be concretely represented as a Fredholm index and takes the form Index(PX_+P) or Index(PuP) depending on whether d is even or odd. It would be desirable to have a similar notion in the real case in order to express the Kasparov product $[H^G]\hat{\otimes}_A[\lambda]$ more concretely. In particular we consider the link to Clifford modules and the Atiyah-Bott-Shapiro constructions in KO-theory (see [ABS64, LM89]).

In order to draw this link, we first must compute the (unbounded) product

$$\big(C\ell_{n,0}, E_A^N, 0, \Gamma\big)\,\hat{\otimes}_{\mathcal{A}} \left(\mathcal{A}\hat{\otimes}C\ell_{0,d},\, \mathcal{H}_b\otimes\bigwedge\nolimits^{*}\mathbb{R}^d,\, \sum_{j=1}^{d} X_j\otimes\gamma^j,\, \gamma_{\bigwedge^*\mathbb{R}^d}\right).$$

Lemma 5.2.18. *The real Kasparov product $[H^G]\hat{\otimes}_A[\lambda]$ can be represented by the unbounded Kasparov module*

$$\left(C\ell_{n,0}\hat{\otimes}C\ell_{0,d},\, (E^N\otimes_A\mathcal{H}_b)\otimes\bigwedge\nolimits^{*}\mathbb{R}^d,\, \sum_{j=1}^{d}(1\otimes_\nabla X_j)\otimes\gamma^j,\, (\Gamma\otimes 1)\hat{\otimes}\gamma_{\bigwedge^*\mathbb{R}^d}\right),$$

where the operators $1\otimes_\nabla X_j$ come from a connection on E_A (cf. Definition 2.2.42).

Proof. In order to take the product

$$KKO(C\ell_{n,0}, A)\times KKO(A\hat{\otimes}C\ell_{0,d},\mathbb{R})\to KKO(C\ell_{n,0}\hat{\otimes}C\ell_{0,d},\mathbb{R}),$$

we first need to take an external product with the identity class in $KKO(C\ell_{0,d}, C\ell_{0,d})$. This class can be represented by the Kasparov module

$$\Big(C\ell_{0,d}, (C\ell_{0,d})_{C\ell_{0,d}}, 0, \gamma_{C\ell_{0,d}}\Big)$$

with right and left actions given by right and left Clifford multiplication (cf. Example 2.2.26). At the level of C^*-modules, the product module is given by

$$\begin{aligned}\big(E_A^N\hat{\otimes}_{\mathbb{R}}C\ell_{0,d}\big)\,\hat{\otimes}_{A\hat{\otimes}C\ell_{0,d}}\Big(\mathcal{H}_b\otimes\bigwedge\nolimits^{*}\mathbb{R}^d\Big) &\cong (E^N\otimes_A\mathcal{H}_b)\hat{\otimes}_{\mathbb{R}}\Big(C\ell_{d,0}\cdot\bigwedge\nolimits^{*}\mathbb{R}^d\Big)\\ &\cong (E^N\otimes_A\mathcal{H}_b)\otimes\bigwedge\nolimits^{*}\mathbb{R}^d\end{aligned}$$

as the action of $C\ell_{0,d}$ on $\bigwedge^*\mathbb{R}^d$ is bijective. Furthermore, the action of $C\ell_{n,0}$ and $C\ell_{0,d}$ on E_A^N and $\bigwedge^*\mathbb{R}^d$ respectively can be extended to an action of $C\ell_{n,0}\hat{\otimes}C\ell_{0,d}$ on $(E^N\otimes_A\mathcal{H}_b)\otimes\bigwedge^*\mathbb{R}^d$.

Next we construct the operator $1\otimes_\nabla X_j$ on $E^N\otimes_A\mathcal{H}_b$ for $j\in\{1,\dots,d\}$. First let $\mathcal{E}_A$ be the submodule of E, which is spanned by elements of the form

$$\sum_{g\in G} a_g R_g = \sum_{g\in G} R_g\alpha_g^{-1}(a_g) = \sum_{g\in G} R_g\tilde{a}_g.$$

On the module of such finite sums, we take the connection

$$\nabla:\mathcal{E}\to\mathcal{E}\otimes_{\mathrm{poly}(a)}\Omega^1(\mathrm{poly}(a)),\qquad \nabla\left(\sum_{g\in G}R_g a_g\right) = \sum_{g\in G}R_g\otimes\delta(a_g),$$

where δ is the universal derivation. We represent 1-forms on $\mathcal{H}_b$ via

$$\tilde{\pi}(a_0\delta(a_1))\,\lambda = a_0[X_j, a_1]\lambda, \quad \lambda \in \mathcal{H}_b,$$

from which we define, for $(e \otimes \lambda) \in E \otimes_A \mathcal{H}_b$,

$$(1 \otimes_\nabla X_j)(e \otimes \lambda) := (e \otimes X_j\lambda) + (1 \otimes \tilde{\pi}) \circ (\nabla \otimes 1)(e \otimes \lambda).$$

We use a connection to correct the naive formula $1 \otimes X_j$ is because $1 \otimes X_j$ is not well-defined on the balanced tensor product. Computing yields that

$$\begin{aligned}(1 \otimes_\nabla X_j)\left(\sum_{g\in G} R_g a_g \otimes \lambda\right) &= \sum_{g\in G} R_g \otimes a_g X_j \lambda + \sum_{g \in G} R_g \otimes [X_j, a_g]\lambda \\ &= \sum_{g\in G} R_g \otimes X_j a_g \lambda.\end{aligned}$$

For the case of $E^N \otimes_A \mathcal{H}_b$ with $N \geq 2$, we can always inflate $\mathcal{H}_b$ to $\mathcal{H}_b^{\oplus N}$ and define the operator $(1 \otimes_\nabla X_j)$ diagonally. The operator $\sum_{j=1}^d (1 \otimes_\nabla X_j) \otimes \gamma^j$ has compact resolvent by entirely analogous arguments to the proof of Proposition 5.2.15.

Combining our results so far, we consider the unbounded tuple

$$\left(C\ell_{n,0}\hat{\otimes} C\ell_{0,d},\, (E^N \otimes_A \mathcal{H}_b) \otimes {\bigwedge}^* \mathbb{R}^d,\, \sum_{j=1}^d (1 \otimes_\nabla X_j) \otimes \gamma^j,\, (\Gamma \otimes 1)\hat{\otimes}\gamma_{\bigwedge^* \mathbb{R}^d} \right).$$

By construction, all Clifford generators have odd grading and graded-commute with the Dirac-type operator. Hence the tuple is a real spectral triple. A simple check shows that the spectral triple satisfies Kucerovsky's criterion [Kuc97, Theorem 13] and so is an unbounded representative of the product. □

We let $\widetilde{X}$ be the product operator $\sum_j (1 \otimes_\nabla X_j) \otimes \gamma^j$. Representing the $\mathbb{Z}_2$-grading as $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, we can express $\widetilde{X} = \begin{pmatrix} 0 & \widetilde{X}_- \\ \widetilde{X}_+ & 0 \end{pmatrix}$, where $\widetilde{X}_\pm$ are real Fredholm operators. The operator $\widetilde{X}$ graded-commutes with a left action of $C\ell_{n,0}\hat{\otimes} C\ell_{0,d} \cong C\ell_{n,d}$. As $\widetilde{X}$ is Fredholm, $\mathrm{Ker}(\widetilde{X}) \cong \mathrm{Ker}(\widetilde{X})^0 \oplus \mathrm{Ker}(\widetilde{X})^1$ is a finite-dimensional $\mathbb{Z}_2$-graded $C\ell_{n,d}$-module. Furthermore, $\mathrm{Ker}(\widetilde{X})^0 \cong \mathrm{Ker}(\widetilde{X}_+)$.

Definition 5.2.19 ([ABS64])**.** Denote by $\hat{\mathfrak{M}}_{r,s}$ the Grothendieck group of equivalence classes of real $\mathbb{Z}_2$-graded modules with an irreducible graded left-representation of $C\ell_{r,s}$.

Using the notation of Clifford modules, $\mathrm{Ker}(\widetilde{X})$ determines a class in the quotient group $\hat{\mathfrak{M}}_{n,d}/i^*\hat{\mathfrak{M}}_{n,d+1}$, where i^* comes from restricting a Clifford action of $C\ell_{n,d+1}$ to $C\ell_{n,d}$. Next, we use the Atiyah-Bott-Shapiro isomorphism [LM89, Theorem I.9.27] to relate

$$\hat{\mathfrak{M}}_{n,d}/i^*\hat{\mathfrak{M}}_{n,d+1} \cong KO^{d-n}(\mathrm{pt}) \cong KO_{n-d}(\mathbb{R}).$$

Definition 5.2.20. The Clifford index of $\widetilde{X}$ is given by

$$\mathrm{Index}_{n-d}(\widetilde{X}) := [\mathrm{Ker}(\widetilde{X})] \in \hat{\mathfrak{M}}_{n,d}/i^*\hat{\mathfrak{M}}_{n,d+1} \cong KO_{n-d}(\mathbb{R}).$$

We remark that Index_k is a generalisation of the usual index. To see this, we first note that $C\ell_{0,0} \cong \mathbb{R}$ and $C\ell_{0,1} \cong \mathbb{C}$. A $\mathbb{Z}_2$-graded $C\ell_{0,0}$-module is given by any $\mathbb{Z}_2$-graded finite-dimensional real vector space $V^0 \oplus V^1$. Next observe that $V \oplus V \cong V \otimes \mathbb{C}$ extends to a graded $C\ell_{0,1}$-module, which implies that $[V \oplus 0] = -[0 \oplus V]$ in $\hat{\mathfrak{M}}_{0,0}/i^*\hat{\mathfrak{M}}_{0,1}$. Hence, given a Dirac-type operator D such that $\mathrm{Ker}(D)$ is a $\mathbb{Z}_2$-graded $C\ell_{0,0}$-module,

$$\begin{aligned}\mathrm{Index}_0(D) &= [\mathrm{Ker}(D)^0 \oplus \mathrm{Ker}(D)^1] \cong [\mathrm{Ker}(D)^0 \oplus 0] - [\mathrm{Ker}(D)^1 \oplus 0] \\ &\cong \dim_{\mathbb{R}} \mathrm{Ker}(D_+) - \dim_{\mathbb{R}} \mathrm{CoKer}(D_+) \in \mathbb{Z} \cong KO_0(\mathbb{R}).\end{aligned}$$

Therefore we see that Index_k reduces to the usual Fredholm index when $k = 0$. We direct the reader to [AS69] and [LM89, Chapter I.9, II.7, III.10] for more details on the Clifford index. A similar viewpoint on expressing the invariants in $KO_{n-d}(\mathbb{R})$ as index-like maps is considered in [DNSB14, GS15].

Lemma 5.2.21. *The unbounded module representing the product from Lemma 5.2.18 does not contribute any topological information outside of* $\mathrm{Ker}(\widetilde{X})$.

Proof. Recall from Lemma 5.2.18 that the real index pairing $[H^G]\hat{\otimes}_A[\lambda]$ is represented by the unbounded module

$$\left(C\ell_{n,0}\hat{\otimes}C\ell_{0,d},\, (E^N \otimes_A \mathcal{H}_b) \otimes \textstyle\bigwedge^* \mathbb{R}^d,\, \displaystyle\sum_{j=1}^{d} (1 \otimes_\nabla X_j) \otimes \gamma^j,\, (\Gamma \otimes 1)\hat{\otimes}\gamma_{\bigwedge^* \mathbb{R}^d} \right). \tag{5.7}$$

We let $\widetilde{X} = \sum_{j=1}^d (1 \otimes_\nabla X_j) \otimes \gamma^j$ and $\mathcal{H} = E^N \otimes_A \mathcal{H}_b$. Associated to the real spectral triple of Equation (5.7) is the real Fredholm module

$$\left(C\ell_{n,d}, \mathcal{H}, \widetilde{F}, \gamma \right),$$

where $\widetilde{F} = \widetilde{X}(1+\widetilde{X}^2)^{-1/2}$ [BJ83]. Because $\widetilde{X}$ is self-adjoint and graded-commutes with the Clifford action, so does $\widetilde{F}$. Hence $[\pi(c), \widetilde{F}]_\pm = \pi(c)(\widetilde{F} - \widetilde{F}^*) = 0$ for any $c \in C\ell_{n,d}$. What stops the Fredholm module being degenerate is that $(1 - \widetilde{F}^2) \in \mathcal{K}(\mathcal{H})$ is not necessarily zero.

We use the (real) polar decomposition of $\widetilde{F} = V|\widetilde{F}|$ from [Li03, Theorem 1.2.5] and note that $\mathrm{Ker}(V) = \mathrm{Ker}(\widetilde{F}) = \mathrm{Ker}(\widetilde{X})$. Because $\mathrm{Ker}(V) = \mathrm{Ker}(\widetilde{F})$, we can take the operator homotopy $F_t = V|\widetilde{F}|^t$, $t \in [0,1]$ to obtain the Fredholm module $(C\ell_{n,d}, \mathcal{H}, V, \gamma)$, which represents a class in $KKO(C\ell_{n,d}, \mathbb{R})$. The partial isometry V is a real Fredholm operator as $1_\mathcal{H} - V^*V$ is a finite-rank projection and so V has a pseudo-inverse.

Finally we write $(C\ell_{n,d}, \mathcal{H}, V, \gamma) = \left(C\ell_{n,d}, \mathrm{Ker}(\widetilde{X}), 0, \gamma\right) \oplus (C\ell_{n,d}, V^*V\mathcal{H}, V, \gamma)$, and the second summand is degenerate. Thus the KK-class and so the index depends only on the former module.

Summing up our discussion, the topological properties of the product spectral triple of Equation (5.7) are wholly contained in the real Fredholm index of V and, hence, are determined by $\mathrm{Ker}(V) = \mathrm{Ker}(\widetilde{X})$. Therefore it suffices to consider the Clifford module properties of $\mathrm{Ker}(\widetilde{X})$. □

5.2.6 Examples

Example 5.2.22 (The quantum spin-Hall effect, Kane-Mele model). We take $d = 2$ and the subgroup $G = \{1, T\}$. We are modelling particles with spin $s = 1/2$ and so the time-reversal involution R_T is such that $R_T^2 = (-1)^{2s} = -1$. The operator R_T is anti-unitary so we will work in the category of real algebras and modules. The time-reversal operator acts on $\mathcal{H} = \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^{2N}$ by the matrix

$$R_T = \begin{pmatrix} 0_N & \mathcal{C} \\ -\mathcal{C} & 0_N \end{pmatrix},$$

where $\mathcal{C}$ is pointwise complex conjugation. A self-adjoint operator that is invariant under conjugation by R_T takes the form $\begin{pmatrix} a & b \\ -\mathcal{C}b\mathcal{C} & \mathcal{C}a\mathcal{C} \end{pmatrix}$, where a and $\mathcal{C}a\mathcal{C}$ are self-adjoint and $b^* = -\mathcal{C}b\mathcal{C}$. Following [KM05, DNSB14], we take the Hamiltonian

$$H_{KM} = \begin{pmatrix} h & g \\ g^* & \mathcal{C}h\mathcal{C} \end{pmatrix},$$

where h is a Haldane Hamiltonian (that is, Hamiltonian of shift operators acting on a honeycomb lattice), and g is the Rashba coupling [KM05]. We either require the Rashba coupling to be such that $g^* = -\mathcal{C}g\mathcal{C}$ or it is sufficiently small so we may take a homotopy of H_{KM} to a Hamiltonian with $g = 0$ [SB13, DNSB14]. Typically h and g are matrices of finite polynomials of the shift operators S_j (see [ASBVB13, Section 5]). Provided h and g are such that $\mu \notin \sigma(H_{KM})$, H_{KM} satisfies Assumption 5.2.11. Therefore we take the algebra $A = M_{2N}(C^*(\mathbb{Z}^2)) \subset \mathcal{B}[\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^{2N}]$ and apply Proposition 5.2.15 to obtain the real spectral triple

$$\left(\mathcal{A}\hat{\otimes}C\ell_{0,2}, \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^{2N} \otimes {\bigwedge}^* \mathbb{R}^2, X_1 \otimes 1_{2N} \otimes \gamma^1 + X_2 \otimes 1_{2N} \otimes \gamma^2, \gamma_{\bigwedge^* \mathbb{R}^2}\right)$$

for a dense subalgebra $\mathcal{A} \subset A$ generated by the shift operators S_1 and S_2. We note that, to obtain a real spectral triple, we are interpreting $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^{2N}$ as a real Hilbert space. The left Clifford action is generated by ρ^1 and ρ^2, whose representation is given by Equation (5.6). We can use the isomorphism ${\bigwedge}^* \mathbb{R}^2 \cong M_2(\mathbb{C})$ to write explicit generators for our Clifford actions as matrices, though the result is independent of the choice of generator. For example, under a suitable identification of i as a 2×2 matrix that squares to -1, we can choose Clifford generators γ^j such that our Dirac-type operator is of the form

$$X = \begin{pmatrix} 0_{2N} & X_1\otimes 1_{2N} - iX_2\otimes 1_{2N} \\ X_1\otimes 1_{2N} + iX_2\otimes 1_{2N} & 0_{2N} \end{pmatrix},$$

which is analogous to the well-known Dirac-type operator of the quantum Hall effect.

We use Proposition 5.2.13 and Table 5.1 to see that a time-reversal invariant Hamiltonian H_{KM} with $R_T^2 = -1$ gives rise to a class $[H_{KM}^G] \in KKO(C\ell_{4,0}, M_{2N}[C^*(\mathbb{Z}^2)])$, which is isomorphic to $KKO(C\ell_{4,0}, C^*(\mathbb{Z}^2))$ by stability. The real index pairing comes from the product $[H_{KM}^G]\hat{\otimes}_A[X]$ (with $[X]$ the KO-homology class represented by the real spectral triple of the system), which is a map

$$\begin{aligned} KKO(C\ell_{4,0}, C^*(\mathbb{Z}^2)) \times KKO(C^*(\mathbb{Z}^2)\hat{\otimes} C\ell_{0,2}, \mathbb{C}) &\to KKO(C\ell_{4,0}\hat{\otimes} C\ell_{0,2}, \mathbb{C}) \\ \big([H_{KM}^G], [X]\big) &\mapsto \mathrm{Index}_{4-2}(\widetilde{X}) \in KO_2(\mathbb{R}) \cong \mathbb{Z}_2 \end{aligned}$$

and so we obtain the well-known $\mathbb{Z}_2$ invariant that arises in such systems. The derived $\mathbb{Z}_2$ invariant is non-trivial provided the spin-orbit coupling in h is sufficiently large and the Rashba coupling g is controlled (see [KM05, DNSB14]).

Example 5.2.23 ($3D$ Topological insulators). Let us now consider some 3-dimensional examples. What we consider does not encompass every possible $3D$-system, but will hopefully give a better understanding of how we apply our general K-theoretic picture.

Consider the space $\mathcal{H} = \ell^2(\mathbb{Z}^3)\otimes\mathbb{C}^{2N}$ and the symmetry operators

$$R_T = \begin{pmatrix} 0_N & \mathcal{C} \\ -\mathcal{C} & 0_N \end{pmatrix}, \qquad R_P = \begin{pmatrix} 0_N & i\mathcal{C} \\ i\mathcal{C} & 0_N \end{pmatrix}. \tag{5.8}$$

These operators correspond to an odd time-reversal involution ($R_T^2 = -1$) and an even particle-hole involution ($R_P^2 = 1$). First, we consider operators of the form

$$h = i\left(\sum_{j=1}^{3}\sum_{k_j}^{\text{finite}} \alpha_{k_j}\left(S_j^{k_j}\otimes 1_N - (S_j^*)^{k_j}\otimes 1_N\right)\right)$$

on $\ell^2(\mathbb{Z}^3)\otimes\mathbb{C}^N$ with $\alpha_{k_j}\in\mathbb{R}$ for all k_j. Using h we define

$$H_{3D} = \begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix}.$$

Because $h = h^*$ and $\mathcal{C}h\mathcal{C} = -h$, one can check that such Hamiltonians are time-reversal and particle-hole symmetric for R_T and R_P given in Equation (5.8). We choose coefficients α_{k_j} such that H_{3D} has a spectral gap at 0. Then H_{3D} satisfies Assumption 5.2.11 and so we can apply our general method. Because H_{3D} is compatible with the full symmetry group $\{1, T, P, PT\}$ with $R_T^2 = -1$ and $R_P^2 = 1$, Proposition 5.2.13 and Table 5.1 imply that the class $[H_{3D}^G] \in KKO(C\ell_{3,0}, C^*(\mathbb{Z}^3))$.

We can use Proposition 5.2.15 and the dense real subalgebra $\mathcal{A} \subset C^*(\mathbb{Z}^3)$ of finite polynomials of shift operators to build the spectral triple

$$\lambda_{3D} = \left(\mathcal{A}\hat{\otimes}C\ell_{0,3}, \ell^2(\mathbb{Z}^3) \otimes \mathbb{C}^{2N} \otimes \bigwedge^* \mathbb{R}^3, \sum_{j=1}^{3} X_j \otimes \gamma^j, \gamma_{\bigwedge^* \mathbb{R}^3} \right)$$

with left and right Clifford actions given by Equation (5.6). Because the pairing $[H_{3D}^G]\hat{\otimes}[\lambda_{3D}] \in KKO(C\ell_{3,3}, \mathbb{R})$, the Clifford index $[\mathrm{Ker}(\widetilde{X})] \in \hat{\mathfrak{M}}_{3-3}/i^*\hat{\mathfrak{M}}_{3-3-1}$ reduces to the usual index

$$\mathrm{Index}_0(\widetilde{X}) = \dim_{\mathbb{R}} \mathrm{Ker}(\widetilde{X}_+) - \dim_{\mathbb{R}} \mathrm{CoKer}(\widetilde{X}_+) \in \mathbb{Z}.$$

Hence, in this example of $d = 3$ with $R_T^2 = -1$ and $R_P^2 = 1$, the invariant of interest is the usual integer-valued index, though now seen as a special case of a much broader framework.

We now consider a different 3-dimensional Hamiltonian, defined by the matrix

$$\breve{H}_{3D} = \begin{pmatrix} h + \breve{h} & 0 \\ 0 & -h + \breve{h} \end{pmatrix}, \qquad \breve{h} = p(S_1, S_2, S_3),$$

where p is a finite polynomial with real coefficients such that $p(S_1, S_2, S_3)$ is self-adjoint. The new Hamiltonian has the property $R_T \breve{H}_{3D} R_T^* = \breve{H}_{3D}$, but is not particle-hole symmetric. Provided $\mu \notin \sigma(\breve{H}_{3D})$, we obtain a class $[\breve{H}_{3D}^G] \in KKO(C\ell_{4,0}, C^*(\mathbb{Z}^3))$ by Table 5.1. We use the same spectral triple

$$\lambda_{3D} = \left(\mathcal{A}\hat{\otimes}C\ell_{0,3}, \ell^2(\mathbb{Z}^3) \otimes \mathbb{C}^{2N} \otimes \bigwedge^* \mathbb{R}^3, \sum_{j=1}^{3} X_j \otimes \gamma^j, \gamma_{\bigwedge^* \mathbb{R}^3} \right)$$

and class $[\lambda_{3D}] \in KKO(C^*(\mathbb{Z}^3)\hat{\otimes}C\ell_{0,3}, \mathbb{R})$, whose product with $[\breve{H}_{3D}^G]$ is such that

$$[\breve{H}_{3D}^G]\hat{\otimes}_{C^*(\mathbb{Z}^3)}[\lambda_{3D}] \cong \mathrm{Index}_{4-3}(\widetilde{X}) \in KO_1(\mathbb{R}) \cong \mathbb{Z}_2.$$

We emphasise that the spectral triples used in the different 3-dimensional examples are the same (up to unitary equivalence) and so represent the same KO-homology class. What differentiates the invariants of interest in the two examples are the different classes represented by $[H^G] \in KKO(C\ell_{n,0}, C^*(\mathbb{Z}^3))$ for changing G and n. Hence the symmetries change but the Dirac type operator of the Brillouin zone $X = \sum_j X_j \otimes \gamma^j$ is the same (up to equivalence of KO-homology classes) in a fixed dimension. Such an occurence also appears in [DNSB14, GS15].

5.3 The bulk-edge correspondence

Now that we have derived the topological invariants of interest for insulator systems, we turn our attention to the case of boundaries and the bulk-edge correspondence. As was the case in Chapter 4, we follow the general picture of [SBKR02, KSB04b, KR08, MT15b] and link a bulk system to a system with boundary via a short exact sequence. We briefly outline this construction.

Let H_b be a bulk Hamiltonian that satisfies Assumption 5.2.11. We associate to H_b the (real or complex) algebra $A = C^*_\phi(\mathbb{Z}^d)$ generated by the shift operators that give H_b. The algebra acts on the boundary-free space $\ell^2(\mathbb{Z}^d)\otimes\mathbb{F}^N$, with $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, as possibly twisted translations $\widehat{S}_j$ (if there is no twist, then $\widehat{S}_j = S_j$). In the case of a constant magnetic field normal to our sample, we may choose the Landau gauge so that $\widehat{S}_j = S_j$ for $j < d$ and $\widehat{S}_d$ is a twisted translation. We introduce a boundary on the Hilbert space, but since there is no priveliged position for the boundary, we take our space to be $\ell^2(\mathbb{Z}^{d-1}\times\mathbb{N}_s)\otimes\mathbb{F}^N$ for $\mathbb{N}_s = \{n\in\mathbb{N} : n\leq s\}$. The Hamiltonian $H_s = \Pi_s H_b \Pi_s$ acts on the (complex) space with boundary, where $\Pi_s : \ell^2(\mathbb{Z}^d)\to\ell^2(\mathbb{Z}^{d-1}\times\mathbb{N}_s)$ is the obvious projection. We choose Dirichlet boundary conditions for H_s (though in the tight-binding picture, our choice of boundary conditions is not so important). We can also consider the (real or complex) algebras $\Pi_s A\Pi_s \cong C^*(\widehat{S}_1,\ldots,\widehat{S}_{d-1},\Pi_s\widehat{S}_d\Pi_s)$ as acting on $\ell^2(\mathbb{Z}^{d-1}\times\mathbb{N}_s)\otimes\mathbb{F}^N$. There is an obvious surjection $\ell^2(\mathbb{Z}^{d-1}\times\mathbb{N}_s)\xrightarrow{s\to\infty}\ell^2(\mathbb{Z}^d)$, which in turn gives a surjective map $q: C^*(\widehat{S}_1,\ldots,\widehat{S}_{d-1},\Pi_s\widehat{S}_d\Pi_s)\to C^*_\phi(\mathbb{Z}^d)$ and short-exact sequence

$$0\to \mathrm{Ker}(q)\to C^*(\widehat{S}_1,\ldots,\widehat{S}_{d-1},\Pi_s\widehat{S}_d\Pi_s)\to C^*_\phi(\mathbb{Z}^d)\to 0. \tag{5.9}$$

The following result is proved for complex algebras in [SBKR02, KSB04b, KR08] and then extended to the real picture in [MT15b].

Proposition 5.3.1. *The map $q: C^*(\widehat{S}_1,\ldots,\widehat{S}_{d-1},\Pi_s\widehat{S}_d\Pi_s)\to C^*_\phi(\mathbb{Z}^d)$ gives rise to the isomorphism $\mathrm{Ker}(q)\cong (C^*(\mathbb{Z})\rtimes\mathbb{Z})\otimes B$, where $B\cong C^*_{\tilde\phi}(\mathbb{Z}^{d-1})$ is a C^*-algebra that acts on $\ell^2(\mathbb{Z}^{d-1})\otimes\mathbb{F}^N$ and carries an action $\alpha(b)=\widehat{S}_d^* b\widehat{S}_d$ such that $C^*_\phi(\mathbb{Z}^d)\cong B\rtimes\mathbb{Z}$. If the Landau gauge is chosen, then $B\cong C^*(\mathbb{Z}^{d-1})$.*

Because $C^*(\mathbb{Z})\rtimes\mathbb{Z}\cong\mathcal{K}$ by Takai duality [Rae88], we obtain the Pimsner-Voiculescu short exact sequence

$$0\to\mathcal{K}\otimes B\to\mathcal{T}\to C^*_\phi(\mathbb{Z}^d)\to 0. \tag{5.10}$$

Equation (5.10) is equivalent to the short exact sequence of Equation (5.9), with $\mathcal{T}$ the real Toeplitz algebra $C^*(\widehat{S}_d\otimes V, B\otimes 1)$ and V the standard shift operator on $\ell^2(\mathbb{N})$ [PV80].

Remark 5.3.2 (Edge algebra). Because B acts on $\ell^2(\mathbb{Z}^{d-1})\otimes\mathbb{F}^N$, the ideal $\mathcal{K}\otimes B$ is interpeted as operators that are concentrated near the edge of our sample in the sense of being compact in the direction normal to the boundary. Therefore, we consider $\mathcal{K}\otimes B$ as our edge algebra describing the system localised near the boundary.

Analogous to the results in Chapter 4, the key result that captures the bulk-edge correspondence in the real setting is the factorisation of (the negative of) the bulk spectral triple as the Kasparov product of the Kasparov module representing the extension of Equation (5.10) and a spectral triple coming from the edge algebra B. By constructing unbounded Kasparov modules explicitly in terms of generators of Clifford algebras, we find that the technical details associated with taking the real intersection product are manageable.

5.3.1 Bulk-edge in KKO

The extension module

As explained in the introduction to this section, we have the bulk algebra A generated by the (twisted) shift operators, $A \cong C^*(\widehat{S}_1, \ldots, \widehat{S}_d) \cong C^*_\phi(\mathbb{Z}^d)$, which is linked to an edge algebra $B \cong C^*_{\tilde{\phi}}(\mathbb{Z}^{d-1})$ by $A \cong B \rtimes_\alpha \mathbb{Z}$ with $\alpha(b) = \widehat{S}_d^* b \widehat{S}_d$. Bulk and edge algebras are also connected by the real Pimsner-Voiculescu short exact sequence

$$0 \to \mathcal{K} \otimes B \to C^*(\widehat{S}_d \otimes V, B \otimes 1) \to C^*_\phi(\mathbb{Z}^d) \to 0,$$

where V is the standard shift operator on $\ell^2(\mathbb{N})$ [PV80]. Under the Landau gauge $\widehat{S}_j = S_j$ for $j < d$ and so $B \cong C^*(\mathbb{Z}^{d-1})$. Of course if there is no external magnetic field (or other twists on the shift operators), then both A and B are commutative.

Given this data, there is a general prescription for constructing the triple (not yet a Kasparov module) $(\mathcal{A}, Z_B, N)$ with $\mathcal{A}$ dense in $C^*_\phi(\mathbb{Z}^d)$ as outlined in Chapter 4.2.3. The space Z_B is a real C^*-module that is the completion of $C^*(\widehat{S}_d \otimes V, B \otimes 1)$ by the B-valued inner product

$$\begin{aligned}\Big(\widehat{S}_d^{l_1-l_2} b_1 \otimes V^{l_1}(V^*)^{l_2} \,\Big|\, \widehat{S}_d^{n_1-n_2} b_2 \otimes V^{n_1}(V^*)^{n_2}\Big)_B &\\ = b_1^* \widehat{S}_d^{(n_1-n_2)-(l_1-l_2)} b_2 \, \Psi\Big[(V^{l_1}(V^*)^{l_2})^* V^{n_1}(V^*)^{n_2}\Big]&\\ = b_1^* b_2 \, \delta_{l_1-l_2, n_1-n_2}.&\end{aligned}$$

The functional $\Psi : C^*(V) \to \mathbb{R}$ is defined as a real analogue to the functional in Chapter 4.2.3, namely

$$\Psi(T) = \lim_{s\to 1}(s-1)\sum_{k=0}^{\infty} \langle e_k, T e_k\rangle (1+k^2)^{-s/2}$$

for any basis $\{e_k\}$ of $\ell^2(\mathbb{N})$. The right-action of B on Z_B is defined from right-multiplication of $B \otimes 1$ on the Toeplitz algebra $C^*(\widehat{S}_d \otimes V, B \otimes 1)$, which is dense in Z_B. We check that

$$\begin{aligned}\left(\widehat{S}_d^{l_1-l_2}b_1 \otimes V^{l_1}(V^*)^{l_2} \,\middle|\, \left(\widehat{S}_d^{n_1-n_2}b_2 \otimes V^{n_1}(V^*)^{n_2}\right)\cdot b\right)_B &= b_1^* b_2 b\,\delta_{n_1-n_2,l_1-l_2} \\ &= \left(\widehat{S}_d^{l_1-l_2}b_1 \otimes V^{l_1}(V^*)^{l_2} \,\middle|\, \widehat{S}_d^{n_1-n_2}b_2 \otimes V^{n_1}(V^*)^{n_2}\right)_B b.\end{aligned}$$

Next, we define a left-action of $A \cong C^*(B, \widehat{S}_d)$ on Z_B to be generated by

$$\begin{aligned}\widehat{S}_d \cdot (\widehat{S}_d^{n_1-n_2}b \otimes V^{n_1}(V^*)^{n_2}) &= \widehat{S}_d^{n_1+1-n_2}b \otimes V^{n_1+1}(V^*)^{n_2} \\ b_1 \cdot (\widehat{S}_d^{n_1-n_2}b_2 \otimes V^{n_1}(V^*)^{n_2}) &= \widehat{S}_d^{n_1-n_2}\alpha_{n_1-n_2}(b_1)b_2 \otimes V^{n_1}(V^*)^{n_2}.\end{aligned}$$

Proposition 5.3.3. *The left action by $C^*_\phi(\mathbb{Z}^d)$ on Z_B is adjointable.*

Proof. We first compute that

$$\begin{aligned}&\left(\widehat{S}_d \cdot \left(\widehat{S}_d^{l_1-l_2}\, b_1 \otimes V^{l_1}(V^*)^{l_2}\right) \,\middle|\, \widehat{S}_d^{n_1-n_2}\, b_2 \otimes V^{n_1}(V^*)^{n_2}\right)_B \\ &\qquad = \left(\widehat{S}_d^{l_1-l_2+1}\, b_1 \otimes \widehat{S}_d^{l_1+1}(V^*)^{l_2} \,\middle|\, \widehat{S}_d^{n_1-n_2}\, b_2 \otimes V^{n_1}(V^*)^{n_2}\right)_B \\ &\qquad = b_1^* b_2\;\delta_{l_1-l_2+1,n_1-n_2} \\ &\qquad = b_1^* b_2\;\delta_{l_1-l_2,n_1-n_2-1} \\ &\qquad = \left(\widehat{S}_d^{l_1-l_2}\, b_1 \otimes V^{l_1}(V^*)^{l_2} \,\middle|\, \widehat{S}_d^{n_1-n_2-1}\, b_2 \otimes V^{n_1}(V^*)^{n_2+1}\right)_B \\ &\qquad = \left(\widehat{S}_d^{l_1-l_2}\, b_1 \otimes V^{l_1}(V^*)^{l_2} \,\middle|\, \widehat{S}_d^{-1} \cdot \left(\widehat{S}_d^{n_1-n_2}\, b_2 \otimes V^{n_1}(V^*)^{n_2}\right)\right)_B\end{aligned}$$

as required. Next, we see that

$$\begin{aligned}\left(b\,\widehat{S}_d^{l_1-l_2}b_1 \otimes V^l \,\middle|\, \widehat{S}_d^{n_1-n_2}b_2 \otimes V^n\right)_B &= \left(\widehat{S}_d^{l_1-l_2}\alpha_{l_1-l_2}(b)b_1 \otimes V^l \,\middle|\, \widehat{S}_d^{n_1-n_2}b_2 \otimes V^n\right)_B \\ &= b_1^*\alpha_{l_1-l_2}(b^*)b_2\,\delta_{l_1-l_2,n_1-n_2} \\ &= b_1^*\alpha_{n_1-n_2}(b^*)b_2\,\delta_{l_1-l_2,n_1-n_2} \\ &= \left(\widehat{S}_d^{l_1-l_2}b_1 \otimes V^l \,\middle|\, \widehat{S}_d^{n_1-n_2}\alpha_{n_1-n_2}(b^*)b_2 \otimes V^n\right)_B \\ &= \left(\widehat{S}_d^{l_1-l_2}b_1 \otimes V^l \,\middle|\, b^*\,\widehat{S}_d^{n_1-n_2}b_2 \otimes V^n\right)_B,\end{aligned}$$

where we have written $V^l = V^{l_1}(V^*)^{l_2}$ in order to save space. Therefore the generating elements of $C^*_\phi(\mathbb{Z}^d)$ are adjointable on the dense span of monomials in Z_B. If $\widehat{S}_d$, b are bounded, then they will generate an adjointable representation of $C^*_\phi(\mathbb{Z}^d)$. To consider the boundedness of $\widehat{S}_d$ and b, we first note that the inner-product in Z_B is defined from multiplication in $C^*(\widehat{S}_d \otimes V, B \otimes 1)$ and the functional Ψ, which has the property $\Psi(T) \leq \|T\|$ by Equation (4.2). These observations imply that

$$\|a\|_{\mathrm{End}_B(Z)} = \sup_{\substack{z\in Z\\ \|z\|=1}} (a\cdot z \mid a\cdot z)_B \leq \sup_{\substack{z\in Z\\ \|z\|=1}} \|aa^*\|\,(z \mid z)_B = \|aa^*\|.$$

Therefore the action of $C^*_\phi(\mathbb{Z}^d)$ is bounded, and so extends to an adjointable action on Z_B. ∎

Finally we define the unbounded operator $N : \mathrm{Dom}(N) \subset Z_B \to Z_B$ such that

$$N(\widehat{S}_d^{n_1-n_2} b \otimes V^{n_1}(V^*)^{n_2}) = (n_1 - n_2)\widehat{S}_d^{n_1-n_2} b \otimes V^{n_1}(V^*)^{n_2},$$
$$\mathrm{Dom}(N) = \left\{ \sum_{k\in\mathbb{Z}} z_{k_1,k_2} b \,:\, \sum_{k\in\mathbb{Z}} k^2 (z_{k_1,k_2} b \mid z_{k_1,k_2} b)_B \text{ well defined} \right\},$$

where $z_{k_1,k_2} b = \widehat{S}_d^{k_1-k_2} b \otimes V^{k_1}(V^*)^{k_2}$ and $k = k_1 - k_2$. We then take the dense subalgebra $\mathcal{A} \subset C^*_\phi(\mathbb{Z}^d)$ of finite polynomials of $\widehat{S}_j$ for $j \in \{1, \ldots, d\}$. Taking gradings into account, we can construct the tuple

$$\left(\mathcal{A}\hat{\otimes} C\ell_{0,1}, Z_B \otimes {\textstyle\bigwedge^*} \mathbb{R}, N \otimes \gamma_{ex}, \gamma_{\bigwedge^* \mathbb{R}}\right),$$

where γ_{ex} generates $C\ell_{1,0}$ and the $C\ell_{0,1}$ action is generated by ρ_{ex} with $\rho_{ex}(\omega) = e_1 \wedge \omega - \iota(e_1)\omega$, $e_1 \in \mathbb{R}$ the unit vector and $\gamma_{\bigwedge^* \mathbb{R}} = -\rho_{ex}\gamma_{ex}$.

Proposition 5.3.4. *If the bulk Hamiltonian satisfies Assumption 5.2.11, then the tuple* $\left(\mathcal{A}\hat{\otimes} C\ell_{0,1}, Z_B \otimes \bigwedge^* \mathbb{R}, N \otimes \gamma_{ex}, \gamma_{\bigwedge^* \mathbb{R}}\right)$ *is a real unbounded Kasparov A-B module that represents the same class in* $KKO(A\hat{\otimes} C\ell_{0,1}, B)$ *as the extension of Equation* (5.10).

Proof. Using the identification $C^*_\phi(\mathbb{Z}^d) \cong C^*(B, \widehat{S}_d)$, we compute that

$$\begin{aligned}[N, \widehat{S}_d^\beta]\widehat{S}_d^{k_1-k_2} b \otimes V^{k_1}(V^*)^{k_2} &= ((\beta + k_1 - k_2) - (k_1 - k_2))\, \widehat{S}_d^{\beta+k_1-k_2} b \otimes V^{k_1}(V^*)^{k_2} \\ &= \beta \widehat{S}_d^\beta \left(\widehat{S}_d^{k_1-k_2} b \otimes V^{k_1}(V^*)^{k_2}\right)\end{aligned}$$

and so $[N, \widehat{S}_d^\beta] = \beta \widehat{S}_d^\beta$. We also note that $[N, b] = 0$. Hence $[N, a] \in \mathrm{End}_B(Z)$ for a a finite polynomial of b and $\widehat{S}_d$ (or infinite polynomial with Schwartz-class coefficients). Next, an easy modification of the proof of Proposition 4.2.13 shows us that $(1+N^2)^{-1/2}$ is compact. The left Clifford action is constructed so that it (graded) commutes with the grading and Dirac-type operator, hence we have an unbounded real Kasparov module.

The proof that the module represents the extension follows same general argument of Proposition 4.2.17 and has been generalised in [RRS15]. By [Kas81, Section 7], the extension class associated to $(\mathcal{A}, Z_B, N)$ comes from the short exact sequence

$$0 \to \mathrm{End}^0_B(PZ) \to C^*(PC^*_\phi(\mathbb{Z}^d)P, \mathrm{End}^0_B(PZ)) \to C^*_\phi(\mathbb{Z}^d) \to 0, \tag{5.11}$$

where $P = \chi_{[0,\infty)}(N)$ is the non-negative spectral projection and we add a degenerate module if necessary to ensure that the Busby map $\varphi : A \to \mathcal{Q}(B)$ is injective.

We have that the map $Q : Z \to \ell^2(\mathbb{Z}) \otimes B$ given by

$$Q\left(\widehat{S}_d^{n_1-n_2} b \otimes V^{n_1}(V^*)^{n_2}\right) = e_{n_1-n_2} \otimes b$$

is an adjointable unitary isomorphism with adjoint

$$Q^*(e_n \otimes b) \mapsto \widehat{V}^{n_1-n_2} b \otimes S^{n_1}(S^*)^{n_2},$$

where n_1, n_2 are *any* natural numbers such that $n = n_1 - n_2$ (cf. Proposition 4.2.11). Conjugation by Q gives an explicit isomorphism $\mathrm{End}^0_B(PZ) \cong \mathcal{K}[\ell^2(\mathbb{N})] \otimes B$. This isomorphism is compatible with the sequence in Equation (5.11) in that the commutators $[P, V^k]$ and $[P, (V^*)^k]$ generate $\mathcal{K}[\ell^2(\mathbb{N})]$. With a suitable identification, the map

$$\mathrm{End}^0_B(PZ) \xhookrightarrow{\iota} C^*(PC^*_\phi(\mathbb{Z}^d)P,\, \mathrm{End}^0_B(PZ))$$

is just inclusion.

Now define the isomorphism $\zeta : C^*(PC^*_\phi(\mathbb{Z}^d)P,\, \mathrm{End}^0_B(PZ)) \to \mathcal{T}$ by

$$\zeta(P\widehat{S}_d^n P) = (\widehat{S}_d \otimes V)^n, \qquad\qquad \zeta(P\widehat{S}_d^{-n} P) = [(\widehat{S}_d \otimes V)^*]^n$$

for $n \geq 0$ and

$$\zeta(PbP) = b \otimes 1, \qquad \zeta(V^j(1 - VV^*)(V^*)^k) = \widehat{S}_d^{k-j} \otimes V^j(1 - VV^*)V^k$$

and then extend accordingly. Then the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \otimes B \arrow[r] & C^*(B \otimes 1, \widehat{S}_d \otimes V) \arrow[r] & C^*_\phi(\mathbb{Z}^d) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{End}^0_B(P\mathcal{T}) \arrow[r] \arrow[u, "\cong"', "\mathrm{Ad}Q"] & C^*(PC^*_\phi(\mathbb{Z}^d)P,\, \mathrm{End}^0_B(PZ)) \arrow[r] \arrow[u, "\cong"', "\zeta"] & C^*_\phi(\mathbb{Z}^d) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

commutes, and so these extensions are unitarily equivalent. $\square$

Edge module and the product

Because the edge algebra B can be represented on $\mathcal{H}_e \cong \ell^2(\mathbb{Z}^{d-1}) \otimes \mathbb{F}^N$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C} \cong \mathbb{R} \oplus i\mathbb{R}$), we can construct a Kasparov module for the edge algebra in the same way as we built the bulk spectral triple in Section 5.2.4. Namely, we use Proposition 5.2.15 to obtain the real spectral triple

$$\lambda_e = \left(\mathcal{B}\hat{\otimes}C\ell_{0,d-1},\, \mathcal{H}_e \otimes \bigwedge\nolimits^* \mathbb{R}^{d-1},\, \sum_{j=1}^{d-1} X_j \otimes \gamma^j,\, \gamma_{\bigwedge^* \mathbb{R}^{d-1}} \right) \tag{5.12}$$

with $\mathcal{B}$ a dense $*$-subalgebra of B generated by the shift operators $\widehat{S}_1, \ldots, \widehat{S}_{d-1}$. If we choose the Landau gauge then $\widehat{S}_j$ is an untwisted translation for any $j \in \{1, \ldots, d-1\}$. The Clifford actions are given analogously to the bulk picture, where ρ^j generate $C\ell_{0,d-1}$ and γ^j generate $C\ell_{d-1,0}$ with

$$\rho^j(\omega) = e_j \wedge \omega - \iota(e_j)\omega, \qquad\qquad \gamma^j(\omega) = e_j \wedge \omega + \iota(e_j)\omega,$$

for $\{e_j\}_{j=1}^{d-1}$ the standard basis of $\mathbb{R}^{d-1}$ and $\omega \in \bigwedge^* \mathbb{R}^{d-1}$.

Because we have used Clifford generators to explicitly construct the various unbounded Kasparov modules, the product

$$\left(\mathcal{A}\hat{\otimes}C\ell_{0,1}, Z_B \otimes \bigwedge^* \mathbb{R}, N \otimes \gamma_{ex}, \gamma_{\bigwedge^* \mathbb{R}}\right)$$
$$\hat{\otimes}_{\mathcal{B}} \left(\mathcal{B}\hat{\otimes}C\ell_{0,d-1}, \mathcal{H}_e \otimes \bigwedge^* \mathbb{R}^{d-1}, \sum_{j=1}^{d-1} X_j \otimes \gamma^j, \gamma_{\bigwedge^* \mathbb{R}^{d-1}}\right)$$

can be computed in KKO. We state the central result.

Theorem 5.3.5. *If the bulk Hamiltonian satisfies Assumption 5.2.11, then the real unbounded Kasparov product of the the extension module from Proposition 5.3.4 with the edge module from Equation* (5.12) *is the inverse of the the bulk module of Proposition 5.2.15.*

Proof. In order to take the internal product, we define $1 \otimes_\nabla X_j$ for any $j \in \{0, \ldots, d-1\}$ acting on $Z \otimes_B \mathcal{H}_e$. First let $\mathcal{Z}_B$ be the submodule of Z given by finite sums of elements $z_{n_1,n_2} b = \widehat{S}_d^{n_1-n_2} b \otimes V^{n_1}(V^*)^{n_2}$ and take the connection

$$\nabla : \mathcal{Z} \to \mathcal{Z} \otimes_{\mathrm{poly}(b)} \Omega^1(\mathrm{poly}(b))$$

given by

$$\nabla \left(\sum_{n_1,n_2} z_{n_1,n_2} b \right) = \sum_{n_1,n_2} z_{n_1,n_2} \otimes \delta(b),$$

where δ is the universal derivation. We represent 1-forms on $\mathcal{H}_e$ via

$$\tilde{\pi}\left(b_0 \delta(b_1)\right) \lambda = b_0 [X_j, b_1] \lambda$$

for $\lambda \in \mathcal{H}_e$. We then define

$$(1 \otimes_\nabla X_j)(z \otimes \lambda) := (z \otimes X_j \lambda) + (1 \otimes \tilde{\pi}) \circ (\nabla \otimes 1)(z \otimes \lambda).$$

One then computes that

$$\begin{aligned}(1 \otimes_\nabla X_j) \left(\sum_{n_1,n_2} z_{n_1,n_2} b \otimes \lambda \right) &= \sum_{n_1,n_2} z_{n_1,n_2} \otimes b X_j \lambda + \sum_{n_1,n_2} z_{n_1,n_2} \otimes [X_j, b] \lambda \\ &= \sum_{n_1,n_2} z_{n_1,n_2} \otimes X_j b \lambda.\end{aligned}$$

In order to take the product of an unbounded $A\hat{\otimes}C\ell_{0,1}$-$B$ Kasparov module with an unbounded $B\hat{\otimes}C\ell_{0,d-1}$-$\mathbb{R}$ module, we need to take an external product with a Kasparov module representing the identity in $KKO(C\ell_{0,d-1}, C\ell_{0,d-1})$. The identity class can be represented by the Kasparov module

$$\left(C\ell_{0,d-1}, (C\ell_{0,d-1})_{C\ell_{0,d-1}}, 0, \gamma_{C\ell_{0,d-1}}\right)$$

with right and left actions given by right and left Clifford multiplication (cf. Example 2.2.26). At the level of C^*-modules, the product module is given by

$$\Big(Z_B \otimes_{\mathbb{R}} \bigwedge\nolimits^* \mathbb{R} \,\hat{\otimes}_{\mathbb{R}} C\ell_{0,d-1}\Big) \,\hat{\otimes}_{B\hat{\otimes}C\ell_{0,d-1}} \Big(\mathcal{H}_e \otimes_{\mathbb{R}} \bigwedge\nolimits^* \mathbb{R}^{d-1}\Big)$$
$$\cong (Z \otimes_B \mathcal{H}_e) \otimes_{\mathbb{R}} \bigwedge\nolimits^* \mathbb{R} \hat{\otimes}_{\mathbb{R}} \Big(C\ell_{0,d-1} \cdot \bigwedge\nolimits^* \mathbb{R}^{d-1}\Big) \cong (Z \otimes_B \mathcal{H}_e) \otimes_{\mathbb{R}} \bigwedge\nolimits^* \mathbb{R} \hat{\otimes}_{\mathbb{R}} \bigwedge\nolimits^* \mathbb{R}^{d-1}$$

as the action of $C\ell_{0,d-1}$ on $\bigwedge^* \mathbb{R}^{d-1}$ by left-multiplication is bijective. Under this identification, we can write the unbounded product module as

$$\Big(\mathcal{A}\hat{\otimes}C\ell_{0,1}\hat{\otimes}C\ell_{0,d-1}, (Z \otimes_B \mathcal{H}_e) \otimes_{\mathbb{R}} \bigwedge\nolimits^* \mathbb{R}\hat{\otimes}_{\mathbb{R}} \bigwedge\nolimits^* \mathbb{R}^{d-1},$$
$$(N \otimes 1) \otimes \gamma_{ex}\hat{\otimes}1 + \sum_{j=1}^{d-1} (1 \otimes_\nabla X_j) \otimes 1\hat{\otimes}\gamma^j, \gamma_{\bigwedge^* \mathbb{R}}\hat{\otimes}\gamma_{\bigwedge^* \mathbb{R}^{d-1}}\Big),$$

where the Clifford actions take the form

$$\rho_{ex}\hat{\otimes}1(\omega_1\hat{\otimes}\omega_2) = (e_1 \wedge \omega_1 - \iota(e_1)\omega_1)\hat{\otimes}\omega_2$$
$$1\hat{\otimes}\rho^j(\omega_1\hat{\otimes}\omega_2) = (-1)^{|\omega_1|}\omega_1\hat{\otimes}(e_j \wedge \omega_2 - \iota(e_j)\omega_2)$$

for $j \in \{1, \ldots, d-1\}$ and $|\omega|$ the degree of the form. It is a simple check to see that the unbounded product module satisfies Kucerovsky's criterion (Theorem 2.2.40 or [Kuc97, Theorem 13]) and therefore represents the product on KK-groups. Our next task is to relate this module back to the bulk system. We first identify $\bigwedge^* \mathbb{R}\hat{\otimes}_{\mathbb{R}} \bigwedge^* \mathbb{R}^{d-1} \cong \bigwedge^* \mathbb{R}^d$ and use the graded isomorphism $C\ell_{p,q}\hat{\otimes}C\ell_{r,s} \cong C\ell_{p+r,q+s}$ on the left and right Clifford generators by the mapping

$$\rho_{ex}\hat{\otimes}1 \mapsto \rho^1, \qquad 1\hat{\otimes}\rho^j \mapsto \rho^{j+1},$$
$$\gamma_{ex}\hat{\otimes}1 \mapsto \gamma^1, \qquad 1\hat{\otimes}\gamma^j \mapsto \gamma^{j+1}.$$

Applying this equivalence gives the unbounded Kasparov module

$$\left(\mathcal{A}\hat{\otimes}C\ell_{0,d},\, (Z \otimes_B \mathcal{H}_e) \otimes \bigwedge\nolimits^* \mathbb{R}^d,\, (N \otimes 1) \otimes \gamma^1 + \sum_{j=2}^{d} (1 \otimes_\nabla X_{j-1}) \otimes \gamma^j,\, \gamma_{\bigwedge^* \mathbb{R}^d}\right)$$

with left Clifford action $\rho^j(\omega) = e_j \wedge \omega - \iota(e_j)\omega$ and right Clifford action $\gamma^j(\omega) = e_j \wedge \omega + \iota(e_j)\omega$ for $\omega \in \bigwedge^* \mathbb{R}^d$ and $\{e_j\}_{j=1}^d$ the standard basis of $\mathbb{R}^d$.

Next we use an analogue of the unitary map $\varrho : Z \otimes_B \mathcal{H}_e \to \mathcal{H}_b$ from Theorem 4.3.3. Starting with a basis element in $Z_B \otimes_B \ell^2(\mathbb{Z}^{d-1}) \otimes \mathbb{F}^N$ we define

$$\begin{aligned}
\widehat{S}_d^{n_1-n_2} b \otimes V^{n_1}(V^*)^{n_2} \otimes e_\rho &= \widehat{S}_d^{n_1-n_2} bS_\rho \otimes V^{n_1}(V^*)^{n_2} \otimes e_0 \\
&= \alpha_{n_2-n_1}(bS_\rho)\widehat{S}_d^{n_1-n_2} \otimes V^{n_1}(V^*)^{n_2} \otimes e_0 \\
&\overset{\varrho}{:\mapsto} \alpha_{n_2-n_1}(bS_\rho) \cdot e_{0,n_1-n_2} \in \ell^2(\mathbb{Z}^d) \otimes \mathbb{F}^N,
\end{aligned}$$

Because S_ρ is a shift operator and $\widehat{S}_d$ is a twisted shift operator, $\alpha_n(S_\rho) = \widehat{S}_d^{-n} S_\rho \widehat{S}_d^n = c_{\rho,n} S_\rho$ with $c_{\rho,n}$ some complex number of modulus 1 (under an appropriate identification in the real category). Hence we can write the map

$$\begin{aligned} \varrho\Big(\widehat{S}_d^{n_1-n_2} b \otimes V^{n_1}(V^*)^{n_2} \otimes e_\rho\Big) &= c_{\rho,n_2-n_1}\, \alpha_{n_2-n_1}(b) S_\rho \cdot e_{0,n_1-n_2} \\ &= c_{\rho,n_2-n_1}\, \alpha_{n_2-n_1}(b) \cdot e_{\rho,n_1-n_2}. \end{aligned} \tag{5.13}$$

To check compatibility with the left-action by A, we compute that for $l \geq 0$

$$\begin{aligned} \varrho\Big(\widehat{S}_d^l \cdot \widehat{S}_d^{n_1-n_2} b \otimes V^{n_1}(V^*)^{n_2} \otimes e_\rho\Big) &= \varrho\Big(\widehat{S}_d^{l+n_1-n_2} b S_\rho \otimes V^{n_1+l}(V^*)^{n_2} \otimes e_0\Big) \\ &= \varrho\Big(\alpha_{n_2-n_1-l}(bS_\rho) \otimes V^{n_1+l}(V^*)^{n_2} \otimes e_0\Big) \\ &= \alpha_{n_2-n_1-l}(bS_\rho) \cdot e_{0,n_1-n_2+l} \end{aligned}$$

and compare to

$$\begin{aligned} \widehat{S}_d^l \cdot \varrho\Big(\widehat{S}_d^{n_1-n_2} b \otimes V^{n_1}(V^*)^{n_2} \otimes e_\rho\Big) &= \widehat{S}_d^l \alpha_{n_2-n_1}(bS_\rho) \cdot e_{0,n_1-n_2} \\ &= \alpha_{n_2-n_1-l}(bS_\rho) \widehat{S}_d^l \cdot e_{0,n_1-n_2} \\ &= \alpha_{n_2-n_1-l}(bS_\rho) \cdot e_{0,n_1-n_2+l}. \end{aligned}$$

The result also holds for $l < 0$ by the same general argument. Next we check

$$\begin{aligned} \varrho\Big(b_1 \widehat{S}_d^{n_1-n_2} b \otimes V^{n_1}(V^*)^{n_2} \otimes e_\rho\Big) &= \varrho\Big(b_1 \widehat{S}_d^{n_1-n_2} b S_\rho \otimes V^{n_1}(V^*)^{n_2} \otimes e_0\Big) \\ &= \varrho\Big(b_1 \alpha_{n_2-n_1}(bS_\rho) \widehat{S}_d^{n_1-n_2} \otimes V^{n_1}(V^*)^{n_2} \otimes e_0\Big) \\ &= b_1 \alpha_{n_2-n_1}(bS_\rho) \cdot e_{0,n_1-n_2}. \end{aligned}$$

Because $\widehat{S}_d$ and b generate A, we see the representation is compatible with ϱ. A basic computation shows that $\varrho(Nz \otimes \lambda) = X_d\, \varrho(z \otimes \lambda)$. To check that the rest of our Dirac operator is compatible with ϱ, it suffices to check that $\varrho((1 \otimes_\nabla X_j)(z \otimes e_\rho)) = X_j\, \varrho(z \otimes e_\rho)$ for e_ρ a basis element of $\ell^2(\mathbb{Z}^{d-1}) \otimes \mathbb{F}^N$. Elements $b \in B$ are made up of shift operators in $\ell^2(\mathbb{Z}^{d-1}) \otimes \mathbb{F}^N$ (under the Landau gauge, we can assume that this remains true with a constant external magnetic field present). Hence we let S_η be some shift operator on $\ell^2(\mathbb{Z}^{d-1}) \otimes \mathbb{F}^N$ and compute

$$\begin{aligned} \varrho\Big[(1 \otimes_\nabla X_j)\left(\widehat{S}_d^{n_1-n_2} S_\eta \otimes V^{n_1}(V^*)^{n_2} \otimes e_\rho\right)\Big] &= \varrho\Big(\widehat{S}_d^{n_1-n_2} \otimes V^{n_1}(V^*)^{n_2} \otimes X_j S_\eta e_\rho\Big) \\ &= (\eta_j + \rho_j)\varrho\Big(\widehat{S}_d^{n_1-n_2} \otimes V^{n_1}(V^*)^{n_2} \otimes e_{\eta+\rho}\Big) \\ &= (\eta_j + \rho_j)\varrho\Big(\alpha_{n_2-n_1}(S_{\eta+\rho}) \widehat{S}_d^{n_1-n_2} \otimes V^{n_1}(V^*)^{n_2} \otimes e_0\Big) \\ &= (\eta_j + \rho_j)\alpha_{n_2-n_1}(S_{\eta+\rho}) \cdot e_{0,n_1-n_2}. \end{aligned}$$

Next we use the characterisation of ϱ from Equation (5.13) to compute

$$\begin{aligned}\varrho\Big[(1\otimes_\nabla X_j)\left(\widehat{S}_d^{n_1-n_2}S_\eta\otimes V^{n_1}(V^*)^{n_2}\otimes e_\rho\right)\Big] &= (\eta_j+\rho_j)c_{\eta+\rho,n_2-n_1}S_{\eta+\rho}\cdot e_{0,n_1-n_2} \\ &= (\eta_j+\rho_j)c_{\eta+\rho,n_2-n_1}e_{\eta+\rho,n_1-n_2} \\ &= X_j\cdot(c_{\eta+\rho,n_2-n_1}e_{\eta+\rho,n_1-n_2}) \\ &= X_j\cdot\varrho\Big(\widehat{S}_d^{n_1-n_2}S_\eta\otimes V^{n_1}(V^*)^{n_2}\otimes e_\rho\Big).\end{aligned}$$

Therefore, applying the unitary map ϱ takes the product module to

$$\left(\mathcal{A}\hat{\otimes}C\ell_{0,d},\,\mathcal{H}_b\otimes{\bigwedge}^*\mathbb{R}^d,\,X_d\otimes\gamma^1+\sum_{j=2}^d X_{j-1}\otimes\gamma^j,\,\gamma_{\bigwedge^*\mathbb{R}^d}\right)$$

with the same Clifford actions as previously. Finally, we consider the permutation $\sigma(i)=(i-1)\bmod d$ for $i\in\{1,\ldots,d\}$, which then gives us the map $e_i\mapsto e_{\sigma_i}$ for $\{e_i\}_{i=1}^d$ the standard basis of $\mathbb{R}^d$. This extends to a unitary operator on $\bigwedge^*\mathbb{R}^d$ by the mapping

$$e_{j_1}\wedge\ldots\wedge e_{j_n}\mapsto(-1)e_{\sigma(j_1)}\wedge\ldots\wedge e_{\sigma(j_n)}.$$

as the permutation σ has odd parity. Applying this unitary transformation to our module, we obtain

$$\left(\mathcal{A}\hat{\otimes}C\ell_{0,d},\,\mathcal{H}_b\otimes{\bigwedge}^*\mathbb{R}^d,\,-\sum_{j=1}^d X_j\otimes\gamma^j,\,-\gamma_{\bigwedge^*\mathbb{R}^d}\right)$$

with Clifford actions $\rho^j(\omega)=-e_j\wedge\omega+\iota(e_j)\omega$ and $\gamma^j(\omega)=-e_j\wedge\omega-\iota(e_j)\omega$. Hence our product module is the KK-inverse of the bulk module. □

Pairings, the bulk-edge correspondence and the edge conductance

To summarise our work, we have factorised the bulk module from Proposition 5.2.15 so that, at the level of KK-classes, $[\lambda_b]=-[\mathrm{ext}]\hat{\otimes}_B[\lambda_e]$. Taking the product with the symmetry KK-class $[H^G]$ from Proposition 5.2.13,

$$C_{n,d}=[H^G]\hat{\otimes}_A[\lambda_b]=-[H^G]\hat{\otimes}_A[\mathrm{ext}]\hat{\otimes}_B[\lambda_e],$$

by Theorem 5.3.5. Therefore we can express the real index pairing as a map

$$\begin{aligned}KKO(C\ell_{n,0},A)\times KKO(A\hat{\otimes}C\ell_{0,1},B)\times KKO(B\hat{\otimes}C\ell_{0,d-1},\mathbb{R})& \\ \to KKO(C\ell_{n,0}\hat{\otimes}C\ell_{0,d},\mathbb{R}).&\end{aligned}$$

By the associativity of Kasparov product, this will either be a pairing

$$KKO(C\ell_{n,0},A)\times KKO(A\hat{\otimes}C\ell_{0,d},\mathbb{R})\to KKO(C\ell_{n,0}\hat{\otimes}C\ell_{0,d},\mathbb{R})\cong KO_{n-d}(\mathbb{R}),$$

the bulk invariant studied in Section 5.2.4, or

$$KKO(C\ell_{n,0}\hat{\otimes}C\ell_{0,1}, B) \times KKO(B\hat{\otimes}C\ell_{0,d-1}, \mathbb{R}) \to KO_{n-d}(\mathbb{R}),$$

an invariant that comes from the edge algebra B of a system with boundary. Theorem 5.3.5 ensures that regardless of our choice of pairing, the result is the same and so we obtain the bulk-edge correspondence. In complex examples, the edge pairing has the interpretation of an 'edge conductance' that is related to currents concentrated at the boundary of the sample $\mathbb{Z}^{d-1} \times \mathbb{N}_s$ [SBKR02, KSB04b, KR08].

Remark 5.3.6 (Is the edge conductance a pairing with an edge Hamiltonian?). A natural question is whether there is a physical interpretation of the class $[H^G]\hat{\otimes}_A[\mathrm{ext}] \in KKO(C\ell_{n-1,0}, B)$, which plays a role in our edge pairing. One might consider the product $[H^G]\hat{\otimes}_A[\mathrm{ext}]$ as the symmetry class $[H_e^{\tilde{G}}]$ of some 'edge Hamiltonian' H_e acting on a $(d-1)$-dimensional system and with symmetry properties giving rise to a class in $KKO(C\ell_{n-1,0}, C^*_{\tilde{\phi}}(\mathbb{Z}^{d-1}))$. That is, we have a lower-dimensional Hamiltonian independent from our bulk Hamiltonian and with different symmetry properties (as a graded representation of $C\ell_{n-1,0}$ represents different symmetries by Table 5.1), but whose pairing with an 'edge spectral triple' $[\tilde{\lambda}_e] \in KKO(C^*_{\tilde{\phi}}(\mathbb{Z}^{d-1})\hat{\otimes}C\ell_{0,d-1}, \mathbb{R})$ gives the same result as the original bulk pairing. Table 5.2 shows that such a situation is possible and one may be able to construct such an edge Hamiltonian. However, we do not think that this is what the bulk-edge short exact sequence of Kellendonk et al. is capturing. Instead, we see the edge conductance as coming from a system with boundary, in which we construct a topological invariant of observables concentrated at a boundary of a higher-dimensional system.

Put another way, the factorisation of the index pairing

$$C_{n,d} = \big([H^G]\hat{\otimes}_A[\mathrm{ext}]\big)\,\hat{\otimes}_B[\mathrm{edge}] = -\,\big([\eta]\hat{\otimes}_B[\mathrm{edge}]\big)$$

suggests that we can in some sense 'forget' the bulk algebra A and instead look for a $(d-1)$-dimensional Hamiltonian H_e with symmetry properties that give rise to a representation of $C\ell_{n-1,0}$ and whose pairing with a spectral triple $\tilde{\lambda}_e$ over $\tilde{\mathcal{B}}\hat{\otimes}C\ell_{0,d-1}$ is such that

$$[H_e^{\tilde{G}}] \otimes_{\tilde{B}} [\tilde{\lambda}_e] = [\eta]\hat{\otimes}_B[\mathrm{edge}].$$

Such Hamiltonians may exist, but we do not claim that their existence is an intrinsic consequence of the bulk-edge factorisation coming from the short-exact sequence of Equation (5.10). Instead, the bulk-edge correspondence links topological invariants of a system without boundary to the same system with an edge (not a different system one dimension lower).

Remark 5.3.7 (Wider applications of Theorem 5.3.5). The bulk-edge correspondence and Theorem 5.3.5 are largely independent of the symmetry considerations in Section

5.2.3. Instead, it is a general property of the unbounded Kasparov module representing the short exact sequence

$$0 \to \mathcal{K} \otimes B \to \mathcal{T} \to C^*_\phi(\mathbb{Z}^d) \to 0$$

and the real spectral triples on the ideal and quotient algebras we have constructed.

In particular, the fact that the factorisation occurs on the K-homological part of the index pairing means other K-theory classes and symmetry types can be considered without changing the result. For example, if we were to consider symmetry compatible Hamiltonians of a group $\tilde{G}$ that included spatial involution or other symmetries, then provided that the symmetry data can be associated to a class in $KKO(C^*(\tilde{G}), C^*_\phi(\mathbb{Z}^d))$, the bulk-edge correspondence of Theorem 5.3.5 will still hold.

The separation of the internal symmetries of the Hamiltonian with the geometry of the Brillouin zone highlights an advantage of using Kasparov theory to study topological systems with internal symmetries. There is a flexibility that allows one to change the symmetry group without affecting the geometric information that is used to obtain the topological invariants of interest and vice versa.

We also briefly comment on the case where $G = \{1, C\}$ or there are no symmetries and, hence, all modules and KK-classes are complex. In such a circumstance, the same general argument to prove Theorem 5.3.5 will extend to complex spaces, algebras and modules without issue. See also Theorem 4.3.3 from Chapter 4 for a 2-dimensional example with magnetic field, where many of the key ideas extend to d-dimensional systems. For brevity of exposition we have focused on the real setting in this chapter.

5.3.2 Examples

Example 5.3.8 (Kane-Mele). We revisit the quantum spin-Hall effect from Example 5.2.22. Recall the bulk Hamiltonian $H_{KM} = \begin{pmatrix} h & g \\ g^* & \mathcal{C}h\mathcal{C} \end{pmatrix}$ with h a Haldane Hamiltonian and g the Rashba coupling such that $g^* = -\mathcal{C}g\mathcal{C}$. In Example 5.2.22, we constructed the real spectral triple

$$\left(\mathcal{A}\hat{\otimes}C\ell_{0,2},\, \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^{2N} \otimes \bigwedge\nolimits^{*} \mathbb{R}^2,\, X_b = \sum_{j=1}^{2} X_j \otimes 1_{2N} \otimes \gamma^j,\, \gamma_{\bigwedge^* \mathbb{R}^2} \right)$$

for a dense subalgebra $\mathcal{A} \subset C^*(\mathbb{Z}^2)$. We now consider the system with edge.

Let H_s^{KM} be the Kane-Mele Hamiltonian compressed to the system with boundary $\ell^2(\mathbb{Z} \times \mathbb{N}_s) \otimes \mathbb{C}^{2N}$ and S_2 the translation along the second coordinate operator in $\ell^2(\mathbb{Z}^2)$. Embedded in the larger space, we have an action η on $C^*(S_1, \Pi_s S_2 \Pi_s)$ by $\eta(a_s) = S_2^* a_s S_2$. We use this action to construct the Pimsner-Voiculescu short exact sequence with ideal $\mathcal{K}\otimes B$ such that $B\cong C^*(S_1)$ acts on $\ell^2(\mathbb{Z})\otimes\mathbb{C}^{2N}$. This extension is represented by the unbounded module

$$\Big(\mathcal{A}\otimes C\ell_{0,1},\, Z_B\otimes \bigwedge\nolimits^{*}\mathbb{R},\, N\otimes\gamma^1_{ex},\,\gamma_{ex}\Big)$$

by the procedure outlined in Section 5.3.1.

The algebra $\mathcal{K}\otimes B$ comes from considering the observables in $C^*(S_1,\Pi_s S_2\Pi_s)$ concentrated on the edge, so self-adjoint operators in B coming from the bulk Hamiltonian are still time-reversal invariant. Running through our bulk-edge argument of Theorem 5.3.5, we get the factorisation of the bulk module and, in particular, the pairing

$$\begin{aligned}&\left[H^G\right]\hat{\otimes}_{C^*(\mathbb{Z}^2)}\left[\Big(\mathcal{A}\hat{\otimes}C\ell_{0,2},\ell^2(\mathbb{Z}^2)\otimes\mathbb{C}^{2N}\otimes\bigwedge\nolimits^{*}\mathbb{R}^2,X_b,\gamma_{\bigwedge^*\mathbb{R}^2}\Big)\right]\\ &=-\left[H^G\right]\hat{\otimes}_{C^*(\mathbb{Z}^2)}\left[\Big(\mathcal{A}\hat{\otimes}C\ell_{0,1},Z_B\otimes\bigwedge\nolimits^{*}\mathbb{R},N\otimes\gamma^1_{ex},\gamma_{ex}\Big)\right]\\ &\qquad\hat{\otimes}_B\left[\Big(\mathcal{B}\hat{\otimes}C\ell_{0,1},\ell^2(\mathbb{Z})\otimes\mathbb{C}^{2N}\otimes\bigwedge\nolimits^{*}\mathbb{R},X_1\otimes 1_{2N}\otimes\gamma^1,\gamma_e\Big)\right].\end{aligned}$$

As we showed in Example 5.2.22, our bulk pairing is the product

$$\begin{aligned}&\left[H^G\right]\hat{\otimes}_{C^*(\mathbb{Z}^2)}\left[\Big(\mathcal{A}\hat{\otimes}C\ell_{0,2},\ell^2(\mathbb{Z}^2)\otimes\mathbb{C}^{2N}\otimes\bigwedge\nolimits^{*}\mathbb{R}^2,X_b,\gamma_{\bigwedge^*\mathbb{R}^2}\Big)\right]\\ &KKO(C\ell_{4,0},C^*(\mathbb{Z}^2))\times KKO(C^*(\mathbb{Z}^2)\hat{\otimes}C\ell_{2,0},\mathbb{R})\to KO_2(\mathbb{R})\cong\mathbb{Z}_2.\end{aligned}$$

By the associativity of the Kasparov product, this is the same as the pairing

$$\begin{aligned}&-\left([H^G]\hat{\otimes}_{C^*(\mathbb{Z}^2)}[\mathrm{ext}]\right)\hat{\otimes}_B\left[\Big(\mathcal{B}\hat{\otimes}C\ell_{0,1},\ell^2(\mathbb{Z})\otimes\mathbb{C}^{2N}\otimes\bigwedge\nolimits^{*}\mathbb{R},X_1\otimes 1_{2N}\otimes\gamma^1,\gamma_e\Big)\right]\\ &KKO(C\ell_{4,0}\hat{\otimes}C\ell_{0,1},B)\times KKO(B\hat{\otimes}C\ell_{0,1},\mathbb{R})\to KO_{4-1-1}(\mathbb{R})\cong\mathbb{Z}_2.\end{aligned}$$

We would like to examine the edge pairing more closely. We first review the what occurs in the complex setting as developed in [SBKR02, KR08]. Let $\Delta\subset\mathbb{R}\setminus\sigma(H_{KM})$ be an open interval of $\mathbb{R}$ such that $\mu\in\Delta$. By considering the image of the spectral projection $P_\Delta=\chi_\Delta(H_s^{KM})$, we are projecting precisely onto the eigenstates that do not exist in the bulk system, namely edge states. One can then define the unitary

$$U(\Delta)=\exp\left(-2\pi i\frac{H_s^{KM}-\inf(\Delta)}{\mathrm{Vol}(\Delta)}P_\Delta\right),$$

It is a key result of [SBKR02, KR08] that $U(\Delta)$ is a unitary in $B_{\mathbb{C}}$ and, furthermore, represents the boundary map in complex K-theory of the Fermi projection. That is, the unitary $[U(\Delta)]\in K_1(B_{\mathbb{C}})$ represents the complex Kasparov product $[P_\mu]\hat{\otimes}_{A_{\mathbb{C}}}[\mathrm{ext}]$ for trivially graded algebras. One then shows that the pairing of $[U(\Delta)]$ with the boundary spectral triple can be expressed as

$$\sigma_e=-\frac{e^2}{h}\hat{\mathcal{T}}\left(U(\Delta)^*i[X_1,U(\Delta)]\right)=-\lim_{|\Delta|\to\mu}\frac{1}{|\Delta|}\hat{\mathcal{T}}(P_\Delta i[X_1,H_s^{KM}])\qquad(5.14)$$

where $\hat{\mathcal{T}} = \mathcal{T}_1 \otimes \mathrm{Tr}_2$ is the trace per unit volume along the boundary and operator trace normal to the boundary [SBKR02]. One recognises the right-hand side of Equation (5.14) as measuring the conductance of an edge current (as P_Δ projects onto edge states). Unfortunately, in the quantum spin-Hall example, the expression $\hat{\mathcal{T}}(P_\Delta i[X_1, H_s^{KM}])$ is zero as there is no net current and the cyclic cocycle cannot detect the $\mathbb{Z}_2$-index we associate to the spin current.

Let us make some preliminary comments about the edge pairing in the real setting. If A is trivially graded, then $[H^G] \in KKO(C\ell_{4,0}, A) \cong KO_0(A\hat{\otimes}C\ell_{0,4})$. The results of Boersema and Loring give us tools to compute an explicit unitary representative of the boundary map $\partial[H^G] \in KO_{-1}(B\hat{\otimes}C\ell_{0,4})$ [BL15].* One can then pair $\partial[H^G]$ with the edge spectral triple λ_e to derive the edge invariant. Unfortunately, the lack of a computationally tractable cyclic formula for the pairing of the edge unitary with the edge spectral triple means that there is not a natural analogue to Equation (5.14) in the real setting.

A concrete representation of the index pairings that give rise to both bulk and edge pairings is a much more difficult task than in the complex case, where invariants can be expressed as the Fredholm index of the operators of interest. This is one reason why we have to consider Kasparov products or the Clifford index. An advantage of unbounded Kasparov theory is that the operators we deal with and the modules we build have geometric or physical motivation and so can be linked to the underlying system. A more physical expression for the edge pairing would be advantageous and remains an open problem in the field. See [GS15] for more on concrete representations of index pairings in the Real category.

5.4 Future work

Our key contribution to the topological insulator problem in this chapter has been to derive the periodic table and prove the bulk-edge correspondence of topological insulators using Kasparov theory. There are many further applications of the methods we have introduced, including:

1. The introduction of disorder into our system as was done in the quantum Hall effect in Chapter 3 and [BvS94]. Related to disorder are localised states and the extension of the (real) index pairing to such states;

2. An adaptation of our argument to the case of continuous models and unbounded Hamiltonians acting on spaces like $L^2(\mathbb{R}^d) \otimes \mathbb{C}^N$;

*The reader should note that the constructions in [BL15] usually require Real C^*-algebras. We can still apply the Kane-Mele example by taking $A_{\mathbb{C}}$ with Real involution $a^\tau = R_T a R_T^*$.

3. A further understanding of the links between the edge pairing of our bulk-edge system and something like an edge conductance as developed in [SBKR02, KSB04b, KR08] and discussed in Example 5.3.8.

The above list gives some immediate problems that the method developed in this chapter can be applied to. In addition, it would be desirable to clarify how the picture we have outlined fits in to the 'duality' of insulator systems studied in [MT15a] and what happens when we consider different symmetry types that are inequivalent to the PT-symmetry group, spatial involution symmetry for example (see Remark 5.2.4 and 5.3.7).

A more thorough investigation of the explicit form of the bulk-edge correspondence in specific models would shed light on the physical interpretation(s) of the edge conductance as discussed in Remark 5.3.6 and Example 5.3.8.

These further research directions are far from exhaustive, but will hopefully open future avenues of discovery into this problem.

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