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Topological string theory and applications

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Abstract

This thesis focuses on various applications of topological string theory based on different types of Calabi-Yau (CY) manifolds. The first type considered is the toric CY manifold, which is intimately related to spectral problems of difference operators. The particular example considered in the thesis closely resembles the Harper-Hofstadter model in condensed matter physics. We first study the non-perturbative sectors in this model, and then propose a new way to compute them using topological string theory. In the second part of the thesis, we consider partition functions on elliptically fibered CY manifolds. These exhibit interesting modular behavior. We show that for geometries which do not lead to non-abelian gauge symmetries, the topological string partition functions can be reconstructed based solely on genus zero Gromov-Witten invariants. Finally, we discuss ongoing work regarding the relation of the topological string partition functions on the so-called Higgsing trees in F-theory.

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Contents

Chapter 1

Introduction

1.1 String theory as a candidate for a "theory of everything"

The Standard Model of fundamental particles is one of the most successful theories in physics. Based on the quantum theory of gauge fields, it provides an extremely precise account of the behavior of nature at length scales down to at least $l \approx 1$ TeV^{-1}. With the discovery of the long-anticipated Higgs particles at the LHC in 2012, it seems that finally the dust has settled.

However, the Standard Model in fact is still far from a microscopically *complete* theory to describe our world. As is well known, it can only incorporate three out of four fundamental forces in nature. In other words, gravity, the fundamental interaction that plays a crucial role in our daily life, is still left untouched. Gravity is known to be best described by Einstein's theory of General Relativity. Therefore, one might be tempted to simply unify these two successful theories. However, naive attempts to construct a field theory model for gravity fail for the following reason. According to our current understanding, the Standard Model is an *effective* theory. This means that we first start from a very high energy, say the Planck scale, then lower the energy level down to the energy scale accessible to large colliders. Through this procedure, high energy modes in the system are integrated out, and if the outcome can be absorbed into redefinition of certain parameters, it is still under control. We call this type of theories *renormalizable*. Nevertheless, when we try to describe gravity by a spin 2 particle going under the name of graviton, we found that more and more new interactions are generated hence it is NOT renormalizable. More severely, at the Planck energy scale, quantum fluctuations become so violent that any possible smooth spacetime structure, a central ingredient to Einstein's theory of gravity, cannot exist. This hints at the fact that there is some serious issue trying to reconcile them naively, and to create a logically self-consistent theory we are forced to go beyond the present framework and embrace new ideas.

The most promising candidate, for a so-called “theory of everything”, is string theory. It underwent several stages of development and we first discuss its earliest form, known as the *Bosonic String Theory*. It starts with the basic assumption that the most fundamental object in our universe is the string, and particles are merely possible excitations of its modes of vibration. Among its excited spectrum, gauge fields as well as graviton naturally appear. Hence instead of considering particles moving in a spacetime which is more intuitive to our mind, we should consider that strings, either open or closed, vibrate and sweep out worldsheets, giving rise to all the fundamental forces. Why the problem of non-renormalizability is cured when we consider strings? To answer this question, let's first go back to the field-theoretic description. The divergence in a field theory of gravitons occurs when all the interaction vertices are coincident. However in string theory the interaction is dictated by summing over Riemann surfaces of different genera,

$$\mathcal{Z}(g_s) = \sum_g \mathcal{Z}_g \, g_s^{2g-2} \,, \tag{1.1.1}$$

when we compute scattering amplitudes. The interaction no longer happens at a definite point but is in some sense smeared out in spacetime, thus avoiding possible singularity at coincident points.

However, there were other problems remaining unsolved. For example, fermions are missing in this picture. For another, it has so-called tachyons which have negative mass squares, rendering the ground state unstable.

Another revolutionary idea is supersymmetry. It postulates the existence of a symmetry with infinitesimal generator Q_α interchanging bosons and fermions. Note that all possible symmetries of the S-matrix in a quantum field theory of dimension larger than two, whose conserved charges transform as Lorentz tensors, are classified in [44] to be either Poincaré symmetry or internal symmetries. However, supersymmetry does not violate this theorem since Q_α transforms as a spinor. Intuitively, the algebra is the “square root” of the Poincaré algebra, as can be seen from the anti-commutation relation in four dimensions,

$$\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \,, \tag{1.1.2}$$

where P_μ is the energy-momentum tensor.

After adding the ingredient of supersymmetry to bosonic string theory, *Superstring Theory* was born. In this set-up, the worldsheet action of the string enjoys superconformal symmetry, which create both bosonic and fermionic vibration modes. A consistency condition, known as the GSO projection [68], was imposed and unstable taychonic states are gone.

Nevertheless, issues are still present. For example, in order that a superstring theory is self-consistent, we must demand that it lives in ten dimensional spacetime, which is much higher than our universe. Where do the extra six dimensions come from? One possible solution is that the extra dimensions are so tiny that any current experiment cannot detect them. That is to say, the ten dimensional spacetime must be compactified on a six dimensional manifold X. The desire to get a supersymmetric four dimensional theory constrains possible types of X. For instance, if we want to obtain a four dimensional system with $\mathcal{N} = 1$ supersymmetry from a ten dimensional $\mathcal{N} = (1, 0)$ superstring theory, the compactified X must be a *Calabi-Yau* manifold, which by definition is a complex Kähler manifold with vanishing first Chern class[1].

Another potential problem is that even after projecting out inconsistent ones, there does not remain a unique "theory of everything", but several possibilities. More specifically, string theorists found that there are five plausible candidates, known as type IIA, type IIB, type I, $E_8 \times E_8$ heterotic and $SO(32)$ heterotic superstrings. At low energies, they reduce to five different ten dimensional supergravity theories. How to show which one is more fundamental than another?

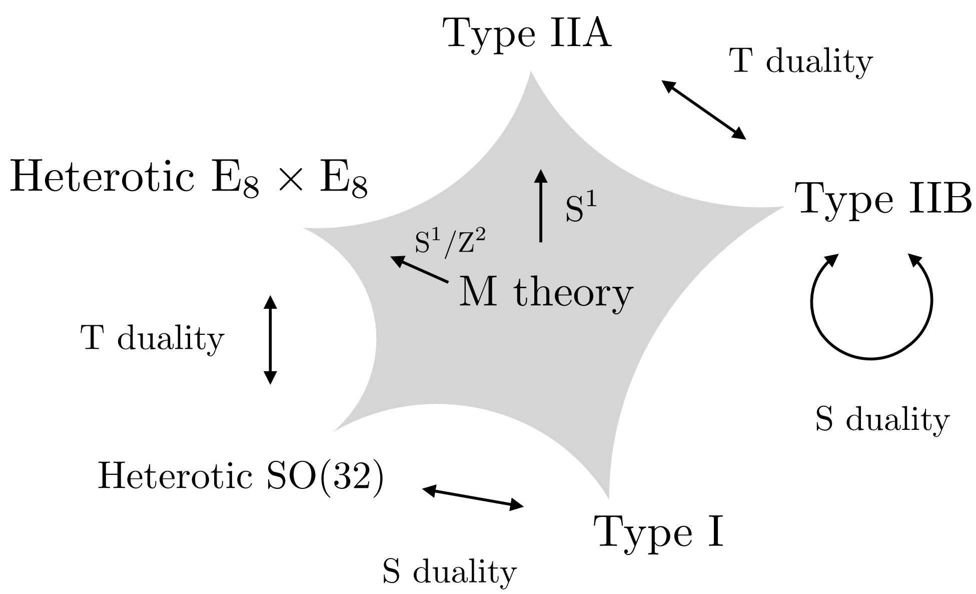


Figure 1.1: Web of some superstring dualities.

A significant breakthrough in the 90's was the discovery that in fact those five theories are equally fundamental [186]. In other words, although they look quite different, they are related to each other via various kinds of dualities. Consider the whole space of coupling constants. When we approach a certain corner and certain coupling becomes small, one of them becomes the most appropriate description. Switching from one corner to another is implemented by dualities. In the center of this space, where the coupling becomes strong, [186] proposed that there exists a theory called *M-theory*, living in eleven dimensions. Its

[1]Due to Yau's theorem [189] for each Kähler class, it has a unique Kähler metric with zero Ricci curvature.

low energy effective theory is the unique eleven-dimensional $\mathcal{N} = 1$ supergravity. An incomplete list of the intricate relations among the five theories can be found in Figure 1.1.

1.2 Topological string theory as a bridge between mathematics and physics

The idea of dualities is extremely profound. For example, the T-duality between type IIA and type IIB superstrings leads ultimately to the notion of mirror symmetry. Mathematically, this predicts that for each Calabi-Yau threefold X, there exists a mirror Calabi-Yau threefold $\tilde{X}$ which satisfies the following relations

$$\begin{aligned} h^{1,1}(X) &= h^{2,1}(\tilde{X}) \,, \\ h^{2,1}(X) &= h^{1,1}(\tilde{X}) \,. \end{aligned} \tag{1.2.1}$$

The nomenclature arises from the fact that the Hodge diamond of $\tilde{X}$ is a mirror reflection through the dashed axis of the Hodge diamond of X, see Figure 1.2.

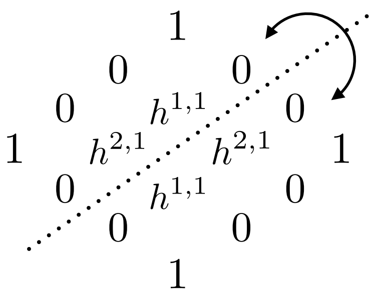


Figure 1.2: Mirror reflection on the Hodge diamond.

The prediction of mirror symmetry is that the type IIA superstring compactified on X should be identical to type IIB superstring compactified on $\tilde{X}$. This is very astonishing at first glance, since the properties of X and $\tilde{X}$ can be drastically different. In the 90's string theorists considered a particular example, i.e. quintic hypersurface in $\mathbb{CP}^4$ and constructed its mirror Calabi-Yau threefold [77]. Exploring the idea of mirror symmetry, [37] found an unexpected way to count rational curves on the quintic hypersurface, which were originally very hard to determine. Many mathematicians, who were at first dubious about string theory which is by no means rigorous, were astonished by this result and began to devote themselves to this area. Shortly afterwards, string theory led to lots of fruitful interactions between mathematics and physics.

Now let's move on to *Topological String Theory*. Type A/Type B topological string theory can be regarded as a simplification of type II A/II B superstring theory compactified on a Calabi-Yau threefold. This is achieved through a "topological twisting" in the worldsheet theory, so that a spinorial symmetry is still present even when the worldsheet is not flat. This remaining symmetry can be utilized to construct topological theories, invariant under continuous deformations of the metric on the worldsheet. If we couple this "topological twisted" theory to gravity, we obtain topological string theory.

The massless fields in four dimensions after compactification are organized into the vector multiplets, hypermultiplets, and a gravity multiplet. Among various terms involving the vector multiplets and the gravity multiplet after the compactification, topological string has access to the kinetic terms of the fields in the vector multiplets and the couplings of the scalars in the vector multiplets to the gravity multiplet. The former are encoded in the genus zero free energy $\mathcal{F}_0$, while the latter are given by higher genera free energies $\mathcal{F}_g$ with $g \geq 1$.

Mirror symmetry introduced above, predicts, in particular, the equivalence of type A topological string theory on X and type B topological string on $\tilde{X}$. From this point of view, the physicist's way of counting the number of rational curves on the quintic hypersurface, which is encoded in $\mathcal{F}_0$ of type A topological string, is via mirror symmetry equal to the $\mathcal{F}_g$ of type B topological string on the mirror quintic, which is much easier to solve. Furthermore, topological string theory generalizes this to higher genera, and the number of rational curves are generalized to the genus g *Gromov-Witten* invariants [78, 184], encoded in the genus g free energy $\mathcal{F}_g$ of type B topological string. Mathematically, Gromov-Witten invariants are enumerative invariants of the manifold, well-defined but notoriously difficult to compute in most cases. However, by exploring various dualities, physicists developed various powerful methods to compute them and left mathematicians with lots of conjectures. For instance, see [69, 70, 190, 123, 129, 157, 147, 148, 156].

Initially formulated between compact Calabi-Yau manifolds, mirror symmetry was later extended to non-compact cases. In some sense, it has a even richer relationship with other areas of mathematics and physics, Chern-Simons theory, matrix models, knot theory and integrable systems etc. To name a few, [185, 69, 53, 5, 4, 82].

Now let's introduce some topics that are discussed in detail in this thesis. In [122], it was pointed out that the non-compact Calabi-Yau manifolds can be used to engineer supersymmetric gauge theories. This is in a sense just the superstring compactification mentioned above, although the compactified manifold is non-compact hence not regarded to be small. In 2002, Nikita Nekrasov [154] employed localization techniques to compute the free energy of $\mathcal{N} = 2$ gauge theories in four and five dimensions, and conjectured the following relation between two partition functions,

$$Z(g_s) = Z_{\text{Nek}}(\epsilon_1, \epsilon_2)|_{-\epsilon_1=\epsilon_2=g_s} \,. \tag{1.2.2}$$

where ϵ_1, ϵ_2 are two formal rotation parameters introduced to perform equivariant localization. The left-hand side refers to the Calabi-Yau manifolds that engineer the supersymmetric gauge theories, and the right hand side is the famous Nekrasov partition function.

The conjecture was proved for a large class of geometries in [113, 114].

Later, it was gradually realized that the full Nekrasov partition function, before specializing $-\epsilon_1 = \epsilon_2 = g_s$, contains more information than $Z(g_s)$. In [3], it was proposed that we can use it to define a refinement of topological string theory for those Calabi-Yau manifolds, known as the *Refined Topological String Theory*. However even until today, we still do not know how to define the refinement from the worldsheet perspective. Aside from the limit $\epsilon_1 = -\epsilon_2 = g_s$ where we recover ordinary topological string theory, we are also interested in the so-called Nekrasov-Shatashvili [155] limit,

$$\epsilon_1 = 0, \qquad \epsilon_2 = \mathrm{i}\hbar\,. \tag{1.2.3}$$

The Nekrasov-Shatashvili limit of refined topological string theory is intimately related to quantization problem in spectral theory. This deep connection benefits both communities: on the one hand, our knowledge in topological string theories can help us to solve the spectrum of certain class of difference operators. On the other hand, we attempt to define non-perturbative topological string theory from the spectral determinant of the corresponding operator. The current literature on this topic is quite vast: see [72, 40, 142, 120, 80, 94, 62, 168, 32, 71] for a partial list and [141] for an excellent review. In chapter 5, our objective is to explore some of these fascinating ideas by focusing on a concrete spectral problem.

To discuss the next topic, let's first go back to Figure 1.1. From there we learn in particular that type IIB theory is self-dual under S-duality. In particular, if we combine the scalar field C_0 with the dilaton ϕ into the axio-dilaton field $\tau = C_0 + \mathrm{i}e^{-\phi}$, the type IIB superstring is invariant under the $SL(2,\mathbb{Z})$ transform

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,. \tag{1.2.4}$$

Note that this is the same redundancy present in specifying the complex structure moduli of a torus. If we want to construct background solutions with varying axio-dilaton field, we can as well consider a family of tori, allowed to be singular at certain loci, fibered over spacetime. This line of reasoning ultimately leads to *F-Theory* [173]. From its very construction F-theory is tightly connected to geometries having an elliptic fibration structure. Roughly speaking, F-theory can be regarded as an auxiliary twelve-dimensional theory that when compactified on an elliptically fibered Calabi-Yau manifold leads to Type IIB string theory compactified on the base of the fibration with the axio-dilaton profile given by the complex structure of the elliptic fiber.

If we take the base B of the elliptic fibration to be complex two dimensional, F-theory compactification gives rise to six dimensional quantum field theories. In the work of Kunihiko Kodaira [132, 133], the possible singular types of an elliptic fiber for a complex surface are systematically classified. With the help of his results, in [98, 49, 97] authors proposed that by choosing the suitable bases, all six dimensional superconformal field theories can be engineered. Furthermore, via F-theory and M-theory duality, this provides new insight to address the topological string partition function on those Calabi-Yau manifolds which are the total spaces of elliptic fibration over B. This will be the topic from chapter 5 to 7.

After all the discussions on non-perturbative results and topological string partition functions to all genera, we hope that this thesis can serve as a small stepping stone towards a better understanding of topological string theory at the level of perturbative expansion, or even beyond that.

The thesis is organized as follows. We first present a gentle introduction to topological string theory in chapter 2 and 3. In chapter 2, we concentrate on the topological field theories, which can be seen as the genus zero part of topological string theory. Next we discuss the coupling to gravity and introduce some basics of topological strings in chapter 3. We also discuss some explicit constructions of mirror Calabi-Yau manifolds. In chapter 4 we focus on the non-perturbative aspects of the Harper-Hofstadter Hamiltonian such as instantons and resurgence, then propose a new way to determine non-perturbative contributions using topological string theory. After that, we switch gears and move on to the topic of elliptic genera and topological strings on elliptically fibered Calabi-Yau manifolds. Chapter 5 is a quick review of the existing literature, preparing the reader for the next two chapters. In chapter 6, we discuss the geometries without codimension-one singular fibers, and prove that their partition functions can be reconstructed using solely genus zero Gromov-Witten invariants. Geometries with codimension-one singular fibers is the subject in chapter 7, where in particular we are searching for the Higgsing tree pattern in the corresponding partition functions. We conclude in chapter 8 and propose some future directions.

Chapter 2

Topological Field Theory

In this and the next chapter, our goal is to partially answer the question: what is topological string theory? In order to achieve that, we need to first introduce the necessary theoretical backgrounds. Section 2.1 is a quick summary of two dimensional $\mathcal{N} = (2,2)$ supersymmetry which is the correct language to describe actions of the string worldsheet. Section 2.2 quickly introduces the notion of a topological field theory (TQFT) and in particular, of Witten type. In section 2.3, we first present a powerful way to produce TQFTs, known as topological twisting, then apply it to the low energy effective theory of the worldsheet. It turns out that there are two possibilities to do the twisting which are both of importance, so we devote subsections 2.3.1 and 2.3.2 to each of them in turn. Section 2.4 serves as an interlude, introducing some basics of moduli space of complex structures in order to answer a question raised at the end of subsection 2.3.2. Useful references for this chapter include [103, 174, 145, 136, 183, 76].

2.1 Two dimensional $\mathcal{N} = (2,2)$ supersymmetry

Considering superstrings propagating inside a target space leads to the notion of supersymmetric nonlinear sigma model mapping a two dimensional worldsheet to a manifold X. In this thesis we consider models having $\mathcal{N} = (2,2)$ supersymmetry. Let's first introduce some rudiments of two dimensional $\mathcal{N} = (2,2)$ supersymmetry, following closely chapter 12 of [103].

We start by constructing the supersymmetry algebra. It necessarily contains the Poincaré algebra,

$$H,\ P,\ M\,, \tag{2.1.1}$$

corresponding to the Hamiltonian, momentum and angular momentum respectively. Moreover, supersymmetry gives us fermionic Noether charges. As the name suggests, we have four supercharges:

$$Q_+ ,\ Q_- ,\ \overline{Q}_+ ,\ \overline{Q}_- \,, \tag{2.1.2}$$

where the subscript $+$ or $-$ means left or right chiralities and the bar means complex conjugation. Since the supersymmetry is extended, we naturally have R-symmetries acting on supercharges. Here, we have two $U(1)$ R-symmetries, whose Noether charges are denoted by F_L and F_R. For convenience, we recombine them into vector and axial R-symmetries,

$$F_V = F_L + F_R ,\ F_A = F_L - F_R \,. \tag{2.1.3}$$

Altogether, they form the $\mathcal{N} = (2,2)$ supersymmetry algebra. More importantly, they satisfy the following (anti-) commutation relations:

$$\{Q_\pm, \overline{Q}_\pm\} = H \pm P \,, \tag{2.1.4}$$

$$Q_+^2 = Q_-^2 = \overline{Q}_+^2 = \overline{Q}_-^2 = 0 \,, \tag{2.1.5}$$

$$\{\overline{Q}_+, \overline{Q}_-\} = \{Q_+, Q_-\} = 0 \,, \tag{2.1.6}$$

$$\{Q_-, \overline{Q}_+\} = \{Q_+, \overline{Q}_-\} = 0 \,, \tag{2.1.7}$$

$$[\mathrm{i}M, Q_\pm] = \mp Q_\pm \,,\ [\mathrm{i}M, \overline{Q}_\pm] = \mp \overline{Q}_\pm \,, \tag{2.1.8}$$

$$[\mathrm{i}F_V, Q_\pm] = -\mathrm{i}Q_\pm \,,\ \left[\mathrm{i}F_V, \overline{Q}_\pm\right] = \mathrm{i}\overline{Q}_\pm \,, \tag{2.1.9}$$

$$[\mathrm{i}F_A, Q_\pm] = \mp \mathrm{i}Q_\pm \,,\ \left[\mathrm{i}F_A, \overline{Q}_\pm\right] = \pm \mathrm{i}\overline{Q}_\pm \,. \tag{2.1.10}$$

In principle, we could have central charges on the right hand side of Eqs. (2.1.6) and (2.1.7),

$$\begin{aligned} \{\overline{Q}_+, \overline{Q}_-\} = Z \,,\ \{Q_+, Q_-\} = Z^* \,, \\ \{Q_-, \overline{Q}_+\} = \tilde{Z} \,,\ \{Q_+, \overline{Q}_-\} = \tilde{Z}^* \,, \end{aligned} \tag{2.1.11}$$

if Z and $\tilde{Z}$ commute with everything else. However, they are necessary zero in the presence of F_V and F_A. For example, the super Jacobi identity gives

$$-2Z = \{\overline{Q}_+, [\overline{Q}_-, \mathrm{i}F_V]\} - \{\overline{Q}_-, [\mathrm{i}F_V, \overline{Q}_+]\} + [\mathrm{i}F_V, \{\overline{Q}_-, \overline{Q}_-\}] = 0 \,. \tag{2.1.12}$$

Next, let's turn to field theories. In order to write down actions that are manifestly $\mathcal{N} = (2,2)$ supersymmetric, it's best to use the language of $\mathcal{N} = (2,2)$ superspace. Superspace has spacetime coordinates x_0, x_1 as bosonic coordinates, as well as four fermionic coordinates

$$\theta^+ \,,\ \theta^- \,,\ \overline{\theta}^+ \,,\ \overline{\theta}^- \,. \tag{2.1.13}$$

Similar to ordinary fields that are functions of spacetime, superfields are functions on the superspace. Due to the anti-commutativity of Grassmann variables, a superfield $\mathcal{F}$ can be expanded in $\theta^\pm$ and $\overline{\theta}^\pm$ with a finite number of terms,

$$\begin{aligned} \mathcal{F}(x_\mu, \theta^\pm, \overline{\theta}^\pm) = {} & f_0(x_\mu) + \theta^+ f_+(x_\mu) \\ & + \theta^- f_-(x_\mu) + \overline{\theta}^+ \overline{f}_+(x_\mu) \\ & + \theta^+\theta^- f_{+-}(x_\mu) + \cdots \,, \end{aligned} \tag{2.1.14}$$

The action of four supercharges (2.1.2) can be represented as differential operators acting on superfields,

$$\begin{aligned} \mathcal{Q}_\pm &= \frac{\partial}{\partial\theta^\pm} + \mathrm{i}\overline{\theta}^\pm\partial_\pm\,, \\ \overline{\mathcal{Q}}_\pm &= -\frac{\partial}{\partial\overline{\theta}^\pm} - \mathrm{i}\theta^\pm\partial_\pm\,, \end{aligned} \tag{2.1.15}$$

where $\partial_\pm$ are partial derivatives with respect to $x^\pm = x^0 \pm x^1$. Based on this fact, if our Lagrangian can be written as integration of superfields over the whole superspace, it is automatically $\mathcal{N} = (2,2)$ supersymmetric.

On the other hand, the $U(1)_A$ and $U(1)_V$ R-symmetries act on a superfield by

$$\begin{aligned} &e^{\mathrm{i}\alpha F_V} : \mathcal{F}(x^\mu, \theta^\pm, \overline{\theta}^\pm) \to e^{i\alpha q_V}\mathcal{F}(x^\mu, e^{-\mathrm{i}\alpha}\theta^\pm, e^{\mathrm{i}\alpha}\overline{\theta}^\pm)\,, \\ &e^{\mathrm{i}\beta F_A} : \mathcal{F}(x^\mu, \theta^\pm, \overline{\theta}^\pm) \to e^{i\beta q_A}\mathcal{F}(x^\mu, e^{\mp\mathrm{i}\beta}\theta^\pm, e^{\pm\mathrm{i}\beta}\overline{\theta}^\pm)\,. \end{aligned} \tag{2.1.16}$$

The q_V and q_A are real numbers known as vector R-charge and axial R-charge of $\mathcal{F}$.

It turns out that often we do not need the most general form of the superfield (2.1.14). In practice, we impose some constraints on $\mathcal{F}$ to remove some degrees of freedom. To achieve that, we first define another set of operators called covariant derivatives,

$$\begin{aligned} D_\pm &= \frac{\partial}{\partial\theta^\pm} - \mathrm{i}\overline{\theta}^\pm\partial_\pm\,, \\ \overline{D}_\pm &= -\frac{\partial}{\partial\overline{\theta}^\pm} + \mathrm{i}\theta^\pm\partial_\pm\,. \end{aligned} \tag{2.1.17}$$

Notice that they just differ from (2.1.15) by signs.

Then a chiral superfield Φ is defined to be

$$\overline{D}_\pm\Phi = 0\,. \tag{2.1.18}$$

Its complex conjugate gives the anti-chiral superfield $\overline{\Phi}$ which is annihilated by $D_\pm$. In terms of components (2.1.14), we can write

$$\Phi = \phi + \theta^\alpha\psi_\alpha + \theta^+\theta^- F\,. \tag{2.1.19}$$

This means that we have a scalar field ϕ, two Weyl fermions $\psi_\pm$ and an auxiliary field F which is not dynamical. The supersymmetric transformations are found by first acting supercharges on Φ then expanding in components,

$$\begin{aligned} \delta\phi &= \epsilon_+\psi_- - \epsilon_-\psi_+\,, \\ \delta\psi_\pm &= \pm 2\mathrm{i}\overline{\epsilon}_\mp\partial_\pm\phi + \epsilon_\pm F\,, \\ \delta F &= -2\mathrm{i}\overline{\epsilon}_+\partial_-\psi_+ - 2\mathrm{i}\overline{\epsilon}_-\partial_+\psi_-\,. \end{aligned} \tag{2.1.20}$$

Now we can construct an $\mathcal{N} = (2,2)$ nonlinear sigma model that describes the mapping of the worldsheet into the n dimensional Calabi-Yau target space X. We simply choose n

copies of chiral superfields Φ^i, which serve as holomorphic coordinates on X. Given the Kähler potential $K(z, \bar{z})$ of X, the action can be written in the following form[1]:

$$\mathcal{S}_{\text{kin}} = \int d^4x \int d^4\theta \, K(\Phi^i, \bar{\Phi}^{\bar{i}}) \,, \tag{2.1.21}$$

with $d^4\theta = d\theta^+ d\theta^- d\bar{\theta}^- d\bar{\theta}^+$. As we mentioned earlier, $\mathcal{S}_{\text{kin}}$ is manifestly $\mathcal{N} = (2,2)$ supersymmetric. This can also be verified directly using the transformation rules . $\mathcal{S}_{\text{kin}}$ is also known as the D-term action.

Apart from it, we can also add an F-term. To start with, we choose a holomorphic function $W(\phi^i)$ on X (therefore X must be non compact for W to be non trivial). Then an F-term looks as follows,

$$\mathcal{S}_F = \frac{1}{2}(\int d^4x \int d^2\theta \, W(\Phi^i) + c.c.) \,, \tag{2.1.22}$$

where we only integrate over half of the four θ: $d^2\theta = d\theta^- d\theta^+$. Thanks to the constraints (2.1.18), it's easy to verify that the Lagrangian is still supersymmetric. The $W(\Phi^i)$ is also called a superpotential. In general, the total Lagrangian is a sum of two terms Eqs. (2.1.21) and (2.1.22).

Next, we would like to know if the two $U(1)$ R-symmetries are preserved or not. Let's start our discussion from the classical level. Since their actions (2.1.16) only change the overall phase hence do not mix the D-term and F-term, we can treat Eqs. (2.1.21) and (2.1.22) separately.

First let's discuss the D-term. Since θ^4 is invariant under both R rotations, S_{kin} is invariant under both both symmetries if both charges of the Kähler potential $K(\Phi^i, \bar{\Phi}^{\bar{i}})$ vanish. Often this is possible just by demanding the q_V and q_A of Φ^i to be zero.

However, if we look at the F-term, the situation is different. Because θ^2 has $U(1)_A$ R-charge 0, setting q_A of Φ^i to zero is still consistent. However, θ^2 has $U(1)_V$ R-charge 2, the superpotential must have $q_V = 2$ if we want to preserve the $U(1)_V$ symmetry. For general W this is impossible to achieve[2].

To summarize, the lesson we draw is as follows. Classically, $U(1)_A$ is always a good symmetry by assigning q_A to zero for all the chiral fields Φ^i. On the other hand, $U(1)_V$ is a symmetry only for very special types of W.

This finishes our discussion at the classical level. What could go wrong in the quantum world? At the quantum level, it's a well-known fact that a chiral symmetry could be anomalous. A detailed analysis requires a full chapter of its own[3], so here we only quote

[1] Although Kähler potential is not globally well-defined and under change of coordinates it picks a purely holomorphic piece and a purely anti-holomorphic piece, it can be easily shown that these extra terms vanish after superspace integration, so $\mathcal{S}_{\text{kin}}$ is well-defined.

[2] However, it's possible that some discrete symmetry is still preserved.

[3] For instance, [28] is a good review on this topic.

the final result in chapter 13 of [103]: to make sure that the $U(1)_A$ is not anomalous, the target space must obey

$$c_1(X) = 0\,. \tag{2.1.23}$$

In other words, X must be a Calabi-Yau manifold.

Another useful subclass of superfields is the twisted chiral superfield U, satisfying

$$\overline{D}_+ U = D_- U = 0\,. \tag{2.1.24}$$

We can repeat the whole story, introducing the Kähler and superpotential and writing down the D-term and F-term. The only difference is that in the latter case, we integrate over $d^2\tilde{\theta} = d\bar{\theta}^- d\theta^+$ instead of $d^2\theta$.

2.2 Topological field theories of Witten type

There exist many definitions of a topological field theory (TQFT). In the mathematics literature, it was M. Atiyah who first axiomatized the topological field theories [14], inspired by works in two dimensional conformal field theories. Being physicists, we prefer to use less rigorous but more physical definitions, which are given below.

Let's first set the stage. We put our physical system on a manifold X with a given metric g, which is usually not flat. In physics, we are interested in partition function Z, physical operators $\mathcal{O}_i$ as well as various correlation functions

$$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle_g\,, \tag{2.2.1}$$

where we use the subscript g to emphasis that they are computed in that given metric. A TQFT simply means that all these quantities are independent of g therefore should be topological.

In the physics literature, there are roughly speaking two types of TQFTs: one is known as the Schwarz type, where there is no explicit dependence on g in the Lagrangian and physical operators. Therefore, it's natural to expect that the theory should be topological. Examples of this sort include, e.g., Chern-Simons theory in three dimensions [182].

In this chapter, we are interested in the second kind of TQFT, known as the Witten type or cohomological type. In this set-up, the Lagrangian and operators do depend explicitly on the metric, but as soon as we pass into the cohomology, the g dependence drops out.

Formally speaking, a TQFT of Witten type has a special fermionic symmetry, whose Noether charge is denoted as Q. The anti-commutator of Q with operators generates the symmetry transformation

$$\delta \mathcal{O}_i = \mathsf{i}\,\{Q, \mathcal{O}_i\}\,. \tag{2.2.2}$$

The first condition we impose is the following:

$$Q^2 = 0\,. \tag{2.2.3}$$

This may look strange at the first sight, but for readers familiar with algebraic topology, this hints at the possibility of defining cohomology. We also assume that the vacuum $|\text{vac}\rangle$ of our system is invariant under the symmetry hence annihilated by Q.

The next condition we impose concerns with the deformation invariance. Recall that the energy momentum tensor is defined as

$$T_{\mu\nu} := \frac{\delta S}{\delta g^{\mu\nu}} . \tag{2.2.4}$$

We suppose that there exists another operator $G_{\mu\nu}$ such that

$$T_{\mu\nu} = \{Q, G_{\mu\nu}\} . \tag{2.2.5}$$

The final assumption is that the physical operators $\mathcal{O}_i$ of interest are all metric-independent and invariant under this symmetry, i.e., annihilated by Q:

$$\{Q, \mathcal{O}_i\} = 0 . \tag{2.2.6}$$

Now a short computation shows why the correlation functions are supposed to be independent of g,

$$\begin{aligned} \frac{\delta}{\delta g^{\mu\nu}}\langle \mathcal{O}_1 \cdots \mathcal{O}_n\rangle &= \frac{\delta}{\delta g^{\mu\nu}}\left(\int D\phi\, \mathcal{O}_1 \cdots \mathcal{O}_n e^{\mathrm{i}S[\phi]}\right) \\ &= \mathrm{i}\int D\phi\, \mathcal{O}_1 \cdots \mathcal{O}_n \frac{\delta S}{\delta g^{\mu\nu}} e^{\mathrm{i}S[\phi]} \\ &= \mathrm{i}\langle \mathcal{O}_1 \cdots \mathcal{O}_n \{Q, G_{\mu\nu}\}\rangle \\ &= \mathrm{i}\langle Q\, \mathcal{O}_1 \cdots \mathcal{O}_n G_{\mu\nu}\rangle + \mathrm{i}\langle \mathcal{O}_1 \cdots \mathcal{O}_n G_{\mu\nu}\, Q\rangle \\ &= 0 , \end{aligned} \tag{2.2.7}$$

where in the third equality we used our assumption (2.2.5), while in the fourth equality we used (2.2.6) and in the final one we used the invariance of our vacuum.

The reason why it is also called a cohomological TQFT is clear: since the operator Q squares to zero, an observable which is Q of another observable (Q-exact) is also annihilated by Q (Q-closed). Furthermore, repeating the argument for the energy-momentum tensor, it is easy to verify that the correlation function vanishes if any of the operators is Q-exact. Thus, as far as correlation functions are concerned, the physical operators are in one-to-one correspondence with the elements in the Q-cohomology,

$$H_Q = \frac{\{Q - \text{closed operators}\}}{\{Q - \text{exact operators}\}} . \tag{2.2.8}$$

Up to now, we only discussed the definitions and some consequences of a TQFT. Beautiful as it is, we still haven't shown how to construct a TQFT in practice. Below we will introduce the powerful topological twisting method first introduced in [180] and show two possible ways to construct a TQFT out of an $\mathcal{N} = (2, 2)$ supersymmetric non-linear sigma model [181, 171].

2.3 Topological twisting

Before discussing topological properties, there are in fact issues with the non-linear sigma model itself when placed on a curved Riemann surface Σ, i.e., the Lagrangian is not necessarily supersymmetric. Under supersymmetry transformation, the D–term action gives a total differential, which integrate to zero on a flat space, but may not integrate to zero on a curved Σ. Another way to look at this problem is to write down the variation of the action under a supersymmetric transformation (2.1)

$$\delta S = \int_\Sigma (\nabla_\mu \epsilon_+ G^\mu_- - \nabla_\mu \epsilon_- G^\mu_+ - \nabla_\mu \bar{\epsilon}_+ \overline{G}^\mu_- + \nabla_\mu \bar{\epsilon}_- \overline{G}^\mu_+)\sqrt{h}\, d^2x\,. \tag{2.3.1}$$

Here $\epsilon_\pm$ and $\bar{\epsilon}_\pm$ are the variational parameters that are spinors on Σ. If Σ is flat, they can be chosen to be constant spinors the above equation tells us that the Lagrangian is supersymmetric. However, for a curved Σ, covariantly constant spinors (satisfying equations $\nabla_\mu \epsilon_\pm = \nabla_\mu \bar{\epsilon}_\pm = 0$) may simply not exist[4]! In other words, we can still formulate a theory with equal amount of bosonic and fermionic degrees of freedom on a curved Riemann surface, but the supersymmetric invariance of the action of may no longer be preserved.

Still, we would like to preserve a fermionic symmetry on Σ, out of the original supersymmetry. One possible solution is topological twisting. Naively, if we can somehow change the spinor to the scalar, then it's possible to find at least one non-trivial covariantly constant solution, namely the constant hence preserve the modified symmetry. Motivated by this observation, our next goal is to learn how to change a spinor into a scalar by the twisting procedure[5].

From now on, we consider the Euclidean version of the theory obtained by performing the Wick rotation $x_0 = -\mathrm{i}x_2$. We also define complex coordinate $z = x_1 + \mathrm{i}x_2$. Then the 2d Lorentz group becomes the Euclidean rotation group $SO(2)_E = U(1)_E$ with the generator

$$M_E = \mathrm{i}M\,. \tag{2.3.2}$$

Accordingly, the commutation relations Eqs. (2.1.8), (2.1.9) and (2.1.10) in the supersymmetry algebra become

$$[M_E, Q_\pm] = \mp Q_\pm\,,\ [M_E, \overline{Q}_\pm] = \mp \overline{Q}_\pm\,, \tag{2.3.3}$$

$$[F_V, Q_\pm] = -Q_\pm\,,\ \left[F_V, \overline{Q}_\pm\right] = \overline{Q}_\pm\,, \tag{2.3.4}$$

$$[F_A, Q_\pm] = \mp Q_\pm\,,\ \left[F_A, \overline{Q}_\pm\right] = \pm \overline{Q}_\pm\,. \tag{2.3.5}$$

We assume that the theory preserves both $U(1)_V$ and $U(1)_A$ R-symmetries under which the R-charges are all integral. From the last part of section 2.1, that is possible if there is

[4] This can be proved from computing the dimension of $H^0(\Sigma, S)$ or $H^0(\Sigma, \overline{S})$, where $S(\overline{S})$ is the spinor (anti-spinor) bundle

[5] This idea first appeared in Edward Witten's seminar work [180], where he found that the topological twisted $\mathcal{N} = 2$ supersymmetry in four dimensions gives a physical realization of Donaldson theory

	$U(1)_V$	$U(1)_A$	M_E	$\mathcal{L}$	M_E^A	$\mathcal{L}$	M_E^B	$\mathcal{L}$
Q_-	-1	1	1	S	0	$\mathbb{C}$	2	K
$\overline{Q}_+$	1	1	-1	$\overline{S}$	0	$\mathbb{C}$	0	$\mathbb{C}$
$\overline{Q}_-$	1	-1	1	S	2	K	0	$\mathbb{C}$
Q_+	-1	-1	-1	$\overline{S}$	-2	$\overline{K}$	-2	$\overline{K}$

Table 2.1: Before and after twisting. $\mathbb{C}$, S and K mean the trivial, spinor and canonical line bundles respectively. The "bar" means conjugate line bundle.

no superpotential W and the target space X is Calabi-Yau. Twisting simply means that we redefine the Euclidean rotation by mixing M_E with R-symmetry charges. There are two possible ways to define it,

$$A - \text{twist} : M_E^A = M_E + F_V \, , \tag{2.3.6}$$

$$B - \text{twist} : M_E^B = M_E + F_A \, . \tag{2.3.7}$$

Another useful point of view is to regard the twisting as modifying the Lagrangian by adding either the $U(1)_V$ or the $U(1)_A$ R-symmetry current into the spin connection, as elucidated in chapter 3 of [145].

Whichever way we choose, the consequence of topological twisting is to change the flavor index to the spinor index, thus change the spin of various fields. For example, if we consider a chiral superfield Φ whose R-charges are both trivial

$$\Phi = \phi + \theta^+\psi_+ + \theta^-\psi_- + \overline{\theta}^+\overline{\psi}_+ + \cdots \, . \tag{2.3.8}$$

Then we know that the Weyl fermion ψ_+ has M_E charge -1, $U(1)_V$ charge $q_V = -1$ and $U(1)_R$ charge $q_R = -1$. The fact that M_E charge is -1 means exactly that it is a anti-spinor, or a section of the anti-spinor bundle $\overline{S}$ over Σ. Now let's see how does it change under two twists. If we perform the A-twist, it has M_E' charge $-1-1=-2$ and it becomes a vector field or an anti-holomorphic one-form; If we perform the B-twist, it has M_E' charge also equal to $-1-1=-2$ and still becomes an anti-holomorphic one-form.

More dramatically, we consider another Weyl fermion ψ_-. After the A-twist it has M_E' charge $1-1=0$ and becomes a scalar field. Namely we successfully change a spinor to a scalar which is our goal! Similarly, we can show that after the B-twist, it is the fermion $\overline{\psi}_+$ that becomes a scalar. For sake of completeness, the result for all possible cases after the twisting is shown in Table 2.1.

Now comes the magic: after either twisting, the modified theory has a fermionic symmetry and turns out to satisfy all the conditions of a TQFT! Let's spell out all the details in turn.

2.3.1 A-Model

The field theory obtained from an A-twist is dubbed the A-model[6]. From Table 2.1, $\overline{Q}_+$ and Q_- become scalars after the A-twist. We choose the fermionic symmetry charge to be

$$Q_A = \overline{Q}_+ + Q_- \, . \tag{2.3.9}$$

Again from Table 2.1 we know that the Weyl fermions ψ_- and $\overline{\psi}_+$ become scalars, while $\overline{\psi}_-$ and $\overline{\psi}_+$ which were spinors and anti-spinors are now holomorphic and anti-holomorphic one-forms. We rename the fields to make this point manifest,

$$\begin{aligned} \chi^i &:= \psi^i_- \, , \; \chi^{\bar{i}} := \overline{\psi}^{\bar{i}}_+ \, , \\ \rho^{\bar{i}}_z &:= \overline{\psi}^{\bar{i}}_- \, , \; \rho^i_{\bar{z}} := \psi^i_+ \, . \end{aligned} \tag{2.3.10}$$

The modified action can be obtained, as mentioned in the previous subsection, by adding a $U(1)_V$ R–symmetry current into the spin connection of the D-term action (2.1.21). It turns out to be

$$S_A = \int d^2z \left(g_{i\bar{j}} \left(h^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^{\bar{j}} - \mathrm{i}\rho^{\bar{j}}_z D_{\bar{z}} \chi^i + \mathrm{i}\rho^i_{\bar{z}} D_z \chi^{\bar{j}} \right) - R_{i\bar{k}j\bar{l}} \rho^i_{\bar{z}} \chi^j \rho^{\bar{k}}_z \chi^{\bar{l}} \right) \, . \tag{2.3.11}$$

Moreover, the symmetry transformation can simply be obtained from the most general supersymmetric transformation laws (2.1.20) by setting $\epsilon_- = \bar{\epsilon}_+ = 0$ and $\epsilon_+ = \bar{\epsilon}_- = \epsilon$,

$$\begin{aligned} \delta\phi^i &= \epsilon\chi^i \, , & \delta\overline{\phi}^{\bar{i}} &= \epsilon\chi^{\bar{i}} \, , \\ \delta\chi^{\bar{i}} &= 0 \, , & \delta\rho^{\bar{i}}_z &= -2\mathrm{i}\epsilon\partial_z\overline{\phi}^{\bar{i}} - \epsilon\chi^{\bar{j}}\Gamma^{\bar{i}}_{\bar{j}\bar{m}}\rho^{\bar{m}}_z \, , \\ \delta\chi^i &= 0 \, , & \delta\rho^i_{\bar{z}} &= 2\mathrm{i}\bar{\epsilon}\partial_{\bar{z}}\phi^i - \epsilon\chi^j\Gamma^i_{jk}\rho^k_{\bar{z}} \, . \end{aligned} \tag{2.3.12}$$

Now we need to show that A-model indeed satisfies all the conditions of a TQFT. The assumption of a symmetric vacuum is satisfied, because there is no spontaneous symmetry breaking. The fermionic symmetry charge Q_A squares to zero since all the supercharges do and $\overline{Q}_+, Q_-$ anti-commute.[7] The most non trivial part is to show that the energy-momentum tensor is Q_A-exact. In fact, after some computations [183], it turns out that even the action itself is Q_A-exact up to a topological term,

$$S_A = \int_\Sigma d^2z \, \{Q_A, V\} + \int_\Sigma \phi^*(\omega) \, , \tag{2.3.13}$$

where

$$V = g_{i\bar{j}} \left(\rho^{\bar{i}}_z \partial_{\bar{z}} \phi^j + \partial_z \phi^{\bar{i}} \rho^j_{\bar{z}} \right) \tag{2.3.14}$$

and $\int_\Sigma \phi^*(\omega)$ is the pull back of the Kähler form of X, depending only on the cohomology class of ω and homotopy type of ϕ. Therefore, it's invariant under continuous deformation of the worldsheet metric g and we know that the condition (2.2.5) is satisfied.

[6]The preface "A" actually has nothing to do with the word "axial". The nomenclature will be explained in section 3.1 of chapter 3.

[7]Thanks to the absence of central charges (2.1.12). This remark holds also for the B-model below.

In fact, for non-linear sigma models, we are also interested in the dependence on the moduli of target space X. (2.3.13) shows us that when passing to cohomology, our theory only depends on the Kähler moduli (i.e. cohomology class of ω) but not the complex structure moduli.

The remaining step is to find the cohomology classes of operators. This can be done by, e.g., direct computations using the symmetry transformations (2.3.12). Let's use another approach, starting from a simple observation. The action of Q_A on a given combination

$$w_{i_1,\cdots i_p\bar{j}_1\cdots\bar{j}_q}(\phi)\chi^{i_1}\cdots\chi^{i_p}\chi^{\bar{j}_1}\cdots\chi^{\bar{j}_q}\,, \tag{2.3.15}$$

is

$$(\partial_k w_{i_1,\cdots i_p\bar{j}_1\cdots\bar{j}_q}\chi^k+\overline{\partial}_{\bar{k}}w_{i_1,\cdots i_p}\chi^{\bar{k}})\,\chi^{i_1}\cdots\chi^{i_p}\chi^{\bar{j}_1}\cdots\chi^{\bar{j}_q}\,. \tag{2.3.16}$$

This means that if we identify χ^i with dz^i and $\chi^{\bar{i}}$ with $dz^{\bar{i}}$, Q_A can be identified with the exterior derivative d. As a result, the Q_A-cohomology is nothing but the de Rham cohomology of the target space X,

$$\{\text{physical operators}\}\simeq\{H_{Q_A}\}\simeq H_{\mathrm{dR}}(X)\,, \tag{2.3.17}$$

which gives the set of operators a very nice geometrical interpretation.

Next let's study the correlation functions. A general correlation function for a given set of operators $\mathcal{O}_i$ takes the form

$$\langle\mathcal{O}_1\cdots\mathcal{O}_l\rangle=\int\mathcal{D}\phi\mathcal{D}\rho\mathcal{D}\chi\mathcal{O}_1\cdots\mathcal{O}_l\,e^{-S_A}\,. \tag{2.3.18}$$

Precisely due to its topological nature, the correlation functions are invariant under continuous deformations so that we can deform the theory to an easy-to-compute point. The upshot is that the computation of them only receives contributions at the vicinities of the fixed loci of Q_A[8]. From (2.3.12), it's clear that the fixed loci are given by

$$\partial_{\bar{z}}\phi^i=\partial_z\overline{\phi}^{\bar{i}}=0\,. \tag{2.3.19}$$

Namely, holomorphic maps $\phi:\Sigma\to X$. In the literature, they are known as *worldsheet instantons*. If we denote the homology class of the image $\phi(\Sigma)$ by $\mathbf{d}\in H_2(X,\mathbb{Z})$, the bosonic part of the action at the fixed loci can be written as

$$\begin{aligned}S_\phi&=\int_\Sigma g_{i\bar{j}}(\partial_z\phi^i\partial_{\bar{z}}\overline{\phi}^{\bar{j}}+\partial_{\bar{z}}\phi^i\partial_z\overline{\phi}^{\bar{j}})\,d^2z\,,\\&=2\int_\Sigma g_{i\bar{j}}\partial_{\bar{z}}\phi^i\partial_z\overline{\phi}^{\bar{j}}\,d^2z+\int_\Sigma\phi^*w=\int_\Sigma\phi^*\omega=\omega\cdot\mathbf{d}\,.\end{aligned} \tag{2.3.20}$$

This shows that the bosonic part of the action at fixed loci just measures the volume of the image of Σ. We can decompose the homology class β in terms of a basis $\{C_i\}$ of the

[8] This is an example of the famous *Localization* phenomenon. See, e.g., [159] for more detailed discussions.

second homology group $H_2(X,\mathbb{Z})$, and rewrite the above as $\sum_i n_i t_i$, with

$$t_i = \int_{C_i} \omega\,, \qquad i = 1,\cdots, b_2(X)\,. \tag{2.3.21}$$

However, when we set out to compute those correlation functions, we will find that most of them vanish. This is due to the two $U(1)$ R-symmetries which impose selection rules for them. Working out all the details [183], we find that the correlation functions must be zero unless

$$\sum_{i=1}^{l} p_i = \sum_{i=1}^{l} q_i = \mathrm{dim}X(1-g) + c_1(X)\cdot\beta\,, \tag{2.3.22}$$

where p_i and q_i are holomorphic and anti-holomorphic degrees of the operators $\mathcal{O}_i(i = 1,\cdots,l)$, and g is the genus of Σ. If X is a Calabi-Yau manifold, $c_1(X)$ is zero and the right hand side can be reduced to

$$\sum_{i=1}^{l} p_i = \sum_{i=1}^{l} q_i = \mathrm{dim}X(1-g)\,. \tag{2.3.23}$$

In order to have non-trivial correlation functions, we have to demand $g = 0$, i.e., the worldsheet Σ is a sphere.

Now let's give some explicit examples. Suppose further that X is a threefold. We choose physical operators $\mathcal{O}_i$ of type (1,1), corresponding to a d-closed form ω_i whose Poincaré dual is denoted by D_i. From (2.3.22), the correlation functions vanish unless we consider three-point functions. In addition, it is shown in chapter 16 of [103] that

$$C_{123} = \langle \mathcal{O}_1\,\mathcal{O}_2\,\mathcal{O}_3\rangle = (D_1\cap D_2\cap D_3) + \sum_{\mathbf{d}\in H_2(X,\mathbb{Z})} I_{0,3,\mathbf{d}}(\omega_1,\omega_2,\omega_3)Q^{\mathbf{d}}\,, \tag{2.3.24}$$

where the first term is the classical intersection number of the three divisors and in the second term $Q^{\mathbf{d}}$ means $e^{-\omega\cdot\mathbf{d}} = e^{-\sum_i n_i t_i}$. The coefficient $I_{0,3,\mathbf{d}}(\omega_1,\omega_2,\omega_3)$ counts the number of holomorphic maps of genus zero worldsheet Σ into a two-cycle of homology class $\mathbf{d}$ such that the three insertion points x_1, x_2, x_3 of the three operators $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ are mapped into the divisors D_1, D_2, D_3 respectively. It is named as the quantum intersection product in [172], which generalizes the classical intersection relation.

The number $I_{0,3,\mathbf{d}}$ can be simplified further to be

$$I_{0,3,\mathbf{d}}(\omega_1,\omega_2,\omega_3) = r_0^{\mathbf{d}}\int_{\mathbf{d}}\omega_1\int_{\mathbf{d}}\omega_2\int_{\mathbf{d}}\omega_3\,, \tag{2.3.25}$$

where $r_0^{\mathbf{d}}$ counts the number of holomorphic maps from a sphere into the homology class $\mathbf{d}$ in X, known as the genus zero *Gromov–Witten* (GW) invariants. They can be collected in the so-called A–model pre-potential

$$\mathcal{F}_0(\mathbf{t}) = \sum_{\mathbf{d}\in H_2(X,\mathbb{Z})} r_0^{\mathbf{d}} Q^{\mathbf{d}}\,. \tag{2.3.26}$$

As a byproduct, we also see that the correlation functions and free energies depend only on the Kähler moduli of X.

Suppose we choose $\{\mathcal{O}_i\}$ such that the corresponding $\{\omega_i\}$ are a set of basis in $H^2(X,\mathbb{Z})$. Dually, we can choose a set of two-cycles $\{S_j\}$ in $H_2(X,\mathbb{R})$ such that,

$$\int_{S_j} \omega_i = \delta_{ij} \,. \tag{2.3.27}$$

Without lost of generality, we can choose $\{S_j\}$ as the basis to expand our β (2.3.21) and find

$$\int_\beta \omega_i = \sum_j n_j \int_{S_j} \omega_i = n_i \,. \tag{2.3.28}$$

Then it's possible to rewrite (2.3.24) in a more elegant form[9],

$$\langle \mathcal{O}_1\, \mathcal{O}_2\, \mathcal{O}_3 \rangle = (D_1 \cap D_2 \cap D_3) - \frac{\partial^3 \mathcal{F}_0}{\partial t_1 \partial t_2 \partial t_3} \,. \tag{2.3.29}$$

If instead we know all the three-point correlation functions in terms of t_i, we can integrate the above equation to get the prepotential $\mathcal{F}_0(\mathbf{t})$.

2.3.2 B-Model

The field theory obtained from a B-twist is dubbed the B-model. Firstly, notice that the target space must be Calabi-Yau in order to be free of the chiral anomaly. Thus in the case of the B-model we will always assume X to be a Calabi-Yau. From Table 2.1, $\overline{Q}_+$ and $\overline{Q}_-$ become scalars after the B-twist. We choose the fermionic symmetry charge to be

$$Q_B = \overline{Q}_+ + \overline{Q}_- \,. \tag{2.3.30}$$

From Table 2.1, we know that the fermions $\overline{\psi}_+$ and $\overline{\psi}_-$ become scalars, while ψ_- and ψ_+ which were spinors and anti-spinors are now holomorphic and anti-holomorphic one-forms. We rename the fields to make this point manifest,

$$\begin{aligned} \rho^i_z &= \psi^i_- \,, & \rho^i_{\bar z} &= \psi^i_+ \,, \\ \eta^{\bar i} &= -(\overline{\psi}^{\bar i}_+ + \overline{\psi}^{\bar i}_-) \,, & g^{\bar i j}\theta_j &= \overline{\psi}^{\bar i}_- - \overline{\psi}^{\bar i}_+ \,. \end{aligned} \tag{2.3.31}$$

The symmetry transformation can be obtained from the most general transformation rules (2.1) by setting $\epsilon_+ = \epsilon_- = 0$ and $\bar\epsilon_+ = \bar\epsilon_- = \bar\epsilon$,

$$\begin{aligned} \delta\rho^i_\mu &= \pm 2\mathrm{i}\bar\epsilon\partial_\mu\phi^i \,, \\ \delta\phi^i &= 0,\ \delta\theta_i = 0 \,, \\ \delta\overline{\phi}^{\bar i} &= \bar\epsilon\eta^{\bar i},\ \delta\eta^{\bar i} = 0 \,. \end{aligned} \tag{2.3.32}$$

[9] It's also customary to redefine $\mathcal{F}_0$ to absorb the first term in the right hand side.

Now let's check the B-model satisfies all the conditions of a TQFT. We can just mimic the argument for the A-model. The assumption of a symmetric vacuum is satisfied, because there is no spontaneous symmetry breaking. The fermionic symmetry charge Q_B squares to zero since all the supercharges do and $\overline{Q}_+, \overline{Q}_-$ anti-commute. The energy-momentum tensor is Q_B-exact because the action is Q_B-exact up to an extra term,

$$L = \int_\Sigma d^2z \, \{Q_B, V\} + \int_\Sigma W \,, \tag{2.3.33}$$

with

$$V = g_{i\bar{j}} \left(\rho^i_z \partial_{\bar{z}} \phi^{\bar{j}} + \rho^i_{\bar{z}} \partial_z \phi^{\bar{j}} \right) \,, \tag{2.3.34}$$

and

$$W = \left(-\theta_i D\rho^i - \frac{i}{2} R_{i\bar{i}j\bar{j}} g^{k\bar{j}} \rho^i \rho^j \eta^{\bar{i}} \theta_k \right) \,. \tag{2.3.35}$$

Here D is the exterior derivative on Σ. W has no dependence on the worldsheet metric g, so the condition (2.2.5) is satisfied. Moreover, it can be shown [183] that under a change of the Kähler metric, $\delta W = \{Q_B, H\}$ for certain H. Therefore, our theory is independent of the Kähler moduli of X. But it depends on the complex structure moduli since the complex structure of X plays a role in the symmetry transformations (2.3.32).

The physical operators can be constructed from $\phi^i, \overline{\phi}^{\bar{i}}, \eta^{\bar{i}}$ and θ_i and are identified with the Q_B-cohomology. Again, we try to map them to geometric quantities on X. It's useful to observe the following: it can be shown that a given combination

$$\omega^{j_1\cdots j_q}_{\bar{i}_1\cdots\bar{i}_p}(\phi)\, \eta^{\bar{i}_1} \cdots \eta^{\bar{i}_p} \theta_{j_1} \cdots \theta_{j_q} \tag{2.3.36}$$

is Q_B-closed if and only if the corresponding anti-holomorphic p-form with values in $\wedge^q T_X$

$$\omega^{j_1\cdots j_q}_{\bar{i}_1\cdots\bar{i}_p}(\phi)\, d\bar{z}^{\bar{i}_1} \cdots d\bar{z}^{\bar{i}_p} \frac{\partial}{\partial z^{j_1}} \cdots \frac{\partial}{\partial z^{j_q}} \tag{2.3.37}$$

is $\overline{\partial}$ closed.

That is to say, if we make the identification,

$$\begin{aligned} \eta^{\bar{i}} &\longleftrightarrow d\bar{z}^{\bar{i}} \,, \\ \theta_i &\longleftrightarrow \frac{\partial}{\partial z^i} \,, \end{aligned} \tag{2.3.38}$$

the Q_B-cohomology is nothing but the Dolbeault cohomology with values in the vector bundle $\wedge^* T_X$,

$$\{\text{physical operators}\} \simeq \bigoplus_{p,q=0}^{n} H^{0,p}(X, \wedge^q T_X) \,, \tag{2.3.39}$$

where n is the dimension of X.

A general correlation function for a given set of operators $\{\mathcal{O}_i\}$ takes the form

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_l \rangle = \int \mathcal{D}\phi \mathcal{D}\eta \mathcal{D}\theta \, e^{-S_B} \mathcal{O}_1 \cdots \mathcal{O}_l \,. \tag{2.3.40}$$

We can repeat the same story in A-model, deform the system and evaluate it at the fixed loci of Q_B. From (2.3.32) we read off the fixed loci

$$\partial_\mu \phi^i = 0\,, \tag{2.3.41}$$

i.e., a constant map $\phi : \Sigma \to X$. Therefore, the path integral in the correlation function is reduced to an integral over X. Here we see an important difference between A-model and B-model: the fixed loci is just the Calabi-Yau manifold X itself, which is much simpler than the moduli space of all holomorphic maps, appeared as the fixed loci in A-model.[10]

Again, before carrying out the actual computation, we can use R-symmetries to derive selection rules for the correlation function. They turn out to impose the following constraints [183]

$$\sum_{i=1}^{l} p_i = \sum_{i=1}^{l} q_i = \dim X(1-g) \tag{2.3.42}$$

for l physical operators $\mathcal{O}_i (i = 1, \cdots, l)$. Once more, in order to have non-trivial correlation functions, the worldsheet must have genus 0.

Now let's specialize X to be a Calabi-Yau threefold. In particular this means that the line bundle $H^{3,0}(X)$ is trivial and generated by a nowhere vanishing section Ω. Then let's consider physical operators corresponding to $\mu_a \in H^{0,1}(X, T_X)$

$$\mu_a = (\mu_a)^k_{\bar{j}} d\bar{z}^{\bar{j}} \frac{\partial}{\partial z^k}\,. \tag{2.3.43}$$

In the mathematical literature, μ_a is known as the *Beltrami differential*, which can be used to infinitesimally deform the complex structure[11]. Taking θ zero modes into account [183], the correlation function is found to be

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \int_X \langle \mu_1 \wedge \mu_2 \wedge \mu_3, \Omega \rangle \wedge \Omega = \int_X (\mu_1)^i_{\bar{i}} (\mu_2)^j_{\bar{j}} (\mu_3)^k_{\bar{k}} \Omega_{ijk} d\bar{z}^{\bar{i}} d\bar{z}^{\bar{j}} d\bar{z}^{\bar{k}} \wedge \Omega\,, \tag{2.3.44}$$

where $\Omega = \sum \Omega_{ijk}\, dz^i dz^j dz^k$ is the nowhere–vanishing holomorphic top form on X[12]. From here we can also see that the correlation functions depend on the complex structure moduli but not the Kähler moduli of X.

In fact, it is also possible to define a pre-potential $\mathcal{F}_0$ such that the three point function is its third derivative. To show that, we need some preliminary knowledge on the moduli space of complex structures on X.

[10] Indeed, as we shall see in the next chapter 3, B-model geometries are much easier to solve, and by using the so-called mirror symmetry we obtain for free the A-model solutions.

[11] For instance, this can be done by deforming the holomorphic one-form: $dz \to dz + \epsilon \mu^z_{\bar{z}} d\bar{z}$.

[12] This is because the canonical line bundle of a Calabi-Yau manifold is trivial.

2.4 Interlude: moduli space of complex structures

Recalled that our target space X is a Calabi-Yau threefold. By definition, the moduli space of complex structures $\mathcal{M}_X$ encodes all possible complex deformations on X. A point on $\mathcal{M}_X$ corresponds to a fixed complex structure on X, where there is a nowhere vanishing holomorphic top form Ω which generates $H^{3,0}(X)$.

What will happen if we start to move on $\mathcal{M}_X$, i.e., vary the complex structure? First of all Ω will still be closed since the exterior derivative does not depend on the complex structure. However, it is no-longer holomorphic and we need to define a new holomorphic form Ω'. In other words, we have a line bundle $\mathcal{L}$ on $\mathcal{M}_X$, where the one-dimensional space above each point is generated by the Ω determined by that complex structure. Moreover, we can define a metric on it,

$$h = |\Omega|^2 = \mathsf{i} \int_X \Omega \wedge \bar{\Omega} \,. \tag{2.4.1}$$

$\Omega \wedge \bar{\Omega}$ is a $(3,3)$ form so this integral is not trivially zero. Also since Ω is defined up to non-zero scaling at each point, we can always multiply Ω by a nowhere vanishing holomorphic function e^f on $\mathcal{M}_X$, hence $h \to |e^f|^2 \, h$. Equivalently, if we consider the function

$$K = -\log |\Omega|^2 = -\log \int_X \Omega \wedge \bar{\Omega}, \tag{2.4.2}$$

then K transforms as a Kähler potential $K \to K - f - \bar{f}$. This means that the quantity $g_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} K$ is globally well-defined, because both f and $\bar{f}$ are annihilated by $\partial\bar{\partial}$ operator. In fact, $g_{a\bar{b}}$ defines a Hermitian metric on $\mathcal{M}_X$, and the above argument shows nothing but the fact that $\mathcal{M}_X$ is Kähler.

The next question is: can we characterize how does Ω change when we start to move on $\mathcal{M}_X$? As a subspace, $H^{3,0}(X)$ is always contained in $H^3(X, \mathbb{C})$. Moreover, the vector spaces $H^3(X, \mathbb{C})$ over each point of $\mathcal{M}_X$ can be glued together to form a vector bundle $\mathcal{H}$[13] on $\mathcal{M}_X$, which contains the line sub-bundle $\mathcal{L}$ defined above. The question is then translated to the following: can we study how $\mathcal{L}$ varies inside the vector bundle $\mathcal{H}$, namely how the Hodge decomposition varies on X? In the mathematical literature, this is also known as the "variation of Hodge structure"(VHS).

The best way to study the position of $\mathcal{L}$ is to parametrize it by suitable coordinates. This leads naturally to the concept of "period". First of all, there is also a Hermitian metric on $\mathcal{H}$ similar to that defined on $\mathcal{L}$:

$$(\mu, \nu) = \mathsf{i} \int_X \mu \wedge \bar{\nu}, \ \forall \, \mu, \nu \in H^3(X, \mathbb{C}) \tag{2.4.3}$$

[13] This vector bundle is known as the "Hodge bundle", and we can be give a flat connection called the Gauss-Manin connection on it.

It's easy to verify that it is indeed Hermitian. Furthermore, since the wedge product is anti-symmetric on $H^3(X,\mathbb{Z})$, we can find a "symplectic-basis" of real and integral three-forms $\alpha_a, \beta^b, a = b = 1, \cdots h^3(X)/2$, such that $(\alpha_a, \alpha_b) = (\beta^a, \beta^b) = 0$, while $(\alpha_a, \beta^b) = \mathsf{i}\delta_a^b$. The basis is unique up to an $Sp(h^3, \mathbb{Z})$ transformation. The Poincaré duals of α_a, β^b are denoted as A^a, B_b.

We can expand ω in terms of the basis above,

$$\Omega = q^a \alpha_a - p_b \beta^b \,, \tag{2.4.4}$$

where $a, b = 1, \cdots, h^3(X)/2 = h^{2,1}(X) + 1$ and the minus sign is introduced for later convenience. If we move on $\mathcal{M}_X$, the basis (α_a, β^b) is unchanged since it does not depend on the complex structure, but (q^a, p_b) depend on the complex structure and are non-trivial functions on $\mathcal{M}_X$. They are the coordinates to parametrize the VHS that we alluded to before. In fact, they even over-determine the point on $\mathcal{M}_X$, because it turns out that $\mathcal{M}_X$ has dimensions $h^{2,1}(X)$ only. [14]

Still we haven't explained why we call them periods. This is due to the following simple fact,

$$q^a = \int_{A^a} \Omega, \quad p_b = \int_{B_b} \Omega, \tag{2.4.5}$$

where we have used the definition of the Poincaré dual. This means that we can express the VHS in terms of integration of Ω over three cycles. In this sense, it is quite similar to integrate one-forms over one-cycles on an elliptic curve, which gives them the name "period".

Furthermore, it can be proved that q^a alone determine the complex structure [36]. In other words, locally we can try to solve p_b as functions of q^a. Then we are left with only one redundant variable. Since Ω is only defined up to an overall scale, we can just regard q^a as homogeneous coordinates thus they determine the points on $\mathcal{M}_X$ without redundancy.

Next let's introduce the famous "Griffith transversality" relation:

$$\int_X \Omega \wedge \frac{\partial \Omega}{\partial p^a} = 0. \tag{2.4.6}$$

This is due to the fact that Ω will only pick up a $(2,1)$ piece to the first order of variation,

$$\begin{aligned} \partial_a \Omega &= (3,0) \text{ form} + (2,1) \text{ form}\,, \\ &= k_a \Omega + \chi_a \,, \end{aligned} \tag{2.4.7}$$

where k_a are holomorphic functions on $\mathcal{M}_X$. As a consequence,

$$(q^a \alpha_a - p_b \beta^b, \alpha_c - \partial_c p_b \beta^b) = q_c - q^a \partial_c p_a = 0, \tag{2.4.8}$$

where $\partial_c := \frac{\partial}{\partial q^c}$. This means that $p_c = q^a \partial_c p_a = \partial_c (q^a p_a) - p_c$. If we define

$$\mathcal{F}_0 := \frac{1}{2} p_a q^a, \tag{2.4.9}$$

[14]This can be seen from computing the dimension of the tangent space at any smooth point.

then we obtain

$$p_c = \partial_c \mathcal{F}_0 \,, \tag{2.4.10}$$

hence q_c is the derivative of the function $\mathcal{F}_0$. Summing with q^c on both sides of (2.4.9), we find $2\mathcal{F}_0 = q^c \partial_c \mathcal{F}_0$ and $\mathcal{F}_0$ is homogeneous of degree 2 in the variables q^c.

Finally, we can also express the metric in terms of $\mathcal{F}_0$,

$$\begin{aligned} h &= \mathsf{i} \int_X \Omega \wedge \bar{\Omega} = \mathsf{i} \int_X (q^a \alpha_a - p_b \beta^b) \wedge (\bar{q}^a \alpha_a - \bar{p}_b \beta^b) \,, \\ &= \mathsf{i}(\bar{q}^{\bar{a}} p_a - q^a \bar{p}_{\bar{a}}) = \mathsf{i}(\bar{q}^a \partial_a \mathcal{F}_0 - q^a \bar{\partial}_a \bar{\mathcal{F}}_0), \end{aligned} \tag{2.4.11}$$

where we have used the (2.4.10).

In short, we find an important function $\mathcal{F}_0$ which encapsulates many, if not all, geometric structures of $\mathcal{M}_X$. The next claim is, the $\mathcal{F}_0$ is exactly the prepotential that we are looking for.

To show that, we need to understand (2.4.7) in more detail. Actually the χ_a are in one-to-one correspondence with the Beltrami differential (2.3.43) by contraction with Ω,

$$\mu_a = (\mu_a)^k_{\bar{j}} d\bar{z}^{\bar{j}} \frac{\partial}{\partial z^k} \longleftrightarrow \chi_a = (\mu_a)^k_{\bar{j}} \Omega_{kmn} d\bar{z}^{\bar{j}} \wedge dz^m \wedge dz^n \,. \tag{2.4.12}$$

Consider the following expression

$$\int_M \Omega \wedge \partial_1 \partial_2 \partial_3 \Omega \,. \tag{2.4.13}$$

Taking (2.4.12) into account, the $\partial_1\partial_2\partial_3\Omega$ in the integrand can be decomposed into four pieces,

$$\partial_1\partial_2\partial_3\Omega = (3,0) \text{ form} + (2,1) \text{ form} + (1,2) \text{ form} + (0,3) \text{ form} \,, \tag{2.4.14}$$

where the $(0,3)$ form is exactly $(\mu_1)^i_{\bar{i}} (\mu_2)^j_{\bar{j}} (\mu_3)^k_{\bar{k}} \Omega_{ijk} d\bar{z}^{\bar{i}} d\bar{z}^{\bar{j}} d\bar{z}^{\bar{k}}$ and is the only piece that contributes to the integral (2.4.13). Namely, (2.4.13) is the same as the three-point function (2.3.44).

On the other hand, from Eqs. (2.4.4) and (2.4.10) we can write it in another form,

$$\begin{aligned} \int_M \Omega \wedge \partial_1\partial_2\partial_3\Omega &= \int_M (q^a \alpha_a - p_b \beta^b) \wedge (-\partial_1\partial_2\partial_3\partial_c \mathcal{F}_0 \, \beta^c) \\ &= -q^c \partial_c \partial_1\partial_2\partial_3 \mathcal{F}_0 \\ &= -\Big(\partial_1\partial_2\partial_3 (q^c \partial_c \mathcal{F}_0) - 3\partial_1\partial_2\partial_3 \mathcal{F}_0\Big) \\ &= -\Big(\partial_1\partial_2\partial_3 (2\mathcal{F}_0) - 3\partial_1\partial_2\partial_3 \mathcal{F}_0\Big) \\ &= \partial_1\partial_2\partial_3 \mathcal{F}_0 \,. \end{aligned} \tag{2.4.15}$$

In the third equality, we use the fact that $\mathcal{F}_0$ is homogeneous of degree 2 in q^a. This proves our claim at the end of subsection 2.3.2,

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \frac{\partial^3 \mathcal{F}_0}{\partial t_1 \partial t_2 \partial t_3} \,. \tag{2.4.16}$$

Chapter 3

Topological String Theory: Coupling to Gravity

After all these preliminaries, we are finally able to define topological string theory in section 3.1. Furthermore, in subsection 3.1.2 we discuss briefly a recent generalization of ordinary topological string theory. In section 3.2, we introduce the important notion of mirror symmetry. It states that the A model topological string theory on a Calabi-Yau threefold X is equivalent to type B topological string theory on a mirror Calabi-Yau threefold $\tilde{X}$. We first discuss Batyrev's construction of mirror pairs in subsection 3.2.1, then we extend to non compact situation in subsection 3.2.2. Useful references for this chapter are, for example, [103, 152, 174, 145].

3.1 Topological string theory

In the discussion above, we always assume that in the topological nonlinear sigma models the worldsheet metric g is fixed, even though after topological twisting we argue that the models are independent of continuous deformation of the metric. An immediate drawback of A and B models is that according to the selection rules Eqs. (2.3.22) and (2.3.41), Σ must be a sphere to allow for non trivial correlation functions. From a string theorist point of view, this is not at all satisfactory. In string theory, we are all familiar with summation over all possible genera in a scattering diagram. Therefore, it would better if the worldsheet Σ of higher genus also plays a role. In other words, we are interested in making the metric on Σ dynamical and coupling the topological sigma model to gravity. Then we should integrate over the space of all possible metrics on Σ in the path integral, just like what we did in ordinary string theory.

The theory constructed from A–twisted and B–twisted topological nonlinear sigma models are called type A and type B topological string theories respectively. As their names suggest, they are related to type IIA and type IIB superstring theories. In fact, the moduli space of type A (type B) topological string are identified with the vector multiplet moduli space of the type IIA (type IIB) superstring compactified on the Calabi–Yau threefold.

At first glance, this may seem trivial: since the theory is topological, the integrand should be independent of the metric and naively we would simply get

$$\int \mathcal{D}g\, Z[g] = \mathrm{Vol(G)}\, Z[g_0]\,, \tag{3.1.1}$$

where the Vol(G) is formally the volume of the "gauge group" which generates the space of metrics. However, even at the level of physical rigor, there are several issues with this line of reasoning:

- The topological symmetry could be anomalous at the quantum level invalidating the conclusion that all the configurations in a gauge group orbit are equivalent.
- Although our theory is invariant under continuous deformation of the metric, there could be metric configurations that cannot be reached from a given metric by continuous changes.

Let's be more careful when talking about the integration over the space of all possible metrics. First of all, just like in ordinary string theory, the two-dimensional sigma models become conformal when we integrate the metric in the path integral, making the energy-momentum tensor traceless. Notice that this has nothing to do with topological twisting. This also means that we can divide the integration over the space of metrics into two steps: we first integrate over all conformally equivalent metrics, then integrate over the quotient space. As we shall see below, since the two-dimensional conformal group is rather large, the quotient is actually finite dimensional.

The first step could in principle leads to problems. From the study of bosonic strings, on a curved worldsheet, there is the notorious conformal anomaly, with coefficient proportional to the central charge. In ordinary string theory, the central charge is canceled by choosing the correct dimension. In our situation, this is done automatically by topological twisting. As detailed in [138], topological twisting amounts to adding a conserved current into the Lagrangian, and the energy-momentum tensor is modified such that the new central charge is zero. The upshot is that a topological nonlinear sigma model coupled to gravity is free of conformal anomaly.

The more interesting part is the second step. By construction, the quotient space is the space of conformally equivalence classes. Since we can associate to each equivalence class a complex structure, it is the same as the moduli space of complex structures on Σ, denoted as $\mathcal{M}_g$, where g is the genus of Σ[1]. It's a known fact that $\mathcal{M}_0$ consists of one point and $\mathcal{M}_1$ is the fundamental domain of a torus, while for $g > 1$, $\mathcal{M}_g$ has complex

[1]This can be proved by computing the dimension of the tangent space at a given point, which turns out to be $\dim H^1(\Sigma, T_\Sigma)$.

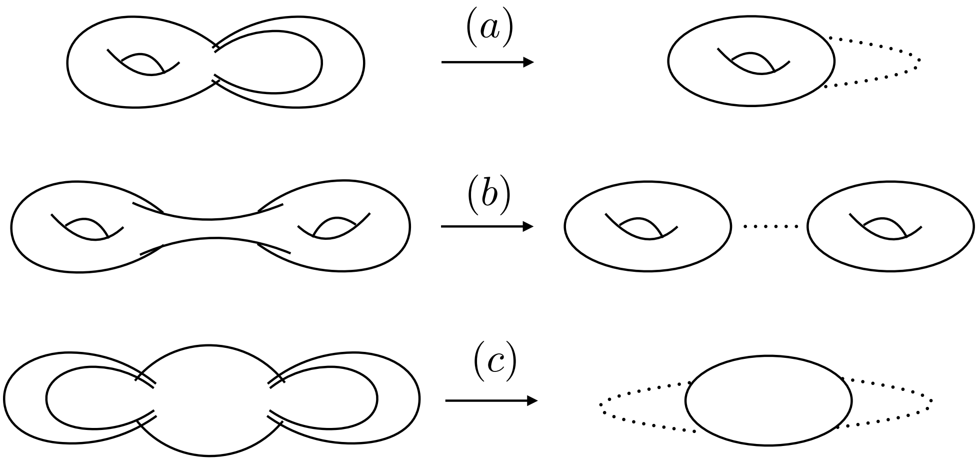


Figure 3.1: Some possible degenerations of genus two stable curves.

dimension $3g - 3$ and is non-compact. They can be compactified, by adding new points that have mild singularities, corresponding to the so-called "stable curve".

- Its only singularities are simple nodes.
- It has a finite number of automorphisms. This means that its genus zero part should have at least three nodal points and genus one part should have at least one nodal point[2].

The passage to its boundary can be represented figuratively, e.g., for genus two, as Figure 3.1. The space $\overline{\mathcal{M}}_g$ is the famous Deligne–Mumford compactification of the moduli space $\mathcal{M}_g$ of Riemann surfaces [52]. Moreover, if we consider correlation functions, we need to consider the complex structure of Riemann surfaces having n marked points, whose compactification leads to $\overline{\mathcal{M}}_{g,n}$.

Note that by borrowing ideas from bosonic string theory, we bypass the first issue and find that the second issue does not occur. We also understand better the space that we integrate over. Now let's look at what quantities to integrate. Recall the first condition (2.2.5) of a TQFT. For the sake of reader's convenience, let's record it here,

$$T_{\mu\nu} = \frac{\delta S}{\delta g^{\mu\nu}} = \{Q, G_{\mu\nu}\}\,. \tag{3.1.2}$$

In addition, the energy-momentum tensor $T_{\mu\nu}$ is traceless because the theory is conformal. Therefore, the only nonzero components of $T_{\mu\nu}$ are T_{zz} and $T_{\bar{z}\bar{z}}$ and they satisfy

$$T_{zz} = \{Q, G_{zz}\}, \qquad T_{\bar{z}\bar{z}} = \{Q, G_{\bar{z}\bar{z}}\}\,. \tag{3.1.3}$$

Since $T_{\mu\nu}$ has axial charge 0 and Q axial charge 1, the G's have axial charge -1. They can be used to define a measure on the moduli space of $\overline{\mathcal{M}}_g$. The tangent space to $\overline{\mathcal{M}}_g$ at

[2] Recall that the automorphic group of a sphere is $PSL(2,\mathbb{C})$ and that of a torus can be identified with itself, acting by translations. While for $g > 1$, it is finite.

a given point Σ corresponds to a choice of Beltrami differential (2.3.43) on the Riemann surface Σ. Let μ_i denote a basis. For genus one there is only one while for higher genera there are $3g-3$ of them, which correspond the dimension of $\overline{\mathcal{M}}_g$. Let's start from the genus one case.

Similar to the genus zero case (2.3.29), the genus one free energy $\mathcal{F}_1$ can be defined through one point function. The measure is given by $\langle G_{zz}(\mu_1) G_{\bar{z}\bar{z}}(\overline{\mu}_1)\rangle$, where

$$G_{zz}(\mu) := \int_\Sigma G_{zz}\mu^z_{\bar{z}} d^2 z\,. \tag{3.1.4}$$

Then we insert one observable in the Kähler class of axial charge (1,1) at one point to cancel the $(-1,-1)$ axial charge from the insertion of $(G_{zz}, G_{\bar{z}\bar{z}})$,

$$\partial_i \mathcal{F}_1 = \int_{\overline{\mathcal{M}}_{1,1}} dm d\overline{m} \langle G_{zz}(\mu_1) G_{\bar{z}\bar{z}}(\overline{\mu}_1)\mathcal{O}_i\rangle\,, \tag{3.1.5}$$

This can be integrated to be

$$\mathcal{F}_1 = \frac{1}{2}\int \frac{d^2\tau}{\tau_2}\mathrm{Tr}(-1)^F F_L F_R q^{H_L}\overline{q}^{H_R}\,. \tag{3.1.6}$$

For higher genera, the measure on $\overline{\mathcal{M}}_g$ is defined as

$$\prod_{i=1}^{3g-3}\left(dm^i d\overline{m}^i \int_\Sigma G_{zz}(\mu_i)^z_{\bar{z}} \int_\Sigma G_{\bar{z}\bar{z}}(\overline{\mu}_i)^{\bar{z}}_z\right)\,. \tag{3.1.7}$$

The G's contribute to the axial charge $(3-3g, 3-3g)$ which saturates the axial charge anomaly, hence the measure is not, a priori, zero.

The genus $g>1$ free energy $\mathcal{F}_g$ is defined by

$$\mathcal{F}_g = \int_{\overline{\mathcal{M}}_g} \prod_{i=1}^{3g-3}\left(dm^i d\overline{m}^i \int_\Sigma G_{zz}(\mu_i)^z_{\bar{z}} \int_\Sigma G_{\bar{z}\bar{z}}(\overline{\mu}_i)^{\bar{z}}_z\right)\,, \tag{3.1.8}$$

where dm^i are the dual one-forms to the μ_i.

Given a set of operators $\{\mathcal{O}_i\}$, the old selection selection rules Eqs. (2.3.23) or (2.3.42) are modified to be [103],

$$\sum_{i=1}^{l} p_i = \sum_{i=1}^{l} q_i = (\mathrm{dim}X - 3)(1-g) + l \tag{3.1.9}$$

In particular, the extra $3(g-1)+l$ in the right-hand side is the contribution of $\overline{\mathcal{M}}_{g,n}$. One particular nice feature topological string theory arises when the Calabi-Yau manifold X has complex dimensions three. Inserting $\mathrm{dim}X = 3$, we see that the g-dependence drops out! For example the new selection rules are trivially satisfied if all the (p_i, q_i) are equal to $(1,1)$. From now on, we will always assume the target space X to be a Calabi-Yau threefold.

One of the central tasks in topological string theory is to understand the structure of the free energies $\mathcal{F}_g$. We first focus on the type A topological string. The type B case will be discussed in the next subsection 3.1.1. Similar to the pre-potential (2.3.26), they also have the following expansion

$$\mathcal{F}_g = \sum_{\mathbf{d}\in H_2(X,\mathbb{Z})} r_g^{\mathbf{d}} Q^{\mathbf{d}}\,, \tag{3.1.10}$$

where similar to (2.3.25), $r_g^{\mathbf{d}}$ are known as the genus g Gromov-Witten invariants. They count, in a mathematically rigorous way, the "number" of holomorphic maps from the genus g worldsheet Σ into the homology class $\mathbf{d} \in H_2(X,\mathbb{Z})$ of X. In fact, due to the possible non trivial isomorphism group of Σ, $r_g^{\mathbf{d}}$ are often rational numbers rather than simply integers.

Furthermore, we can package all the $\mathcal{F}_g$ into a single free energy,

$$\mathcal{F}(g_s,\mathbf{t}) = \sum_{g=0}^{\infty} \mathcal{F}_g(\mathbf{t}) g_s^{2g-2}\,, \tag{3.1.11}$$

where g_s is a formal parameter. In connection with ordinary string theory, g_s can be identified with the vev of the self-dual part of the gravi-photon field strength [3].

Later, it was shown by Rajesh Gopakumar and Cumrun Vafa [69, 70] that we can actually carry out a partial resummation over the genus g and rewrite (3.1.11) as

$$\mathcal{F}(g_s,\mathbf{t}) = \sum_{w=1}^{\infty}\sum_{g=0}^{\infty}\sum_{\mathbf{d}\in H_2(X,\mathbb{Z})} \frac{I_g^{\mathbf{d}}}{w}\left(2\sin\frac{wg_s}{2}\right)^{2g-2} Q^{w\mathbf{d}}\,. \tag{3.1.12}$$

with integer numbers $I_g^{\mathbf{d}}$ known as the *Gopakumar-Vafa* (GV) invariants. In the M-theory picture, they are certain traces over the Hilbert space of five dimensional BPS states in the spacetime after compactification.

If we perform a Laurent expansion in g_s, we can find relations between these two types of invariants. For instance[4], we have

$$\mathcal{F}_0(\mathbf{t}) = \sum_{\mathbf{d}\in H_2(X,\mathbb{Z})} r_0^{\mathbf{d}} Q^{\mathbf{d}} = \sum_{\mathbf{d}\in H_2(X,\mathbb{Z})}\sum_{w=1}^{\infty} \frac{I_0^{\mathbf{d}}}{w^3} Q^{w\mathbf{d}}\,. \tag{3.1.13}$$

3.1.1 Holomorphic anomaly

In this part, we will discuss the type B topological string theory. Let's first say a few words about the genus zero case, which is already quite interesting.

The B model pre-potential is closely related to the moduli space of complex structures $\mathcal{F}_0$, as detailed in the section 2.4 of the first chapter, which is much easier to determine

[3] For more details, we refer the reader to [11].

[4] The $1/w^3$ contribution for genus zero GW invariants was first noticed by [37] and proved rigorously in [139]. It is also known as the Aspinwall-Morrison formula [13].

than the A model. In particular, the periods (p_a, q^b) defined in (2.4.5) satisfy certain differential equations known as the Picard-Fuchs (PF) equations. Instead of showing how to find them[5], we choose to explain the heuristic idea which is in fact very simple. Recall from (2.4.7), we obtain a $(2,1)$ piece when we take the derivative of Ω with respect to q_a. If we take further derivatives, we can also generate $H^{1,2}$ and $H^{0,3}$ pieces in turn. Since the dimension of H^3 is finite, some linear combinations of them, after carefully choosing the derivatives, must be zero in the cohomology, i.e., $L\Omega = d\eta$ for a moduli dependent linear operator L. Therefore, after integrating over corresponding closed three cycle, we obtain $L \circ F = 0$ for F equal to either p_a or q^b. Accidentally, this also proves that they should satisfy the same PF equations.

To summarize, for the genus zero case, we only need to consider the holomorphic structure as well as their variation in $\mathcal{M}_X$, and essentially everything reduces to a set of linear partial differential equations. However, for higher genera, it was first pointed out in [25] that the partition function does not remain holomorphic anymore.

To understand this point, let's first see why in the genus zero case the anti-holomorphic part decouples. Recall that we have two types of topological twisting on a given Calabi-Yau manifold, giving rise to A and B models. We also denote their conjugate twisting by $\overline{\text{A}}$ and $\overline{\text{B}}$. If we consider the B model for example, the $\overline{\text{B}}$ observables are all cohomologically trivial. In more details, since we known that the correlation functions can be obtained from $\bar{t}$-derivatives of the perturbed partition function $Z[\bar{t}]$ at $\bar{t} = 0$, in which we added the following term into the Lagrangian,

$$\bar{t}^{\bar{a}} \int_\Sigma d^2\bar{\theta}\, \bar{\mathcal{O}}_a = \bar{t}^{\bar{a}} \int_\Sigma \bar{\mathcal{O}}_a^{(2)}\,, \tag{3.1.14}$$

where $\mathcal{O}_a^{(2)}$ is the coefficient of $\bar{\theta}^+\bar{\theta}^-$ term in the expansion of $\bar{\mathcal{O}}$. Up to a possible minus sign depending on whether $\bar{\mathcal{O}}_a^{(2)}$ is bosonic or fermionic, this is equal to $\{\bar{Q}_+, [\bar{Q}_-, \bar{\mathcal{O}}_a^{(0)}]\}$ [103].

Using the nilpotency of $\overline{Q}_-$, we find

$$\begin{aligned}\{\bar{Q}_+, [\bar{Q}_-, \bar{\mathcal{O}}_a^{(0)}]\} &= \{\bar{Q}_+ + \bar{Q}_-, [\bar{Q}_-, \bar{\mathcal{O}}_a^{(0)}]\} \\ &= \{Q_B, [\bar{Q}_-, \bar{\mathcal{O}}_a^{(0)}]\}\,.\end{aligned} \tag{3.1.15}$$

Clearly this means the insertion is Q_B-exact and naively decouples from all the correlation functions. From the worldsheet perspective, the dependence on $\bar{t}^{\bar{a}}$ comes from inserting $\overline{\text{B}}$ observables, so we know that $\mathcal{F}_0$ must be holomorphic.

However, why after coupling to gravity, $\overline{\text{B}}$ observables can contribute non-trivially in higher genera? Let's first consider some heuristic idea. The crucial difference is that now we have to integrate over the compactified moduli space $\overline{\mathcal{M}}_g$ mentioned earlier. Recall

[5] From a computational point of view, if we only consider the toric geometry, their PF operators can be constructed just from its polytope [105].

that we have the following insertions (3.1.4) in the path integral,

$$G_{zz}(\mu_i) := \int d^2z\, G_{zz}\mu^z_{\bar{z}}\,, \tag{3.1.16}$$

with μ the Beltrami differential (2.3.43). During the process of commuting Q_B with operators until it hits the vacuum, we need to compute the commutator of Q_B and $G_{++}(\mu_i)$. The result turns out to be non-zero,

$$\{Q_B, G_{zz}(\mu_i)\} = T \cdot \mu_i\,. \tag{3.1.17}$$

From the very definition of energy-momentum tensor $T_{\mu\nu} = \frac{\delta S}{\delta g^{\mu\nu}}$, we have

$$T \cdot \mu_i = \frac{\delta S}{\delta m^i}\,, \tag{3.1.18}$$

so the anti-commutator is expressed as a total derivative of the action along certain direction in $\overline{\mathcal{M}}_g$.

More precisely, the computation carried out in [25] shows the following result,

$$\frac{\partial \mathcal{F}_g}{\partial t^{\bar{a}}} = \int_{\overline{\mathcal{M}}_g} \prod_{i=1}^{3g-3} dm^i d\bar{m}^i \sum_{j,k} \frac{\partial^2}{\partial m^j \partial \bar{m}^k} \left\langle \left(\prod_{l\neq j} \int \mu_l \cdot G \right) \left(\prod_{l\neq k} \int \bar{\mu}_l \cdot \bar{G} \right) \int \bar{\mathcal{O}}^{(2)}_a \right\rangle, \tag{3.1.19}$$

where $\mathcal{F}_g$ is the genus g free energy. Note that the final integrand is a total derivative, so if $\overline{\mathcal{M}}_g$ has no boundary, this would just be zero due to Stokes theorem. However exactly because of compactification, the moduli space does have boundaries. More precisely, as discussed above, those boundaries correspond to genus g surfaces that have nodal singularities. Some examples of genus two curves are shown in Figure 3.1. In general, this happens in two ways: a non-separating cycle of the surface can be pinched, leaving a single surface of genus $g-1$, see process (a) in Figure 3.1, or a separating cycle of the surface can be pinched, splitting it up into two surfaces of genus g_1 and $g_2 = g - g_1$, see process (b) in Figure 3.1.

By carefully analyzing the boundary contributions to the integral for these two types of boundaries, [25] shows that we get,

$$\frac{\partial \mathcal{F}_g}{\partial t^{\bar{i}}} = \frac{1}{2}\overline{C}^{jk}_{\bar{i}} \left(D_j D_k \mathcal{F}_{g-1} + \sum_{r=1}^{g-1} D_j \mathcal{F}_r D_k \mathcal{F}_{g-r} \right), \quad g > 2\,. \tag{3.1.20}$$

where $\overline{C}^{jk}_{\bar{i}} = \overline{C}_{\bar{i}\bar{j}\bar{k}} g^{j\bar{j}} g^{k\bar{k}}$ with $g^{j\bar{j}}, \overline{C}_{\bar{i}\bar{j}\bar{k}}$ the two-point, three-point function on the sphere and D_i is the covariant derivative[6].

Finally, we would like to make some remark concerning this set of holomorphic anomaly equations[7]. A particularly nice feature is that it is recursive. This means that in principle

[6] This is related to the fact that actually $\mathcal{F}_g$ is not a function but a section of the line bundle $\mathcal{L}^{2g-2}$ on $\mathcal{M}_X$.

[7] They first term in the right-hand side comes from the degeneration process appeared in figure (a), while the second term comes from the degeneration process appeared in figure (b).

we are able to determine the left-hand side inductively in terms of right-hand side which consists of free energies of strictly lower genera. However, note that the left-hand side only involves anti-holomorphic derivative. This means that we can always add a holomorphic piece $f(t_i)$ to $\mathcal{F}_g$ without violating the equality. This ambiguity is known as the *holomorphic ambiguity*. One possible way to determine $f(t_i)$ is to impose sufficiently many boundary conditions at special points of $\mathcal{M}_X$ [84]. We will work out one example explicitly in chapter 4.

3.1.2 Refined topological string theory

Finally, we introduce the notion of refined topological string theory. Since presenting all the details would lead us too far, we choose to be brief and only list some important results.

As mentioned in the introduction, for a non-compact Calabi-Yau manifold X there exists a refinement of ordinary topological string theory, whose partition function is denoted by $Z(\epsilon_1, \epsilon_2)$. If we take the log of $Z(\epsilon_1, \epsilon_2)$, we get the refined free energy $\mathcal{F}_{\text{ref}}(\epsilon_1, \epsilon_2, \mathbf{t})$. It admits the following expansion,

$$\mathcal{F}_{\text{ref}}(\epsilon_1, \epsilon_2, \mathbf{t}) = \sum_{g,n=0}^{\infty} (\epsilon_1\epsilon_2)^{g-1}(\epsilon_1+\epsilon_2)^{2n} F_{\text{ref}}^{g,n}(\mathbf{t})\,. \tag{3.1.21}$$

Note that by setting $\epsilon_1 = -\epsilon_2 = g_s$, we indeed recover (3.1.11) after identifying $F_{\text{ref}}^{g,0}(\mathbf{t})$ to be $\mathcal{F}_g(\mathbf{t})$. In the next chapter, we are instead interested in a sort of opposite limit, by taking ϵ_1 to be zero. This is known as the Nekrasov-Shatashvili limit [155].

There also exists a formula similar to (3.1.12),

$$F(\epsilon_{1,2}, \mathbf{t}) = \sum_{g_{L,R}\geq 0}\sum_{w\geq 1}\sum_{\mathbf{d}\in H_2(X,\mathbb{Z})} \frac{n^{\mathbf{d}}_{g_L,g_R}}{w}\frac{(2\sin\frac{w\epsilon_L}{2})^{2g_L}(2\sin\frac{w\epsilon_R}{2})^{2g_R}}{2\sin\frac{w(\epsilon_R+\epsilon_L)}{2}\,2\sin\frac{w(\epsilon_R-\epsilon_L)}{2}}\mathbf{Q}^{w\mathbf{d}}\,, \tag{3.1.22}$$

where $\epsilon_{L/R} = \frac{\epsilon_1\pm\epsilon_2}{2}$. The integer numbers $n^{\mathbf{d}}_{g_L,g_R}$ are called the refined GV invariants [102, 116]. They encode the numbers of BPS states of the M-theory compactification on X with five dimensional Ω-background $(\mathbb{R}^4\times S^1)_{\epsilon_1,\epsilon_2}$, where the parameters ϵ_1, ϵ_2 describe how the $\mathbb{R}^4$ is twisted along S^1.

Last but not least, in B type topological string theory, there exists a refined version of holomorphic anomalies [109, 107],

$$\bar{\partial}_{\bar{i}}\mathcal{F}^{(n,g)} = \frac{1}{2}\overline{C}^{jk}_{\bar{i}}\left(D_jD_k\mathcal{F}^{(n,g-1)} + {\sum_{m,h}}' D_j\mathcal{F}^{(m,h)}D_k\mathcal{F}^{(n-m,g-h)}\right), \qquad n+g>1. \tag{3.1.23}$$

where the prime in the sum means $(m,h)\neq(0,0)$. Direct integration method of the refined holomorphic anomaly equations can also be found in [109, 107].

3.2 Mirror symmetry

Mirror symmetry is an extremely vast subject enjoying rich mathematical and physical structures. [45] is a thorough monograph dedicated to mirror symmetry, aimed primarily at mathematicians. [103] and [12] are two monographs written by both mathematicians and physicists, which contain surely much more than we can cover in this thesis. Rather, we choose to take a more down-to-earth approach and focus on the construction of mirror pairs of toric Calabi-Yau threefolds.

3.2.1 Batyrev's Construction

Before we start, it is necessary to review some basic knowledge of toric geometry. For convenience of the reader, a brief introduction can be found in appendix B, which also sets all our conventions. Even though there are several ways to construct a toric variety, we will start from the polytope. The reason is that this will allow for an elegant description of the mirror pairs, first found by Batyrev [16].

In order to describe his proposal, we need to single out a subclass of polytopes known as reflexive polytopes.

Definition 1. *An integral polytope Δ is reflexive if*

- *For each codimension 1 face $F \subset \Delta$, there exists an $n_F \in N$ such that $F = \{m \in \Delta | \langle m, n_F \rangle = -1\}$,*
- $0 \in \text{int}(\Delta)$.

If we take the convex hull of the n_F in $N_{\mathbb{R}}$, we get a dual polytope named as the polar polytope Δ^*. Note that comparing with the definition of a normal fan in Appendix B, we see that the n_F correspond exactly to the one dimensional rays of the fan for $\mathbb{CP}_\Delta$.

Theorem 1. *Recall that $\mathbb{CP}_\Delta$ is the toric variety defined by Δ.*

- *A polytope Δ is reflexive if and only if $\mathbb{CP}_\Delta$ is Gorenstein and Fano.*
- *A polytope Δ is reflexive if and only if Δ^* is reflexive, and we clearly have $(\Delta^*)^* = \Delta$.*

Proof. [16]. □

The Gorenstein condition means that the geometry has at worst Gorenstein singularities. Thus even we may not have a good notion of top degree holomorphic differential form, we can still define the canonical line bundle. The Fano condition means that the anti-canonical line bundle is ample. This in particular demands the anti-canonical divisor intersects effective curves with non-negative values[8].

Now we can describe how to construct mirror pairs of Calabi-Yau hypersurfaces inside four dimensional toric varieties. Suppose we are given a reflexive polytope Δ embedded in a four dimensional space. Denoting its only inner point by α_0 and integral points lying on Δ by α_i $(i = 1, \ldots, s)$, we consider the zero locus of the Laurent polynomial[9]

$$f_\Delta = t_0 - \sum_{i \neq 0} t_i Y_1^{\alpha_i^1} \ldots Y_4^{\alpha_i^4}, \quad \alpha_i \in \mathbb{Z}^4 , \tag{3.2.1}$$

inside the algebraic torus $(\mathbb{C}^*)^4 \subset \mathbb{P}_\Delta$. Its closure Z_Δ in $\mathbb{P}_\Delta$ defines a hypersurface. Note that by rescaling the four coordinates Y_i and adjusting the overall normalization we can set five of the parameters t_i to one.

In general, $\mathbb{P}_\Delta$ and hence Z_Δ are singular. As mentioned above, [16] shows that Z_Δ can be resolved into a non singular Calabi-Yau manifold if and only if $\mathbb{P}_\Delta$ has only Gorenstein singularities. According to Theorem 1, this is equivalent to Δ being reflexive, which is exactly our assumption. We will still denote the resolved hypersurface by Z_Δ.

Theorem 1 also tells us that Δ^* is reflexive as well, so we can just apply the same construction to Δ^*. We choose the inner and integer points to be α_i^* $(i = 0, \cdots, s^*)$, construct the hypersurface and resolve it into a smooth manifold Z_{Δ^*}.

The pair of hypersurfaces $(\hat{Z}_\Delta, \hat{Z}_{\Delta^*})$ forms a mirror pair, thanks to the following combinatorial identities for the Hodge numbers first observed in [16],

$$\begin{aligned} h^{1,1}(Z_{\Delta^*}) &= h^{2,1}(Z_\Delta) \\ &= l(\Delta) - 5 - \sum_{\operatorname{codim} F = 1} l'(F) + \sum_{\operatorname{codim} F = 2} l'(F) l'(F^*) \\ h^{1,1}(Z_\Delta) &= h^{2,1}(Z_{\Delta^*}) \\ &= l(\Delta^*) - 5 - \sum_{\operatorname{codim} F^* = 1} l'(F^*) + \sum_{\operatorname{codim} F^* = 2} l'(F^*)\, l'(F) , \end{aligned} \tag{3.2.2}$$

where F stands for the face of Δ and F^* means its dual face. If a k dimensional F is specified by vertices $m_1, \cdots m_k$, then F^* is a $3-k$ dimensional face in Δ^* defined by $\{\nu \in \Delta^* | (\nu, m_1) = (\nu, m_k) = 0\}$. $l(P)$ and $l'(P)$ for any convex set P are the number of integral points on P and in its interior respectively.

Therefore, each reflexive pair of polytopes gives us automatically a mirror pair of Calabi-Yau manifolds. In fact, for low dimensions we can even list out all of them. It was found that there are 16 reflexive polytopes in two dimensions, 4319 reflexive polytopes in three dimensions, 473,800,776 reflexive polytopes in four dimensions, etc[10].

[8] This is a useful criterion since it's not difficult to compute those numbers in toric geometry.

[9] More invariantly, this is a generic section of the anti-canonical bundle $\mathcal{O}_{X_\Delta}(\sum_i D_i)$.

[10] Their integral points, vertices, Picard and Hodge numbers can be found on the website [1].

Most of those toric geometries are singular. As mentioned above, by blowing up at the singular point we are guaranteed to make Z_Δ smooth. Now comes an important remark. Because of the way we define the dual polytope Δ^*, there is a very convenient way to smoothen our geometry. Recalled that Δ^* has a unique interior point p. Note that the face fan of Δ^*, defined as the fan whose top dimensional cones are generated by the rays connecting p to the points on all the faces of Δ^*, coincides with the normal fan of Δ. Desingularization of Δ is the same as star triangulations of Δ^* with regard to p. The same mechanism also works for Δ.

Moreover, as we mentioned in the appendix B, for threefolds we can have different ways to resolve singularities, related to each other by flops. In terms of toric geometries, it means that the rays in the fan have different ways to combine into two-dimensional cones. This naturally explains why we have different possible triangulations of Δ^*. Note that GW invariants are reshuffled under different choices of triangulations.

Example 1. *As an example, let's consider perhaps the most famous mirror pair of Calabi-Yau manifolds: the quintic hypersurface in $\mathbb{CP}^4$ and its mirror. The mirror manifold constructed by [75] lives in the space $\mathbb{CP}^4/\mathbb{Z}_5^3$, where $\mathbb{Z}_5^3$ is the group of automorphism*

$$(\lambda_1, \cdots, \lambda_5), \quad \prod_{i=1}^{5} \lambda_i = 1,\ \lambda_i^5 = 1,\ 1 \le i \le 5\,, \tag{3.2.3}$$

acting on $\mathbb{CP}^4$ by coordinate multiplication.

The mirror quintic is also a quintic hypersurface

$$\psi \prod_{i=1}^{5} x_i - \sum_{i=1}^{5} t_i x_i^5 = 0\,. \tag{3.2.4}$$

It's easy to verify that the equation is invariant under all the $\mathbb{Z}_5^3$ action so the mirror quintic is well-defined. As mentioned before, five of the parameters are redundant, so there is only one parameter ψ which parametrizes the moduli space of complex structure.

We start from the fan of $\mathbb{CP}^4$. The toric data can be presented as follows,

$$\left(\begin{array}{cccc|c} -1 & -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \tag{3.2.5}$$

We refer readers to appendix B for our convention. Here we just point out that the last column represents the single relation among five edges in a four dimensional space.

From our correspondence between polytopes and fans (theorem 4 in appendix B), we can find out the polytope Δ for $\mathbb{CP}^4$, represented schematically as the right figure in Figure 3.2. We use the anti-canonical divisor to obtain a projective embedding of $\mathbb{CP}^4$. The points on

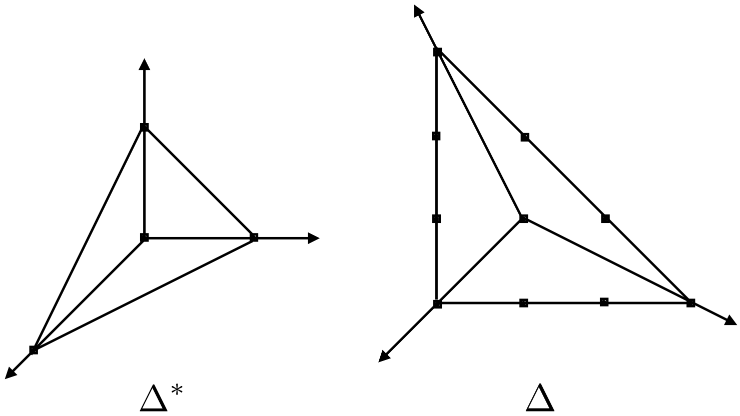


Figure 3.2: Dual polytopes and their face fans.

and inside Δ *are exactly all the possible quintic monomials. Summing them up, we get the quintic hypersurface. It's not difficult to see that this coincides with* Z_Δ.

Next, we take the dual polytope Δ^*, *schematically depicted as the left figure of Figure 3.2. Precisely due to its definition, it's the same as the convex set formed by the end points of the five edges of fan for* $\mathbb{CP}^4$. *Moreover, based on our theorem 5 in appendix B, we readily see that* $\mathbb{CP}_{\Delta^*}$ *must be an orbifold. The five end points exhaust all the integral points lying on* Δ^* [11], *together with a unique inner point which is the origin. As mentioned in section B.2 of the appendix B, the* $\mathbb{CP}_{\Delta^*}$ *itself can be represented as a hypersurface inside* $\mathbb{CP}^5$, *defined by*

$$\prod_{i=1}^{5} y_i = y_0^5 \,. \tag{3.2.6}$$

The variables Y_i *can be related to* y_i *by the map*

$$[1, Y_1, Y_2, Y_3, Y_4, \frac{1}{Y_1 Y_2 Y_3 Y_4}] = [1, \frac{y_1}{y_0}, \frac{y_2}{y_0}, \frac{y_3}{y_0}, \frac{y_4}{y_0}, \frac{y_5}{y_0}] \,. \tag{3.2.7}$$

According to the Batyrev construction, the mirror hypersurface Z_{Δ^*} *in* $\mathbb{CP}_{\Delta^*}$ *should take the following form,*

$$t_0 y_0 - \sum_{i=1}^{5} t_i y_i = 0. \tag{3.2.8}$$

This is precisely the mirror quintic, provided that we reparametrize the coordinates

$$y_0 = \prod_{i=1}^{5} x_i, \quad y_i = x_i^5, \;\; 1 \le i \le 5 \tag{3.2.9}$$

to identify $\mathbb{CP}_{\Delta^*}$ *with* $\mathbb{CP}^4/\mathbb{Z}_5^3$. *Plugging them into (3.2.8), we recover (3.2.4).*

3.2.2 Local Mirror Symmetry

Although originally mirror symmetry was formulated in terms of compact Calabi-Yau threefolds (hypersurfaces inside toric varieties), it was later extended to the non-compact

[11] This can be checked explicitly using computer software such as [170].

case, dubbed "local mirror symmetry" [39]. In this set-up, we consider X itself to be a three dimensional toric variety. As we will discuss shortly, this means that X must be non-compact. It can be studied from decompactifying a compact hypersurface, or start directly from its fan, which is the approach adopted here.

Let the set of 1-cones $\Sigma(1)$ of X be $\{\nu_\alpha\}$ for $\alpha = 1, \cdots, n_\Sigma + 3$. The Calabi-Yau condition implies that there are n_Σ vectors $\ell_\alpha^{(i)}$ such that

$$\sum_{\alpha=1}^{n_\Sigma+3} \ell_\alpha^{(i)} = \sum_{\alpha=1}^{n_\Sigma+3} \ell_\alpha^{(i)} \nu_\alpha = 0\,, \qquad i = 1, \ldots, n_\Sigma\,. \tag{3.2.10}$$

From the above constraints, some entries of $\ell_\alpha^{(i)}$ must be negative. They are responsible for the non-compactness of X [12]. Furthermore, we can rotate Σ in such a way that all the $\{\nu_\alpha\}$ lie on a hyperplane

$$v_\alpha = (1, m_\alpha, n_\alpha), \qquad \alpha = 1, \ldots, n_\Sigma + 3\,. \tag{3.2.11}$$

Hence it is enough to just write down its planar support, which is the convex set N_Σ with vertices given by

$$v'_\alpha = (m_\alpha, n_\alpha), \qquad \alpha = 1, \ldots, n_\Sigma + 3\,. \tag{3.2.12}$$

Base on the above information, we can write down its mirror Calabi-Yau manifold $\tilde{X}$, whose proof can be found in [104],

$$uv = W(x, y)\,, \tag{3.2.13}$$

for $u, v \in \mathbb{C}^*$, $x, y \in \mathbb{C}$, where

$$W(x, y) = \sum_{\alpha=1}^{n_\Sigma+3} a_\alpha e^{m_\alpha x + n_\alpha y}\,. \tag{3.2.14}$$

Note that the directions u and v only enter in the left-hand side and do not carry much geometric information. In fact, all the geometry can be reduced to the curve $\mathcal{C}_\Sigma$: $W(x, y) = 0$. $\mathcal{C}_\Sigma$ is also known as the *mirror curve*.

The moduli space of $\mathcal{C}_\Sigma$ is parametrized by the coefficients a_α, modulo the $\mathbb{C}^*$ actions on $\mathrm{e}^x, \mathrm{e}^y$ and the overall $\mathbb{C}^*$ rescaling. Furthermore, even after modulo this group action, not all the remaining coefficients correspond to the complex structure. It can be shown that only those corresponding to inner points of N_Σ contribute[13].

[12] This can be seen, e.g., from presenting X as the vacua of a gauged linear sigma model.
[13] This was in particular emphasized in [110, 111].

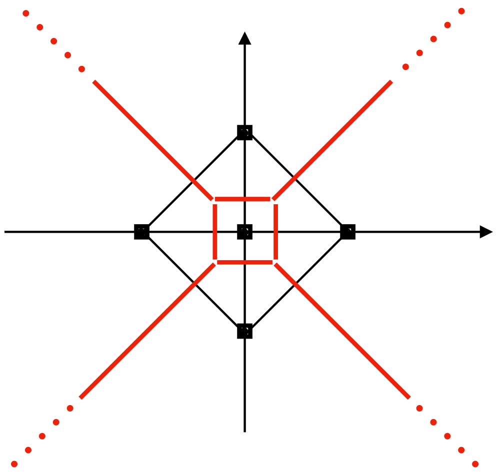


Figure 3.3: "Amoeba" of the mirror curve (3.2.16).

Example 2. *We choose our example to be the local Calabi-Yau threefold* $X = \mathcal{O}(K) \to \mathbb{F}_0$[14]*. The toric data of its fan is given by*

$$\left(\begin{array}{ccc|cc} 1 & 0 & 0 & -2 & -2 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 1 \end{array}\right) \tag{3.2.15}$$

Again, we refer to appendix B for our convention. Note that if we get rid of the first coordinate, we recover the fan for $\mathbb{F}_0$*. This just means that there exists a projective map* $\pi : X \to \mathbb{F}_0$*, as it should be.*

According to our general recipe, we can immediately write down the mirror curve C_Σ*,*

$$W(x, y) = e^x + me^{-x} + e^y + e^{-y} - u = 0\,, \tag{3.2.16}$$

where we use our freedom to set three of the coefficients in (3.2.14) to one. Moreover, since the parameter m *corresponds to a vertex on the boundary of* N_Σ*, it does not contribute to the true moduli. Only* u *parametrizes the moduli space of complex structures* $\mathcal{M}_{\mathcal{C}_\Sigma}$*. This also means that* $\mathcal{C}_\Sigma$ *is a genus one curve. Another way to see this is to consider its tropical limit, which is the shape in red in Figure 3.3.*

[14]Its toric diagram is Figure B.5 in the appendix B.

Chapter 4

Resurgence and Quantum Mirror Curve: A Case Study

Historically, the spectral problem for electrons on a two-dimensional square lattice in a uniform magnetic field was originally considered by Harper in 1955 [89], where an elegant difference equation was derived. More than 20 years later, in 1976 Hofstadter derived a recursive equation which allowed him to plot the spectrum as a function of the magnetic field, now known as the Hofstadter butterfly [100]. Due to the magnetic effect, the electron spectrum shows a rich structure. Recently, a novel link between a two-dimensional electron lattice system and a Calabi-Yau geometry was found in [93]. It was pointed out in [93] that the Hofstadter's spectral problem is related to another spectral problem appearing in the mirror geometry of the toric Calabi-Yau manifold known as local $\mathbb{F}_0$ [72]. The interesting point of this relation is that the magnetic effect is interpreted as a kind of *quantum deformation* of the Calabi-Yau geometry.

Let's briefly summarize the content of this chapter. Our central goal is a more quantitative understanding of this relation as well as of the non-perturbative and resurgent structure of the spectrum. We here focus on the band structure of the Harper-Hofstadter problem in the weak magnetic limit. In this regime, we can treat the magnetic flux perturbatively. The perturbative expansion of the energy spectrum can explain the position (the center) of the band for each Landau level. However, it does not explain the width of bands because the band width is *non-perturbative* in the weak magnetic flux limit. Such non-perturbative corrections are caused by quantum mechanical tunneling effects. We will demonstrate that the non-perturbative band width is explained by instanton effects in the path integral formalism. This was observed long ago in [63] (see also [178] for the WKB approach to the problem). Nevertheless, here we will focus on the resurgent properties intimately related to these instantons, or the multi-instanton contributions which we discuss in some details.

Moreover, we have a very efficient way to compute the perturbative expansion of the energy spectrum around the trivial saddle [167, 81], but this efficient way is not applicable for the computation of semi-classical expansion around the other nontrivial saddles. To our knowledge, there are no systematic ways to compute the semi-classical expansions around the instanton saddles in the Harper-Hofstadter model. We employ several approaches to extract this information. One is a brute force numerical approach, which we use as a check. The second is a path-integral approach, where we find the exact saddle of the path-integral action and the one-loop fluctuation. The instanton analysis is performed only to the leading-order in perturbation theory, and is not easily extended to perturbative corrections around the instanton saddles.

To extract higher corrections around instanton saddles, we propose a rather unconventional approach. We use the connection with a toric Calabi-Yau threefold, local $\mathbb{F}_0$, and find that the non-perturbative band width is captured by the free energy of the refined topological string on this geometry. Using this remarkable connection, we can efficiently compute the semi-classical fluctuation around the 1-instanton saddle by using a string theory technique, called the refined holomorphic anomaly equations [25, 135, 109]. Our approach here is conceptually very similar to the previous works [42] on certain quantum mechanical systems[1]. We would like to emphasize that here we have a realistic electron system where string theory techniques can be applied.

The structure of this chapter is as follows. In section 4.1 we quickly review the eigenvalue problem of the Harper-Hofstadter model and its exact solutions when the magnetic flux ϕ is 2π times a rational number. In section 4.2, we make a trans-series ansatz for the energy in the small ϕ limit. We then compute the leading order contribution in the 1-instanton sector for the ground state energy by a path integral calculation, and find that it agrees with the numerical results. In section 4.3, we perform further path integral calculations in the 2-instanton sector. The imaginary part of the instanton–anti-instanton sector is extracted numerically using the well-known relation to the large order growth of the perturbative energy. Inspired by [41, 42, 61], we also find in section 4.4 the fluctuations in the 1-instanton and instanton–anti-instanton sector can be computed from topological string on local $\mathbb{F}_0$.

This chapter is mostly based on the article *Instantons in the Hofstadter butterfly: difference equation, resurgence and quantum mirror curves* [55] by Jie Gu, Yasuyuki Hatsuda, Tin Sulejemanpasic and the author, with various changes for pedagogical reasons.

4.1 The Harper-Hofstadter problem

To prepare for the other sections, we quickly review in this section the classic results on the Harper-Hofstadter model [89, 100], including the formulation of its eigenvalue problem,

[1] In fact, the results in [42] correspond to the special case, the midpoint of each sub-band in our analysis.

and the exact solutions when the magnetic flux is 2π times a rational number. We make the careful distinction that there are two Bloch angles in this case while only one of them can be turned on if the value of the magnetic flux is generic.

4.1.1 The eigenvalue problem of the Harper-Hofstadter equation

The Harper-Hofstadter model describes an electron in a two dimensional lattice potential with a uniform magnetic flux in the perpendicular direction. Let the lattice spacing be a, and suppose the electron momentum has components k_x and k_y in the two directions. The energy of the electron before turning on the magnetic flux is, up to a normalization

$$E = -\frac{1}{2}(\mathrm{e}^{\mathrm{i}k_x a} + \mathrm{e}^{-\mathrm{i}k_x a} + \mathrm{e}^{\mathrm{i}k_y a} + \mathrm{e}^{-\mathrm{i}k_y a}) + 2\ . \tag{4.1.1}$$

We have chosen for later convenience a particular normalization so that the energy vanishes for zero electron momentum. In this convention, the energy forms a single band $0 \leq E \leq 4$.

After we turn on the magnetic flux, quantum mechanically we get the Hamiltonian operator by replacing the momentum $\vec{k}$ by the operator[2] $\vec{\pi} := \vec{p} - \vec{A}$. Notice that $\vec{p}$ is the *canonical momentum*. Upon the gauge transformation $\vec{A} \to \vec{A} + \vec{\nabla}\Lambda$, the Hamiltonian is only invariant up to a canonical transformation $\vec{p} \to \vec{p} + \vec{\nabla}\Lambda$. Under such a canonical transformation, the state of the Hilbert space transforms as $|\Psi\rangle \to e^{i\Lambda(x,y)}|\Psi\rangle$. Notice that the momentum $\vec{\pi}$ generally depends on the coordinates. Indeed this is reflected in the fact that the commutator

$$[\pi_x, \pi_y] = \mathrm{i}F_{xy}(x, y) \tag{4.1.2}$$

where $F_{xy}(x, y) = \partial_x A_y - \partial_y A_x$ is the xy component of the field-strength tensor of $\vec{A}$, i.e. the magnetic field through the xy-plane at the point (x, y). Henceforth, we consider the case where the magnetic field is uniform: $F_{xy}(x, y) = B$.

Replacing[3] $\mathsf{x} = \pi_x a, \mathsf{y} = \pi_y a$, we have that the lattice Hamiltonian becomes

$$H = -\frac{1}{2}(\mathrm{e}^{\mathrm{i}\mathsf{x}} + \mathrm{e}^{-\mathrm{i}\mathsf{x}} + \mathrm{e}^{\mathrm{i}\mathsf{y}} + \mathrm{e}^{-\mathrm{i}\mathsf{y}}) + 2\ . \tag{4.1.3}$$

with the commutation relation

$$[\mathsf{x}, \mathsf{y}] = \mathrm{i}\phi, \tag{4.1.4}$$

where $\phi = Ba^2$ is the flux of the magnetic field through the plaquette. We will also use the exponentiated notation

$$\mathsf{T_x} = \mathrm{e}^{\mathrm{i}\mathsf{x}}\ , \quad \mathsf{T_y} = \mathrm{e}^{-\mathrm{i}\mathsf{y}} \tag{4.1.5}$$

with the commutation relation

$$\mathsf{T_x}\mathsf{T_y} = \mathrm{e}^{\mathrm{i}\phi}\mathsf{T_y}\mathsf{T_x} \tag{4.1.6}$$

[2] We work in $\hbar = c = 1$ units.

[3] Despite the notation, x and y are not the original coordinates of the system, but are proportional to the magnetic translation operators.

so that the Hamiltonian can be written as

$$H = -\frac{1}{2}(\mathsf{T}_\mathsf{x} + \mathsf{T}_\mathsf{x}^{-1} + \mathsf{T}_\mathsf{y} + \mathsf{T}_\mathsf{y}^{-1}) + 2 \ . \tag{4.1.7}$$

We regard x and y as the canonical operators, and can now look at eigenstates $|\psi\rangle$ in the x-representation, i.e. define $\psi(x) = \langle x|\psi\rangle$ where $|x\rangle$ is an eigenstate of x with eigenvalue x, so that

$$H|\psi\rangle = E|\psi\rangle \Rightarrow -\frac{1}{2}\left(\psi(x+\phi) + \psi(x-\phi)\right) - \cos(x)\psi(x) = (E-2)\psi(x) \tag{4.1.8}$$

which is a difference equation.

4.1.2 Symmetries and θ-angles

The Hamiltonian (4.1.3) clearly commutes with the symmetry operators[4]

$$\tilde{\mathsf{T}}_\mathsf{y} = \mathrm{e}^{\mathrm{i}\frac{2\pi\mathsf{x}}{\phi}} \ , \tilde{\mathsf{T}}_\mathsf{x} = \mathrm{e}^{-\mathrm{i}\frac{2\pi\mathsf{y}}{\phi}} \ , \tag{4.1.9}$$

each of which generates a group $\mathbb{Z}$. The labelling above is because

$$\tilde{\mathsf{T}}_\mathsf{y}\, \mathsf{y}\, \tilde{\mathsf{T}}_\mathsf{y}^\dagger = \mathsf{y} - 2\pi \ , \qquad \tilde{\mathsf{T}}_\mathsf{x}\, \mathsf{x}\, \tilde{\mathsf{T}}_\mathsf{x}^\dagger = \mathsf{x} - 2\pi \ . \tag{4.1.10}$$

But we generally have

$$\tilde{\mathsf{T}}_\mathsf{x}\tilde{\mathsf{T}}_\mathsf{y} = \mathrm{e}^{-\mathrm{i}\frac{4\pi^2}{\phi}}\tilde{\mathsf{T}}_\mathsf{y}\tilde{\mathsf{T}}_\mathsf{x} \ . \tag{4.1.11}$$

Since for a generic value of $\phi \in \mathbb{R}$ the operators commute up to a phase, we can say that the physical symmetry group $\mathbb{Z} \times \mathbb{Z}$ acts projectively.

Let us first choose

$$\phi = 2\pi/Q \ , \quad Q \in \mathbb{Z} \ . \tag{4.1.12}$$

In this case the two operators commute, and the symmetry $\mathbb{Z} \times \mathbb{Z}$ is no longer acting projectively. Now we can project to simultaneous eigenstates of the operators $\tilde{\mathsf{T}}_\mathsf{x}$ and $\tilde{\mathsf{T}}_\mathsf{y}$, i.e. we can demand that

$$\tilde{\mathsf{T}}_\mathsf{x}|\Psi\rangle = \mathrm{e}^{\mathrm{i}\theta_x}|\Psi\rangle \ , \tilde{\mathsf{T}}_\mathsf{y}|\Psi\rangle = \mathrm{e}^{\mathrm{i}\theta_y}|\Psi\rangle \tag{4.1.13}$$

The angles θ_x and θ_y are Bloch angles for the x and y translations. Notice however that they can only be defined in this way if $2\pi/\phi \in \mathbb{Z}$.

Next, we consider the more general case that

$$\phi/(2\pi) = P/Q \in \mathbb{Q} \ , \tag{4.1.14}$$

[4] Similar operators also play an important role in the context of quantum mechanics associated with toric Calabi-Yau threefolds [93].

where P, Q are coprime integers. Then we have that

$$\tilde{\mathsf{T}}_{\mathsf{x}}\tilde{\mathsf{T}}_{\mathsf{y}} = \mathrm{e}^{-\mathrm{i}\frac{2\pi Q}{P}}\tilde{\mathsf{T}}_{\mathsf{y}}\tilde{\mathsf{T}}_{\mathsf{x}} \,. \tag{4.1.15}$$

Clearly, if $P \neq 1$, the generators $\tilde{\mathsf{T}}_{\mathsf{x}}, \tilde{\mathsf{T}}_{\mathsf{y}}$ must be supplemented by the generator[5] $\mathsf{I}_P = \mathrm{e}^{\mathrm{i}\frac{2\pi}{P}}$, and the $\mathbb{Z}\times\mathbb{Z}$ must be centrally extended by $\mathbb{Z}_P$.

What about θ-angles? In this case we have that $[(\tilde{\mathsf{T}}_{\mathsf{x}})^P, (\tilde{\mathsf{T}}_{\mathsf{y}})^P] = 0$, and we can define θ_x, θ_y angles by the simultaneous eigenstate of $(\tilde{\mathsf{T}}_{\mathsf{x}})^P$ and $(\tilde{\mathsf{T}}_{\mathsf{y}})^P$. Alternatively, in this case we also have $[(\tilde{\mathsf{T}}_{\mathsf{x}}), (\tilde{\mathsf{T}}_{\mathsf{y}})^P] = 0$, so we could equally define the two θ-angles as eigenstates of these two operators. Finally if $P = n^2$ is a perfect square, we have that $[(\tilde{\mathsf{T}}_{\mathsf{x}})^n, (\tilde{\mathsf{T}}_{\mathsf{y}})^n] = 0$ and we can define θ-angels accordingly as well. In most cases we will only consider $P = 1$. Then, all these definitions of θ-angles coincide, and we are back to the scenario (4.1.12).

Finally if $\phi/2\pi$ is irrational, then

$$\tilde{\mathsf{T}}_{\mathsf{x}}\tilde{\mathsf{T}}_{\mathsf{y}} = \mathrm{e}^{\mathrm{i}\alpha}\tilde{\mathsf{T}}_{\mathsf{y}}\tilde{\mathsf{T}}_{\mathsf{x}} \,, \tag{4.1.16}$$

where $\alpha/(2\pi) = -2\pi/\phi$ is irrational as well. The additional generator $\mathsf{I}_\alpha = \mathrm{e}^{\mathrm{i}\alpha}$ generates the group $\mathbb{Z}$, so the $\mathbb{Z}\times\mathbb{Z}$ is centrally extended by $\mathbb{Z}$. In this case we are allowed only one θ-angle, which we can get as an eigenstate of either $\tilde{\mathsf{T}}_{\mathsf{x}}$ or $\tilde{\mathsf{T}}_{\mathsf{y}}$ but not both simultaneously.

4.1.3 Exact solutions for rational magnetic flux

It is well-known that the eigenvalue problem (4.1.8) can be solved exactly if the rationality condition (4.1.14) is satisfied [100]. Let us set

$$\phi = 2\pi P/Q \tag{4.1.17}$$

where P, Q are two coprime integers and $Q > 0$. The underlying reason of the exact solvability is that in the case of (4.1.17) we can project onto simultaneous eigenstates of the powers $\tilde{\mathsf{T}}_{\mathsf{x}}^P$ and $\tilde{\mathsf{T}}_{\mathsf{y}}^P$, as these two operators commute. This will allow, as we shall see, for a finite-dimensional representation of the operators T_{x} and T_{y}, in which the Hamiltonian (4.1.7) is written, and give us an algebraic equation for the eigenvalue problem. Note that in this case $\mathsf{T}_{\mathsf{x}}, \mathsf{T}_{\mathsf{y}}$ are also shift operators, as

$$\mathsf{T}_{\mathsf{x}}\,\mathsf{y}\,\mathsf{T}_{\mathsf{x}}^\dagger = \mathsf{y} - 2\pi P/Q \,, \quad \mathsf{T}_{\mathsf{y}}\,\mathsf{x}\,\mathsf{T}_{\mathsf{y}}^\dagger = \mathsf{x} - 2\pi P/Q \,. \tag{4.1.18}$$

Recall that in this case we can define θ-angles as eigenvalues of $\tilde{\mathsf{T}}_{\mathsf{x}}^P = \mathsf{T}_{\mathsf{y}}^Q$ and $\tilde{\mathsf{T}}_{\mathsf{y}}^P = \mathsf{T}_{\mathsf{x}}^Q$. Now let us for the moment choose $\theta_x = \theta_y = 0 \bmod 2\pi$, i.e.

$$(\mathsf{T}_{\mathsf{x}}^{(0)})^Q = (\mathsf{T}_{\mathsf{y}}^{(0)})^Q = \mathbf{1} \,, \tag{4.1.19}$$

[5] I_P is equivalent to $\mathrm{e}^{-\frac{2\pi\mathrm{i}Q}{P}}$ because there always exists an integer k such that $\mathrm{e}^{-\frac{2\pi\mathrm{i}Q}{P}k} = \mathsf{I}_P$

In other words we impose periodic boundary conditions on physical states under the shift $\mathsf{x} \to \mathsf{x} - 2\pi P$ and $\mathsf{y} \to \mathsf{y} - 2\pi P$. The algebra (4.1.6), which now reads

$$\mathsf{T}_{\mathsf{x}}^{(0)}\mathsf{T}_{\mathsf{y}}^{(0)} = \mathrm{e}^{\frac{2\pi \mathrm{i} P}{Q}}\mathsf{T}_{\mathsf{y}}^{(0)}\mathsf{T}_{\mathsf{x}}^{(0)} \tag{4.1.20}$$

has a finite dimensional representation in terms of the clock and shift matrices

$$\mathsf{T}_{\mathsf{x}}^{(0)} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & q & 0 & \dots & 0 \\ 0 & 0 & q^2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & q^{b-1} \end{pmatrix}, \quad \mathsf{T}_{\mathsf{y}}^{(0)} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \tag{4.1.21}$$

where $q = \mathrm{e}^{\mathrm{i}\phi} = \mathrm{e}^{\frac{2\pi \mathrm{i} P}{Q}}$. Note also that $(\mathsf{T}_{\mathsf{x}}^{(0)})^Q = (\mathsf{T}_{\mathsf{y}}^{(0)})^Q = \mathbb{I}_{Q\times Q}$, as it should.

Now let us introduce the twisted boundary condition through the replacement $(\mathsf{T}_{\mathsf{x}}^{(0)}, \mathsf{T}_{\mathsf{y}}^{(0)}) \to (\mathsf{T}_{\mathsf{x}}, \mathsf{T}_{\mathsf{y}}) = (\mathsf{T}_{\mathsf{x}}^{(0)}\mathrm{e}^{\mathrm{i}\frac{\theta_x}{Q}}, \mathsf{T}_{\mathsf{y}}^{(0)}\mathrm{e}^{\mathrm{i}\frac{\theta_y}{Q}})$. Then we have that

$$(\mathsf{T}_{\mathsf{x}})^Q = \mathrm{e}^{\mathrm{i}\theta_x}\mathbf{1}, \quad (\mathsf{T}_{\mathsf{y}})^Q = \mathrm{e}^{\mathrm{i}\theta_y}\mathbf{1}, \tag{4.1.22}$$

while the algebra (4.1.6) is intact. Alternatively, the twisted boundary condition is equivalent to a deformation of the Hamiltonian. Using the notation $k_x = \theta_x/Q, k_y = \theta_y/Q$, we can write the Hamiltonian operator depending on k_x and k_y as

$$H(k_x, k_y) = -\frac{1}{2}(e^{\mathrm{i}k_x}\mathsf{T}_{\mathsf{x}}^{(0)} + e^{-\mathrm{i}k_x}\mathsf{T}_{\mathsf{x}}^{(0)-1} + e^{\mathrm{i}k_y}\mathsf{T}_{\mathsf{y}}^{(0)} + e^{-\mathrm{i}k_y}\mathsf{T}_{\mathsf{y}}^{(0)-1}) + 2, \tag{4.1.23}$$

while keeping the boundary condition periodic. Now we are finally ready to write the eigenvalue equation for the operator (4.1.3). Plugging the matrix representation of $\mathsf{T}_{\mathsf{x}}^{(0)}$ and $\mathsf{T}_{\mathsf{y}}^{(0)}$ into (4.1.23), the Hamiltonian becomes

$$\begin{bmatrix} 2-\cos(k_x) & -\frac{1}{2}\mathrm{e}^{-\mathrm{i}k_y} & 0 & \dots & 0 & -\frac{1}{2}\mathrm{e}^{\mathrm{i}k_y} \\ -\frac{1}{2}\mathrm{e}^{\mathrm{i}k_y} & 2-\cos\left(k_x + \frac{2\pi P}{Q}\right) & -\frac{1}{2}\mathrm{e}^{-\mathrm{i}k_y} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & & -\frac{1}{2}\mathrm{e}^{-\mathrm{i}k_y} \\ -\frac{1}{2}\mathrm{e}^{-\mathrm{i}k_y} & 0 & 0 & \dots & -\frac{1}{2}\mathrm{e}^{\mathrm{i}k_y} & 2-\cos\left(k_x + \frac{2\pi(Q-1)P}{Q}\right) \end{bmatrix} \tag{4.1.24}$$

so that the characteristic equation $\det(H - E\,\mathbb{I}_{Q\times Q}) = 0$ is given by

$$F_{P/Q}(E, k_x, k_y) = \det\begin{bmatrix} M_0 & -\mathrm{e}^{-\mathrm{i}k_y} & 0 & \dots & 0 & 0 & -\mathrm{e}^{\mathrm{i}k_y} \\ -\mathrm{e}^{\mathrm{i}k_y} & M_1 & -\mathrm{e}^{-\mathrm{i}k_y} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -\mathrm{e}^{\mathrm{i}k_x} & M_{Q-2} & -\mathrm{e}^{-\mathrm{i}k_y} \\ -\mathrm{e}^{-\mathrm{i}k_y} & 0 & 0 & \dots & 0 & -\mathrm{e}^{\mathrm{i}k_y} & M_{Q-1} \end{bmatrix} = 0, \tag{4.1.25}$$

with

$$M_n = 2(2-E) - 2\cos(2\pi nP/Q + k_x) \ . \tag{4.1.26}$$

As in [91], it is straightforward to check that

$$F_{P/Q}(E, k_x, k_y) = F_{P/Q}(E, k_x, 0) - 2\cos(Qk_y) + 2 \ . \tag{4.1.27}$$

Using the symmetry under the mapping $(k_x, k_y) \mapsto (k_y, -k_x,)$, one finds that the equation (4.1.25) can be simplified to

$$F_{P/Q}(E, 0, 0) + 4 = 2(\cos(\theta_x) + \cos(\theta_y)) \ , \tag{4.1.28}$$

with the Bloch angels $\theta_x = Qk_x, \theta_y = Qk_y$. It is then a simple job to get eigen-energy E by solving (4.1.28).

We notice that the equation (4.1.28) depends on the value of P only through the polynomial $F_{P/Q}(E, 0, 0)$. Note (4.1.28) indicates that the minimal ranges for the Bloch angles θ_x, θ_y are

$$0 \le \theta_x < 2\pi \ , \quad 0 \le \theta_y < 2\pi \ , \tag{4.1.29}$$

as they should. By varying the values of θ_x, θ_y, the eigen-energies $E(\theta_x, \theta_y)$ form bands. The two edges of a energy band correspond to $(\theta_x, \theta_y) = (0, 0), (\pi, \pi)$. If we turn off one Bloch angle, the energy band width is reduced to its one half. We reproduce in Figure 4.1 the famous plot of the Hofstadter butterfly, which is a plot of the energy bands as a function of the magnetic flux ϕ when $\phi/2\pi$ is rational.

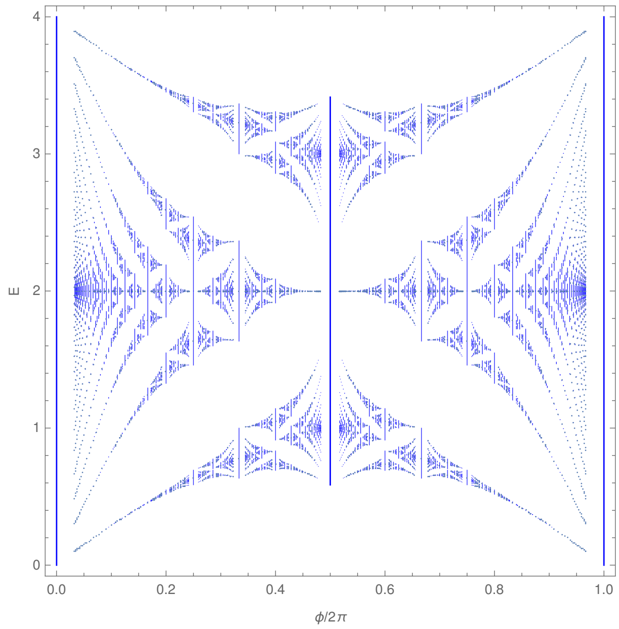


Figure 4.1: The Hofstadter butterfly plots energy levels $E_{k_x,k_y}(N, \phi)$ with $0 \le k_x, k_y \le 2\pi/Q$ against magnetic flux $\phi \in 2\pi\mathbf{Q}$ for the Harper-Hofstadter model. We take $\phi/2\pi$ to be P/Q for any coprime pairs of positive integers such that $P \le Q$ and $Q \le 30$.

4.2 Trans-series expansion and one-instanton sector

4.2.1 Why trans-series expansion?

We are interested in the energy spectrum of the Harper-Hofstadter model in the weak flux limit $\phi \to 0$. As discussed in section 4.1, with generic values of ϕ, we should use the Hamiltonian operator (4.1.3) for the twisted boundary condition with only one Bloch angle. Throughout this part, however, we consider the weak flux limit with the specific form

$$\phi = \frac{2\pi}{Q}, \qquad Q \to \infty, \tag{4.2.1}$$

for which we can introduce two distinct Bloch angles θ_x and θ_y simultaneously.

We want to understand the spectral behavior in the limit given in (4.2.1). To do so, it is useful to treat ϕ as a continuous parameter even in the specific case (4.2.1). Since the Hamiltonian is a Laurent polynomial of $\mathrm{e}^{\mathrm{i}\mathsf{x}}$ and $\mathrm{e}^{\mathrm{i}\mathsf{y}}$, we can use the `Mathematica` package `BenderWu` [167, 81] to compute its perturbative energy[6]. The first few orders are as follows

$$E^{\text{pert}}(N) = \frac{2N+1}{2}\phi - \frac{2N^2+2N+1}{16}\phi^2 + \frac{2N^3+3N^2+3N+1}{384}\phi^3 + \mathcal{O}(\phi^4) \;, \tag{4.2.2}$$

where N is the Landau level of the eigen-energy. We note the agreement with earlier studies [22, 64].

The perturbative energy (4.2.2) or even its Borel resummation cannot be the full answer. First of all, the higher order terms of the perturbative series have the same sign, and thus its Borel transform of the perturbative series has poles on the positive axis, leading to ambiguity in the Borel resummation. This ambiguity is an indication that the energy receives non-perturbative corrections. We discuss the ambiguity in detail in section 4.3.2. Second, the perturbative series clearly does not depend on Bloch angles, thus by itself cannot explain the energy bands. As a result, the band spectrum should have the trans-series expansion, with the explicit dependence of θ_x and θ_y in instanton sectors. The trans-series expansion of the spectrum should take the following form:

$$E_{(\theta_x,\theta_y)}(N) = E^{\text{pert}}(N) + E^{\text{1-inst}}_{(\theta_x,\theta_y)}(N) + E^{\text{2-inst}}_{(\theta_x,\theta_y)}(N) + \cdots \quad (\phi \to 0) \;. \tag{4.2.3}$$

The leading perturbative contribution is given by (4.2.2). The k-instanton sector is expo-

[6] The Hamiltonians considered in [81] consist of operators e^{x} and e^{y} with $[\mathsf{x},\mathsf{y}] = \mathrm{i}\hbar$. To translate it into our case here, one has to identify $\hbar = -\phi$. See subsection 4.4 for detail. This package can compute the perturbative spectrum for difference operators of the exponential-polynomial type. We have furthermore updated the `BenderWu` package version 2.2 with a function `BWDifferenceArray`, which allows of mixed inclusion of terms $e^x, e^p, e^{\mathrm{i}x}, e^{\mathrm{i}p}$. The package is available on Wolfram Package site.

nentially suppressed by a factor $e^{-kA/\phi}$ with a constant A[7]. The reason why it is called a trans-series is due to the presence of exponentially small terms. One may wonder why those terms are important, given that they are invisible in the semi-classical analysis. However, as we shall see below, they are crucial to understanding the non-perturbative phenomena.

In other words, our goal in this part is to reveal this trans-series structure in the spectrum. In particular, we will show explicit forms for a few instanton sectors. In the subsequent subsections, we will first compute the leading (1-loop) order contribution to the 1-instanton for the ground state energy by an honest path integral computation, and then compare them with the prediction from the numerical analysis. We further argue that the quantum fluctuations in the one-instanton sector for any energy level can be read off from the topological string theory on local $\mathbb{F}_0$. In the next section, we will investigate the 2-instanton sector.

4.2.2 Path integral in one-instanton sector

The problem of instantons in the Harper/Hofstadter problem was first discussed in [63] where the authors computed the one-instanton and its one-loop determinant numerically. Here we will re-derive these instanton solutions and compute analytically the one-loop fluctuations in the instanton sectors of the ground state energy.

To begin with, let's reproduce the Hamiltonian operator (4.1.3) for convenience

$$\mathsf{H}(\mathsf{x},\mathsf{y}) := -\cos\mathsf{x} - \cos\mathsf{y} + 2\ , \qquad [\mathsf{x},\mathsf{y}] = \mathrm{i}\phi\,. \tag{4.2.4}$$

The cosine potential has infinitely many degenerate vacua located at

$$x = 2\pi n_x\ ,\ y = 2\pi n_y\ , \quad n_x, n_y \in \mathbb{Z}\ . \tag{4.2.5}$$

Classically we have complete freedom of whether to identify different vacua as physically equivalent. This is not possible quantum mechanically for generic values of ϕ as we shall see.

Treating ϕ as the Planck constant, the above Hamiltonian can be associated with the Euclidean path integral

$$Z = \mathrm{tr}\ e^{-\frac{\beta}{\phi}\mathsf{H}(\mathsf{x},\mathsf{y})} = \int \mathcal{D}x\mathcal{D}y \exp\left[-\frac{1}{\phi}\int_{-\beta/2}^{\beta/2} \mathrm{d}t\,(H(x,y) - \mathrm{i}\dot{x}y)\right]\ . \tag{4.2.6}$$

with boundary conditions for x and y to be specified momentarily. The partition function above is related to the eigen-energies $E(N)$ with levels $N = 0, 1, 2, \ldots$ of the Hamiltonian H by

$$Z = \sum_{N=0}^{\infty} e^{-\beta E(N)/\phi}\ , \tag{4.2.7}$$

[7]More precisely, beyond the one-instanton order, in general logarithmic corrections of the form $\log^\ell \phi$ also appear. Therefore, the full trans-series expansion consists of three kinds of trans-monomials: ϕ, $e^{-A/\phi}$ and $\log\phi$.

so that the ground state energy $E(0)$ can be obtained through the Euclidean path integral in the large β limit.

Before we continue we should emphasize that the action of the above path integral is similar to that of the phase-space quantum mechanical system where x is identified with a coordinate, and y is identified with a momentum. The difference is that here we do not have a purely Gaussian dependence on the "momentum" y. For this reason we cannot integrate it out. Still one may hope to analyze the problem semi-classically. But there are several issues here. Firstly the semi-classics of path-integrals is to this day not a completely understood subject, but it has become clear recently that the correct interpretation of it is via the Picard-Lefschetz (PL) theory [187, 188, 88, 21, 20, 18, 17, 134, 19, 153]. The PL theory analysis is by far not a straightforward matter, and requires the identification of saddles which contribute in the semi-classical expansion. As we shall see all such saddles of the action above will be on complex x, y trajectories. We do not a priori know whether such saddles should contribute. To determine it we should compute the so-called intersection number of the co-thimble (we refer the reader to the cited literature for details). This is a difficult task way beyond our current understanding. We will find some instanton solutions and argue that they must contribute on physical grounds. We will check quantitatively their contribution against numerics and find exact agreement.

Secondly it is not clear whether a continuum limit of the above path-integral exists. The path-integral is typically obtained by slicing the Boltzmann weight into N pieces, and inserting a complete set of states in between. This amounts to a lattice discretization of the path-integral, with a lattice spacing $\epsilon = \beta/N$. Upon integration over the momentum, the resulting path-integral has a Gaussian suppression factors $\mathrm{e}^{-(\dots)\frac{(x_{i+1}-x_i)^2}{\epsilon}}$. As we take the continuum limit $\epsilon \to 0$ the path of x is forced to be smoother and smoother. No such smoothness seems to be justified in the continuum-limit of the phase-space path integral above. Still as we shall see the semiclassical analysis passes many non-trivial checks against the numerical brute-force calculation.

The boundary conditions of the path integral can be made strictly periodic. This amounts to saying that values of coordinates (x, y) and $(x + 2\pi n_x, y + 2\pi n_y)$ are physically distinct for any $n_x, n_y \in \mathbb{Z}$. In this case the above Lagrangian has a shift symmetry which takes $x \to x + 2\pi n_x$ and $y \to y + 2\pi n_y$, with $n_{x,y} \in \mathbb{Z}$.

Now let us consider the values of x and $x + 2\pi$ to be physically equivalent. In other words we are gauging the shift symmetry of the scenario above, projecting the full Hilbert space down to eigenstates of a shift symmetry operator. Without a θ_x-term, the projection will be to singlets of the shift operator. Gauging the symmetry amounts to saying that the boundary conditions must be relaxed to include periodicity of $x(t)$ up to a 2π shift, i.e. $x(t + \beta) = x(t) + 2\pi m_x$, where m_x is to be summed over. The integers m_x can be viewed as holonomies of the $\mathbb{Z}$-valued gauge field which we have to sum over in order to project to a subspace of singlets under the shift symmetry $x \to x + 2\pi$.

Notice however that after gauging the x-shift symmetry, shifting y to $y + 2\pi n_y$ we get an additional phase in the partition function

$$e^{\frac{i}{\phi}(2\pi)^2 n_y m_x} . \tag{4.2.8}$$

The above is only unity if $\phi = 2\pi/Q$, where $Q \in \mathbb{Z}$. Hence if we insist that $x \sim x + 2\pi$ (i.e. x-shift symmetry is gauged) and that $y \to y + 2\pi$ is a global symmetry we must have that[8] $\phi = 2\pi/Q$. This is of course evident from the Hilbert space picture, but it is satisfying to see it in the path-integral. Incidentally we can say that there is a 't Hooft anomaly between the two $(\mathbb{Z})_x$ and $(\mathbb{Z})_y$ shift symmetries, so that the system must break at least one of the two to saturate the anomaly.

Since we are assuming that $\phi = 2\pi/Q$, we can insert the two θ-angles by introducing the terms $\theta_y \frac{\dot{y}}{2\pi}$ and $\theta_x \frac{\dot{x}}{2\pi}$. The path integral can be treated by the saddle-point approximation if ϕ is small. The main contribution comes from the perturbative saddle for which $x = y = 0$ at any time t. This solution does not break the translational symmetry on the time-circle, and all its modes are Gaussian. The perturbative partition function can be expanded in powers of ϕ using the Feynman diagrams. The result will be the perturbative partition function which we denote as Z_0. In turn this is related to the perturbative energies as follows

$$Z_0 = \sum_{N=0}^{\infty} e^{-\beta E^{\text{pert}}(N)/\phi} . \tag{4.2.9}$$

where $E^{\text{pert}}(N)$ is the perturbative energy at level N.

On the other hand, the contributions of the partition function can be classified by their topological winding number, i.e.

$$Z(\beta, \theta) = Z_0 + Z_1 + Z_{-1} + Z_2 + Z_{-2} \cdots = Z_0 \left(1 + \sum_{n\neq 0}^{\infty} \hat{Z}_n\right) \ , \quad \hat{Z}_n = Z_n/Z_0 \ , \tag{4.2.10}$$

where Z_0 is the expansion around the trivial saddle point (i.e. the perturbative vacuum), and it is responsible for perturbative contributions $E^{\text{pert}}(0)$

$$Z_0 \approx C e^{-\beta E^{\text{pert}}(0)/\phi} \ , \qquad \beta \to \infty \ , \tag{4.2.11}$$

while $Z_{n\neq 0}$ come from different instanton sectors (n counts the instanton number). The constant C above may be UV divergent, and may be removed by the appropriate definition of the path integral measure. Further all Z_n-s are UV divergent. However all the UV divergences are the same, and so $\hat{Z}_n$ is UV finite. The constant C therefore factorizes, and is of no physical consequence as it cancels in the observables.

[8] From the point of view of the Hilbert space this means that if $x \to x + 2\pi$ is a gauge symmetry, the operator which shifts $\mathsf{y} \to \mathsf{y} + 2\pi$, given by $e^{i2\pi \mathsf{x}/\phi}$, is not a gauge invariant operator unless $2\pi/\phi \in \mathbb{Z}$, and even though it commutes with the Hamiltonian, it is not a valid generator of the symmetry transformation.

The *dilute instanton gas approximation* makes now the following assumption: the multi-instanton contributions factorize to 1-instanton contributions. So

$$\hat{Z}_n = \sum_{m-\bar{m}=n} \frac{\hat{Z}_1^m}{m!} \frac{\hat{Z}_{-1}^{\bar{m}}}{\bar{m}!} \ . \tag{4.2.12}$$

Summing over n we simply have

$$Z(\beta, \theta) \approx Z_{\text{dilute instanton gas}} = Z_0 \, \mathrm{e}^{\hat{Z}_1 + \hat{Z}_{-1}} \ . \tag{4.2.13}$$

Now the $\hat{Z}_{\pm 1}$ is given by

$$\hat{Z}_{\pm 1} = -\int_{-\beta/2}^{\beta/2} \mathrm{d}t \ K \mathrm{e}^{-A/\phi \pm \mathrm{i}\theta} = -\beta K \mathrm{e}^{-A/\phi \pm \mathrm{i}\theta} \tag{4.2.14}$$

where K is the measure of the 1-instanton configurations, including the perturbative corrections, and θ is the relevant θ-angle coupling to the instantons[9]. Therefore the 1-instanton correction to the ground state energy is given by

$$E_\theta^{\text{1-inst}}(0) = E_I + E_{\bar{I}} = 2\phi K \mathrm{e}^{-A/\phi} \cos\theta \ . \tag{4.2.15}$$

To get this correction we need to compute K.

Let us first consider the partition function $Z(\beta, \theta)$ in the trivial vacuum given by $x = y = 0$ by expanding in x and y up to quadratic terms and performing the Gaussian integral to get

$$Z_0(\beta) \approx \frac{1}{(\det \mathsf{O}_0)^{1/2}} \ , \tag{4.2.16}$$

with

$$\mathsf{O}_0 = -\partial_t^2 + 1 \ , \tag{4.2.17}$$

Now we consider the 1-instanton sector. For this purpose, we need to solve for the 1-instanton configuration. The equations of motion for the partition function (4.2.6) is

$$\mathrm{i}\dot{x} - \sin y = 0 \ , \tag{4.2.18a}$$

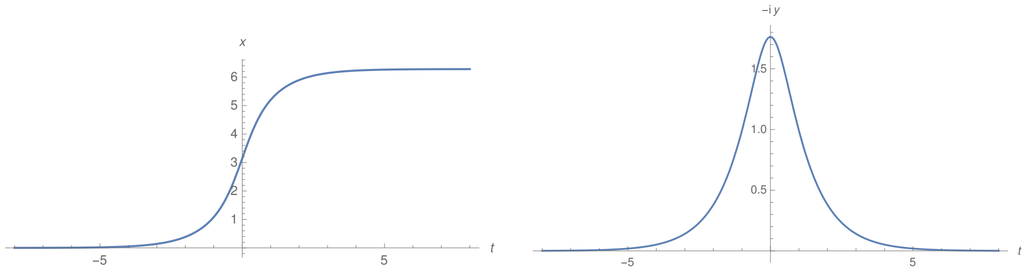
$$\mathrm{i}\dot{y} + \sin x = 0 \ . \tag{4.2.18b}$$

We solve these equations in the appendix C.1 to give the 1-instanton solution

$$x_1(t) = 2\cos^{-1}\left(-\frac{\sqrt{2}\tanh(t-t_0)}{\sqrt{1+\tanh^2(t-t_0)}} \right) \ , \quad y_1(t) = \cos^{-1}\left(1 + \frac{2}{\cosh 2(t-t_0)} \right) \ , \tag{4.2.19}$$

where t_0 is a free parameter interpreted as the center of the instanton. Note that $x_1(t)$ starts from 0 in $t = -\infty$ and reaches 2π in $t = +\infty$, and thus it indeed has topological

[9] In the Harper-Hofstadter problem we will have two types of instantons which tunnel in x- and y-directions respectively. So we may have two θ-angles: $\theta = \theta_x$ or $\theta = \theta_y$ coupling to the tunneling events $x \to x + 2\pi$ and $y \to y + 2\pi$. Recall that these θ angles can only be defined when $2\pi/\phi \in \mathbb{Z}$.



(a) x-profile **(b)** y-profile

Figure 4.2: The x- and y-profiles of 1-instanton in the Harper-Hofstadter model. The value of y is purely imaginary.

charge 1, while $y_1(t)$ is always imaginary and its imaginary value reaches the maximum $\cos^{-1}(3)$ at $t = t_0$. This means that we are considering the instanton tunneling in the x-direction. We call it an x-instanton. We plot $x_1(t)$ and $-\mathrm{i}y_1(t)$ in Figure 4.2. The profile of an anti-instanton is obtained by simply the time-reversal transformation[10]. We also notice that the Hamiltonian is constant

$$\cos y_1 + \cos x_1 = 2 \ , \tag{4.2.20}$$

as it should be, with the help of the e.o.m. (4.2.18a), which will be of use later.

There exists in fact another type of 1-instanton due to the fact that the Hamiltonian function is also periodic in y. In the example of the Harper-Hofstadter model, one can easily find the new instanton due to the symmetry of the theory under the map

$$(x(t), y(t)) \to (-y(t), x(t)) \ . \tag{4.2.21}$$

Applying this map to the instanton solution (4.2.19), we get a new instanton solution with the x- and y-profiles exchanged (up to a minus sign). We call it a y-instanton, since it has a non-trivial topological charge in the y-direction, but a trivial topological charge in the x-direction. This instanton does not couple to θ_x. Instead it couples to the θ_y-angle.[11]

Let us compute the action of the 1-instanton configuration (4.2.19), in the limit $\beta \to \infty$. The action of the instanton is computed analytically in appendix C.1 and it reads

$$A = 8\mathrm{C} \ , \tag{4.2.22}$$

where C is the Catalan's constant.

Now we compute the one-loop partition function in the 1-instanton sector, by performing the expansion

$$x = x_1 + \delta x \ , \quad y = y_1 + \delta y \ , \tag{4.2.23}$$

[10]The time reversal transformation takes $T : (x(t), y(t)) \to (x(-t), -y(-t))$. In addition we have a parity transformation which takes $P : (x(t), y(t)) \to (-x(t), -y(t))$.

[11]We remind the reader that both θ_x and θ_y are only possible if the $2\pi/\phi \in \mathbb{Z}$, which we assume here. However much of the results will hold for generic ϕ, as we shall comment later.

and keeping only terms up to quadratic orders. Using the conservation law (4.2.20) as well as the e.o.m. (4.2.18), we have

$$Z_1(\beta) \approx \mathrm{e}^{-A/\phi+\mathrm{i}\,\theta} \int \mathcal{D}(\delta x)\mathcal{D}(\delta y) \exp\left[-\frac{1}{2\phi}\int_{-\beta/2}^{\beta/2} \mathrm{d}t \left(\cos x_1 \cdot \delta x^2 + \cos y_1 \cdot \delta y^2 - 2\mathrm{i}\,\delta\dot{x}\delta y\right)\right] . \tag{4.2.24}$$

We can first integrate out δy. However notice that in doing so we will get a nontrivial factor in front of the path-integral, because the coefficient of δy^2 is not a constant. To avoid this, let us first replace $\delta\tilde{y} = \sqrt{\cos y_1}\delta y$ and $\delta\tilde{x} = \delta x/\sqrt{\cos y_1}$.[12] Notice that this replacement keeps the measure invariant i.e. $\mathcal{D}(\delta\tilde{x})\mathcal{D}(\delta\tilde{y}) = \mathcal{D}(\delta x)\mathcal{D}(\delta y)$. Upon integrating out the $\delta\tilde{y}$, we get

$$\begin{aligned} Z_1(\beta) \approx & \mathrm{e}^{-A/\phi+\mathrm{i}\theta} \\ & \times \int \mathcal{D}(\delta\tilde{x}) \exp\left[-\frac{1}{2\phi}\int_{-\beta/2}^{\beta/2} \mathrm{d}t \left(\frac{\left[\partial_t\left(\delta\tilde{x}\sqrt{\cos y_1}\right)\right]^2}{\cos y_1} + \cos x_1 \cos y_1 \delta\tilde{x}^2\right)\right] \\ = & \frac{\mathrm{e}^{-A/\phi+\mathrm{i}\,\theta}}{\sqrt{\det \tilde{\mathsf{O}}}} \end{aligned} \tag{4.2.25}$$

where the operator $\tilde{\mathsf{O}}$ is

$$\tilde{\mathsf{O}} = -\sqrt{\cos y_1}\partial_t \frac{1}{\cos y_1}\, \partial_t\sqrt{\cos y_1} + \cos y_1 \cos x_1 \; . \tag{4.2.26}$$

The operator $\tilde{\mathsf{O}}$ has a zero mode given by $\psi_0(t) = N^{-1}\frac{\dot{x}_1(t)}{\sqrt{\cos y_1}}$, as can be checked. Here

$$N = \sqrt{(\dot{x}_1/\sqrt{\cos y_1}, \dot{x}_1/\sqrt{\cos y_1})} \tag{4.2.27}$$

is the normalization factor. So the above expression of the one loop weight of the instanton cannot be correct. The zero mode originates from the time-translation symmetry of the theory. In other words, field fluctuations which only change the location of the instanton do not change the action, and the modes in this direction must be treated exactly (i.e. beyond the Gaussian approximation).

To find the measure of the instanton we must first separate out the zero mode, which we denote by t_0. We will get that

$$Z_1 = \mathrm{e}^{-A/\phi+\mathrm{i}\theta} \int \mathrm{d}t_0 \; \frac{\mu}{\sqrt{\det' \tilde{\mathsf{O}}}} \; , \tag{4.2.28}$$

where the prime indicates that the zero mode has been excluded from the determinant. The μ above is the measure of the instanton moduli t_0 (or the moduli space metric). It is given by (see appendix C.2)

$$\mu = \sqrt{\frac{N^2}{2\pi\phi}} \; , \tag{4.2.29}$$

[12] Note that $\cos y_1 > 0$, because y_1 is purely imaginary on the instanton trajectory.

so that the one-loop instanton contribution to the partition function is given by

$$Z_1(\beta) \approx \int \frac{\mathrm{d}t_0}{\sqrt{2\pi\phi}} \sqrt{(\dot{x}_1/\sqrt{\cos y_1(t)}, \dot{x}_1/\sqrt{\cos y_1(t)})} \frac{\mathrm{e}^{-A/\phi+\mathrm{i}\,\theta}}{(\det' \mathsf{O})^{1/2}} \,. \tag{4.2.30}$$

The contribution is of course divergent, as the functional determinant is infinite in the continuum. We therefore normalize it with respect to the perturbative partition function. The normalized 1-instanton partition function is given by

$$\hat{Z}_1(\beta) = \frac{Z_1(\beta)}{Z_0(\beta)} \approx \mathrm{e}^{-A/\phi+\mathrm{i}\,\theta} \frac{\beta\sqrt{(\dot{x}_1/\sqrt{\cos y_1}, \dot{x}_1/\sqrt{\cos y_1})}}{\sqrt{2\pi\phi}} \left(\frac{\det \mathsf{O}_0}{\det' \tilde{\mathsf{O}}}\right)^{1/2} \,. \tag{4.2.31}$$

Comparing with (4.2.14), we find that the prefactor K entering formula (4.2.15) is given by[13]

$$K = \frac{\sqrt{(\dot{x}_1/\sqrt{\cos y_1}, \dot{x}_1/\sqrt{\cos y_1})}}{\sqrt{2\pi\phi}} \left(\frac{\det \mathsf{O}_0}{\det' \tilde{\mathsf{O}}}\right)^{1/2} \,. \tag{4.2.32}$$

As we show in the appendix C.3, the ratio of determinants is given by

$$\frac{\det' \tilde{\mathsf{O}}}{\det \mathsf{O}_0} = \frac{\dot{x}_1(-\beta/2)\dot{x}_1(\beta/2)}{\sinh\beta\cos y_1(-\beta/2)} \int_{-\beta/2}^{\beta/2} \mathrm{d}t\, \frac{\dot{x}_1^2(t)}{\cos y_1(t)} \int_{-\beta/2}^{t} \mathrm{d}t' \frac{\cos y_1(t')}{\dot{x}_1^2(t')} \int_{t}^{\beta/2} \mathrm{d}t'' \frac{\cos y_1(t'')}{\dot{x}_1^2(t'')} \,. \tag{4.2.33}$$

Note that in obtaining the above result, we have used Dirichlet boundary conditions for the space of function acted on by the operators O and O_0. Since we will only be interested in the limit $\beta \to \infty$, the boundary conditions will not matter. However if one is interested in computing the instanton contributions to higher energy levels, a computation with periodic boundary conditions is necessary.

Further since we only care about the limit $\beta \to \infty$, we can make convenient approximations. We notice that the 1-instanton configuration (C.1.6), (C.1.7) has the following asymptotic form

$$\dot{x}_1(t) \sim A_\pm \mathrm{e}^{\mp\omega t} \,, \ \cos y_1(t) \sim 1 + B_\pm \mathrm{e}^{\mp 2\omega t} \,, \quad t \to \pm\infty \,, \tag{4.2.34}$$

where

$$A_\pm = 2\sqrt{2} \,, \ B_\pm = 4 \,, \ \omega = 1 \,. \tag{4.2.35}$$

Besides, the integrand of the integral over t is small when t is close to $\pm\beta/2$, so the integral over t is saturated away from them. So regarding the two integrals over t' and t'', only the $-\frac{\beta}{2} \ll t \ll \frac{\beta}{2}$ region is important. The two integrals can be approximated by

$$\begin{aligned} &\int_{-\beta/2}^{t} \mathrm{d}t' \frac{\cos y_1(t')}{\dot{x}_1^2(t')} \sim \int_{-\beta/2}^{t} \mathrm{d}t' \frac{\mathrm{e}^{-2\omega t'}}{A_-^2} \sim \frac{\mathrm{e}^{\omega\beta}}{2\omega A_-^2} \,, \\ &\int_{t}^{\beta/2} \mathrm{d}t'' \frac{\cos y_1(t'')}{\dot{x}_1^2(t'')} \sim \int_{t}^{\beta/2} \mathrm{d}t'' \frac{\mathrm{e}^{2\omega t''}}{A_+^2} \sim \frac{\mathrm{e}^{\omega\beta}}{2\omega A_+^2} \,. \end{aligned} \tag{4.2.36}$$

[13] A possible minus sign can be absorbed into the θ angle

Pulling these two integrals out of the integral of t, the latter becomes $(\dot{x}_1/\sqrt{\cos y_1}, \dot{x}_1/\sqrt{\cos y_1})$. Apply (4.2.34) in the remaining part of the determinant evaluation, we find in the end

$$\frac{\det' \tilde{\mathsf{O}}}{\det \mathsf{O}_0} = \frac{(\dot{x}_1/\sqrt{\cos y_1}, \dot{x}_1/\sqrt{\cos y_1})}{16} \ . \tag{4.2.37}$$

we get that the factor K in (4.2.32) is given by

$$K = 4\left(\frac{1}{2\pi\phi}\right)^{1/2} . \tag{4.2.38}$$

The anti-instanton partition function is the same but with the opposite topological charge. Therefore using (4.2.15), the leading order 1-instanton correction to the ground state energy given by x-instanton coupled to θ_x is

$$E_{I_x}(0) + E_{\bar{I}_x}(0) = 8\cos\theta_x \left(\frac{\phi}{2\pi}\right)^{1/2} \mathrm{e}^{-A/\phi} \ . \tag{4.2.39}$$

Since we have two kinds of instantons coupled to θ_x and θ_y respectively, the full 1-instanton correction is finally given by

$$\begin{aligned} E^{\text{1-inst}}_{(\theta_x,\theta_y)}(0) &= E_{I_x}(0) + E_{\bar{I}_x}(0) + E_{I_y}(0) + E_{\bar{I}_y}(0) \\ &= 8(\cos\theta_x + \cos\theta_y)\left(\frac{\phi}{2\pi}\right)^{1/2} \mathrm{e}^{-A/\phi} \ . \end{aligned} \tag{4.2.40}$$

We can check that it indeed agrees with the numerical results [55].

4.3 Two-instanton sector

4.3.1 Two-instanton calculation

Now we wish to go beyond the dilute instanton gas approximation, and compute the contributions of the two-instanton sector to the leading order in semi-classics. Recall that we have two types of instantons, which we will call I_x and I_y, where I_x is a tunneling event in x, i.e. it takes $x \to x + 2\pi$, while I_y is a tunneling event in $y \to y + 2\pi$.

We will consider all kinds of two-instanton events, ranging from "pure" correlations

$$\begin{aligned} &[I_x\bar{I}_x], [\bar{I}_x I_x], [I_y\bar{I}_y], [\bar{I}_y I_y], \\ &[I_x I_x], [I_y I_y], [\bar{I}_x\bar{I}_x], [\bar{I}_y\bar{I}_y], \end{aligned} \tag{4.3.1}$$

to "mixed" ones

$$\begin{aligned} &[I_x I_y], [I_y I_x], [\bar{I}_x\bar{I}_y], [\bar{I}_y\bar{I}_x], \\ &[\bar{I}_x I_y], [I_y\bar{I}_x], [I_x\bar{I}_y], [\bar{I}_y I_x] \ . \end{aligned} \tag{4.3.2}$$

Before computing their interactions, we should stress that the contribution of such events has long been subject to debates. Particularly tricky is the instanton–anti-instanton contribution $[I\bar{I}]$, which is a priori ill-defined. This is because when instanton and anti-instanton are close to each other the configuration is indistinguishable from the perturbative vacuum, and it is not clear how such configurations should be taken into account (see [19] for an incomplete list of references on the topic).

If we naively superpose the well-separated instanton and anti-instanton, where we label their separation by τ, the action will be an increasing function of τ. Such a configuration spends most of the time in one of the vacua (say $x = 0$) and then tunnels to the other vacuum ($x = 2\pi$), lingering there for the time τ, and then returns back to the original ($x = 0$) vacuum. The action of such a configuration is approximately

$$S_2 \approx 2A + B\,\mathrm{e}^{-\tau} \tag{4.3.3}$$

where the exponential contribution is the "classical" interaction[14] of the instanton–anti-instanton pair. The contribution of such a class of configurations to the partition function would then be[15]

$$\int \mathrm{d}t_0 \int \mathrm{d}\tau K^2\,\mathrm{e}^{-\frac{2A}{\phi}} \left(\mathrm{e}^{-\frac{B}{\phi}e^{-\tau}} - 1\right) , \tag{4.3.4}$$

where ϕ is the coupling constant, t_0 is the "center of mass" location of the pair, and K is the one-loop measure of the individual (anti-)instantons. The integral over t_0 will simply produce one power of β, while the rest of the expression will be related to the $I\bar{I}$ contribution to the energy. The integral over their separation is, however, an awkward operation. As can be seen from path integral calculations [55], the interaction constant B is negative, so the integral is saturated by its lower limit $\tau \sim 0$, where the approximations of the above expression are invalid, and where the notion of the instanton–anti-instanton is ill defined.

Bogomolny [29] and Zinn-Justin [192, 195] argued long ago that the ill-defined $I\bar{I}$ amplitude is connected with the ambiguity of the Borel sum of the perturbation theory. They correctly argued that the definition of the $I\bar{I}$ amplitude must be ambiguous in the same way that the perturbation theory is. A prescription which is now dubbed the Bogomolny–Zinn-Justin (BZJ) prescription, is to take the coupling ϕ to be negative, so that the above integral is saturated away from $\tau \sim \log(1/\phi) \gg 1$, where the approximations are valid. The above integral over τ is then performed to produce a correction to the energy

$$E_0^{I\bar{I}} = \phi K^2 \mathrm{e}^{-2A/\phi}(-\gamma_E - \log(B/\phi) - \Gamma(0, B/\phi)) , \tag{4.3.5}$$

[14] The term "classical" is used to reflect the $1/\phi$ dependence of the interaction, but it is a bit of a misnomer, because an instanton–anti-instanton event is in fact a large-quantum fluctuation, and is in no way classical.

[15] The subtracted unity is to control the IR divergence due to the uncorrelated instantons. Since uncorrelated instantons have already been taken into account by the instanton gas approximation it should be subtracted here to avoid double counting.

where γ_E is the Euler's constant, and $\Gamma(\bullet, \bullet)$ the incomplete gamma function. The last term is exponentially small when $\phi < 0$ so it is normally dropped. Further the expression is ambiguous if we now send ϕ from negative to positive values in the upper or lower complex half-plane, because of the appearance of the log. Moreover the ambiguity is exactly canceled by the ambiguity in the Borel sum of the perturbation theory. This was one of the great successes of resurgence in quantum mechanics and our understanding of its relationship with path-integrals.

The BZJ prescription, however revolutionary, causes some unease. Perhaps the most uncomfortable aspect is that it requires dropping a factor which is exponentially small when $\phi < 0$, but becomes exponentially large when the correct limit $\phi > 0$ is taken. In recent years it became increasingly evident that at the heart of the correct interpretation of the BZJ result is the Picard-Lefschetz theory – a generalization of the steepest decent method to multi-integral (or indeed path-integral) cases. In fact it was only recently that a resolution of this puzzle was proposed by the interpretation of the instanton–anti-instanton pair as a saddle point at infinity [19], which establishes a concrete method for a systematic calculation of the semi-classical expansion in path integrals. The procedure is roughly as follows (We refer the reader to [19] for details.):

1) Consider an instanton–anti-instanton configuration for the case of finite time β.

2) Note that if the instanton and the anti-instanton are at opposite ends of a temporal circle, the configuration becomes a saddle point. Since the action can be decreased by bringing the pair closer together, the saddle point in question is "unstable".

3) Treat the saddle point with Picard-Lefschetz theory, i.e. instead of integrating over a cycle of real instanton–anti-instanton separation, replace the cycle with the Lefschetz thimble integral (i.e. the "steepest decent cycle"), along which the action is monotonically increasing.

4) Note that the imaginary part of the thimble integral is ambiguous depending on whether Im ϕ is greater or smaller than zero, and that the ambiguity cancels the Borel sum ambiguity of the path-integral, while the real part is identical to the BZJ result above, provided that we drop the incomplete-gamma term.

In particular the ambiguity, which comes from the imaginary part, is given by

$$\text{Im}\, E_0^{I_x \bar{I}_x} = \pm\pi\phi\, K^2\, e^{-2A/\phi} = \pm 8\, \mathrm{e}^{-2A/\phi} \tag{4.3.6}$$

where we used our result (4.2.38). We would like to point out that the ambiguity does not contain the interaction term for the ground-state energy, i.e. it is independent of the constant B which parametrizes the instanton–anti-instanton interactions. This is in fact clarified by the thimble integration procedure in [19], summarized above. The ambiguity comes from the vicinity of the critical point at infinity, which, for a finite temporal extent, is the instanton–anti-instanton pair at opposing ends of the temporal circle. Since the saddle is "unstable" with regards to the perturbations in the real field space, the proper thimble integration will force us to integrate along the direction of imaginary separation[16], inducing an imaginary factor in the result. This is the ambiguity, and in this case it is saturated in the vicinity of the $I\bar{I}$ saddle. When we take $\beta \to \infty$, this vicinity of the $I\bar{I}$ saddle moves to infinity, where the instanton and anti-instanton are decorrelated, and all dependence on the interactions vanishes.

4.3.2 Large order growth and ambiguity of energy

According to the resurgence theory, the large order growth of the perturbative energy expansion is controlled by the ambiguity (imaginary part) of energy, which receives contributions from instanton sectors with topological charge zero (see for instance [54]). The first such sector is the instanton–anti-instanton sector $[I\bar{I}]$ including all four events listed in the first line of (4.3.1). The imaginary energy correction from this sector is

$$\mathrm{Im}E^{I\bar{I}}(N,\phi) = \pm\, \mathrm{e}^{-2A/\phi}(S_{(N)}/2)\cdot \phi^{b_N} \sum_{n=0}^{\infty} a_n^{(1,1)}(N)\phi^n$$

where $S_{(N)}$ is the Stoke's constant, related to the ambiguity of the lateral Borel resummation of the perturbative expansion, and b_N is the leading exponent of ϕ in the instanton–anti-instanton sector. Let us denote the perturbative expansion by

$$E^{\mathrm{pert}}(N) = \sum_{n=1}^{\infty} a_n^{(0)}(N)\phi^n \tag{4.3.7}$$

The resurgent analysis then suggests the following relation

$$a_n^{(0)}(N) = \frac{S_{(N)}}{2\pi}\frac{(n-b_N-1)!}{(2A)^{n-b_N}}\left(1+\frac{a_1^{(1,1)}(N)2A}{n-b_N-1}+\frac{a_2^{(1,1)}(N)(2A)^2}{(n-b_N-1)(n-b_N-2)}+\cdots\right). \tag{4.3.8}$$

We will use this relation to compute numerically the imaginary part of $E^{I\bar{I}}$.

We start with the ground state with $N=0$. We compute $a_n^{(0)}$ up to $n=320$ using the `BenderWu` package. With the help of (4.3.8), we found that

$$b_0 = 0\ , \tag{4.3.9}$$

and we also extracted the following numerical values of A and $S_{(0)}$

$$2A^{\mathrm{num}} = 14.6554495068355\ldots\ , \qquad S_{(0)}^{\mathrm{num}} = 63.9999999999999\ldots\ . \tag{4.3.10}$$

[16]The contour $I\bar{I}$ separation parameter τ along the thimble eventually bends and becomes parallel to the real axis in the complex τ-plane, which gives the real contribution.

In this process, it is convenient to use the Richardson transformation to accelerate the convergence (see for instance [146] for details). It is easy to check that these numerical estimations reproduce the exact values

$$2A = 16\mathrm{C}, \qquad S_{(0)} = 64 \ , \tag{4.3.11}$$

so that in the leading order, we have

$$\mathrm{Im}E^{I\bar{I}}(0,\phi) = \pm 32\mathrm{e}^{-16\mathrm{C}/\phi} \ , \tag{4.3.12}$$

which can be checked to agree with the path integral calculation.

As in the 1-instanton sector, once the analytic values of $S_{(0)}$ and A are fixed, numerically we can go beyond the leading order and further extract the values of $a_n^{(1,1)}(0)$ using (4.3.8). For instance, we find

$$\begin{gathered} a_1^{(1,1)}(0) = -\frac{13}{48} \ , \quad a_2^{(1,1)}(0) = \frac{115}{4608} \ , \\ a_3^{(1,1)}(0) = -\frac{12209}{3317760} \ , \quad a_4^{(1,1)}(0) = -\frac{355687}{637009920} \ , \cdots \end{gathered} \tag{4.3.13}$$

These coefficients should give the perturbative fluctuation around the instanton–anti-instanton saddle.

We repeat the same computation for higher energy levels. Observing the general structure (4.3.8), we find that

$$b_N = -2N \ , \quad S_{(N)} = \frac{2^{8N+6}}{(N!)^2} \ . \tag{4.3.14}$$

In addition, we extract the coefficients $a_n^{(1,1)}(N)$ for various energy levels N and fit them as functions of N. As a result, we find the fluctuation around the $[I\bar{I}]$ saddle point to be

$$\begin{aligned} \log \mathcal{P}^{I\bar{I}}_{\text{fluc}} :=& \log\left(\sum_{n=0}^{\infty} a_n^{(1,1)}(N)\phi^n\right) \\ =& -\frac{6N^2+18N+13}{48}\phi - \frac{20N^3+66N^2+100N+27}{2304}\phi^2 \\ & - \frac{210N^4+900N^3+2190N^2+1980N+653}{184320}\phi^3 + \mathcal{O}(\phi^4). \end{aligned} \tag{4.3.15}$$

From these data, we could construct the $[I\bar{I}]$ contribution to the imaginary part of the eigen-energy

$$\mathrm{Im}E^{I\bar{I}}(N,\phi) = \pm\,\mathsf{i}\,\mathrm{e}^{-2A/\phi}(S_{(N)}/2)\cdot\phi^{b_N}\sum_{n=0}^{\infty}a_n^{(1,1)}(N)\phi^n = \ \pm\mathsf{i}\,\mathrm{e}^{-2A/\phi}\frac{2^{8N+5}}{(N!)^2}\phi^{-2N}\cdot\mathcal{P}^{I\bar{I}}_{\text{fluc}} \ .$$

Before we conclude this section, we point out that there is an interesting empirical relation between $\mathcal{P}^{I\bar{I}}_{\text{fluc}}$ and $\mathcal{P}^{\text{1-inst}}_{\text{fluc}}$

$$\frac{\mathcal{P}^{I\bar{I}}_{\text{fluc}}}{(\mathcal{P}^{\text{1-inst}}_{\text{fluc}})^2} = \left(\frac{1}{\phi}\frac{\partial E^{\text{pert}}}{\partial N}\right)^{-1} \ . \tag{4.3.16}$$

which indicates that we can can cast the 1-instanton fluctuation and $[I\bar{I}]$ fluctuation as

$$\mathcal{P}^{\text{1-inst}}_{\text{fluc}} = \frac{1}{\phi}\frac{\partial E^{\text{pert}}(N)}{\partial N} e^{-\mathcal{A}(N,\phi)} \, , \tag{4.3.17}$$

$$\mathcal{P}^{I\bar{I}}_{\text{fluc}} = \frac{1}{\phi}\frac{\partial E^{\text{pert}}(N)}{\partial N} e^{-2\mathcal{A}(N,\phi)} \, . \tag{4.3.18}$$

where the function $\mathcal{A}(N,\phi)$ is nothing else but the "non-perturbative" A-function appearing in the Zinn-Justin–Jentschura exact quantization conditions [193, 194] in conventional quantum mechanics. In our example, the first few terms of $\mathcal{A}(N,\phi)$ read

$$\begin{aligned} \mathcal{A}(N,\phi) = &\left(\frac{\nu^2}{16}+\frac{11}{192}\right)\phi + \left(\frac{5\nu^3}{1152}+\frac{49}{4608}\right)\phi^2 \\ &+ \left(\frac{7\nu^4}{12288}+\frac{77\nu^2}{24576}+\frac{889}{2949120}\right)\phi^3 + \mathcal{O}(\phi^4) \, . \end{aligned} \tag{4.3.19}$$

where $\nu = N + 1/2$.

4.4 Instanton fluctuation from topological string

4.4.1 An application of the topological string/spectral theory correspondence

Here we reveal an interesting connection between the fluctuation parts $\mathcal{P}^{\text{1-inst}}_{\text{fluc}}, \mathcal{P}^{I\bar{I}}_{\text{fluc}}$ and topological string theory.

Before our analysis, we would like to remind the reader that the Harper-Hofstadter model is closely related to a Calabi-Yau threefold called the canonical bundle of $\mathbb{F}_0$, also known as local $\mathbb{F}_0$ in the string theory community, as first pointed out in [93]. According to local mirror symmetry, all the Gromov-Witten invariants of local $\mathbb{F}_0$ are encoded in an algebraic curve, called mirror curve which is defined in section 3.2.2. Actually its equation was already worked out in example 2 in the last chapter[17].

$$\mathrm{e}^{x} + \mathrm{e}^{-x} + \mathrm{e}^{y} + \mathrm{e}^{-y} = u \, . \tag{4.4.1}$$

Clearly the Hamiltonian of the Harper-Hofstadter model (4.1.3) can be obtained by rotating (x,y) in complex plane to $(\mathrm{i}x, \mathrm{i}y)$, and promoting them to operators satisfying the commutation relation (4.1.4). Then the free parameter u is related to the energy by $u = 4-2E$. One can obtain another QM model by promoting x and y without the rotation, i.e., one considers the Hamiltonian

$$\mathcal{H}^{\mathbb{F}_0} = -\frac{1}{2}\left(\mathrm{e}^{\mathsf{x}} + \mathrm{e}^{-\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-\mathsf{y}}\right) + 2 \, , \tag{4.4.2}$$

[17] We have set one coefficient of the curve equation, the so-called mass parameter, to be 1. This mass parameter corresponds to anisotropy of the 2d lattice.

with

$$[\mathsf{x}, \mathsf{y}] = \mathsf{i}\hbar \ , \quad \hbar \in \mathbb{R}_+ \ . \tag{4.4.3}$$

We choose a normalization of $\mathcal{H}^{\mathbb{F}_0}$ slightly different from that in the literature to match the normalization of (4.1.3) we use in this paper. Motivated by topological string considerations [4, 3], this QM model has been thoroughly studied, both its spectrum [118, 72, 119, 176, 168, 71, 73] and its wave functions [144, 143, 191] (see also [121, 163]). This has led to exciting development of non-perturbative completion of topological string theory and topological string / spectral theory duality [72, 142, 120, 141, 40, 32, 30, 31], which in turn inspired a new procedure to solve non-perturbatively QM models [41, 42, 61], as well as the discovery of a new class of exactly solvable deformed QM models [74].

We would like to point out that on the one hand, the Hamiltonian (4.4.2) and that of the Harper-Hofstadter model are rather different in nature. The former is confining and has a discrete spectrum, while the Harper-Hofstadter Hamiltonian is periodic and thus has a rich band structure. On the other hand, the spectra of the two Hamiltonians are closely related in the semi-classical regime. In fact, the perturbative eigen-energies of $\mathcal{H}^{\mathbb{F}_0}$ was computed in [42], also using the `BenderWu` package [167, 81], and it is easy to check that they are related to the perturbative eigen-energies of $\mathcal{H}(0,0)$ by the map

$$\hbar \to -\phi \ . \tag{4.4.4}$$

We will see in later sections that many results [42] also apply for the Harper-Hofstadter model as well with appropriate modification.

The large order growth of the perturbative energy of $\mathcal{H}^{\mathbb{F}_0}$ has been analyzed in detail in [42], and it is incorporated in the leading non-perturbative correction[18] to the perturbative series. It is revealed in [42] that this non-perturbative correction can be obtained from the refined free energies in the Nekrasov-Shatashvili limit of topological string theory on the Calabi-Yau threefold local $\mathbb{F}_0$. We will demonstrate that we can obtain the 1-instanton correction (and the instanton–anti-instanton correction) of the Harper-Hofstadter model from their data by applying the map $\hbar \to -\phi$. This is not obvious at first glance because the 1-instanton correction here is the half order of the non-perturbative correction in [42]. This is a consequence of the fact that the 1-instanton sector and the instanton–anti-instanton sector are closely interrelated, as suggested in [92].

Let us quickly review the results of [42] concerning the spectrum of $\mathcal{H}^{\mathbb{F}_0}$. The perturbative eigen-energy can be computed also by using the `BenderWu` package [167, 81], and

[18] This is what is called the 1-instanton correction in [42]. We refer to it as the "instanton–anti-instanton" correction because of the similarity to the Harper-Hofstadter model. More precisely, the situation in [42] corresponds to the special Bloch angles $(\theta_x, \theta_y) = (\pi/2, \pi/2)$, which is just the midpoint (or the Van Hove singularity) of each subband. At this point, the one-instanton correction vanishes, and the leading non-perturbative correction starts from the two-instanton order.

the first few terms read

$$E_{\mathbb{F}_0}^{\text{pert}}(\nu,\hbar) = -\nu\hbar - \frac{4\nu^2+1}{32}\hbar^2 - \frac{4\nu^3+3\nu}{768}\hbar^3 - \frac{16\nu^4+72\nu^2+13}{49152}\hbar^4 + \mathcal{O}(\hbar^5) \ , \tag{4.4.5}$$

with

$$\nu = N + 1/2 \ . \tag{4.4.6}$$

Indeed, this agrees with the perturbative energy of the Harper-Hofstadter model (4.2.2) by the replacement (4.4.4). Note we have adapted the series of $E_{\mathbb{F}_0}^{\text{pert}}(\nu,\hbar)$ to be consistent with the normalization of $\mathcal{H}^{\mathbb{F}_0}$ used in this paper. To formulate the results of the formal "instanton–anti-instanton" correction, we need some terminology from topological string theory on a local Calabi-Yau manifold and its mirror curve.

The coefficient u in the equation of mirror curve (4.4.1) parametrizes the complex structure moduli space of the curve. The moduli space has several singular points, one of which of particular interest is called the conifold singularity and it is located at $u = 4$, as it corresponds to the semi-classical limit $E_{\mathbb{F}_0} = 0$ of the QM model $\mathcal{H}_{\mathbb{F}_0}$. Let us introduce

$$z = \frac{1}{u^2} \ . \tag{4.4.7}$$

Then the classical periods of the mirror curve are

$$\begin{aligned} \partial_z t_c &= -\frac{2}{\pi z}\mathbf{K}(1-16z) \ , \\ \partial_z t_c^D &= \frac{2}{z\sqrt{1-16z}}\mathbf{K}\left(\frac{16}{16z-1}\right) \ , \end{aligned} \tag{4.4.8}$$

of which t_c can serve as a good local coordinate on the moduli space near the conifold singularity. Here $\mathbf{K}$ is the complete elliptic integral of the first kind. Furthermore, for the topological string theory on a local Calabi-Yau threefold X, an important quantity is the refined free energy $\mathcal{F}(t,\epsilon_1,\epsilon_2)$ defined in (3.1.23). In the application to the spectrum of $\mathcal{H}^{\mathbb{F}_0}$, one is in particular interested in the so-called Nekrasov-Shatashvili limit [155]

$$F^{\text{NS}}(t,\hbar) = \lim_{\epsilon_1\to 0} \mathsf{i}\epsilon_1 \mathcal{F}_{\text{ref}}(t,\epsilon_1,\mathsf{i}\hbar) \ . \tag{4.4.9}$$

In this limit, we also need to promote x and y to be operators with commutation relation $[x,y] = \mathsf{i}\hbar$. Then the mirror curve also becomes an operator, dubbed *quantum mirror curve*. In particular, we notice that this gives us exactly (4.4.2) for local $\mathbb{F}_0$ geometry. This observation marks the starting point of the whole story involving topological string theory.

The free energy in the NS limit enjoys a genus expansion

$$F^{\text{NS}}(t,\hbar) = \sum_{n=0}^{\infty} F_n^{\text{NS}}(t)\hbar^{2n} \ . \tag{4.4.10}$$

Near the conifold singularity, the NS free energies F_n^{NS} are functions of t_c with at most logarithmic singularity, and we will use the notation

$$F^{\text{NS}}(t,\hbar) = F^C(t_c,\hbar) = \sum_{n=0}^{\infty} F_n^C(t_c)\hbar^{2n} \ . \tag{4.4.11}$$

They can be computed recursively by the so-called refined holomorphic anomaly equations [25, 135, 109] in the NS limit, as explained in detail in [41, 42, 61]. For local $\mathbb{F}_0$, the first few NS free energies are

$$\begin{aligned}F_0^C(t_c) &= \frac{1}{2}t_c^2\left(\log\left(\frac{t_c}{16}\right)-\frac{3}{2}\right)-\frac{t_c^3}{48}+\frac{5t_c^4}{4608}-\frac{7t_c^5}{61440}+\frac{733t_c^6}{44236800}+\mathcal{O}(t_c^7)\ .\\ F_1^C(t_c) &= -\frac{1}{24}\log t_c-\frac{11t_c}{192}+\frac{49t_c^2}{9216}-\frac{77t_c^3}{73728}+\frac{2213t_c^4}{8847360}-\frac{607t_c^5}{9437184}+\mathcal{O}(t_c^6)\ ,\\ F_2^C(t_c) &= -\frac{7}{5760t_c^2}-\frac{889t_c}{2949120}+\frac{181981t_c^2}{707788800}-\frac{16157t_c^3}{113246208}+\frac{2194733t_c^4}{32614907904}+\mathcal{O}(t_c^5)\ .\end{aligned} \tag{4.4.12}$$

We stress that these results are obtained purely in the framework of topological string theory. We do not need any knowledge of the corresponding quantum mechanics. Our goal is to relate these quantities to the eigen-energy in quantum mechanics.

It turns out, the formal “instanton–anti-instanton” correction to the eigen-energy of $\mathcal{H}^{\mathbb{F}_0}$, which controls the asymptotic growth of the coefficients of $E^{\text{pert}}_{\mathbb{F}_0}(\nu,\hbar)$, is given by [42]

$$E^{I\bar{I}}_{\mathbb{F}_0}(\nu,\hbar) = \pm\mathrm{i}\,2f^{(1)}\mathrm{e}^{16\mathrm{C}/\hbar}\frac{\partial E^{\text{pert}}_{\mathbb{F}_0}(\nu,\hbar)}{\partial\nu}\exp\left(-\frac{2}{\hbar}\frac{\partial F^C(t_c,\hbar)}{\partial t_c}\right)\Big|_{t_c\to\hbar\nu}\ , \tag{4.4.13}$$

where C is the Catalan’s constant, and $f^{(1)}$ a free constant. $t_c=\hbar\nu$ is the all-order WKB quantization condition, which is an all-order generalization of the famous “Bohr-Sommerfeld” quantization and bridges the topological string theory and spectral theory[19]. The exponential factor is $\mathrm{e}^{16\mathrm{C}/\hbar}=\mathrm{e}^{2A/\hbar}$, and this indeed corresponds to the 2-instanton sector in our terminology. Using the NS free energies of local $\mathbb{F}_0$, one can write down the terms in the exponential

$$\begin{aligned}-\frac{1}{\hbar}\frac{\partial F^C}{\partial t_c}\Big|_{t_c\to\hbar\nu} &= \nu-\nu\log\left(\frac{\nu}{16}\right)+\frac{1}{24\nu}-\frac{7}{2880\nu^3}+\mathcal{O}(\nu^{-5})\\ &\quad -\nu\log\hbar+\frac{12\nu^2+11}{192}\hbar-\frac{20\nu^3+49\nu}{4608}\hbar^2+\frac{1680\nu^4+9240\nu^2+889}{2949120}\hbar^3+\mathcal{O}(\hbar^4)\ .\end{aligned} \tag{4.4.14}$$

Interestingly, the terms independent of $\hbar$ can be resummed to

$$\log\left(\frac{\sqrt{2\pi}16^\nu}{\Gamma(\frac{1}{2}+\nu)}\right)\ . \tag{4.4.15}$$

Furthermore, let us denote the power series in $\hbar$ starting from $\mathcal{O}(\hbar)$ by

$$\left[-\frac{1}{\hbar}\frac{\partial F^C}{\partial t_c}\right]\ . \tag{4.4.16}$$

Then the “instanton–anti-instanton” correction can be written as

$$E^{I\bar{I}}_{\mathbb{F}_0}(\nu,\hbar) = \pm\mathrm{i}f^{(1)}\frac{2^{8\nu+2}\pi}{\Gamma(\frac{1}{2}+\nu)^2}\hbar^{1-2\nu}\mathrm{e}^{16\mathrm{C}/\hbar}\cdot\frac{1}{\hbar}\frac{\partial E^{\text{pert}}_{\mathbb{F}_0}(\nu,\hbar)}{\partial\nu}\exp\left[-\frac{2}{\hbar}\frac{\partial F^C}{\partial t_c}\right]\Big|_{t_c\to\hbar\nu}\ , \tag{4.4.17}$$

[19] This deep observation first appeared in four dimensional gauge theory. For instance see [155, 150].

where the components after $\cdot$ is a power series starting from constant 1.

We observe here that this result in terms of topological string free energies also reproduces the imaginary part of the instanton–anti-instanton correction for the Harper-Hofstadter model after applying the map (4.4.4), if we choose

$$f^{(1)} = \frac{1}{2\pi} \ . \tag{4.4.18}$$

Note that this normalization constant can also be fixed through the path integral calculation in section 4.3.1. Comparing the remaining part with the numerical result (4.3.18), one conjectures then the A-function should be identified with the opposite of the derivative of the NS free energy for local $\mathbb{F}_0$, i.e.

$$\mathcal{A}(N,\phi) = \left[+\frac{1}{\hbar}\frac{\partial F^C}{\partial t_c}\right]\Big|_{\substack{\hbar\to-\phi \\ t_c\to-\phi\nu}} \ . \tag{4.4.19}$$

We follow the calculation in [42] of the NS free energies for local $\mathbb{F}_0$ by solving the NS holomorphic anomaly equations and push it to a few orders higher than what is explicitly given in [42]. We find

$$\begin{aligned}\left[+\frac{1}{\hbar}\frac{\partial F^C}{\partial t_c}\right]\Big|_{\substack{\hbar\to-\phi \\ t_c\to-\phi\nu}} = & \left(\frac{\nu^2}{16}+\frac{11}{192}\right)\phi \\ & + \left(\frac{5\nu^3}{1152}+\frac{49\nu}{4608}\right)\phi^2 \\ & + \left(\frac{7\nu^4}{12288}+\frac{77\nu^2}{24576}+\frac{889}{2949120}\right)\phi^3 \\ & + \left(\frac{733\nu^5}{7372800}+\frac{2213\nu^3}{2211840}+\frac{181981\nu}{353894400}\right)\phi^4 \\ & + \left(\frac{47\nu^6}{2359296}+\frac{3035\nu^4}{9437184}+\frac{16157\nu^2}{37748736}+\frac{112573}{3170893824}\right)\phi^5 \\ & + \left(\frac{35921\nu^7}{8323596288}+\frac{2443337\nu^5}{23781703680}+\frac{2194733\nu^3}{8153726976}+\frac{652008227\nu}{7990652436480}\right)\phi^6 + \mathcal{O}(\phi^7)\,,\end{aligned} \tag{4.4.20}$$

and it agrees completely with the A-function (4.3.19) from the numerical fit.

Finally, since the power series in the 1-instanton sector is given by the A-function as shown in (4.3.17), we claim that the 1-instanton sector can also be expressed in terms of the NS free energy of local $\mathbb{F}_0$. In fact, by plugging (4.4.19) into (4.3.17) for the fluctation and comparing the prefactor with the component (4.4.15), we find an expression similar to (4.4.13)

$$E^{\text{1-inst}}_{(\theta_x,\theta_y)}(N,\phi) = \frac{\cos\theta_x+\cos\theta_y}{\pi}\mathrm{e}^{-A/\phi}\frac{\partial E^{\text{pert}}(N)}{\partial N}\mathrm{Im}\exp\left(+\frac{1}{\phi}\frac{\partial F^C}{\partial t_c}\right)\Big|_{\substack{\hbar\to-\phi \\ t_c\to-\phi(N+1/2)}} \ . \tag{4.4.21}$$

Note that after mapping $\hbar \to -\phi$ the exponential becomes purely imaginary, and we take its imaginary value in the expression above. This indeed agrees with numerical results [55], and recover (4.2.40) in the leading order for the ground state.

4.5 Interlude: holomorphic anomaly at work

This section provides details for the calculations underlying Eqs. (4.4.12). It is mostly based on [42].

Since here we are only interested in the NS free energies part $F_n^{\rm NS} := \mathcal{F}^{0,n}_{\rm ref}$ in (3.1.21), we can take the NS limit of the full set of equations (3.1.23),

$$\frac{\partial F_n^{\rm NS}}{\partial \bar{t}} = \frac{1}{2}\overline{C}_{\bar{t}}^{tt} \sum_{r=1}^{n-1} D_t F_r^{\rm NS} D_t F_{n-r}^{\rm NS}, \quad n \geq 2, \tag{4.5.1}$$

where we use a single coordinate t to parametrize the one dimensional moduli space of complex structure for (4.4.1)[20]. In this situation, all the anti-holomorphic dependence can be encapsulated in a single function known as the propagator,

$$\overline{C}_{\bar{t}}^{tt} = \partial_{\bar{t}} S^{tt} \,. \tag{4.5.2}$$

This means we can rewrite (4.5.1) to be,

$$\frac{\partial F_n^{\rm NS}}{\partial S} = \frac{1}{2\gamma} \sum_{r=1}^{n-1} D_\tau F_r^{\rm NS} D_\tau F_{n-r}^{\rm NS}, \quad n \geq 2 \,, \tag{4.5.3}$$

where γ takes into account normalization of free energies. Starting from [2], we realized that holomorphic anomaly is correlated with modular anomaly. In our situation, since the moduli space of complex structures is one dimensional, this is actually easy to explain. We parametrize all the quantities in terms of elliptic modular forms introduced in appendix A. If we want to maintain the modularity, we have to choose the non-holomorphic modular form $\hat{E}_2(\tau,\bar{\tau})$ defined in (A.1.14),

$$\hat{E}_2(\tau,\bar{\tau}) = E_2(\tau) - \frac{3}{\pi {\rm Im}\tau} \,, \tag{4.5.4}$$

making the free energies non-holomorphic[21]. The identification in terms of modular forms also gives us a very convenient way to solve holomorphic anomaly equations.

In our case, the propagator which encapsulates all the non-holomorphicity can be found as

$$S^{tt} = \frac{1}{6}\hat{E}_2(\tau,\bar{\tau}) \,. \tag{4.5.5}$$

[20] It is one dimensional since the mirror curve has genus one.

[21] If we instead choose E_2, everything is holomorphic but we lose modularity.

The covariant derivative D_t can be identified with the *Maass derivative* acting on (almost-holomorphic) modular forms of weight k,

$$D_\tau = \frac{1}{2\pi\mathrm{i}}\frac{d}{d\tau} - \frac{k}{4\pi\mathrm{Im}\tau}\,. \tag{4.5.6}$$

We further introduce some auxiliary modular forms[22],

$$b(\tau) = \vartheta_2^4(q), \quad c(\tau) = \vartheta_3^4(q), \quad d(\tau) = \vartheta_4^4(q)\,. \tag{4.5.7}$$

In terms of those modular forms, the only component of the Yukawa coupling is

$$Y = \frac{2}{d\sqrt{c}}. \tag{4.5.8}$$

This also gives a map between the modular parameter τ and the moduli z,

$$z = \frac{1}{16}\frac{b}{c}. \tag{4.5.9}$$

Given $\gamma = 1$ and the initial condition

$$F_1^{\mathrm{NS}} = -\frac{1}{24}\log(\frac{1-16z}{z^2}) = -\frac{1}{24}\log(\frac{256\;c\;d}{b^2}), \tag{4.5.10}$$

the holomorphic anomaly equations can determine the free energy at each order n up to a holomorphic function f_n, known as the holomorphic ambiguity, which can be parametrized in the following form

$$f_n = \sum_{i=0}^{3n-3} \alpha_{n,i}\; b^i\; d^{3n-3-i}, \tag{4.5.11}$$

with $\alpha_{n,i}$ some constants. This ambiguity can be fixed by the boundary conditions. More precisely, the moduli space has three distinguished points: the large radius point corresponding to $z = 0$, the conifold point corresponding to $z = 1/16$ and the orbifold point orresponding to $z = \infty$. In order to have a well-behaved expansion around these points, we choose the vanishing periods rather than z as good local coordinates.

At the large radius point, t has the asymptotic behavior $t = -\log z - 4z + \cdots$. we expand the free energies in terms of $Q = e^t$ which is a good local coordinate, and we demand the absence of constant terms.

At the conifold point, we use the local coordinate

$$t_c = \frac{1}{\pi}(\frac{\partial F_0^{\mathrm{NS}}}{\partial t} - \frac{\pi^2}{3}), \tag{4.5.12}$$

and demand the expansion of free energies satisfy the so-called gap condition [109],

$$F_n^{\mathrm{NS}} = \frac{(2^{2n-2} - 2^{-1})B_{2n}}{(2n)(2n-1)(2n-2)}\frac{1}{t_c^{2n-2}} + \mathcal{O}(t_c^0). \tag{4.5.13}$$

[22] They can be obtained from the corresponding $\Theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\tau, z)$ in section A.2 by setting z to zero.

At the orbifold point, we use the local coordinate

$$t_o = \frac{\mathrm{i}}{4\pi}(t - 2\frac{\partial F_0^{\mathrm{NS}}}{\partial t}), \tag{4.5.14}$$

and demand the expansion of free energies satisfy another gap condition [109],

$$F_n^{\mathrm{NS}} = \frac{(-1)^{n-1}(1-2^{1-2n})B_{2n}}{(n)(2n-1)(2n-2)}\frac{1}{t_o^{2n-2}} + \mathcal{O}(t_o^0). \tag{4.5.15}$$

These are already enough to completely fix the holomorphic ambiguity.

The first few terms of the free energy are found as follows,

$$\begin{aligned}
F_2^{\mathrm{NS}} &= \frac{(c+d)^2}{1728cd^2}\hat{E}_2 - \frac{37b^3+51b^2d+18bd^2+20d^3}{8640cd^2},\\
F_3^{\mathrm{NS}} &= -\frac{(c+d)^3\hat{E}_2^3}{1119744c^2d^4} + \frac{(c+d)^2(5c^2-5cd+2d^2)\hat{E}_2^2}{373248c^2d^4}\\
&+ \frac{(c+d)(449c^4-799c^3d+360c^2d^2-91cd^3+101d^4)\hat{E}_2}{1866240c^2d^4}\\
&+ \frac{110539b^6+373926b^5d+467142b^4d^2+259765b^3d^3+72690b^2d^4+4260bd^5+280d^6}{39191040c^2d^4},\\
F_4^{\mathrm{NS}} &= \frac{(c^4+4c^3d+6c^2d^2+4cd^3+d^4)\hat{E}_2^5}{322486272c^3d^6}\\
&+ \frac{(-21c^5-44c^4d-14c^3d^2+12c^2d^3-5cd^4-8d^5)\hat{E}_2^4}{322486272c^3d^6}\\
&+ \frac{(869c^6+409c^5d-1159c^4d^2-310c^3d^3+319c^2d^4+101cd^5+171d^6)\hat{E}_2^3}{806215680c^3d^6}\\
&+ \frac{(-12173c^7+8647c^6d+13647c^5d^2-7978c^4d^3)\hat{E}_2^2}{806215680c^3d^6}\\
&+ \frac{(-1927c^3d^4+363c^2d^5+253cd^6-1232d^7)\hat{E}_2^2}{806215680c^3d^6}\\
&+ \frac{(19160307c^8-39788474c^7d+9431827c^6d^2)\hat{E}_2}{56435097600c^3d^6}\\
&+ \frac{(27005574c^5d^3-17935348c^4d^4+2537054c^3d^5)\hat{E}_2}{56435097600c^3d^6}\\
&- \frac{(411333c^2d^6+1087154cd^7-1101547d^8)\hat{E}_2+335441623b^9}{56435097600c^3d^6}\\
&+ \frac{-1736667437b^8d-3719865425b^7d^2+4231474589b^6d^3+2720317301b^5d^4}{56435097600c^3d^6}\\
&+ \frac{+9643365356b^4d^5+1595883344b^3d^6+12231464b^2d^7-408496bd^8+2800d^9}{56435097600c^3d^6}.
\end{aligned} \tag{4.5.16}$$

The non-holomorphic piece does not play a role in our comparison with (4.3.19). So then we just take $\overline{\tau} \to 0$ to decouple the anti-holormorphic part, which also means replacing $\hat{E}_2$ by E_2 in F_n^{NS}.

Chapter 5

Elliptic Genera and Topological Strings: Overview

Starting from this chapter, we will discuss another application of topological string theory, which is related to six dimensional super-conformal field theories. More concretely, a recent revival of interest in those theories has led to a better understanding of the topological string partition function Z_{top} on an elliptically fibered Calabi-Yau threefold X. An incomplete list of techniques and strategies is [110, 85, 126, 83, 108, 128, 66, 117, 125, 50, 101, 79, 96, 87, 15, 48, 127, 137].

A particularly elegant way of encoding Z_{top} in this setting is in terms of modular expressions for the coefficients $\mathcal{Z}_{\boldsymbol{k}}(\tau, z, \boldsymbol{m})$ of the expansion of Z_{top} in suitably shifted exponentiated Kähler moduli $\tilde{\boldsymbol{Q}}_{\boldsymbol{B}}$ of base classes,

$$Z_{\text{top}} = Z_0 \cdot \left(1 + \sum_{\boldsymbol{k} \neq 0} \mathcal{Z}_{\boldsymbol{k}} \tilde{\boldsymbol{Q}}_{\boldsymbol{B}}^{\boldsymbol{k}}\right) . \tag{5.0.1}$$

Here and throughout this note, we use the notation $\boldsymbol{k} = (k_1, k_2, \ldots)$, $k_i \geq 0$, to denote a curve class in the base B. $\mathcal{Z}_{\boldsymbol{k}}(\tau, z, \boldsymbol{m})$ is a Jacobi form whose modular parameter is the Kähler modulus τ of the elliptic fiber, and with elliptic parameters the string coupling $z = \frac{g_s}{2\pi}$, as well as the Kähler moduli $\boldsymbol{m}$ of the fibral curve classes.

The Gopakumar-Vafa form [69, 70] of Z_{top} reveals that $\mathcal{Z}_{\boldsymbol{k}}$ must exhibit poles; $\mathcal{Z}_{\boldsymbol{k}}$ as a Jacobi form must hence be meromorphic. Unlike the ring of weak Jacobi forms, whose elements are holomorphic, the ring of meromorphic Jacobi forms is not finitely generated. In [108, 79, 48, 127], progress hinged on expressing $\mathcal{Z}_{\boldsymbol{k}}$ as a quotient of weak Jacobi forms[1],

$$\mathcal{Z}_{\boldsymbol{k}} = \frac{\sum c_i \phi_{\boldsymbol{k},i}(\tau, z, \boldsymbol{m})}{\phi_{\boldsymbol{k}}^D(\tau, z, \boldsymbol{m})} . \tag{5.0.2}$$

The denominator takes a universal form depending only on the knowledge of the classical intersection numbers of the divisors of the elliptically fibered Calabi-Yau manifold X.

[1] There are lots of ϕ in this thesis, so we use the subscript D to indicate that $\phi_{\boldsymbol{k}}^D$ is the denominator of $\mathcal{Z}_{\boldsymbol{k}}$.

This data also fixes the weight and indices of the numerator, allowing an expansion in appropriate ring generators. The expansion coefficients c_i must be determined by imposing additional constraints on $\mathcal{Z}_{\boldsymbol{k}}$.

However, before discussing possible ways to determine them, let's first set the stage and introduce necessary background. Thus we devote this chapter to a quick review of some known results in the literature, which prepares the reader for the next two chapters.

Its structure is the following. In section 5.1, we review the classes of 4-cycles that occur in elliptically fibered Calabi-Yau manifolds, and explain the role the corresponding Kähler classes play in $\mathcal{Z}_{\boldsymbol{k}}$. During the analysis, we find that it's necessary to separate the geometry into two classes, depending on whether it gives rise to non-abelian gauge symmetries or not. We also give one example for each class. In section 5.2 we review the ansatz (5.0.2), in particular the denominator $\phi^D_{\boldsymbol{k}}(\tau, z, \boldsymbol{m})$, which plays a crucial role in later analysis.

This chapter is based on the section 2 and 3 of the article *Computing the elliptic genus of higher rank E-strings from genus 0 GW invariants* [56] by Jie Gu, Amir-Kian Kashani-Poor and the author, with various changes for pedagogical reasons.

5.1 Elliptic fibrations and four-cycles

Compact elliptically fibered Calabi-Yau manifolds can be constructed as hypersurfaces in a projective bundle $\mathbb{P}^{1,2,3}(\mathcal{O} \oplus 2K_B \oplus 3K_B)$ over a compact Kähler base manifold B. The hypersurface is cut out via a Weierstrass equation in variables $[x : y : z]$,

$$y^2 = 4x^3 - f_4 x z^4 - g_6 z^6 \,, \tag{5.1.1}$$

where f_4 and g_6 are sections of particular line bundles over the base surface B,

$$f_4 \in -4K_B \,, \quad g_6 \in -6K_B \,. \tag{5.1.2}$$

Another important quantity is the so-called discriminant of the Weierstrass equation,

$$\Delta = 4f^3 + 27g^2 \in -12K_B. \tag{5.1.3}$$

The order of vanishing of (f, g, Δ) must be strictly smaller than (4, 6, 12) along any divisor, in order that that the Calabi-Yau condition is satisfied. In other words, the allowed singularities along any divisor must be of Kodaira type.

The Kodaira singularity structure of the elliptic fiber depends on the vanishing orders of these sections. When we resolve singularities along a given divisor, extra 2-cycles occur and can give rise to gauge groups in F theory compactification. The precise dictionary is worked out by the "Tate algorithm" [26]. We summarize the result in table 5.1. Note that when there are multiple choices, we need to also supply the so-called monodromy cover equation.

ord(f)	ord(g)	ord(Δ)	Kodaira type	singularity	non-abelian algebra
≥ 0	≥ 0	0	I_0	none	none
0	0	1	I_1	none	none
0	0	$n \geq 2$	I_n	$\mathfrak{a}_{n-1}$	$\mathfrak{su}_n$ or $\mathfrak{sp}_{[n/2]}$
≥ 1	1	2	II	none	none
1	≥ 2	3	III	$\mathfrak{a}_1$	$\mathfrak{su}_2$
≥ 2	2	4	IV	$\mathfrak{a}_2$	$\mathfrak{su}_3$ or $\mathfrak{su}_2$
≥ 2	≥ 3	6	I_0^*	$\mathfrak{d}_4$	$\mathfrak{so}_8$ or $\mathfrak{so}_7$ or $\mathfrak{g}_2$
2	3	$n \geq 7$	I_{n-6}^*	$\mathfrak{d}_{n-2}$	$\mathfrak{so}_{2n-4}$ or $\mathfrak{so}_{2n-5}$
≥ 3	4	8	IV^*	$\mathfrak{e}_6$	$\mathfrak{e}_6$ or $\mathfrak{f}_4$
3	≥ 5	9	III^*	$\mathfrak{e}_7$	$\mathfrak{e}_7$
≥ 4	5	10	II^*	$\mathfrak{e}_8$	$\mathfrak{e}_8$

Since our primary interest here is six dimensional gauge theory, we decouple gravity by decompactifying the geometries. For instance, decompactification along the fiber direction can lead to local Calabi-Yau manifolds which are the total space of the canonical bundle of a surface, while decompactification perpendicular to the fiber direction yields elliptic fibration over a non-compact surface, such as the geometries appearing as building blocks in the classification scheme of 6d SCFTs via F-theory [97].

The topological string partition function depends on the topological string coupling constant g_s (or, in the case of refinement, on two parameters $\epsilon_{1,2}$ which can be organized as $g_s^2 = -\epsilon_1\epsilon_2$ and $s = (\epsilon_1+\epsilon_2)^2$) and (in the A-model perspective) on Kähler parameters associated to homology classes in $H_2(X)$ of the Calabi-Yau manifold X. In the generic Gopakumar-Vafa formula, which we will review in the next section, all Kähler parameters enter the partition function on the same footing. When X is elliptically fibered, different Kähler parameters are distinguished by the action of the monodromy group on the associated curve classes [108]. The curve classes in the base B are essentially invariant under this action (see the discussion around equations (5.2.8) and (5.2.9) for the precise statement). As we shall see later, the associated Kähler parameters are treated as expansion parameters. The coefficients $\mathcal{Z}_{\boldsymbol{k}}$ of the expansion in appropriately shifted exponentiated base classes, with $\boldsymbol{k}$ indicating the base class, are Jacobi forms. All remaining Kähler parameters as well as the string coupling play the role of the modular and elliptic parameters of these Jacobi forms.

To understand the roles played by different curve classes in the topological string partition function on an elliptic Calabi-Yau threefold, we focus on the case of compact Calabi-Yau manifolds, and argue, via Poincaré duality, in terms of 4-cycles rather than 2-cycles. The non-compact case can then be obtained via degeneration. We can distinguish between 4 classes of divisors [177, 151]:

1) The pullbacks B_α of divisors (curves) of the base B to X via the projection $\pi : X \to$

B, i.e. $B_\alpha = \pi^* H_\alpha$.

2) The zero section of the fibration, which is topologically the base B.

3) Divisors $T_{\kappa,I}$ consisting of fiber components $\alpha_{\kappa,I}$ arising from the resolution of singularities of the fibration over curves b_κ in B, fibered over b_κ.

4) Divisors S_i associated to other than the zero section of the fibration.

By [177], all 4-cycles in X fall into one of these four classes. Upon replacing the 4-cycles S_i by their images under the threefold version of the Shioda map [158], the Poincaré dual 2-cycles to these 4 classes are

1) Curve classes H_α of the base B.

2) f, the fiber class of the fibration.

3) Fiber components $\alpha_{\kappa,I}$.

4) Isolated rational curves s_i in the fiber.

In the F-theory compactification on X, curve classes of type 1 give rise to tensor multiplets, those of type 3 to non-abelian vector multiplets, while those of type 4 give rise to abelian vector multiplets as well as hypermultiplets charged under them.

The corresponding Kähler classes play different roles in Z_{top}, depending on the monodromy action on the associated curve classes:

1) The exponentials $\boldsymbol{Q_B} = (Q_1, \ldots, Q_{b_2(B)})$ of the base Kähler classes $t_\alpha = \int_{H_\alpha} J$ can be rendered invariant upon an appropriate shift explained below (see equations (5.2.8) and (5.2.9)). Z_{top} is expanded in terms of the shifted variables $\tilde{\boldsymbol{Q}}_{\boldsymbol{B}}$, see (5.0.1), to yield the Jacobi forms $\mathcal{Z}_{\boldsymbol{k}}$ as coefficients.

2) $\tau = \int_f J$ is the modular parameter of the Jacobi forms $\mathcal{Z}_{\boldsymbol{k}}$.

3) $c_{\kappa,I} = \int_{\alpha_{\kappa,I}} J$ are elliptic parameters on which the position of the poles of $\mathcal{Z}_{\boldsymbol{k}}$ as a function of g_s (or $\epsilon_{1,2}$ in the refined case) depends.[2]

4) $m_i = \int_{\sigma(S_i)} J$ give rise to elliptic parameters which do not modify the position of the poles of $\mathcal{Z}_{\boldsymbol{k}}$ as a function of g_s (or $\epsilon_{1,2}$ in the refined case).

[2]Note that F-theory, and therefore the 6 dimensional theory obtained by compactifying on such geometries, is not sensitive to these Kähler parameters. Upon compactifying on a circle, however, the resulting vector multiplets exhibit real scalar fields whose VEV keeps track of the blow-up cycle size, and which collectively transform under the global symmetry given by the Weyl group of the gauge group.

We refer to the curve classes of type 3 and 4 as fibral, and denote the corresponding Kähler parameters collectively as t_ι.

Now let's consider some canonical examples to illustrate our points above. We choose the complex base B to be Hirzebruch surfaces $\mathbb{F}_n$ which itself is a fibration of $\mathbb{CP}^1$ over another $\mathbb{CP}^1$[3]. Then we decompatify the base by sending the volume of the fiber to infinity, in such a way that the non-compact space is total space $\mathcal{O}(-n) \to \mathbb{P}^1$. Schematically, the procedure can be represented as the graph 5.1.4,

$$\begin{array}{ccc} \mathcal{E} \to X & & \mathcal{E} \to \check{X} \\ \downarrow \pi_1 & & \downarrow \check{\pi}_1 \\ F = \mathbb{P}^1 \to B = \mathbb{F}_n & \to & \mathcal{O}(-n) \to \check{B} \\ \downarrow \pi_2 & & \downarrow \check{\pi}_2 \\ b = \mathbb{P}^1 & & b = \mathbb{P}^1 \end{array} . \tag{5.1.4}$$

After the decompactification, the only one compact divisor is the base $\mathbb{CP}^1$. Whether it is a divisor of type 3 or 4 depends on n. It turns out that it is of type 4 if $n \leq 2$, while of type 3 if $n > 2$. Below we will give one example for each, and further explore them in the next two chapters.

Example 3. *Let's first consider n to be 1. The toric data of the ambient toric variety is given in table 5.1.5 [4]. Moreover, there are two ways to resolve the singularity (triangulate the face fan), related by the flop. Both phases have $h^{2,1} = 243$ and $h^{1,1} = 3$ as can be checked from our general formula (3.2.2). However, their intersection rings are different, as they should be. We consider the phase I and send the volume of F to infinity. The vanishing degree of (f, g, Δ) along the base $\mathbb{CP}^1$ is (0,0,0). Hence according to table 5.1, the elliptic fiber over it is generically non-singular, i.e., b is a divisor of type 4. It is also known as the E string geometry upon decompatification.*

Div.	ν_i				$l_I^{(e)}$	$l_I^{(f)}$	$l_I^{(b)}$	$l_{II}^{(e')}$	$l_{II}^{(h)}$	$l_{II}^{(-b)}$
D_0	0	0	0	0	−6	0	0	−6	0	0
D_1	−1	0	0	0	2	0	0	2	0	0
D_2	0	−1	0	0	3	0	0	3	0	0
E	2	3	0	−1	0	1	−1	−1	0	1
K	2	3	0	0	1	−2	−1	0	−3	1
F	2	3	−1	−1	0	0	1	1	1	−1
H	2	3	0	1	0	1	0	0	1	0
F	2	3	1	0	0	0	1	1	1	−1

. (5.1.5)

[3]For its toric construction, see example 5.

[4]See appendix B for our convention.

Example 4. *Next, let's consider the case when $n = 3$. The toric data of the ambient toric variety is given in table 5.1.6. There are altogether sixteen possible triangulations, and we only list one of them. Again, we send the volume of F to infinity to decompactify the geometry. The vanishing degree of (f, g, Δ) along the base $\mathbb{CP}^1$ is (2, 2, 4). Hence according to table 5.1, the elliptic fiber over it has singularity of Kodaira type IV. When we resolve it, we obtain the gauge algebra $\mathfrak{su}_3$. This also means that b is a divisor of type 3.*

Div.	ν_i				$l^{(1)}$	$l^{(2)}$	$l^{(3)}$	$l^{(4)}$	$l^{(5)}$
D_0	0	0	0	0	−1	0	0	0	0
D_1	−1	0	0	0	0	1	0	−1	0
D_2	0	−1	0	0	1	0	0	0	−1
D_3	1	1	0	−1	−1	0	0	0	2
D_4	1	2	0	−1	1	−3	0	3	0
E	2	3	0	−1	0	0	1	−3	1
K	2	3	0	0	0	0	−2	1	0
F	2	3	−1	−3	0	1	0	0	−1
H	2	3	0	1	0	0	1	0	0
F	2	3	1	0	0	1	0	0	−1

. (5.1.6)

5.2 The base degree k partition function as Jacobi form

We now turn to the discussion of the structure of the partition function at base-wrapping degree $\boldsymbol{k}$, $\mathcal{Z}_{\boldsymbol{k}}$. At the end of the section, we point out the modifications necessary to generalize to the case of refinement.

One path to the identification of the transformation properties of $\mathcal{Z}_{\boldsymbol{k}}$ runs via the holomorphic anomaly equations of the topological string [26, 106, 108]. These can be rewritten in the form

$$\left(\frac{\partial}{\partial E_2} + \frac{\pi^2}{3} M_{\boldsymbol{k}}\right) \mathcal{Z}_{\boldsymbol{k}} = 0\,. \tag{5.2.1}$$

$M_{\boldsymbol{k}}$, dubbed the index bilinear form in [48], depends quadratically on the string coupling $z = \frac{g_s}{2\pi}$ and all fibral Kähler parameters,

$$M_{\boldsymbol{k}} = i_z(\boldsymbol{k}) z^2 + \sum i_{t_\iota}(\boldsymbol{k}) t_\iota^2. \tag{5.2.2}$$

It is easy to show that a convergent power series

$$\sum_{\ell_z, \boldsymbol{\ell}} a_{\ell_z, \boldsymbol{\ell}} z^{2\ell_z} \bar{\tau}^{2\boldsymbol{\ell}} \tag{5.2.3}$$

with coefficients $a_{\ell_z,\boldsymbol{\ell}}$ of weight $w+2(\ell_z+\sum_\iota \ell_\iota)$ in the ring $\mathbb{C}[E_2,E_4,E_6]$ of quasi-modular forms which satisfies the differential equation (5.2.1) is a weak Jacobi form[5] with elliptic parameters z, t_ι of index $i_z(\boldsymbol{k})$, $i_{t_\iota}(\boldsymbol{k})$ respectively.

The program of rewriting the holomorphic anomaly equations of [25] in the form (5.2.1) has been carried out for local 1/2 K3 in [106] and for the elliptic fibration over the base $B=\mathbb{F}_1$ in [130], though some details remain to be ironed out.

A currently more efficient path towards determining the indices i_{t_ι} for all fibral curve classes proceeds via F-theory compactifications on the elliptically fibered manifold X. The resulting 6d theory exhibits non-critical strings which arise via D3 branes wrapping curves $C_{\boldsymbol{k}}$ in the base manifold B. $\mathcal{Z}_{\mathbf{k}}$ for $\boldsymbol{k}\neq 0$ can be identified with the elliptic genus of these strings [131, 83]. The transformation properties of elliptic genera under modular transformations have been argued for in [23, 24]. One can use this vantage point to fix the index bilinear $M_{\boldsymbol{k}}$ in terms of the anomaly polynomial of the worldsheet theory[6] of these strings [50, 79, 48, 137], which has been computed in [125, 166]. For our purposes, the important characteristic is that all indices other than i_z depend linearly on the base-wrapping degree $\boldsymbol{k}$; i_z, which takes the elegant form

$$i_z(\boldsymbol{k})=\frac{C_{\boldsymbol{k}}\cdot(C_{\boldsymbol{k}}+K_B)}{2}\,, \tag{5.2.4}$$

is quadratic in $\boldsymbol{k}$.

An important ingredient in solving for $\mathcal{Z}_{\boldsymbol{k}}$, once its transformation properties under the modular group have been identified, is determining its pole structure. The Gopakumar-Vafa form of the free energy motivates the ansatz

$$\phi^D_{\boldsymbol{k}}=\prod_{i=1}^{b_2(B)}\prod_{s=1}^{k_i}\phi_{-2,1}(\tau,sz) \tag{5.2.5}$$

for the denominator in (5.0.2). In the absence of curves of type 3, this ansatz has been verified for numerous examples in [108, 79]. Indeed, isolated rational curves, such as the curves of type 4, are locally modeled by the conifold. Having a trivial moduli space, they are not expected to give rise to a contribution (hence $\boldsymbol{m}$ dependence) in the denominator of $\mathrm{Z}_{\mathrm{top}}$. On the other hand, rational curves of self-intersection number less than -1, such as the curves of type 3, do necessarily exhibit a non-trivial moduli space, and are expected to modify the pole structure of $\mathrm{Z}_{\mathrm{top}}$ (cf. the discussion in section 3.4 of [115] juxtaposing -1 and -2 curves). This expectation is born out by localization computations [50, 137] and topological string computations [79, 48, 137], the latter primarily in the case of local geometries.

To describe how the ansatz (5.2.5) must be modified in the presence of curves of type 3, recall that these arise upon resolution of singularities over a divisor C in the base B.

[5]For a quick introduction to Jacobi modular forms, see section A.2.

[6]See [51] for an analysis of these worldsheet theories beyond the minimal SCFTs.

Those curves $\{E_i\}$ are related to the gauge algebra according to table 5.1. This point will be discussed in more detail in chapter 7. The upshot is that the Kähler parameters m_i associated to a choice of basis of the exceptional curves are assembled into an element of the complexified root lattice $\boldsymbol{m} = \sum_i m_i \boldsymbol{\omega_i}$, the ω_i denoting the fundamental weights of $\mathfrak{g}$. There is also a monodromy action on $\{E_i\}$ captured by the Weyl group of $\mathfrak{g}$. The invariance of $\mathcal{Z}_{\boldsymbol{k}}$ under this Weyl group action can be easily seen from the worldsheet expression of elliptic genus[7].

The denominator in (5.0.2) in the case $b_2(B) = 1$ given in [48] reduces in the unrefined case (up to a sign) to

$$\prod_{s=1}^{k} \left(\phi_{-2,1}(\tau, sz) \prod_{\ell=0}^{s-1} \prod_{\alpha \in \Delta_+} \phi_{-2,1}\left(\tau, (s-1-2\ell)z + m_\alpha\right) \right) , \tag{5.2.6}$$

where we have defined

$$m_\alpha = (\boldsymbol{m}, \alpha^\vee) \tag{5.2.7}$$

for any root $\alpha \in \Delta$. The naive generalization beyond $b_2(B) = 1$ should be correct, but no computations have yet been performed in this case.

Finally, we explain the relation between the exponentiated expansion parameters $\tilde{\boldsymbol{Q}}_{\boldsymbol{B}}$ and the exponentiated base Kähler classes $\boldsymbol{Q_B}$. As argued in [108], based on genus zero observations in [38], for the case of non-singular elliptic fibrations, the base class shifted by an appropriate multiple of the fiber class is invariant under the $SL(2,\mathbb{Z})$ monodromy action, up to a sign corresponding to the same multiplier system as an appropriate power of $\eta^{12}(\tau)$. The modular properties of $\mathrm{Z}_{\mathrm{top}}$ are thus manifest when expanding in

$$\tilde{\boldsymbol{Q}}_{\boldsymbol{B}} = \left(\frac{\sqrt{q}}{\eta^{12}(\tau)} \right)^{-C_{\boldsymbol{k}} \cdot K_B} \boldsymbol{Q_B} \, . \tag{5.2.8}$$

From the identification of the topological string with the elliptic genus of the worldsheet theory of non-critical strings, this result was generalized to the case of singular fibrations in [83] to

$$\tilde{\boldsymbol{Q}}_{\boldsymbol{B}} = \left(\frac{\sqrt{q}}{\eta^{12}(\tau) \prod_{i=1}^{r} Q_i^{a_i^\vee}} \right)^{-C_{\boldsymbol{k}} \cdot K_B} \boldsymbol{Q_B} \, . \tag{5.2.9}$$

Note that the appropriate expansion parameter to extract the enumerative data encoded in $\mathrm{Z}_{\mathrm{top}}$ remains $\boldsymbol{Q_B}$; this is also the parameter that occurs in the Gopakumar-Vafa presentation of the free energy which we shall review in section 6.1. We thus introduce straight letters Z and F to denote the corresponding expansion coefficients, such that

$$Z_{\boldsymbol{k}} = \left(\frac{\sqrt{q}}{\eta^{12}(\tau) \prod_{i=1}^{r} Q_i^{a_i^\vee}} \right)^{-C_{\boldsymbol{k}} \cdot K_B} \mathcal{Z}_{\boldsymbol{k}} \, , \quad F_{\boldsymbol{k}} = \left(\frac{\sqrt{q}}{\eta^{12}(\tau) \prod_{i=1}^{r} Q_i^{a_i^\vee}} \right)^{-C_{\boldsymbol{k}} \cdot K_B} \mathcal{F}_{\boldsymbol{k}} \, , \tag{5.2.10}$$

[7] For a quick proof of this important result, see section 2.5 in [48]

where the product in the denominator ranges over divisors of the geometry of type 3.

The refined version of the results discussed in the section is also known [50, 79, 48]. Introducing the variables

$$z_{1,2} = \frac{\epsilon_{1,2}}{2\pi} \quad , \qquad z_{L,R} = \frac{\epsilon_1 \mp \epsilon_2}{4\pi} = \frac{\epsilon_{L,R}}{2\pi} \,, \tag{5.2.11}$$

the refined partition function takes the general form

$$\mathcal{Z}_{\boldsymbol{k}} = \frac{\mathcal{N}_{\boldsymbol{k}}^{\text{refined}}(\tau, z_L, z_R, \boldsymbol{m})}{\phi_{\boldsymbol{k}}^{D}(\tau, z_1, z_2, \boldsymbol{m})} \,, \tag{5.2.12}$$

where $\mathcal{N}_{G,\boldsymbol{k}}^{\text{refined}}(\tau, z_L, z_R, \boldsymbol{m}) = \sum c_i \phi_{\boldsymbol{k},i}(\tau, z_L, z_R, \boldsymbol{m})$ a linear combination of weak Jacobi modular forms with $z_{L,R}$ (or equivalently $z_{1,2}$) serving as elliptic parameters. The denominator in the absence of divisors of type 3 is given by [50, 79]

$$\phi_{\boldsymbol{k}}^{D} = \prod_{i=1}^{b_2(B)} \prod_{s=1}^{k_i} \phi_{-1,1/2}(\tau, s z_1)\phi_{-1,1/2}(\tau, s z_2) \,, \tag{5.2.13}$$

where $\phi_{-1,1/2}^2 = \phi_{-2,1}$. When such divisors are present, the denominator (in the case $b_2(B) = 1$) is [50, 127]

$$\prod_{s=1}^{k} \prod_{j=\pm 1} \left(\phi_{-1,1/2}(s(z_R + j z_L)) \prod_{\ell=0}^{s-1} \prod_{\alpha \in \Delta_+} \phi_{-1,1/2}((s+1)z_R + (s-1-2\ell)z_L + j m_\alpha) \right) . \tag{5.2.14}$$

Note that this specializes to (5.2.6) for $z_R = 0, z = z_L$ up to an irrelevant sign.

The different forms of the denominators Eqs. (5.2.5), (5.2.13) and (5.2.14) may seem hard to digest. Actually there exists a uniform way to argue them from the gauge theory perspective. For completeness, we include the basic idea here. It can be safely skipped for the uninterested reader. The rationale is due to [50].

To start with, we can assume $b_2(B) = 1$ without lost of generality. From [50], when $q \to 0$, the following relation holds,

$$Z_{\text{top}}|_{q=0} = Z_0 \cdot \left(1 + \sum_{k \neq 0} Z_k Q^k \right) |_{q=0} = Z_0' \left(1 + \sum_{k \neq 0} \mathcal{H}(\mathcal{M}_{G,k}) Q^k \right) , \tag{5.2.15}$$

where $\mathcal{H}\left(\mathcal{M}_{G,k}\right)$ is the Hilbert series of the moduli space of k G-instantons. This can teach us some important lesson about Z_k. In particular, counting the generators [47], we find the explicit expression for the denominator $D_{G,k}(\epsilon_1, \epsilon_2, m_\alpha)$ of $\mathcal{H}\left(\mathcal{M}_{G,k}\right)$,

$$D_{G,k}(\epsilon_L, \epsilon_R, m_\alpha) = \prod_{i=1}^{k} \left(\prod_{\substack{j=-i \\ j-i \text{ even}}}^{i} (1 - v^i x^j) \right) \left(\prod_{\substack{j=-i+1 \\ j-i \text{ odd}}}^{i-1} \prod_{\alpha \in \widetilde{\Delta}_G} (1 - v^{i+1} x^j e^{2\pi i m_\alpha}) \right) , \tag{5.2.16}$$

where $\widetilde{\Delta}_G$ includes all the roots of G as well as Cartan elements, while $x = e^{2\pi i \epsilon_L}$ and $v = e^{2\pi i \epsilon_R}$.

On the other hand, based on the form of refined topological string partition function (6.1.21), only poles at $\epsilon_1 = 0$ or $\epsilon_2 = 0$ are possible for Z_k. Combining these two pieces of information, we are led to speculate the q-independent part of the denominator of Z_k,

$$\phi_k^D(\tau, z_L, z_R, \boldsymbol{m})|_{q=0} = \prod_{s=\pm 1} \prod_{i=1}^{k} \left((1-(vx^s)^i) \prod_{l=0}^{i-1} \prod_{\alpha \in \Delta_+} (1 - v^{i+1} x^{i-1-2l} e^{2\pi i s m_\alpha}) \right), \quad (5.2.17)$$

where Δ_+ only includes all the positive roots.

In order to guess its full form, we should examine each individual term more carefully. First notice that they can all be written as $(1 - e^{2\pi i z})$, with z a linear combination of $\epsilon_{L,R}$ and m_α. Besides, from the elliptic genus point of view, the final expression should behave reasonably under modular transformations. Since it already vanishes when z is an integer, it should also be zero at $z = n\tau + m$ after $n\tau$ translations. Thus naively, $(1-e^{2\pi i z})$ can be completed as $(1 - e^{2\pi i z}) \prod_{j=1}^{\infty}(1 - q^j e^{2\pi i z}) \prod_{j=1}^{\infty}(1 - q^j e^{-2\pi i z})$. Actually there is a more modular covariant way to package it, if one invokes the infinite product expansion of $\varphi_{-1,1/2}(z,\tau)$,

$$\varphi_{-1,1/2}(z,\tau) = i e^{-\pi i z}(1 - e^{2\pi i z}) \prod_{j=1}^{\infty} \frac{(1 - q^j e^{2\pi i z})(1 - q^j e^{-2\pi i z})}{(1-q^j)^2}. \quad (5.2.18)$$

To summarize, we replace $(1-(vx^s)^i)$ by $\varphi_{-1,2}(\tau, i(\epsilon_R + s\epsilon_L))$ and $(1 - v^{i+1}x^{i-1-2l}e^{2\pi i s m_\alpha})$ by $\varphi_{-1,2}(\tau, (i+1)\epsilon_R + (i-1-2l)\epsilon_L + s m_\alpha)$, hence obtain most general denominator (5.2.14).

Chapter 6

Geometries without codimension-one singular fibers: Reconstruction

Recall that our goal is to compute $\mathcal{Z}_{\boldsymbol{k}}$, for which we have the following ansatz,

$$\mathcal{Z}_{\boldsymbol{k}} = \frac{\sum c_i \phi_{\boldsymbol{k},i}(\tau, z, \boldsymbol{m})}{\phi^D_{\boldsymbol{k}}(\tau, z, \boldsymbol{m})} \,. \tag{6.0.1}$$

In this chapter, the main result is to show that for geometries without divisors of type 3 (no non-abelian gauge symmetry), e.g., higher rank E-strings obtained from stacks of M5 branes embedded in the end of the world M9 brane, genus zero Gromov-Witten invariants provide sufficient boundary conditions to determine the c_i in (6.0.1). In other words, we can compute the topological string to all orders in g_s. Recall that the geometries underlying the higher rank E-strings are elliptic fibrations over a non-compact base consisting of a chain of -2 curves ending on a -1 curve [97]. The six dimensional theory engineered by considering F-theory on these geometries generalize the E-string as originally studied in [67, 164, 131, 149]. The topological string on the higher rank E-strings was solved based on the ansatz (6.0.1) in [79] by imposing vanishing conditions on Gopakumar-Vafa invariants – a target space perspective. We show here that boundary conditions arising from a worldsheet perspective are also sufficient to solve these models.

Our analysis relies on determining the principal parts (the negative degree terms in the Laurent expansion) of $\mathcal{Z}_{\boldsymbol{k}}$ around all of its poles. We achieve this by combining modular and multi-wrapping properties of $\mathrm{Z}_{\mathrm{top}}$. While the modular properties are manifest in the form (6.0.1), the multi-wrapping properties, as captured by the Gopakumar-Vafa formula, are formulated more naturally in terms of the topological string free energy $\mathcal{F}_{\boldsymbol{k}}$, related to $\mathcal{Z}_{\boldsymbol{k}}$ via

$$Z_{\mathrm{top}} = Z_0 \cdot \exp\left(\sum_{\boldsymbol{k}\neq 0} \mathcal{F}_{\boldsymbol{k}} \tilde{\boldsymbol{Q}}_{\boldsymbol{B}}^{\boldsymbol{k}}\right) \,. \tag{6.0.2}$$

We thus begin our analysis by asking to what extent the modularity properties of $\mathcal{Z}_{\boldsymbol{k}}$ carry over to $\mathcal{F}_{\boldsymbol{k}}$. In a nutshell, the answer is that the string coupling as elliptic parameter is lost. However, the coefficients of $\mathcal{F}_{\boldsymbol{k}}$ in a g_s expansion are elements of the ring of Jacobi forms with modular parameter τ and elliptic parameters $\boldsymbol{m}$, tensored with the ring of quasi-modular forms. We next study the pole structure of $\mathcal{F}_{\boldsymbol{k}}$ as a function of z. In the case of elliptic fibrations with only isolated fibral curves (geometries that engineer SCFTs in F-theory with at most abelian gauge symmetry are in this class), the denominator $\phi_{\boldsymbol{k}}^D$ in (6.0.1) in fact does not depend on fibral Kähler classes. This gives rise to a tractable pole structure for the free energy, and ultimately allows us to reduce the computation of all negative index Laurent coefficients of $\mathcal{Z}_{\boldsymbol{k}}$ to the knowledge of genus 0 Gromov-Witten invariants. The analogous considerations in the case of more general elliptic fibrations fail, as we explain.

We also discuss the case of the refined topological string. Here, the elliptic parameter $z = \frac{g_s}{2\pi}$ is replaced by two elliptic parameters $z_{1,2} = \frac{\epsilon_{1,2}}{2\pi}$. The role of the genus zero Gromov-Witten invariants in determining the partition function is played in the refined case by the Nekrasov-Shatashvili data

$$n_g^{\mathbf{d},\mathrm{NS}} = \sum_{g_L+g_R=g} n_{g_L,g_R}^{\mathbf{d}} \,, \tag{6.0.3}$$

with n_{g_L,g_R} indicating the refined Gopakumar-Vafa invariants.

This chapter is organized as follows. We first study the modularity and pole structure of the topological free energy $\mathcal{F}_{\boldsymbol{k}}$ in section 6.1. Section 6.2 explains how to reconstruct $\mathcal{Z}_{\boldsymbol{k}}$ from all its principal parts. In section 6.3 we show how to reduce the computation of the principal parts of $\mathcal{Z}_{\boldsymbol{k}}$ in the case of elliptic fibrations with only isolated fibral curves to the knowledge of genus zero Gromov-Witten invariants. We first work out one particular example in sufficient detail to motivate the general procedure, then we give the proof in full generality.

It is mostly based on section 3, 4 and section 5 of the article *Computing the elliptic genus of higher rank E-strings from genus 0 GW invariants* [56] by Jie Gu, Amir-Kian Kashani-Poor and the author, with various changes for pedagogical reasons.

6.1 The structure of the topological string free energy

6.1.1 Setting up the analysis

Depending on the computational approach one takes, the topological string partition function or its free energy moves to the fore. The considerations in section 5.2 centered around Z_{top}, as this quantity has the better transformation properties under the modular group. From the Gopakumar-Vafa presentation, it is clear however that the partition function contains redundant information due to multi-wrapping contributions. This redundancy is easily identified at the level of the free energy. Recall that it takes the general form

$$F(g_s,\bar{\tau}) = \sum_{\substack{w\geq 1\\ g\geq 0}} \sum_{\mathbf{d}\in H_2(X,\mathbb{Z})} n_g^{\mathbf{d}} \left(2\sin\left(\frac{wg_s}{2}\right)\right)^{2g-2} \frac{\mathbf{Q}^{w\mathbf{d}}}{w}\,, \qquad n_g^{\mathbf{d}}\in\mathbb{Z}\,, \tag{6.1.1}$$

with single-wrapping contributions

$$f(g_s,\bar{\tau}) = \sum_{g\geq 0} \sum_{\mathbf{d}\in H_2(X,\mathbb{Z})} n_g^{\mathbf{d}} \left(2\sin\left(\frac{g_s}{2}\right)\right)^{2g-2} \mathbf{Q}^{\mathbf{d}}\,. \tag{6.1.2}$$

We thus turn to the study of the free energy on elliptic Calabi-Yau manifolds in this section. These considerations will play an important role when determining the interdependence of the principal parts of the base degree $\boldsymbol{k}$ contributions $Z_{\boldsymbol{k}}$ to the partition function in section 6.3.

As in the case of the partition function, it will be convenient to consider coefficients $F_{\boldsymbol{k}}$ of the free energy in an expansion in base degree classes $\boldsymbol{Q_B}$ (recall that $z=\frac{g_s}{2\pi}$),

$$Z_{\text{top}} = Z_0\left(\sum_{\boldsymbol{k}>0} Z_{\boldsymbol{k}}\boldsymbol{Q}_{\boldsymbol{B}}^{\boldsymbol{k}}\right) = Z_0\exp\left(\sum_{\boldsymbol{k}>0} F_{\boldsymbol{k}}\boldsymbol{Q}_{\boldsymbol{B}}^{\boldsymbol{k}}\right)\,, \tag{6.1.3}$$

with single-wrapping contribution

$$f_{\boldsymbol{k}}(z,\bar{\tau}) = \sum_{g,d_\tau,\boldsymbol{d}_m} n_g^{\boldsymbol{k},d_\tau,\boldsymbol{d}_m}(2\sin\pi z)^{2g-2} q^{d_\tau}\mathbf{Q_m}^{\boldsymbol{d}_m}\,. \tag{6.1.4}$$

We can invert

$$Z_{\boldsymbol{k}} = \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\substack{\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n>\mathbf{0}\\ \sum\boldsymbol{k}_i=\boldsymbol{k}}}\prod_{i=1}^{n} F_{\boldsymbol{k}_i} \tag{6.1.5}$$

to obtain the free energy at base wrapping $\boldsymbol{k}$ in terms of $Z_{\boldsymbol{k}'}$, $|\boldsymbol{k}'|\leq|\boldsymbol{k}|$,

$$F_{\boldsymbol{k}} = \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\substack{\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n>\mathbf{0}\\ \sum\boldsymbol{k}_i=\boldsymbol{k}}} a^{(\boldsymbol{k})}_{\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n}\prod_{i=1}^{n} Z_{\boldsymbol{k}_i} = \sum_n M_{\boldsymbol{k},n}(\{Z_{\boldsymbol{k}'}:|\boldsymbol{k}'|\leq|\boldsymbol{k}|\})\,. \tag{6.1.6}$$

Note that the sums over n in (6.1.5) and (6.1.6) are effectively finite due to the constraint $\boldsymbol{k}_i>\mathbf{0}$ on the n summation coefficients $\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n$. This constraint also implies that the monomials $M_{\boldsymbol{k},n}$ are homogeneous in the index $\boldsymbol{k}_i$ of is arguments, $\sum_i\boldsymbol{k}_i=\boldsymbol{k}$.

The integer coefficients $a^{(\boldsymbol{k})}_{\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n}$ are easily computed iteratively. In particular, $a^{(\boldsymbol{k})}_{\boldsymbol{k}}=1$. We give the corresponding monomial the index 1,

$$M_{\boldsymbol{k},1} = Z_{\boldsymbol{k}}\,. \tag{6.1.7}$$

In the case of $b_2(B)=1$, e.g., we have

$$F_1 = Z_1\,, \quad F_2 = Z_2 - \frac{1}{2}Z_1^2\,, \quad F_3 = Z_3 - Z_1Z_2 + \frac{1}{3}Z_1^3\,, \quad \ldots \tag{6.1.8}$$

and

$$M_{1,1} = Z_1 \ , \tag{6.1.9}$$

$$M_{2,1} = Z_2 \,, \quad M_{2,2} = -\frac{1}{2} Z_1^2 \ , \tag{6.1.10}$$

$$M_{3,1} = Z_3 \,, \quad M_{3,2} = -Z_1 Z_2 \,, \quad M_{3,3} = \frac{1}{3} Z_1^3 \ , \quad \ldots . \tag{6.1.11}$$

To compute the single-wrapping contribution $f_{\boldsymbol{k}}$, it suffices to subtract from $F_{\boldsymbol{k}}$ appropriate linear combinations of $F_{\boldsymbol{k}'}$, $|\boldsymbol{k}'| < |\boldsymbol{k}|$, evaluated at integer multiples of all of their arguments. Symbolically, we write this as

$$f_{\boldsymbol{k}}(z,\bar{\tau}) = F_{\boldsymbol{k}}(z,\bar{\tau}) + P_{\boldsymbol{k}}^{F\to f}(F_{|\boldsymbol{k}'|<|\boldsymbol{k}|}(*z,*\bar{\tau})) \ , \tag{6.1.12}$$

where $*$ is a placeholder for a possible multi-wrapping factor, and $P_{\boldsymbol{k}}^{F\to f}$ is a polynomial with the property that every contributing monomial $\prod_i F_{\boldsymbol{k}_i}(w_i z, w_i\bar{\tau})$ satisfies $\sum_i w_i \boldsymbol{k}_i = \boldsymbol{k}$. In the case $b_2(B) = 1$, the first few expressions are

$$\begin{aligned}
f_1(z,\bar{\tau}) &= F_1(z,\bar{\tau}) \,, \\
f_2(z,\bar{\tau}) &= F_2(z,\bar{\tau}) - \frac{1}{2} F_1(2z,2\bar{\tau}) \,, \\
f_3(z,\bar{\tau}) &= F_3(z,\bar{\tau}) - \frac{1}{3} F_1(3z,3\bar{\tau}) \\
&\ldots .
\end{aligned} \tag{6.1.13}$$

Combining this with (6.1.6), we immediately obtain

$$f_{\boldsymbol{k}}(z,\bar{\tau}) = Z_{\boldsymbol{k}}(z,\bar{\tau}) + P_{\boldsymbol{k}}^{Z\to f}(Z_{|\boldsymbol{k}'|<|\boldsymbol{k}|}(*z,*\bar{\tau})) \ . \tag{6.1.14}$$

Again in the case $b_2(B) = 1$, the first few relations are

$$\begin{aligned}
f_1(z,\bar{\tau}) &= Z_1(z,\bar{\tau}) \ , \\
f_2(z,\bar{\tau}) &= Z_2(z,\bar{\tau}) - \frac{1}{2} Z_1(2z,2\bar{\tau}) - \frac{1}{2} Z_1(z,\bar{\tau})^2 \ , \\
f_3(z,\bar{\tau}) &= Z_3(z,\bar{\tau}) - \frac{1}{3} Z_1(3z,3bt) - \frac{1}{6} Z_1(z,\bar{\tau})^3 \ , \\
&\ldots .
\end{aligned} \tag{6.1.15}$$

We of course can also invert these relation to obtain

$$Z_{\boldsymbol{k}}(z,\bar{\tau}) = f_{\boldsymbol{k}}(z,\bar{\tau}) + P_{\boldsymbol{k}}^{f\to Z}(f_{|\boldsymbol{k}'|<|\boldsymbol{k}|}(*z,*\bar{\tau})) \,. \tag{6.1.16}$$

When $b_2(B) = 1$,

$$\begin{aligned}
Z_1(z,\bar{\tau}) &= f_1(z,\bar{\tau}) \\
Z_2(z,\bar{\tau}) &= f_2(z,\bar{\tau}) + \frac{1}{2} f_1(2z,2\bar{\tau}) + \frac{1}{2} f_1(g_s,\bar{\tau})^2 \\
Z_3(z,\bar{\tau}) &= f_3(z,\bar{\tau}) + \frac{1}{3} f_1(3z,3\bar{\tau}) + \frac{1}{6} f_1(z,\bar{\tau})^3 \\
&\ldots .
\end{aligned} \tag{6.1.17}$$

In section 6.3, it will be more natural to reorder (6.1.14),

$$Z_{\boldsymbol{k}}(z,\bar{\tau}) = f_{\boldsymbol{k}}(z,\bar{\tau}) - P^{Z\to f}_{\boldsymbol{k}}(Z_{|\boldsymbol{k}'|<|\boldsymbol{k}|}(*z,*\bar{\tau}))\,. \tag{6.1.18}$$

This equation encodes the fact that when increasing the base degree by one step to $\boldsymbol{k}$, all the new information required to reconstruct $Z_{\boldsymbol{k}}$ is captured by the single-wrapping contribution to the free energy $f_{\boldsymbol{k}}$. Anticipating our discussion in section 6.3, we also introduce the monomials $m_{\boldsymbol{k},i}$ constituting $P^{Z\to f}_{\boldsymbol{k}}$,

$$Z_{\boldsymbol{k}} = f_{\boldsymbol{k}} + \sum m_{\boldsymbol{k},i}\,, \tag{6.1.19}$$

where each $m_{\boldsymbol{k},i}$ is of the form

$$m_{\boldsymbol{k},i} \propto \prod_j Z_{\boldsymbol{k}_{i,j}}(s_j\tau, s_j z, s_j\boldsymbol{m})\,, \quad \sum_j s_j \boldsymbol{k}_{i,j} = \boldsymbol{k}\,. \tag{6.1.20}$$

All of these formulae generalize straightforwardly to the refined case, given the refined Gopakumar-Vafa expansion (6.1.21)

$$F(\epsilon_{L,R},\bar{\tau}) = \sum_{g_{L,R}\geq 0}\sum_{w\geq 1}\sum_{\mathbf{d}\in H_2(X,\mathbb{Z})} \frac{n^{\mathbf{d}}_{g_L,g_R}}{w}\frac{(2\sin\frac{w\epsilon_L}{2})^{2g_L}(2\sin\frac{w\epsilon_R}{2})^{2g_R}}{2\sin\frac{w(\epsilon_R+\epsilon_L)}{2}2\sin\frac{w(\epsilon_R-\epsilon_L)}{2}}\mathbf{Q}^{w\mathbf{d}}\,, \quad n^{\mathbf{d}}_{g_L,g_R}\in\mathbb{Z} \tag{6.1.21}$$

as starting point. In terms of the equivariant parameters $\epsilon_{1,2}$, we have $\epsilon_{L,R} = \frac{\epsilon_1\mp\epsilon_2}{2}$. The conventional (non-refined) topological string is obtained by setting $\epsilon_2 = -\epsilon_1$, $g_s^2 = \epsilon_1^2$.

6.1.2 Transformation behavior

While the object with the best transformation behavior under the modular group is $\mathcal{Z}_{\boldsymbol{k}}$, the partition function at fixed base-wrapping, some of this behavior survives to the level of the free energies $\mathcal{F}_{\boldsymbol{k}}$. Perhaps somewhat surprisingly, the unrefined and the refined situations are qualitatively different. We discuss these two cases separately in this subsection.

The unrefined case

From the relation (6.1.6) between free energy and partition function, we can conclude that $\mathcal{F}_{\boldsymbol{k}}$ has the general form

$$\mathcal{F}_{\boldsymbol{k}}(\tau,z,\boldsymbol{m}) = \sum_n M_{\boldsymbol{k},n}(\mathcal{Z}_{\boldsymbol{k}'} : |\boldsymbol{k}'| \leq |\boldsymbol{k}|\})\,. \tag{6.1.22}$$

Note that the $M_{\boldsymbol{k},n}(\mathcal{Z}_{\boldsymbol{k}'} : |\boldsymbol{k}'| \leq |\boldsymbol{k}|\})$ as monomials in Jacobi forms are again Jacobi forms. They have identical $\boldsymbol{m}$-index, as this quantity is linear in the base-wrapping degree $\boldsymbol{k}$, and the $M_{\boldsymbol{k},n}$ are homogeneous with regard to base-wrapping. By contrast, the z-index depends quadratically on the base-wrapping degree, hence varies with the index n. $\mathcal{F}_{\boldsymbol{k}}$ is hence not a Jacobi form of fixed z-index. To make progress, we Laurent expand in the variable z,

$$\mathcal{F}_{\boldsymbol{k}}(\tau,z,\boldsymbol{m}) = \sum_{g=0}^{\infty}\mathcal{F}_{\boldsymbol{k},g}(\tau,\boldsymbol{m})\,z^{2g-2}\,. \tag{6.1.23}$$

The coefficients $\mathcal{F}_{\boldsymbol{k},g}(\tau,\boldsymbol{m})$ are the genus g contributions to the free energy at base degree $\boldsymbol{k}$.

In the absence of divisors of type 3 (i.e. non-abelian gauge symmetry in the case of SCFTs), $\mathcal{Z}_{\boldsymbol{k}}$ is of the form

$$\mathcal{Z}_{\boldsymbol{k}} = \frac{\sum c_i \phi_{\boldsymbol{k},i}(\tau,z,\boldsymbol{m})}{\phi_{\boldsymbol{k}}^D(\tau,z)}\,, \tag{6.1.24}$$

with

$$\phi_{\boldsymbol{k}}^D = \prod_{i=1}^{b_2(B)} \prod_{s=1}^{k_i} \phi_{-1,1/2}(\tau,sz_1)\phi_{-1,1/2}(\tau,sz_2)\,, \tag{6.1.25}$$

and $\phi_{\boldsymbol{k},i}(\tau,z,\boldsymbol{m})$ an element of the tensor product of the ring of Jacobi forms in z and in $\boldsymbol{m}$, and with $\phi_{\boldsymbol{k}}^D(\tau,z)$ having τ-independent leading coefficient in z. The expansion in z thus yields Jacobi forms in $\boldsymbol{m}$ with quasi-modular forms (see appendix A), as coefficients. By the argument above, the $\boldsymbol{m}$-index of $\mathcal{F}_{\boldsymbol{k},g}$ is equal to that of $\mathcal{Z}_{\boldsymbol{k}}$. Restoring the η-dependence, the weight of $F_{\boldsymbol{k},g}$ can be read off from the power of z it multiplies; it is equal to $2g-2$.

The structure of $\mathcal{F}_{\boldsymbol{k},g}$ for these cases can be further constrained. Let us restrict for simplicity to the case $b_2(B)=1$. Fix k and g. The contributions of the second Eisenstein series $E_2(\tau)$, the source of the "quasi" in the quasi-modularity of $\mathcal{F}_{\boldsymbol{k},g}$, stem from the Taylor expansion of the Jacobi forms A and B in z. From the Taylor series of A and B, given in (A.2.10), we can infer that the highest power of E_2 at given order in z will arise from the contribution to $\mathcal{F}_{\boldsymbol{k},g}$ with the highest power in B. At genus g, this highest power is $2k+2g-2$; the contribution $2k$ stems from the leading power z^{2k} in the universal denominator (6.1.25) of $\mathcal{Z}_{\boldsymbol{k}}$ in the absence of divisors of type 3. The bound on the highest power in E_2 is thus $k+g-1$, which lies below the bound provided by the weight alone.

The structure of $\mathcal{F}_{\boldsymbol{k},g}$ in the presence of divisors of type 3 (i.e. of non-abelian gauge symmetry in the case of SCFTs) is similar. To arrive at this conclusion, consider again the general form

$$\mathcal{Z}_{\boldsymbol{k}} = \frac{\sum c_i \phi_{\boldsymbol{k},i}(\tau,z,\boldsymbol{m})}{\phi_{\boldsymbol{k}}^D(\tau,z,\boldsymbol{m})}\,. \tag{6.1.26}$$

of $\mathcal{Z}_{\boldsymbol{k}}$. The denominator, as given in (5.2.6), now has $\boldsymbol{m}$-dependence. Naively, the occurrence of linear combinations of z and $\boldsymbol{m}$ as elliptic parameters invalidates the above argument for the structure of $\mathcal{F}_{\boldsymbol{k},g}$. Upon rewriting the denominator in the form

$$\left(\prod_{s=1}^{k}\phi_{-2,1}(\tau,sz)\right)\prod_{\alpha\in\Delta_+}\phi_{-2,1}(\tau,m_\alpha)^{\alpha_k(0)}\left(\prod_{j=1}^{k-1}\prod_{\alpha\in\Delta_+}\Big(\phi_{-2,1}(\tau,jz+m_\alpha)\phi_{-2,1}(\tau,jz-m_\alpha)\Big)^{\alpha_k(j)}\right)\,, \tag{6.1.27}$$

where

$$\alpha_k(j) = \left\lceil\frac{k-j}{2}\right\rceil\,, \tag{6.1.28}$$

we note however that each occurrence of the linear combination $jz+m_\alpha$ as elliptic parameter of a $\phi_{-2,1}$ factor in $\phi_{\boldsymbol{k}}^D$ is balanced by the occurrence of $jz-m_\alpha$. The index bilinear form of the denominator thus disentangles the contribution of z and of $\boldsymbol{m}$ as elliptic parameters. The product over positive roots guarantees invariance under the action of the Weyl group. This is consistent with $\phi^D_{\boldsymbol{k}}$ taking value in the tensor product of the rings of Jacobi forms with elliptic parameter z and of Weyl invariant Jacobi forms with elliptic parameter $\boldsymbol{m}$. Indeed, the identity

$$\phi_{-2,1}(z_1)\phi_{-2,1}(z_2) = \frac{1}{144}\left(\phi_{-2,1}(z_+)\phi_{0,1}(z_-) - \phi_{-2,1}(z_-)\phi_{0,1}(z_+)\right)^2 , \quad z_\pm = \tfrac{z_1 \pm z_2}{2} , \quad (6.1.29)$$

can be used to eliminate linear combinations of z and m_α as elliptic arguments. The leading coefficient of $\phi^D_{\boldsymbol{k}}(\tau, z, \boldsymbol{m})$ in z is a Weyl invariant Jacobi form. Laurent expanding $\mathcal{F}_{\boldsymbol{k}}$ in z thus yields $\mathcal{F}_{\boldsymbol{k},g}$ as meromorphic Weyl invariant Jacobi forms in $\boldsymbol{m}$ with coefficients in the ring of quasi-modular forms.

As an example, we list low genus results for the free energy at base wrapping degree 2 for the geometry of example 4, based on the calculation in [48]:

$$\begin{aligned}
\mathcal{F}_{2,0} &= \frac{1}{(2\pi \mathsf{i})^2}\frac{5\phi^4_{-2,3} - 1792\phi_{0,3}\phi_{-2,3}\phi_{-6,6} + 32E_2\phi^2_{-2,3}\phi_{-6,6} + 512E_4\phi^2_{-6,6}}{4096\phi^3_{-6,6}} , \\
\mathcal{F}_{2,1} &= \frac{1}{98304\phi^4_{-6,6}}(3\phi^6_{-2,3} - 1568\phi_{0,3}\phi^3_{-2,3}\phi_{-6,6} + 50E_2\phi^4_{-2,3}\phi_{-6,6} + 73728\phi^2_{0,3}\phi^2_{-6,6} \\
&\quad -17920E_2\phi_{0,3}\phi_{-2,3}\phi^2_{-6,6} + 224E_2^2\phi^2_{-2,3}\phi^2_{-6,6} + 672E_4\phi^2_{-2,3}\phi^2_{-6,6} + 5120E_2E_4\phi^3_{-6,6} \\
&\quad +2048E_6\phi^3_{-6,6}) , \\
\mathcal{F}_{2,2} &= \frac{(2\pi \mathsf{i})^2}{754974720\phi^5_{-6,6}}(495\phi^8_{-2,3} - 341760\phi_{0,3}\phi^5_{-2,3}\phi_{-6,6} + 9600E_2\phi^6_{-2,3}\phi_{-6,6} \\
&\quad +44892160\phi^2_{0,3}\phi^2_{-2,3}\phi^2_{-6,6} - 5017600E_2\phi_{0,3}\phi^3_{-2,3}\phi^2_{-6,6} + 80000E_2^2\phi^4_{-2,3}\phi^2_{-6,6} \\
&\quad +149120E_4\phi^4_{-2,3}\phi^2_{-6,6} + 235929600E_2\phi^2_{0,3}\phi^3_{-6,6} - 28672000E_2^2\phi_{0,3}\phi_{-2,3}\phi^3_{-6,6} \\
&\quad -22839296E_4\phi_{0,3}\phi_{-2,3}\phi^3_{-6,6} + 266240E_2^3\phi^2_{-2,3}\phi^3_{-6,6} + 2158592E_2E_4\phi^2_{-2,3}\phi^3_{-6,6} \\
&\quad +548864E_6\phi^2_{-2,3}\phi^3_{-6,6} + 8192000E_2^2E_4\phi^4_{-6,6} + 3342336E_4^2\phi^4_{-6,6} + 6553600E_2E_6\phi^4_{-6,6}) .
\end{aligned}$$

This result is expressed in terms of the generators of the ring $J^D_{*,*}(\mathfrak{a}_2)$ introduced in appendix A. The map $\mathcal{F}_{k,g} \to F_{k,g}$ is implemented by replacing these generators by their images in $J^{\widehat{D}}_{*,*}(\mathfrak{a}_2)$ under the map (A.3.17), and dividing by $\eta(\tau)^{12k}$.

The refined case

In topological string applications, the refined free energy is sometimes usefully written as a function of the parameters $g_s^2 = -\epsilon_1\epsilon_2$ and $s = \epsilon_1 + \epsilon_2$. The symmetry under exchange of ϵ_1 and ϵ_2, apparent in (6.1.21), is manifest with this choice. For our purposes of retaining some of the transformation properties of the partition function in passing to the free energy, this choice at first glance appears propitious, as the index bilinear written in terms of these variables decomposes into a sum of squares, with the coefficient of s^2 depending linearly on $\boldsymbol{k}$. This naively suggests that the expansion coefficients of $\mathcal{F}_{\boldsymbol{k}}$ in g_s should have good transformation properties with regards to the remaining parameter s. This however is not the case, as unlike the superficially analogous case of $\boldsymbol{m}$-dependence in the unrefined case, expansion in g_s breaks the periodicity in the variable s as well; $\mathcal{F}_{\boldsymbol{k},g}(\tau,s)$ hence cannot be expanded in Jacobi forms in the elliptic parameter s. Expanding $\mathcal{F}_{\boldsymbol{k}}(\tau,z_L,z_R,\boldsymbol{m})$ therefore in both z_L and z_R (recall that $z_\bullet = \frac{\epsilon_\bullet}{2\pi}$), we obtain

$$\mathcal{F}_{\boldsymbol{k}}(\tau,z_L,z_R,\boldsymbol{m}) = \frac{1}{z_1 z_2}\sum_{g=0}^{\infty}\mathcal{F}_{\boldsymbol{k},g_L,g_R}(\tau,\boldsymbol{m})\, z_L^{2g_L} z_R^{2g_R}\,. \tag{6.1.30}$$

The same arguments as in the unrefined case show that $\mathcal{F}_{\boldsymbol{k},g_L,g_R}(\tau,\boldsymbol{m})$ are elements of the ring of Jacobi forms with elliptic index $\boldsymbol{m}$, tensored over the ring of quasi-modular forms. The $\boldsymbol{m}$-index is that of $\mathcal{Z}_{\boldsymbol{k}}$, and the weight increases with the indices g_L and g_R.

6.1.3 Pole structure of single-wrapping contributions to free energy

Divisors of type 3 modify the pole structure of $Z_{\boldsymbol{k}}$, and therefore $f_{\boldsymbol{k}}$ drastically, as can be seen in (5.2.6) or (6.1.27). We will thus discuss geometries with and without such divisors separately in this section.

Absence of divisors of type 3 (no non-abelian gauge symmetry) – unrefined

We will argue that $f_{\boldsymbol{k}}(z,\tau,\boldsymbol{m})$ as a function of $z = \frac{g_s}{2\pi}$ has the following pole structure:

(I) A second order pole at all integers, and no further poles on the real line. The Laurent coefficients at these poles are determined by the single-wrapping genus zero Gromov-Witten invariants at base degree $\boldsymbol{k}$.

(II) All the other poles are at non-real s-torsion points for all integers $s \le \max\{k_i\}$. These poles are of order 2ℓ and less, where $\ell = \sum_i \lfloor k_i/s \rfloor$. The Laurent coefficients of these poles are determined by the single-wrapping free energies $f_{\boldsymbol{k}'}$, $|\boldsymbol{k}'| < |\boldsymbol{k}|$, together with the data specified in **(I)**.

To show **(I)**, notice that the Gopakumar-Vafa presentation (6.1.2) of the single-wrapping contributions in the unrefined case becomes

$$f_{\boldsymbol{k}}(z,\tau,\boldsymbol{m}) = \sum_{g,d_\tau,\boldsymbol{d}_m \ge 0} n_g^{\boldsymbol{k},d_\tau,\boldsymbol{d}_m}\,(2\sin\pi z)^{2g-2}\, q^{d_\tau}\mathbf{Q}_m^{\boldsymbol{d}_m}\,. \tag{6.1.31}$$

The second order poles at integer values of z are visible from the $\sin(\pi z)$ factor with $g=0$. Their Laurent coefficients depend only on the single-wrapping genus zero Gromov-Witten invariants; for example at $z \to 0$

$$f_{\boldsymbol{k}}(z,\tau,\boldsymbol{m}) = \frac{\sum_{d_\tau,\boldsymbol{d}_m\ge 0} n_0^{\boldsymbol{k},d_\tau,\boldsymbol{d}_m} q^{d_\tau}\mathbf{Q}^{\boldsymbol{d}_m}}{(2\pi)^2 z^2} + \text{regular terms in } z\,. \tag{6.1.32}$$

Note that we prefer speaking of single-wrapping genus zero Gromov-Witten invariants, rather than the identical genus zero Gopakumar-Vafa invariants, as we are concerned with the z expansion of $f_{\boldsymbol{k}}$, rather than the expansion in $\sin \pi z$.

Regarding the possibility of other real poles, the infinite sum in q cannot lead to poles at τ-independent (hence real) points, and the infinite sum in $\mathbf{Q}_m$ does not lead to additional poles in the absence of divisors of type 3. $f_{\boldsymbol{k}}$ hence has no other poles on the real line.

To argue for **(II)**, consider the expression (6.1.14) of the single-wrapping function $f_{\boldsymbol{k}}(z, \bar{\tau})$ in terms of the partition function $Z_{\boldsymbol{k}'}$ at base wrappings $|\boldsymbol{k}'| \leq |\boldsymbol{k}|$.

$f_{\boldsymbol{k}}(\tau, z, \boldsymbol{m})$ will at best share the poles of the base-wrapping $\boldsymbol{k}'$ partition functions $Z_{\boldsymbol{k}'}$ appearing on the RHS of (6.1.14). These poles lie at s-torsion points for all integers $s \leq \max\{k_i\}$. The maximal order of a pole at an s-torsion point will arise from a contribution to (6.1.14) with a maximal number ℓ of $Z_\bullet$ factors at multi-wrapping s. This number is $\ell = \sum_i \lfloor k_i/s \rfloor$, and the order of the corresponding pole is 2ℓ.

By **(I)**, a certain number of cancellations must take place between the poles stemming from the monomials contributing to $f_{\boldsymbol{k}}$:

- All but the second order pole at integer values of z cancel.
- All real poles at real s torsion points, $s > 1$, cancel.

We observe by explicit computation that generically, no other cancellations occur. For all non-real s-torsion points, $s > 1$, this follows from the following observation: as we will see in detail in section 6.3, all poles at s-torsion points for a given s are related via modular transformations. Given that the monomials contributing to $f_{\boldsymbol{k}}$ have different indices, the vanishing of the poles at one element of this $SL(2,\mathbb{Z})$ orbit, the real s-torsion point, generically excludes the vanishing at all others.

We will show in section 6.3 that the principal part of the partition function $Z_{\boldsymbol{k}}$ at base-wrapping $\boldsymbol{k}$ can be computed from knowledge of the partition function at lower base-wrapping, in conjunction with knowledge of the single-wrapping genus zero Gromov-Witten invariants at base-wrapping $\boldsymbol{k}$. The central ingredient in the argument is that all poles at s-torsion points for fixed s are related by an $SL(2,\mathbb{Z})$ action, and each $SL(2,\mathbb{Z})$ orbit has a distinguished real representative; the principal part of the Laurent expansion of $Z_{\boldsymbol{k}}$ around this point is determined by the data proposed. This property of $Z_{\boldsymbol{k}}$ implies the final claim of **(II)** regarding the pole structure of $f_{\boldsymbol{k}}(\tau, z, \boldsymbol{m})$.

Absence of divisors of type 3 (no non-abelian gauge symmetry) – refined

As in the unrefined limit given by $\epsilon_1 = -\epsilon_2$, $\epsilon_L \propto g_s$ while $\epsilon_R = 0$, we will study the pole structure of $f_{\boldsymbol{k}}(z_L, z_R, \bar{\tau})$ as a function of z_L. This choice is not necessarily canonical. Arguing in direct analogy to the unrefined case, we find the following:

(I) $f_{\boldsymbol{k}}(z_L, z_R, \bar{\tau})$ as a function of z_L has simple poles at $z_L = \pm z_R + \mathbb{Z}$ and no further poles along the axes $\pm z_R + \mathbb{R}$. At these poles, the Laurent coefficients depend only on the refined GV invariants $n_g^{\mathbf{d},\mathrm{NS}}$ appearing in the NS limit $\epsilon_1 \to 0$ (or equivalently $\epsilon_2 \to 0$) of the free energy, which are given by

$$n_g^{\mathbf{d},\mathrm{NS}} = \sum_{g_L+g_R=g} n_{g_L,g_R}^{\mathbf{d}} \,. \tag{6.1.33}$$

(II) There are further poles at $\pm z_R$ shifted by non-real s-torsion points for all integers $s \leq \max\{k_i\}$. These poles are of order $\ell = \sum_i \lfloor k_i/s \rfloor$ and less.

The argument proceeds in precise analogy to the unrefined case in section 6.1.3. To see that the invariants (6.1.33) play the role of the genus zero Gromov-Witten invariants in the unrefined case, note that expanded around $z_L = \pm z_R$,

$$\begin{aligned} f_{\boldsymbol{k}}(z_L, z_R, \bar{\tau}) = \mp \frac{\sum_{d_\tau, \boldsymbol{d}_m, g \geq 0} n_g^{n, d_\tau, \boldsymbol{d}_m, \mathrm{NS}} (2\sin \pi z_R)^{2g} / (2 \sin 2\pi z_R) q^{d_\tau} \mathbf{Q}_m^{\boldsymbol{d}_m}}{2\pi (z_L \mp z_R)} \\ + \text{regular terms in } z_L \mp z_R \,. \end{aligned} \tag{6.1.34}$$

Presence of divisors of type 3 (gauge symmetry) - unrefined

In geometries that exhibit divisors of type 3, the denominator $\phi_{\boldsymbol{k}}^D$ exhibited in (6.0.1) depends on the associated Kähler parameters, and $Z_{\boldsymbol{k}}$ as a function of z exhibits poles that depend on these parameters. These poles are inherited by the free energies $F_{\boldsymbol{k}}$.

We will focus on the case $b_2(B) = 1$. From the explicit form of the denominator $\phi_{\boldsymbol{k}}^D$ given in (6.1.27), we can read off the poles of the partition function Z_k at fixed base degree k:

1) Poles of order $2\lfloor k/s \rfloor$ at s-torsion points for $1 \leq s \leq k$.

2) Poles at
$$z_j = \frac{m_\alpha}{j} + j\text{-torsion point} \tag{6.1.35}$$
for $1 \leq j \leq k-1$ and $m_\alpha = (\boldsymbol{m}, \alpha)$, $\alpha \in \Delta$, of order $2\alpha_k(j)$ as defined in (6.1.28).

The poles of the first kind are independent of the presence of divisors of type 3, and can be treated as in subsection 6.1.3. The poles of the second kind are qualitatively different, as no representative in their $SL(2,\mathbb{Z})$ orbit is related to simply accessible data: they are all invisible in the Gopakumar-Vafa presentation (6.1.1) of the free energy. This limitation reduces the utility of analyzing the pole structure of Z_k via knowledge of the pole structure of f_k. Having come this far, we nevertheless want to make some observations regarding the latter.

Let us first consider F_k and the possibility that the poles of the monomials $M_{k,n}$ on the RHS of the expression (6.1.6) for F_k in terms of $Z_{k'}$, $k' \leq k$ cancel. For $k-j$ odd, this is ruled out by the fact that $\alpha_k(j) > \alpha_{k'}(j)$ for $k' < k$. Hence, the order of the pole at z_j is highest amongst the monomial $M_{k,n}$ in the one equaling Z_k, the pole at this order can therefore not be canceled. For $k - j$ even, $\alpha_k(j) = \alpha_{k-1}(j)$ and $\alpha_{k-1}(j) > \alpha_{k'-1}(j)$ for $k' < k$, so the Z_k and the $\propto Z_{k-1}Z_1$ contribution to F_k have the same order pole at z_j. We have checked via explicit computation that these do not cancel.

Turning now to f_k, by the multi-wrapping structure (6.1.2) of the free energy, the single-wrapping quantities f_k must exhibit poles beyond those of F_k. To see the origin of such poles, consider e.g. the base degree $k = 4$. We have

$$F_4(\tau, z, m_\alpha) = f_4(\tau, z, m_\alpha) + \frac{1}{2} f_2(2\tau, 2z, 2m_\alpha) + \frac{1}{4} f_1(4\tau, 4z, 4m_\alpha)\,. \tag{6.1.36}$$

As $f_2(\tau, z, m_\alpha)$ exhibits poles at $z = r + s\tau \pm m_\alpha$, $f_2(2\tau, 2z, 2m_\alpha)$ will exhibit poles at $z = \frac{r}{2} + s\tau \pm m_\alpha$. But as no the $Z_\bullet$ exhibits poles at this location, neither can F_4. $f_1 = Z_1$, hence does not exhibit a pole at this location either. The pole must hence be canceled by f_4.

6.2 $Z_{\mathbf{k}}$ of negative index

In the following subsections, we first explain why the negativity of the index is crucial to determining the partition functions. We then review the technique presented in [34] to explicitly construct a negative index Jacobi form solely from its principal part.

6.2.1 Why negative index is simpler

The observation that negative index Jacobi forms are determined by their principal parts holds simply because holomorphic Jacobi forms of negative index do not exist.

Indeed, consider two meromorphic Jacobi forms of equal weight, index, and principal parts. Their difference is again a Jacobi form of same weight and index as before, but with vanishing principle parts. If the index is negative, this difference, by the non-existence of non-trivial holomorphic Jacobi forms of negative index, is hence zero. The statement follows.

Let us also point out why the argument fails for elliptic fibrations in the presence of type 3 curves. The denominator in this case is given by (6.1.27). The mass dependent poles are not visible in the Gopakumar-Vafa form, and as we say in section 6.1, they do not distribute nicely with regard to multi-wrapping considerations. Hence, the difference

(6.2.1) in the singular case takes the more complicated form

$$Z_k^1 - Z_k^2 = \frac{\phi_{\boldsymbol{k}}(\tau, z, \boldsymbol{m})}{\phi_{-2,1}(\tau, z) \prod_{\alpha\in\Delta_+} \phi_{-2,1}(\tau, m_\alpha)^{\alpha_k(0)} \left(\prod_{j=1}^{k-1} \prod_{\alpha\in\Delta_+} \left(\phi_{-2,1}(\tau, jz+m_\alpha)\phi_{-2,1}(\tau, jz-m_\alpha)\right)^{\alpha_k(j)}\right)}\,. \tag{6.2.1}$$

To argue for vanishing as above, the sum of the index of the LHS and the index of the denominator of the RHS must be negative, a much weaker constraint than previously.

6.2.2 Explicit reconstruction of Jacobi forms from their principal parts

The construction presented in [34] relies on the existence of a function $F_M(\tau, z, u)$ for any $M \in \frac{1}{2}\mathbb{N}$, which satisfies the following two properties:

1) It is quasi-periodic as a function of u, i.e. it satisfies the relation

$$F_M(\tau, z, u + \lambda\tau + \mu) = (-1)^{2M\mu}\mathbf{e}\left[-M(\lambda^2\tau + 2\lambda u)\right] F_M(\tau, z, u)\,. \tag{6.2.2}$$

   For $M \in \mathbb{N}$, this is the elliptic transformation property of a Jacobi form of index M.

2) Again as a function of u, it is meromorphic with only simple poles. These lie at $z + \mathbb{Z}\tau + \mathbb{Z}$. The residue at $u = z$ is $\frac{1}{2\pi\mathsf{i}}$.

Now consider a Jacobi form $\phi_{-M}(\tau, u)$ of index $-M$, $M \in \mathbb{N}$. The product

$$\phi_{-M}(\tau, u)F_M(\tau, z, u) \tag{6.2.3}$$

is one- and τ-periodic in the variable u. The integral around the boundary of a fundamental parallelogram for the lattice $\mathbb{Z}\tau + \mathbb{Z}$, chosen to avoid all poles of the integrand (6.2.3) along the integration path, thus vanishes,

$$\oint_{\square} \phi_{-M}(\tau, u)F_M(\tau, z, u)\, du = 0\,. \tag{6.2.4}$$

Now choose z away from the poles of $\phi_{-M}(\tau, u)$. Evaluating the LHS of (6.2.4) by the residue theorem yields

$$\phi_{-M}(\tau, z) = -2\pi\mathsf{i} \sum_i \mathrm{Res}_{u=u_i}(\phi_{-M}(\tau, u)F_M(\tau, z, u))\,, \tag{6.2.5}$$

with the sum ranging over all poles of ϕ_{-M} in the chosen fundamental domain. This is the desired result, as the RHS of this expression is completely determined by F_M and the negative index Laurent coefficients at the poles of ϕ_{-M}.

Explicitly, F_M is a level 2M Appell-Lerch sum, given by the expression

$$F_M(\tau, z, u) = (\zeta\omega^{-1}) \sum_{n\in\mathbb{Z}} \frac{\omega^{-2Mn} q^{Mn(n+1)}}{1 - q^n \zeta \omega^{-1}}\,, \quad q = \mathbf{e}[\tau],\, \omega = \mathbf{e}[u],\, \zeta = \mathbf{e}[z]\,. \tag{6.2.6}$$

In terms of the principal parts D_{n,u_i} of $\phi_{-M}(\tau, z)$ at the poles u_i,

$$\phi_{-M}(\tau, z) = \sum_{n<0} D_{n,u_i}(z - u_i)^n + \text{holomorphic}\,, \tag{6.2.7}$$

the RHS of (6.2.5) can be evaluated to

$$\phi_{-M}(\tau, z) = \sum_i \sum_{n>0} \frac{D_{-n,u_i}}{(n-1)!} \left[\left(\frac{1}{2\pi \mathrm{i}} \frac{\partial}{\partial u} \right)^{n-1} F_M(\tau, z, u) \right]_{u=u_i} . \tag{6.2.8}$$

6.3 Partition function from genus zero GW Invariants

Based on the above discussion, we learn that $\mathcal{Z}_{\boldsymbol{k}}$ is completely determined by its principal parts at all poles inside the fundamental domain, and we have explicit way to reconstruct it given those data. One natural question then arises: since determining all the principal parts is still a huge amount of work, can we reduce the necessary data further to a minimal subset? As we will show, in fact what we only need is sufficient numbers of genus zero GW invariants[1].

The proof goes by induction. Instead of rushing to show the general proof which is somehow tedious, let's first look at one particular example: E-string theory. Without lost of generality and for simplicity, we set all the flavor masses to zero. Once we understand how does the induction work for base degree 1 and 2, we are readily able to understand how does it work in general.

6.3.1 Motivating Example: the massless E-string

In this subsection, we will work out the details of massless E-string at base degree one and two. Its index bilinear is

$$i_z(k) = -\frac{k(k+1)}{2}\,. \tag{6.3.1}$$

We introduce the following notation: $D_{z_0}\, g$ will stand for the principal part of $g(z)$ at $z = z_0$, and $D_{z_0,k}\, g$ for the coefficient of $(z - z_0)^{-k}$ in this expansion. Thus, if g has a pole at z_0 of maximal order ℓ,

$$D_{z_0}\, g = \sum_{n=-\ell}^{-1} a_n (z - z_0)^n\,, \quad D_{z_0,k}\, g = a_k\,. \tag{6.3.2}$$

[1]The precise number is the same as the dimension of elliptic modular forms at certain weight, which is surely always finite.

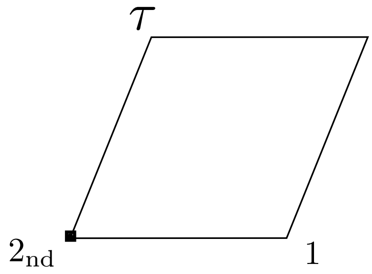


Figure 6.1: The position and the order of pole for $\mathcal{Z}_1$ inside the fundamental parallelogram.

The genus zero Gromov-Witten invariants required as input for our calculation can be determined e.g. with the help of mirror symmetry. For example, a partial list of the first few base degrees can be found in appendix E. This approach naturally yields the full genus zero free energy $\mathcal{F}_{k,0} = D_{0,2}\mathcal{F}_k$, even though the new data at each new base order is of course captured by the single-wrapping invariants, generated by f_k. We report here the results up to base degree 3:

$$D_{0,2}\,\mathcal{F}_1 = -(2\pi\mathsf{i})^{-2}E_4, \tag{6.3.3}$$

$$D_{0,2}\,\mathcal{F}_2 = \frac{(2\pi\mathsf{i})^{-2}}{24}(E_2E_4^2 + 2E_4E_6), \tag{6.3.4}$$

$$D_{0,2}\,\mathcal{F}_3 = \frac{(2\pi\mathsf{i})^{-2}}{15552}(4E_2^2E_4^3 + 109E_4^4 + 216E_2E_4^2E_6 + 197E_4E_6^2). \tag{6.3.5}$$

Here E_i are elliptic (quasi) modular forms whose explicit expressions can be found for instance in the appendix A. Note in particular the bound $k-1$ on the power of E_2 occurring in $\mathcal{F}_k$, as derived in section 6.1.2.

Base degree one: for partition function $\mathcal{Z}_1$, the general ansatz (6.0.1) together with the denominator (6.1.25) and the indices

$$\mathcal{Z}_1 = \frac{\phi_1(\tau,z)}{\phi_{-2,1}(\tau,z)}\,. \tag{6.3.6}$$

Inside the fundamental parallelogram for z spanned by 1 and τ, $\mathcal{Z}_{\boldsymbol{k}}$ only has a double pole at the origin, with coefficient (6.3.3). The position and the order of pole is shown in Figure 6.1.

The weight of $\mathcal{Z}_1$ fix the numerator $\phi_1(\tau,z)$ to have weight 4. This uniquely fixes it to be

$$\mathcal{Z}_1 = -\frac{E_4}{\phi_{-2,1}(\tau,z)}\,, \tag{6.3.7}$$

up to the proportionality factor -1, which can be determined from a single genus zero Gromov-Witten invariant.

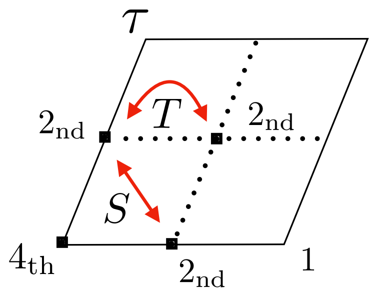


Figure 6.2: The position and the order of pole for $\mathcal{Z}_2$ inside the fundamental parallelogram.

Base degree two: next, assuming we have already determined $\mathcal{Z}_2$, let's see in detail how to determine $\mathcal{Z}_1$. For the case of base degree two, we have[2]

$$\begin{aligned} \mathcal{Z}_2(\tau, z) &= \mathcal{F}_2(\tau, z) + \frac{1}{2}\mathcal{Z}_1^2(\tau, z) \\ &= \mathfrak{f}_2(\tau, z) + \frac{1}{2}\frac{\eta^{24}(\tau)}{\eta^{12}(2\tau)}\mathcal{Z}_1(2\tau, 2z) + \frac{1}{2}\mathcal{Z}_1^2(\tau, z) . \end{aligned} \tag{6.3.8}$$

For reader's convenience, let's name the last two terms[3],

$$\mathfrak{m}_{2,1} = \frac{1}{2}\mathcal{Z}_1(2\tau, 2z) , \quad \mathfrak{m}_{2,2} = \frac{1}{2}\mathcal{Z}_1^2(\tau, z) . \tag{6.3.9}$$

From the general form (6.1.25) of the denominator of $\mathcal{Z}_k$, we can read off that within a fundamental parallelogram of z, $\mathcal{Z}_2$ exhibits poles at all 2-torsion points

$$z = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2} . \tag{6.3.10}$$

They are summarized in Figure 6.2.

To find the corresponding principal parts, we adopt the following strategy: we first determine the expansions around the real poles, and then relate the Laurent data at the non-real torsion points to these.

$\mathbf{z = 0}$ The structure of the denominator (6.1.25) when $k = 2$ tells us that there are a fourth and a second order pole at $z = 0$. The fourth order pole is due to the contribution $\frac{1}{2}\mathcal{Z}_1^2$ in (6.3.8). Its Laurent coefficient can be determined as

$$D_{0,4}\,\mathcal{Z}_2 = \frac{1}{2}\left(D_{0,2}\,\mathcal{Z}_1\right)^2 . \tag{6.3.11}$$

The same term also contributes to the second order pole, together with the genus zero part of $\mathcal{F}_2$,

$$D_{0,2}\,\mathcal{Z}_2 = (D_{0,2}\,\mathcal{Z}_1)(D_{0,0}\,\mathcal{Z}_1) + D_{0,2}\,\mathcal{F}_2 . \tag{6.3.12}$$

[2]Note that the $\mathfrak{f}_2(\tau, z)$ in (6.3.8) differs from $f_2(\tau, z)$ by some overall factor, so we use a different font.
[3]The notations are consistent with those appear in the next subsection.

$\mathbf{z}=\frac{\mathbf{1}}{\mathbf{2}}$ By property **(I)** of f_k and the presentation (6.1.16) of Z_k in terms of $f_\bullet$, the second order pole at $z=\frac{1}{2}$ is due to the multi-covering contribution $\frac{1}{2}\mathcal{Z}_1(2\tau,2z)$ to $\mathcal{F}_2$,

$$D_{\frac{1}{2},2}\,\mathcal{Z}_2=\frac{1}{2}\frac{\eta^{24}(\tau)}{\eta^{12}(2\tau)}D_{\frac{1}{2},2}\,\mathcal{Z}_1(2\tau,2z)\,. \tag{6.3.13}$$

$\mathbf{z}=\frac{\tau}{\mathbf{2}}$ The pole at $\frac{\tau}{2}$ is due to the contribution $\mathfrak{f}_2(\tau,z)$, which, a prior, we have no access to. To determine its Laurent data, we use the following crucial observation. The non-real torsion point $\frac{\tau}{2}$ is mapped to the real torsion point $\frac{1}{2}$ via the $SL(2,\mathbb{Z})$ matrix $S=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$!

We now compare the two Laurent expansions

$$\mathcal{Z}_2=\sum_{n=-2}^{\infty}\mathfrak{c}_{2,n}(\tau)(z-\tfrac{\tau}{2})^n=\sum_{n=-2}^{\infty}\mathfrak{b}_{2,n}(\tau)(z-\tfrac{1}{2})^n\,. \tag{6.3.14}$$

$\mathcal{Z}_2$ has weight $w=12$ and indices $i_z(2)=-3$. Invoking the modular transformation of a Jacobi Modular form for $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)=S$, i.e., (A.2.2), we obtain

$$\mathcal{Z}_2(\tau,z)=\tau^{-w}\mathbf{e}\left[-\tfrac{i_z(2)\,z^2}{\tau}\right]\mathcal{Z}_2(-\frac{1}{\tau},\frac{z}{\tau})=\tau^{-w}\mathbf{e}\left[-\tfrac{i_z(2)\,z^2}{\tau}\right]\sum_{n=-2}^{\infty}\frac{\mathfrak{b}_{2,n}(-\frac{1}{\tau})}{\tau^n}(z-\tfrac{\tau}{2})^n\,. \tag{6.3.15}$$

Comparing with (6.3.14) yields

$$\sum_{n=-2}^{\infty}\mathfrak{c}_{2,n}(\tau)(z-\tfrac{\tau}{2})^n=\tau^{-w}\mathbf{e}\left[-\tfrac{i_z(2)\,z^2}{\tau}\right]\sum_{n=-2}^{\infty}\frac{\mathfrak{b}_{2,n}(-\frac{1}{\tau})}{\tau^n}(z-\tfrac{\tau}{2})^n\,, \tag{6.3.16}$$

Our next task is to rewrite $\mathfrak{b}_{2,n}(-\frac{1}{\tau})$ for n negative so that it's more convenient to use. Of the two monomials $_{2,i}(\tau,z)$ in (6.3.9), only $\mathfrak{m}_{2,1}$ contributes to the poles at $\frac{1}{2}$. We have

$$D_{\frac{1}{2}}\,\mathfrak{m}_{2,1}(\tau,z)=\sum_{n=-2}^{-1}\mathfrak{b}_{2,n,1}(\tau)(z-\frac{1}{2})^n\,, \tag{6.3.17}$$

with $\mathfrak{b}_{2,n}=\mathfrak{b}_{2,n,1}$ for $n<0$, such that

$$\left(\frac{\eta^{24}(-\frac{1}{\tau})}{\eta^{12}(-\frac{2}{\tau})}\right)D_{\frac{\tau}{2}}\mathfrak{m}_{2,1}(-\tfrac{1}{\tau},\tfrac{z}{\tau})=\sum_{n=-2}^{-1}\frac{\mathfrak{b}_{2,n}(-\frac{1}{\tau})}{\tau^n}(z-\tfrac{\tau}{2})^n, \tag{6.3.18}$$

with

$$\mathfrak{m}_{2,1}(-\tfrac{1}{\tau},\tfrac{z}{\tau})=\frac{1}{2}\mathcal{Z}_1(-\tfrac{2}{\tau},\tfrac{2z}{\tau})\,, \tag{6.3.19}$$

Naively, the matrix $\left(\begin{smallmatrix}0&-2\\1&0\end{smallmatrix}\right)$ would map the modular argument $\tau'=-\frac{2}{\tau}$ on the RHS of (6.3.19) back to τ, which is the form that we want, but it is clearly not an element of $SL(2,\mathbb{Z})$. Here we use a small trick. We regard $\tilde{\tau}=\frac{\tau}{2}$ instead of τ as our modular parameter. Now the element $\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\in SL(2,\mathbb{Z})$ transforms the modular argument $\tilde{\tau}$ to $\frac{\tau}{2}$ thus gets rid of τ on the denominator. Acting with this transformation on the RHS of (6.3.19), we obtain

$$\mathfrak{m}_{2,1}(-\tfrac{1}{\tau},\tfrac{z}{\tau})=\frac{1}{2}\left(\frac{\tau}{2}\right)^{w/2}\mathbf{e}\left[2\tfrac{i_z(1)z^2}{\tau}\right]\mathcal{Z}_1(\tfrac{\tau}{2},z)\,. \tag{6.3.20}$$

Using this result to obtain a q-expansion of the RHS of (6.3.17) and plugging into (6.3.16) yields

$$\begin{aligned}
&\sum_{n=-2}^{-1} \mathfrak{c}_n(\tau)(z-\tfrac{\tau}{2})^n = \\
&\quad= D_{\frac{\tau}{2}} \tau^{-w} \mathbf{e}\left[-\tfrac{i_z(2)\,z^2}{\tau}\right] \left(\frac{\eta^{24}(-\frac{1}{\tau})}{\eta^{12}(-\frac{2}{\tau})}\right) \mathfrak{m}_{2,1}(-\tfrac{1}{\tau},\tfrac{z}{\tau}) \\
&\quad= \frac{1}{2} D_{\frac{\tau}{2}} \left(\frac{1}{2\tau}\right)^{w/2} \mathbf{e}\left[-\tfrac{i_z(2)-2i_z(1)}{\tau} z^2\right] \left(\frac{\eta^{24}(-\frac{1}{\tau})}{\eta^{12}(-\frac{2}{\tau})}\right) \mathcal{Z}_1(\tfrac{\tau}{2},z) \\
&\quad= -\frac{1}{2} D_{\frac{\tau}{2}} \mathbf{e}\left[-\tfrac{i_z(2)-2i_z(1)}{\tau} z^2\right] \left(\frac{\eta^{24}(\tau)}{\eta^{12}(\frac{\tau}{2})}\right) \mathcal{Z}_1(\tfrac{\tau}{2},z).
\end{aligned} \tag{6.3.21}$$

The overall minus sign is due to the non-trivial multiplier system of the Dedekind η function. Expanding the RHS around $z=\frac{\tau}{2}$ and comparing coefficients finally yields

$$\begin{aligned}
D_{\frac{\tau}{2},2}\,\mathcal{Z}_2(\tau) &= \mathfrak{c}_{-2}(\tau) = -\frac{1}{2} q^{\frac{1}{4}} \left(\frac{\eta^{24}(\tau)}{\eta^{12}(\frac{\tau}{2})}\right) D_{\frac{\tau}{2},2}\mathcal{Z}_1(\tfrac{\tau}{2},z)\,, \\
D_{\frac{\tau}{2},1}\,\mathcal{Z}_2(\tau) &= \mathfrak{c}_{-1}(\tau) = -(2\pi \mathrm{i}) i_z(2)\mathfrak{c}_{-2}(\tau)\,,
\end{aligned} \tag{6.3.22}$$

where we have evaluated $i_z(2)-2i_z(1)=-1$, and used the fact that $\mathcal{Z}_1$ only has a second order pole at $z=\frac{\tau}{2}$, as can be seen from its explicit form (6.3.7). The above transformation is presented schematically in Figure 6.2.

$\mathbf{z=\frac{\tau+1}{2}}$ We can determine the Laurent coefficients at $z=\frac{\tau+1}{2}$ by invoking the periodicity of $\mathcal{Z}_2$ in τ:

$$\begin{aligned}
\mathcal{Z}_2(\tau,z) &= \sum_{n=-2}^{\infty} \mathfrak{c}_n(\tau)(z-\tfrac{\tau}{2})^n = \sum_{n=-2}^{\infty} \mathfrak{d}_n(\tau)(z-\tfrac{\tau+1}{2})^n \\
&= \mathcal{Z}_2(\tau+1,z) = \sum_{n=-2}^{\infty} \mathfrak{c}_n(\tau+1)(z-\tfrac{\tau+1}{2})^n\,.
\end{aligned} \tag{6.3.23}$$

Hence

$$\mathfrak{d}_n(\tau) = \mathfrak{c}_n(\tau+1)\,, \tag{6.3.24}$$

yielding

$$D_{\frac{\tau+1}{2},2}\,\mathcal{Z}_2 = D_{\frac{\tau}{2},2}\,\mathcal{Z}_2(\tau+1,z)\,, \tag{6.3.25}$$

and

$$D_{\frac{\tau+1}{2},1}\,\mathcal{Z}_2 = D_{\frac{\tau}{2},1}\,\mathcal{Z}_2(\tau+1,z)\,. \tag{6.3.26}$$

The above transformation is also presented schematically in Figure 6.2.

To summarize, we have thus expressed the principal parts of $\mathcal{Z}_2$ around all of its poles (6.3.10) using the knowledge of $\mathcal{Z}_1$ and of genus zero Gromov-Witten data at base degree two.

Below, we list explicitly the Laurent data of base degree one and two:

$$D_{0,2}\,\mathcal{Z}_1 = -(2\pi\mathsf{i})^{-2}E_4\,.$$

$$D_{0,4}\,\mathcal{Z}_2 = \frac{(2\pi\mathsf{i})^{-4}}{2}E_4^2\,.$$

$$D_{0,2}\,\mathcal{Z}_2 = -\frac{(2\pi\mathsf{i})^{-2}}{24}\left(3E_2E_4^2+2E_4E_6\right)\,,$$

$$D_{\frac{1}{2},2}\,\mathcal{Z}_2 = -\frac{(2\pi\mathsf{i})^{-2}}{8}\left(\frac{\eta^{24}(\tau)}{\eta^{12}(2\tau)}\right)E_4(2\tau)\,,$$

$$D_{\frac{\tau}{2},2}\,\mathcal{Z}_2 = \frac{(2\pi\mathsf{i})^{-2}}{2}\left(\frac{\eta^{24}(\tau)}{\eta^{12}(\frac{\tau}{2})}\right)q^{3/4}E_4(\tfrac{\tau}{2})\,,$$

$$D_{\frac{\tau}{2},1}\,\mathcal{Z}_2 = 3\frac{(2\pi\mathsf{i})^{-1}}{2}\left(\frac{\eta^{24}(\tau)}{\eta^{12}(\frac{\tau}{2})}\right)q^{3/4}E_4(\tfrac{\tau}{2})\,,$$

$$D_{\frac{\tau+1}{2},2}\,\mathcal{Z}_2 = -\mathsf{i}\frac{(2\pi\mathsf{i})^{-2}}{2}\left(\frac{\eta^{24}(\tau)}{\eta^{12}(\frac{\tau+1}{2})}\right)q^{3/4}E_4(\tfrac{\tau+1}{2})\,,$$

and

$$D_{\frac{\tau+1}{2},1}\,\mathcal{Z}_2 = -3\mathsf{i}\frac{(2\pi\mathsf{i})^{-1}}{2}\left(\frac{\eta^{24}(\tau)}{\eta^{12}(\frac{\tau+1}{2})}\right)q^{3/4}E_4(\tfrac{\tau+1}{2})\,.$$

We extend these calculations to base degree 3 and 4, restricting to the massless case for simplicity, in appendix D.

6.3.2 General Proof

Based on the relation (6.1.18) relating $Z_{\boldsymbol{k}}(\tau,z,\boldsymbol{m})$ to the single-wrapping free energy $f_{\boldsymbol{k}}(\tau,z,\boldsymbol{m})$ and $Z_{\boldsymbol{k}'}$, $|\boldsymbol{k}'|<|\boldsymbol{k}|$ and the properties of $f_{\boldsymbol{k}}(\tau,z,\boldsymbol{m})$ that we have established in the previous section, we will now demonstrate that the principal parts of $Z_{\boldsymbol{k}}(\tau,z,\boldsymbol{m})$ can be computed from the knowledge of $Z_{\boldsymbol{k}'}$ with $|\boldsymbol{k}'|<|\boldsymbol{k}|$, complemented with the single-wrapping genus zero Gromov-Witten invariants at base degree $\boldsymbol{k}$.

$Z_{\boldsymbol{k}}$ has poles at all s-torsion points

$$z = \frac{c\tau+d}{s} \tag{6.3.27}$$

for $s\leq\max\{k_i\}$. From property **(I)** of $f_{\boldsymbol{k}}$ and (6.1.18), it follows that the knowledge of genus zero Gromov-Witten invariants for base wrapping degree $|\boldsymbol{k}'|\leq|\boldsymbol{k}|$ is sufficient to fix the Laurent coefficients of $Z_{\boldsymbol{k}}$ at all real torsion points. Our task is therefore to reduce the discussion at non-real torsion points to this case. As the following discussion relies on the $SL(2,\mathbb{Z})$ transformation behavior of $\mathrm{Z}_{\mathrm{top}}$, it will be convenient to work with the quantity $\mathcal{Z}_{\boldsymbol{k}}$, which differs from $Z_{\boldsymbol{k}}$ by a rescaling, see (5.2.10).

We consider hence a non-real s-torsion point

$$z = r\frac{c\tau+d}{s}\,,\quad c\neq 0, \tag{6.3.28}$$

where, for $d \neq 0$, we enforce $(c,d) = 1$ by including the factor r. We can relate the Laurent coefficients $\mathfrak{c}_{\boldsymbol{k},n}$ of an expansion around this point,

$$\mathcal{Z}_{\boldsymbol{k}}(\tau, z, \boldsymbol{m}) = \sum_{n=-2\ell}^{\infty} \mathfrak{c}_{\boldsymbol{k},n}(\tau, \boldsymbol{m}) \left(z - r\frac{c\tau + d}{s}\right)^n , \tag{6.3.29}$$

to those around the real torsion point $\frac{r}{s}$,

$$\mathcal{Z}_{\boldsymbol{k}}(\tau, z, \boldsymbol{m}) = \sum_{n=-2\ell}^{\infty} \mathfrak{b}_{\boldsymbol{k},n}(\tau, \boldsymbol{m})(z - \tfrac{r}{s})^n . \tag{6.3.30}$$

by considering the modular transformation $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ of this latter expression. We obtain

$$\Phi_{w,i_z(\boldsymbol{k}),i_{\boldsymbol{m}}(\boldsymbol{k})}\mathcal{Z}_{\boldsymbol{k}}(\tau, z, \boldsymbol{m}) = \mathcal{Z}_{\boldsymbol{k}}(\tfrac{a\tau+b}{c\tau+d}, \tfrac{z}{c\tau+d}, \tfrac{\boldsymbol{m}}{c\tau+d}) = \sum_{n=-2\ell}^{\infty} \frac{\mathfrak{b}_{\boldsymbol{k},n}(\frac{a\tau+b}{c\tau+d}, \frac{\boldsymbol{m}}{c\tau+d})}{(c\tau+d)^n} \left(z - r\frac{c\tau + d}{s}\right)^n , \tag{6.3.31}$$

where

$$\Phi_{w,i_z(\boldsymbol{k}),i_{\boldsymbol{m}}(\boldsymbol{k})} = (c\tau+d)^w \mathbf{e}\left[\tfrac{i_z(\boldsymbol{k})cz^2}{c\tau+d}\right] \epsilon\left[\tfrac{i_{\boldsymbol{m}}(\boldsymbol{k})c(\boldsymbol{m},\boldsymbol{m})}{2(c\tau+d)}\right] , \tag{6.3.32}$$

with the notation $\mathbf{e}[x] = \exp(2\pi \mathrm{i} x)$. Note that unlike $Z_{\boldsymbol{k}}$, $\mathcal{Z}_{\boldsymbol{k}}$ has non-trivial weight w. Substituting in the expression (6.3.29) yields

$$\sum_{n=-2\ell}^{\infty} \mathfrak{c}_{\boldsymbol{k},n}(\tau, \boldsymbol{m}) \left(z - r\frac{c\tau + d}{s}\right)^n = (c\tau+d)^{-w} \mathbf{e}\left[-\tfrac{i_z(\boldsymbol{k})cz^2}{c\tau+d}\right] \mathbf{e}\left[-\tfrac{i_{\boldsymbol{m}}(\boldsymbol{k})c(\boldsymbol{m},\boldsymbol{m})}{2(c\tau+d)}\right]$$
$$\sum_{n=-2\ell}^{\infty} \frac{\mathfrak{b}_{\boldsymbol{k},n}(\frac{a\tau+b}{c\tau+d}, \frac{\boldsymbol{m}}{c\tau+d})}{(c\tau+d)^n} \left(z - r\frac{c\tau + d}{s}\right)^n . \tag{6.3.33}$$

The Laurent coefficients $\mathfrak{c}_{\boldsymbol{k},n}$ of $\mathcal{Z}_{\boldsymbol{k}}$ around the non-real torsion point (6.3.28) can now be expressed in terms of the coefficients $\mathfrak{b}_{\boldsymbol{k},n}$ around the real torsion point $\frac{r}{s}$ by expanding the remaining z dependence in the exponential on the RHS of (6.3.33).

Equation (6.3.33) is difficult to use for explicit calculations away from $s = 1$, as the expansion coefficients of Jacobi forms around s-torsion points do not behave well under modular transformations for $s > 1$. For computations, we must obtain all expressions in an expansion in $q = \epsilon[\tau]$. To put $\mathfrak{b}_{\boldsymbol{k},n}(\frac{a\tau+b}{c\tau+d}, \frac{\boldsymbol{m}}{c\tau+d})$ in this form, we proceed by induction. Assume that we have determined all $Z_{\boldsymbol{k}'}$ for $|\boldsymbol{k}'| < |\boldsymbol{k}|$. From equation (6.1.18) and property **(I)** of the single-wrapping free energy $f_{\boldsymbol{k}}$, we see that the principal parts of $Z_{\boldsymbol{k}}$ around real s-torsion points for $s > 1$ are determined completely by the monomials $m_{\boldsymbol{k},i}$ in $Z_{\boldsymbol{k}'}$, $|\boldsymbol{k}'| < |\boldsymbol{k}|$ introduced in (6.1.19), i.e. do not involve $f_{\boldsymbol{k}}$. The negative index Laurent coefficients $\mathsf{b}_{\boldsymbol{k},n}$ of $Z_{\boldsymbol{k}}$ around the s-torsion point $\frac{r}{s}$, $s > 1$, relate to those of each monomial $m_{\boldsymbol{k},i}$ around this point, which we denote as $\mathsf{b}_{\boldsymbol{k},i,n}$, as

$$\mathsf{b}_{\boldsymbol{k},n} = \sum_i \mathsf{b}_{\boldsymbol{k},i,n} , \quad n < 0 . \tag{6.3.34}$$

For each monomial, we introduce the product of Jacobi forms $\mathfrak{m}_{\boldsymbol{k},i}$, which is $m_{\boldsymbol{k},i}$ evaluated on the set $\{\mathcal{Z}_{\boldsymbol{k}'}\}$ rather than $\{Z_{\boldsymbol{k}'}\}$, with the corresponding Laurent coefficients $\mathfrak{b}_{\boldsymbol{k},i,n}$,

$$\mathfrak{m}_{\boldsymbol{k},i}(\tau, z, \boldsymbol{m}) = \sum_{n=-2\ell}^{\infty} \mathfrak{b}_{\boldsymbol{k},i,n}(\tau, \boldsymbol{m}) \left(z - \frac{r}{s}\right)^n . \tag{6.3.35}$$

The coefficient $\mathfrak{b}_{\boldsymbol{k},i,n}$ evaluated at the point of interest satisfy

$$\mathfrak{m}_{\boldsymbol{k},i}(\tfrac{a\tau+b}{c\tau+d}, \tfrac{z}{c\tau+d}, \tfrac{\boldsymbol{m}}{c\tau+d}) = \sum_{n=-2\ell}^{\infty} \frac{\mathfrak{b}_{\boldsymbol{k},i,n}(\frac{a\tau+b}{c\tau+d}, \frac{\boldsymbol{m}}{c\tau+d})}{(c\tau+d)^n} \left(z - r\frac{c\tau+d}{s}\right)^n . \tag{6.3.36}$$

We cannot, in analogy with (6.3.31), directly relate the LHS to $\mathfrak{m}_{\boldsymbol{k},i}(\tau, z, \boldsymbol{m})$ via a modular transformation. This is because $\mathfrak{m}_{\boldsymbol{k},i}$ will generically contain factors of Jacobi forms evaluated at arguments $(t\tau, tz, t\boldsymbol{m})$, $t > 1$, hence not be a Jacobi form for the full modular group $SL(2,\mathbb{Z})$. To obtain a q-expansion of $\mathfrak{b}_{\boldsymbol{k},i,n}(\frac{a\tau+b}{c\tau+d}\frac{\boldsymbol{m}}{c\tau+d})$ nonetheless, we study the factors contributing to $\mathfrak{m}_{\boldsymbol{k},i}(\tau, z, \boldsymbol{m})$ individually. A generic such factor is of the form $\mathcal{Z}_{\boldsymbol{k}_{i,j}}(s_j\tau, s_j z, s_j\boldsymbol{m})$. We will write $t = s_j$ in the following argument to lighten the notation. On the LHS of (6.3.36), this factor occurs evaluated at the arguments

$$\mathcal{Z}_{\boldsymbol{k}_{i,j}}(t\tfrac{a\tau+b}{c\tau+d}, t\tfrac{z}{c\tau+d}, t\tfrac{\boldsymbol{m}}{c\tau+d}) . \tag{6.3.37}$$

Our goal is to obtain a form amenable to q-expansion of all such factors contributing to the monomial $\mathfrak{m}_{\boldsymbol{k},i}(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}, \frac{\boldsymbol{m}}{c\tau+d})$. This will allow us to express the negative index coefficients on the RHS of (6.3.36) in terms of a q-expansion, which via (6.3.34) will yield the desired q-expansion of the coefficients $\mathfrak{c}_{\boldsymbol{k},n}(\tau, \boldsymbol{m})$ in (6.3.33) .

The transformation

$$\begin{pmatrix} d & -t\,b \\ -c & t\,a \end{pmatrix} \ : \quad t\,\frac{a\tau+b}{c\tau+d} \mapsto \tau \tag{6.3.38}$$

would remove the τ-dependence of the denominator of the modular argument of (6.3.37), but is not an element of $SL(2,\mathbb{Z})$ for $t \neq 1$. We can correct for this by adjusting the two top entries of the matrix, as only the bottom entries enter in removing the τ-dependence in the denominator of $t\,\frac{a\tau+b}{c\tau+d}$. For a solution to this problem to exist, the bottom entries must be mutually prime. Let therefore $u = \gcd(c,t)$. Then we can find α and β such that

$$\mathrm{SL}(2,\mathbb{Z}) \ni \begin{pmatrix} \alpha & \beta \\ -\frac{c}{u} & \frac{t}{u}\,a \end{pmatrix} \ : \quad t\,\frac{a\tau+b}{c\tau+d} \mapsto \frac{p\,\tau+q}{t} , \quad p,\, q \in \mathbb{Z} , \tag{6.3.39}$$

with

$$p = u^2 \ , \quad q = u(\alpha b t + \beta d) \ . \tag{6.3.40}$$

Setting

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\frac{c}{u} & \frac{t}{u}\,a \end{pmatrix} , \quad (\tau', z', \boldsymbol{m}') = (t\tfrac{a\tau+b}{c\tau+d}, t\tfrac{z}{c\tau+d}, t\tfrac{\boldsymbol{m}}{c\tau+d}) , \tag{6.3.41}$$

we obtain

$$\Phi_{w,i_z(\boldsymbol{k}_{i,j}),i_{\boldsymbol{m}}(\boldsymbol{k}_{i,j})}\mathcal{Z}_{\boldsymbol{k}_{i,j}}(\tau',z',\boldsymbol{m}') = \mathcal{Z}_{\boldsymbol{k}_{i,j}}\left(\tfrac{\alpha\tau'+\beta}{\gamma\tau'+\delta},\tfrac{z'}{\gamma\tau'+\delta},\tfrac{\boldsymbol{m}'}{\gamma\tau'+\delta}\right) = \mathcal{Z}_{\boldsymbol{k}_{i,j}}\left(\tfrac{p\tau+q}{t},u\,z,u\,\boldsymbol{m}\right), \tag{6.3.42}$$

where

$$\Phi_{w,i_z(\boldsymbol{k}),i_{\boldsymbol{m}}(\boldsymbol{k})} = \left(\frac{t}{u(c\tau+d)}\right)^{w_{i,j}} \mathbf{e}\left[-t\frac{i_z(\boldsymbol{k}_{i,j})cz^2}{c\tau+d}\right]\epsilon\left[-t\frac{i_{\boldsymbol{m}}(\boldsymbol{k}_{i,j})c(\boldsymbol{m},\boldsymbol{m})}{2(c\tau+d)}\right]. \tag{6.3.43}$$

Recalling $t=s_j$ and the constraint (6.1.20) on s_j, we note that by linearity of the index $i_{\boldsymbol{m}}(\boldsymbol{k})$ in $\boldsymbol{k}$, the q-expansion of the expansion coefficients $\mathfrak{b}_{\boldsymbol{k},i,n}(\frac{a\tau+b}{c\tau+d},\frac{\boldsymbol{m}}{c\tau+d})$ of the monomials $m_{\boldsymbol{k},i}$ for all i will exhibit the same prefactor $\mathbf{e}\left[\frac{i_{\boldsymbol{m}}(\boldsymbol{k})(\boldsymbol{m},\boldsymbol{m})}{2(c\tau+d)}\right]$. The expression for $\mathfrak{b}_{\boldsymbol{k},n}(\frac{a\tau+b}{c\tau+d},\frac{\boldsymbol{m}}{c\tau+d})$ $(n<0)$ obtained from these by summing over i therefore also exhibits this prefactor. In determining the Laurent coefficients $\mathfrak{c}_{\boldsymbol{k},n}$ via (6.3.33), this prefactor will hence cancels against $\mathbf{e}\left[-\frac{i_{\boldsymbol{m}}(\boldsymbol{k})(\boldsymbol{m},\boldsymbol{m})}{2(c\tau+d)}\right]$.

Above, we have shown that in the case of elliptically fibered Calabi-Yau manifolds without divisors of type 3, the negative index Laurent coefficients of $Z_{\boldsymbol{k}}$ at all poles can be reconstructed from the knowledge of $Z_{\boldsymbol{k}'}$, $|\boldsymbol{k}'|<|\boldsymbol{k}|$, complemented by the genus zero Gromov-Witten at base degree $\boldsymbol{k}$. By the discussion in section 6.2, when the z-index of $Z_{\boldsymbol{k}}$ is negative, the knowledge of these coefficients is sufficient to reconstruct all of $Z_{\boldsymbol{k}}$ [34, 86]. Thus we finish our proof.

6.3.3 Geometries on which Z_k is completely determined by genus zero Gromov-Witten invariants

The z-index of $Z_{\boldsymbol{k}}$ is given by the formula

$$i_z(\boldsymbol{k}) = \frac{C_{\boldsymbol{k}}\cdot(C_{\boldsymbol{k}}+K_B)}{2}, \tag{6.3.44}$$

with the $C_{\boldsymbol{k}}$ denoting divisors of the base B of the elliptically fibered Calabi-Yau manifold X. For this to be negative for all $\boldsymbol{k}$, we need $C_{\boldsymbol{k}}\cdot C_{\boldsymbol{k}}<0$, as the second contribution to $i_z(k)$ grows only linearly in $\boldsymbol{k}$. In particular, we have to exclude base surfaces with divisors of positive self-intersection number, hence all compact projective surfaces. Luckily, we are often interested in considering non-compact base surfaces, e.g. when engineering 6d SCFTs in F-theory.

To allow for the determination of the principal parts of $Z_{\boldsymbol{k}}$ from genus zero Gromov-Witten invariants, we need to exclude divisors of type 3 in the Calabi-Yau manifold X. This leaves us with geometries containing only curves of self-intersection number -2 or -1. By the analysis of [97], the most general such configuration leading to a minimal SCFT is a chain of -2 curves ending with a -1 curve, with neighboring curves intersecting once. This is the higher rank E-string. Its z-index is given in [79]. Considering a chain of length $n+1$, with the 0^{th} node indicating the -1 curve, and denoting the corresponding intersection matrix as C_{IJ}, the index can be written as

$$i_z(\boldsymbol{k}) = \frac{1}{2}\left(\sum_{I,J=0}^{n} k_I C_{IJ} k_J - k_0\right) , \tag{6.3.45}$$

which is clearly negative definite.

6.3.4 Laurent coefficients of the refined partition functions

The discussion for the refined partition function follows the same pattern as in the unrefined case; the additional data upon increasing the base-wrapping degree to $\boldsymbol{k}$ are now the NS-invariants (6.1.33) at base degree $\boldsymbol{k}$.

$Z_{\boldsymbol{k}}$ as a function of z_L has poles at $\pm z_R$ shifted by all s-torsion points

$$z_L = \pm z_R + \frac{c\tau + d}{s} \tag{6.3.46}$$

for $s \leq \max\{k_i\}$. Due to the property **(I)** of single wrapping contributions $f_{\boldsymbol{k}}$ to the refined free energy, and the relation (6.1.18) adapted by replacing z everywhere by z_L, z_R, the Laurent coefficients of $Z_{\boldsymbol{k}}$ at all the poles $z_L = \pm z_R +$ real torsion can be fixed by the knowledge of the NS-invariants at base wrapping degree $|\boldsymbol{k}'| \leq |\boldsymbol{k}|$. Poles at $\pm z_R$ shifted by non-real torsion points, on the other hand, can be related to the former by a modular transformation. As in the unrefined case, we will for the rest of the discussion switch to the quantity $\mathcal{Z}_{\boldsymbol{k}}$, related to $Z_{\boldsymbol{k}}$ by the rescaling given in (5.2.10), as $\mathcal{Z}_{\boldsymbol{k}}$ has better modular transformation properties.

The Laurent coefficients $\mathfrak{c}_{\boldsymbol{k},n}(\tau, z_R, \boldsymbol{m})$ of an expansion around a nonreal-torsion-shifted singular point,

$$\mathcal{Z}_{\boldsymbol{k}}(\tau, z_L, z_R, \boldsymbol{m}) = \sum_{n=-\ell}^{\infty} \mathfrak{c}_{\boldsymbol{k},n}(\tau, z_R, \boldsymbol{m}) \left(z_L \mp z_R - r\frac{c\tau + d}{s}\right)^n , \tag{6.3.47}$$

can be expressed in terms of the coefficients $\mathfrak{b}_{\boldsymbol{k},n}(\tau, z_R, \boldsymbol{m})$ around the real-torsion-shifted point $\pm z_R + \frac{r}{s}$

$$\mathcal{Z}_{\boldsymbol{k}}(\tau, z_L, z_R, \boldsymbol{m}) = \sum_{n=-\ell}^{\infty} \mathfrak{b}_{\boldsymbol{k},n}(\tau, z_R, \boldsymbol{m}) \left(z_L \mp z_R - \frac{r}{s}\right)^n . \tag{6.3.48}$$

by expanding the following identity

$$\begin{aligned}\sum_{n=-\ell}^{\infty} \mathfrak{c}_{\boldsymbol{k},n}(\tau, z_R, \boldsymbol{m}) \left(\hat{z}_L - r\frac{c\tau + d}{s}\right)^n &= (c\tau+d)^{-w} \mathbf{e}\left[-\tfrac{i_L(\boldsymbol{k})c(\hat{z}_L^2 \pm 2 z_R \hat{z}_L)}{c\tau+d}\right] \mathbf{e}\left[-\tfrac{(\boldsymbol{k})c z_R^2}{c\tau+d}\right] \\ &\mathbf{e}\left[-\tfrac{i_m(\boldsymbol{k})c(\boldsymbol{m},\boldsymbol{m})}{2(c\tau+d)}\right] \sum_{n=-\ell}^{\infty} \frac{\mathfrak{b}_{\boldsymbol{k},n}\left(\frac{a\tau+b}{c\tau+d}, \frac{z_R}{c\tau+d}, \frac{\boldsymbol{m}}{c\tau+d}\right)}{(c\tau+d)^n} \left(\hat{z}_L - r\frac{c\tau+d}{s}\right)^n . \end{aligned} \tag{6.3.49}$$

Here we have introduced $(\boldsymbol{k}) = i_L(\boldsymbol{k}) + i_R(\boldsymbol{k})$ and the variable

$$\hat{z}_L = z_L \mp z_R \ , \tag{6.3.50}$$

evaluated in the neighborhood of $r(c\tau + d)/s$. As in the unrefined case, we also enforce $\gcd(c, d) = 1$ by introducing r. Beside, we need to put $\mathfrak{b}_{\boldsymbol{k},n}(\frac{a\tau+b}{c\tau+d}, \frac{z_R}{c\tau+d}, \frac{\boldsymbol{m}}{c\tau+d})$ in a form amenable to expansion in $q = \mathbf{e}[\tau]$. We first note that the coefficients $\mathsf{b}_{\boldsymbol{k},n}$ are related to those of the monomials $m_{\boldsymbol{k},i}$ introduced in (6.1.18) at the same torsion point by

$$\mathsf{b}_{\boldsymbol{k},n} = \sum_i \mathsf{b}_{\boldsymbol{k},i,n} \ , \quad n < 0 \ . \tag{6.3.51}$$

The monomials $\mathfrak{m}_{\boldsymbol{k},i}$, which are $m_{\boldsymbol{k},i}$ evaluated on the set $\{\mathcal{Z}_{\boldsymbol{k}'}\}$ rather than $\{Z_{\boldsymbol{k}'}\}$, are products of Jacobi forms. Their expansion coefficients $\mathfrak{b}_{\boldsymbol{k},i,n}$ around $\hat{z} = \frac{r}{s}$, evaluated at the point of interest, satisfy

$$\mathfrak{m}_{\boldsymbol{k},i}(\tfrac{a\tau+b}{c\tau+d}, \tfrac{z_L}{c\tau+d}, \tfrac{z_R}{c\tau+d}, \tfrac{\boldsymbol{m}}{c\tau+d}) = \sum_{n=-\ell}^{\infty} \frac{\mathfrak{b}_{\boldsymbol{k},i,n}(\frac{a\tau+b}{c\tau+d}, \frac{z_R}{c\tau+d}, \frac{\boldsymbol{m}}{c\tau+d})}{(c\tau+d)^n} \left(\hat{z}_L - r\frac{c\tau+d}{s}\right)^n . \tag{6.3.52}$$

To obtain a q-expansion of the negative order coefficients, we need to consider the individual factors contributing to the monomial $\mathfrak{m}_{\boldsymbol{k},i}$, which are of the form

$$\mathcal{Z}_{\boldsymbol{k}_{i,j}}(s_j \tfrac{a\tau+b}{c\tau+d}, s_j \tfrac{z_L}{c\tau+d}, s_j \tfrac{z_R}{c\tau+d}, s_j \tfrac{\boldsymbol{m}}{c\tau+d}) \ . \tag{6.3.53}$$

with possible multi-wrapping factors s_j. With

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\frac{c}{u} & \frac{t}{u}a \end{pmatrix} , \quad (\tau', z'_L, z'_R, \boldsymbol{m}') = (s_j \tfrac{a\tau+b}{c\tau+d}, s_j \tfrac{z_L}{c\tau+d}, s_j \tfrac{z_R}{c\tau+d}, s_j \tfrac{\boldsymbol{m}}{c\tau+d}) , \tag{6.3.54}$$

and $u = \gcd(c, t)$, we find

$$\begin{aligned} \Phi_{w,i_L(\boldsymbol{k}_{i,j}),i_R(\boldsymbol{k}_{i,j}),i_{\boldsymbol{m}}(\boldsymbol{k}_{i,j})} \mathcal{Z}_{\boldsymbol{k}_{i,j}}(\tau', z'_L, z'_R, \boldsymbol{m}') &= \mathcal{Z}_{\boldsymbol{k}_{i,j}}(\tfrac{\alpha\tau'+\beta}{\gamma\tau'+\delta}, \tfrac{z'_L}{\gamma\tau'+\delta}, \tfrac{z'_R}{\gamma\tau'+\delta}, \tfrac{\boldsymbol{m}'}{\gamma\tau'+\delta}) \\ &= \mathcal{Z}_{\boldsymbol{k}_{i,j}}(\tfrac{p\tau+q}{s_j}, u z_L, u z_R, u\,\boldsymbol{m}) , \end{aligned} \tag{6.3.55}$$

where

$$p = u^2 \ , \quad q = u(\alpha b s_j + \beta d) \ , \tag{6.3.56}$$

and

$$\Phi_{w,i_L(\boldsymbol{k}),i_R(\boldsymbol{k}),i_{\boldsymbol{m}}(\boldsymbol{k})} = \left(\frac{s_j}{u(c\tau+d)}\right)^w \mathbf{e}\left[-s_j \tfrac{i_L(\boldsymbol{k}_{i,j})c(\hat{z}_L^2 \pm 2z_R\hat{z}_L)}{c\tau+d}\right] \mathbf{e}\left[-s_j \tfrac{(\boldsymbol{k}_{i,j})cz_R^2}{(c\tau+d)}\right] \mathbf{e}\left[-s_j \tfrac{i_{\boldsymbol{m}}(\boldsymbol{k}_{i,j})c(\boldsymbol{m},\boldsymbol{m})}{2(c\tau+d)}\right] . \tag{6.3.57}$$

We note that both $(\boldsymbol{k})$ and $i_{\boldsymbol{m}}(\boldsymbol{k})$ are linear in $\boldsymbol{k}$. As a result, the expansion coefficients $\mathfrak{b}_{\boldsymbol{k},i,n}(\frac{a\tau+b}{c\tau+d}, \frac{z_R}{c\tau+d}, \frac{\boldsymbol{m}}{c\tau+d})$ of the monomials $\mathfrak{m}_{\boldsymbol{k},i}$ for all i and thus their sum $\mathfrak{b}_{\boldsymbol{k},n}(\frac{a\tau+b}{c\tau+d}, \frac{z_R}{c\tau+d}, \frac{\boldsymbol{m}}{c\tau+d})$ will exhibit the same prefactors $\mathbf{e}\left[\frac{(\boldsymbol{k})cz_R^2}{(c\tau+d)}\right]$, $\mathbf{e}\left[\frac{i_{\boldsymbol{m}}(\boldsymbol{k})c(\boldsymbol{m},\boldsymbol{m})}{2(c\tau+d)}\right]$, which then cancel against $\mathbf{e}\left[-\frac{(\boldsymbol{k})cz_R^2}{(c\tau+d)}\right]$, $\mathbf{e}\left[-\frac{i_{\boldsymbol{m}}(\boldsymbol{k})c(\boldsymbol{m},\boldsymbol{m})}{2(c\tau+d)}\right]$ in (6.3.49) when one computes the Laurent coefficients $\mathfrak{c}_{\boldsymbol{k},n}$.

Chapter 7

Geometries with codimension-one singular fibers: Higgsing Trees

In this chapter, we turn to geometries having codimension-one singular fibers (leading to non-abelian gauge symmetries). As we mentioned in chapter 6, their topological string partition functions have a more complicated denominator structure, and the partition functions cannot be solely determined from genus zero GW invariants. Instead, we need to determine them using other boundary data. Besides, in the classification list of [97], there exists a set of minimal geometries, upon which the F-theory compactification gives rise to theories that cannot be Higgsed further. By specializing the complex structure, we obtain geometries with more severe singularity. This naturally arranges all the geometries into different branches. Within a given branch, the geometries are related via fine tuning of their complex structure moduli, or so-called "Higgs branch flow" of the underlying SCFTs. We would like to see if one can recover this pattern from the topological string partition functions, e.g., through mappings of the corresponding Weyl-invariant Jacobi forms.

This chapter is organized as follows. In section 7.1, we discuss resolution of singular fibers, Higgsing tree structure that are necessary for later sections. In section 7.2, we will discuss the $\mathfrak{a}_2$ model [83, 48], which lies at the bottom of a Higgsing tree. In section 7.3, we climb up one level on the Higgsing tree and discuss the $\mathfrak{g}_2$ model.

It is based on some work in progress by Amir-Kian Kashani-Poor and the author.

7.1 Higgsing trees

To begin with, since we assume the existence of divisor of type 3, we need to discuss in detail how to desingularize the elliptic fiber. The geometry that we consider here is $\check{X}$ in 5.1.4. This yields $n \geq 3$ in order that b is the divisor we want. Since we are only interested in how the singularity is resolved along b, we can forget the non-compact direction and focus on the elliptic surface S

$$\begin{array}{rl} \mathcal{E} \to & S \\ & \Big\downarrow \pi' \\ b = & \mathbb{P}^1 \end{array} \tag{7.1.1}$$

The exceptional curves C_i that resolve the singularities in the elliptic fiber are all rational curves, in one to one correspondence with a choice of simple roots α_i for the Lie algebra $\mathfrak{g}$. They intersect within S according to the negative of its Cartan matrix. What's more, if we denote the original fiber by F_0, then it serves as the affine node α_0. In other words, F_0 and all the exceptional curves C_i intersect according to the negative of the Cartan matrix for the corresponding affine Lie algebra $\hat{\mathfrak{g}}$. Furthermore, the resolved elliptic fiber F can be written as a combination of those curves,

$$F = F_0 + \sum_i a_i^\vee E_i \,, \tag{7.1.2}$$

with $a_i^\vee$ equal to the comark of the simple root that E_i is identified with. One way to understand this is the following. The correspondence $F_0 \to \alpha_0$ and $E_i \to \alpha_i$ maps F to the imaginary root δ, which means

$$F \cdot F = \delta \cdot \delta = 0 \,. \tag{7.1.3}$$

Therefore, the resolved fiber F has zero self-intersection, as it should be.

So far, we haven't talked about flavor groups. In fact, they can be represented as non-compact divisors inside the base B. This can be argued for as follows. We can regard flavor group as the limit of gauge group when the gauge coupling is tuned to zero. Given a divisor of type 3, the gauge group is determined by the singularity, and the square of the inverse coupling is proportional to the volume of that divisor. Divisors with infinite volume then have zero gauge coupling, thus correspond to flavor symmetries. The intersection of flavor divisors and gauge divisors give rise to the matter. A general picture of the divisor structure inside B can be presented schematically as Figure 7.1.

Since we have the charged matter, we can give vevs to its scalar and invoke Higgs mechanism to the gauge field. The broken part then becomes massive and is decoupled from our theory. In terms of the elliptic Calabi-Yau geometry, this means deforming the complex structure and making the singularities of the elliptic fiber less and less severe.

We can draw a graph to represent the Higgs branch flows. We first label the theory by the gauge algebra and the representation of matters. Meanwhile whenever there is a flow between any two theories, we draw a line between them with an arrow indicating the direction of the flow. This type of graph is called the *Higgsing tree*. An example can be found in Figure 7.1.

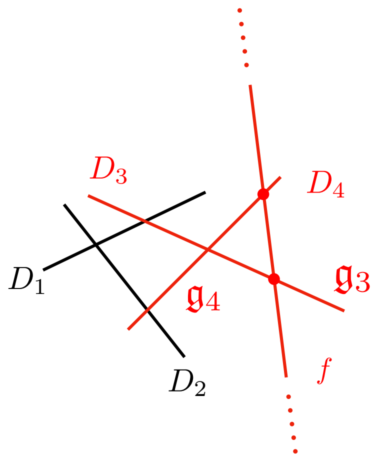


Figure 7.1: Divisorial structure of the base B. The divisors in red are divisors of type 3 (over which the elliptic fiber is singular), and f labels the non-compact flavor divisor. Matters live on their intersect points.

$$\mathfrak{e}_7 \oplus (\tfrac{1}{2}\mathbf{56})^{\oplus 5}$$
$$\downarrow$$
$$\mathfrak{e}_6 \oplus \mathbf{27}^{\oplus 3}$$
$$\downarrow$$
$$\mathfrak{f}_4 \oplus \mathbf{26}^{\oplus 2}$$
$$\downarrow$$
$$\mathfrak{so}_8 \oplus \mathbf{8}_v \oplus \mathbf{8}_s \oplus \mathbf{8}_c$$
$$\downarrow$$
$$\mathfrak{so}_7 \oplus \mathbf{8}^{\oplus 2}$$
$$\downarrow$$
$$\mathfrak{g}_2 \oplus \mathbf{7}$$
$$\downarrow$$
$$\mathfrak{a}_2$$

At the bottom of this Higgsing tree, we recover the system we discussed before: one compact divisor of type 3 with no matter example 4.

7.2 $\mathfrak{a}_2$ model

Let's first discuss the $\mathfrak{a}_2$ model lying at the bottom of the Higgsing tree in Table 7.1.

Recalled our general ansatz (5.2.12). For reader's convenience, we reproduce it here,

$$\mathcal{Z}_{\boldsymbol{k}} = \frac{\mathcal{N}_{\boldsymbol{k}}^{\text{refined}}(\tau, z_L, z_R, \boldsymbol{m})}{\phi_{\boldsymbol{k}}^{D}(\tau, z_1, z_2, \boldsymbol{m})}\,. \tag{7.2.1}$$

The denominator $\phi^D_{\boldsymbol{k}}$ takes a universal form thus only the numerator $\mathcal{N}^{\text{refined}}_{\boldsymbol{k}}$ is unknown.

Let's specialize the above to our case. This means $\boldsymbol{k} = 1$ and $\boldsymbol{m}$ takes values in the Cartan subalgebra of $\mathfrak{a}_2$. Throughout this section for simplicity we also sometimes use G to denote $SU(3)$.

The weight and index of $\mathcal{N}^{\text{refined}}_{G,k}$ follow by subtracting the weight and index of the denominator from those of the elliptic genus, which is of vanishing total weight, and index bilinear form $i_{G,k}$ given by [50]

$$i_{G,k}(\epsilon_L, \epsilon_R, m) = i_L(G,k)\epsilon_L^2 + i_R(G,k)\epsilon_R^2 + i_f(G,k)\frac{(\boldsymbol{m}_G, \boldsymbol{m}_G)_{\mathfrak{g}}}{2}\,, \tag{7.2.2}$$

with coefficients

$$\begin{aligned} i_L &= -\frac{3}{2}k^2 + \frac{1}{2}k\,, && (7.2.3)\\ i_R &= \frac{3}{2}k^2 - \frac{5}{2}k\,, && (7.2.4)\\ i_f &= -3k\,. \end{aligned}$$

Based on the general form of denominator (5.2.12), the modular weight $w(G,k)$ of the numerator then follows as

$$w(G,k) = -3k^2 + k\;, \tag{7.2.5}$$

and the index bilinear form of the numerator is

$$d_{G,k}(\epsilon_L, \epsilon_R, m) = d_L(G,k)\epsilon_L^2 + d_R(G,k)\epsilon_R^2 + d_g(G,k)\frac{(\boldsymbol{m}_G, \boldsymbol{m}_G)_{\mathfrak{g}}}{2}\;, \tag{7.2.6}$$

with coefficients

$$\begin{aligned} d_L(G,k) &= \frac{1}{4}k^4 + \frac{5}{6}k^3 - \frac{5}{4}k^2 + \frac{1}{6}k\,, && (7.2.7)\\ d_R(G,k) &= \frac{3}{4}k^4 + \frac{23}{6}k^3 + \frac{29}{4}k^2 + \frac{1}{6}k\,, && (7.2.8)\\ d_g(G,k) &= 3k^2\,. && (7.2.9) \end{aligned}$$

Together with the relation

$$\mathcal{N}^{\text{refined}}_{G,k} \in J_{*,d_L}(\epsilon_L) \otimes J_{*,d_R}(\epsilon_R) \otimes J_{*,d_g}(\mathfrak{a}_2)\,, \tag{7.2.10}$$

determining the partition function is reduced to a finite dimensional problem. The unrefined case can be obtained by setting $\epsilon_L = 0,\;\epsilon_R = g_s$.

However, in practice, the unknown coefficients grow very fast as we increase the base degree. Table 7.1 gives the numbers of possible terms for the first few cases.

In section 5.2 of chapter 5, we mentioned the Weyl group invariance of $\mathcal{Z}_{\boldsymbol{k}}$. But in the last section we learned that we have more, there is actually an affine Lie algebra structure lurking inside. Naturally, we would like to ask: can we explore the affine Lie algebra

k	w	d_L	d_R	d_f	unrefined	refined
1	-2	0	12	3	4	126
2	-10	6	72	12	583	192859
3	-24	230	32	27	33154	38331108

Table 7.1: Numbers of possible terms in the numerator of the elliptic genus of k strings for $\mathfrak{a}_2$ in terms of $J_{*,*}(\mathfrak{a}_2)$.

k	w	d_L	d_R	d_f	unrefined	refined
1	-2	0	12	3	1	5
2	-10	6	72	12	38	30362
3	-24	32	220	27	2299	4904253
4	-44	98	554	48	57378	237021553

Table 7.2: Numbers of possible terms in the numerator of the elliptic genus of k strings for $\mathfrak{a}_2$ in terms of $J^{\hat{D}}_{*,*}(\mathfrak{a}_2)$..

symmetry to simplify our calculation? After careful analyses, in [48] it is argued that $\mathcal{N}^{\text{refined}}_{G,k}$ not only is invariant under the Weyl group of $\mathfrak{g}$, but also should transform in a simple way (in the sense of being invariant up to some powers of Q_i) under the symmetry group $D(\hat{\mathfrak{g}})$ of the affine Dynkin diagram of $\hat{\mathfrak{g}}$. This also explains nicely the slight miss-match between elliptic genus and topological string partition (5.2.10): once we include the prefactor, Z_k, i.e., the topological string partition function becomes invariant under $\hat{\mathfrak{g}}$ symmetry[1]!

Weyl invariant Jacobi modular forms invariant under $D(\hat{\mathfrak{g}})$ will be denoted as $J^{\hat{D}}_{*,*}(\mathfrak{g})$. In [48], a set of basis is conjectured to be

$$\phi_0 \in J^{\hat{D}}_{0,3}(\mathfrak{a}_2)\ , \quad \phi_2 \in J^{\hat{D}}_{-2,3}(\mathfrak{a}_2)\ , \quad \phi_6 \in J^{\hat{D}}_{-6,6}(\mathfrak{a}_2)\ , \tag{7.2.11}$$

Their explicit expressions can be found in appendix A.

This extra symmetry greatly simplifies the computation, see Table 7.2 and compare it with Table 7.1.

Then in [48] it was found that for base degree up to 3, $\mathcal{Z}_{\boldsymbol{k}}$ can be completely solved by imposing the so-called precise vanishing conditions for GV invariants. Then we can extract GV invariants from the form of the topological string partition function. Some examples are given in appendix E.

Finally, in order to compare the above results with those from topological string theory, we need to choose the correct basis of two-cycles that have the desired intersection matrix

[1] Namely, the map (A.3.17) in section A.3.

inside S. The basis is listed in Table 7.2.12.

Div.	ν_i				$l^1_{\hat{\mathfrak{a}}_2}$	$l^2_{\hat{\mathfrak{a}}_2}$	$l^3_{\hat{\mathfrak{a}}_2}$	l_{b}	l_{de}
D_0	0	0	0	0	−4	−1	−1	1	−5
D_1	−1	0	0	0	1	1	0	0	$\frac{14}{9}$
D_2	0	−1	0	0	3	0	0	0	$\frac{7}{3}$
D_3	1	1	0	−1	−2	1	1	−1	$\frac{1}{3}$
D_4	1	2	0	−1	1	−2	1	−1	$\frac{1}{3}$
E	2	3	0	−1	1	1	1	−1	$\frac{1}{3}$
K	2	3	0	0	0	0	−2	0	$-\frac{8}{9}$
F	2	3	−1	−3	0	0	1	0	0
H	2	3	0	1	0	0	0	0	1
F	2	3	1	0	0	0	0	1	0

. (7.2.12)

7.3 $\mathfrak{g}_2$ model

In this subsection, we mention briefly the $\mathfrak{g}_2$ model. From Graph 5.1.6, we see that our theory has gauge algebra $\mathfrak{g}_2$ with charged matter in the fundamental representation. The flavor symmetry can be determined to be $SU(2)$. Throughout this section for simplicity we sometimes also use G to denote G_2 and F for $SU(2)$.

Recall again our ansatz for $\mathcal{Z}_{\boldsymbol{k}}$,

$$\mathcal{Z}_{\boldsymbol{k}} = \frac{\mathcal{N}^{\text{refined}}_{\boldsymbol{k}}(\tau, z_L, z_R, \boldsymbol{m})}{\phi^D_{\boldsymbol{k}}(\tau, z_1, z_2, \boldsymbol{m})}\,. \tag{7.3.1}$$

In our case, $\boldsymbol{k} = 1$ and $\boldsymbol{m}$ takes values in the Cartan subalgebra of $\mathfrak{g}_2$, since only the gauge mass parameters appear in the denominator according to our discussion in chapter 5. The expression of $\phi^D_{\boldsymbol{k}}$ can be found in (5.2.14).

We record here the weight and index bilinear form of $\mathcal{N}^{\text{refined}}_{G,k}$, followed by subtracting the weight and index of the denominator from those of the elliptic genus.

The result is that $\mathcal{N}^{\text{refined}}_{G,k}$ has weight $w(G,k) = -6k^2 - 2k$ and index bilinear

$$d^F_{G,k}(\epsilon_+, \epsilon_-, m) = d_L(G,k)\epsilon_L^2 + d_R(G,k)\epsilon_R^2 + d_g(G,k)\frac{(\boldsymbol{m}_G, \boldsymbol{m}_G)_{\mathfrak{g}}}{2} + d_f(F,k)\frac{(\boldsymbol{m}_F, \boldsymbol{m}_F)_{\mathfrak{g}}}{2}\,, \tag{7.3.2}$$

where

$$d_L(k) = \tfrac{1}{2}k^4 + \tfrac{4}{3}k^3 - \tfrac{3}{2}k^2 - \tfrac{1}{3}k, \tag{7.3.3}$$
$$d_R(k) = \tfrac{3}{2}k^4 + \tfrac{22}{3}k^3 + \tfrac{25}{2}k^2 + \tfrac{5}{3}k, \tag{7.3.4}$$
$$d_g(G,k) = 4k^2 + k. \tag{7.3.5}$$
$$d_f(G,k) = k. \tag{7.3.6}$$

Together with the relation,

$$\mathcal{N}^{\text{refined}}_{G,k} \in J_{*,d_L}(\epsilon_L) \otimes J_{*,d_R}(\epsilon_R) \otimes J_{*,d_g}(\mathfrak{g}_2) \otimes J_{*,d_f}(\mathfrak{a}_1)\,, \tag{7.3.7}$$

we reduce the calculation to a finite dimensional problem.

However, the numbers of possible terms are huge. For example, we turn off the chemical potential for the flavor group F, so that the elliptic genus doesn't depend on $\boldsymbol{m}_F$. For base degree 1, we have 664 unknown coefficients to determine. For base degree 2, the number is 2291820.

For gauge groups $SU(3)$ as in [48], we can significantly reduce the number of terms using Jacobi forms invariant under $D(\hat{\mathfrak{a}}_2)$, the symmetry of affine Dynkin digram $\mathfrak{a}_2$. However, here $D(\hat{\mathfrak{g}}_2)$ is trivial so it does not help.

In [48], it is conjectured that the so-called precise vanishing conditions of GV invariants can fix all the unknown coefficients for the $\mathfrak{a}_2$ model, which was verified explicitly for base degrees up to three. For the $\mathfrak{g}_2$ model, we are now searching for the precise vanishing conditions, hoping that they can fix all the unknown coefficients. Meanwhile, since the generators of $J_{*,*}(\mathfrak{g}_2)$ and $J_{*,*}(\mathfrak{a}_2)$ are closely related to each other (see section A.3), it is plausible to conjecture that their partition functions can be identified upon mappings of Weyl-invariant Jacobi forms, which we have verified [57] for base degree one. In general, we expect the following diagram to hold,

$$\mathcal{Z}^{\mathfrak{g}_2}_k|_{\boldsymbol{m}_F=0} \xrightarrow{\text{(A.3.19)}} \mathcal{Z}^{\mathfrak{a}_2}_k\,. \tag{7.3.8}$$

Meanwhile, an expression of $\mathcal{Z}^{\mathfrak{g}_2}_{\boldsymbol{k}}$ that holds for all base degrees was found using localization techniques [124]. We can use their result to extra GV invariants and speculate possible vanishing conditions. Some examples are given in appendix E. If we would like to compare the above results with those from topological string theory, we again need to choose the correct basis of two-cycles that have the desired intersection matrix inside S. The toric data and corresponding two-cycles for $\mathfrak{g}_2$ model are listed in Table 7.3.9.

Div.	ν_i				$l^1_{\mathfrak{g}_2}$	$l^2_{\mathfrak{g}_2}$	l_{fiber}	l_{b}	l_{de}
D_0	0	0	0	0	−3	0	0	0	0
D_1	−1	0	0	0	1	0	0	0	0
D_2	0	−1	0	0	0	3	0	0	0
D_3	1	1	0	−1	3	−6	0	0	0
D'	2	3	0	−2	−2	3	1	−1	0
E	2	3	0	−1	1	0	−2	−1	1
K	2	3	0	0	0	0	1	0	−2
F	2	3	−1	−3	0	0	0	1	0
H	2	3	0	1	0	0	0	0	1
F	2	3	1	0	0	0	0	1	0

. (7.3.9)

Chapter 8

Conclusion and Outlook

In this thesis we have presented various results about topological string theory on Calabi-Yau manifolds and its relation with spectral theory and six dimensional super-conformal field theories.

In the first part, we explore the non-perturbative aspects in the Harper-Hofstadter model. In particular, we employed various techniques to compute the trans-series structure of the spectrum. The perturbative series is computed very conveniently using the extended `BenderWu` package [167, 81]. The 1-loop contributions to the 1-, 2-instanton sectors, and the energy ambiguity are obtained by a path integral calculation, albeit restricted to the ground state level. Higher order corrections in the 1-instanton sector and in the ambiguity (imaginary contributions of the instanton–anti-instanton sector) are computed using refined topological string techniques in connection with the local $\mathbb{F}_0$ geometry inspired by a similar work [42]. All these results can be checked against numerical results, which can be computed exactly when the magnetic flux is 2π times a rational number, and they all agree perfectly. This validates all our techniques.

Clearly there are still many open questions. For example, we have checked that the perturbative–non-perturbative relation[1] relating the perturbative sector and the 1-instanton sector is not satisfied, which is not that surprising since the Schrödinger equation of the Harper-Hofstadter model is a difference rather than a second-order differential equation. On the other hand there still exists a curious relation between the three sectors: perturbative, 1-instanton, instanton–anti-instanton. Also, we can ask ourselves how far can we extend this correspondence between topological string theory and condensed matter physics. For instance, another real world model that describes electrons on a triangular lattice, is revealed to be connected to the topological string theory, with the target space being the canonical bundle of the three-point blow-up of $\mathbb{P}^2$ [95]. One can explore whether a similar analysis can be applied in that model as well.

In the second part of thesis, we discuss elliptic genera of six dimensional super-conformal

[1] This relation was first proposed in [10, 6, 7, 8, 9] and later rediscovered by [59].

field theories and topological string partition functions on Calabi-Yau manifolds that engineer them. After careful analysis of their pole structure, we found that for geometries without codimension one singular fibers, their partition functions can be reconstructed based solely on genus zero Gromov-Witten invariants. Then we move on to geometries having codimension one singular fibers and discuss the Higgs tree structure.

There are lots of future directions waiting to be explored. First of all, for the geometries leading to non-abelian gauge symmetries, it's not clear what set of boundary conditions suffices to reconstruct the topological string partition functions. For the $\mathfrak{a}_2$ model, the precise vanishing conditions for GV invariants [48] were conjectured to be sufficient. We would like to find conditions of this sort for other geometries. Furthermore, once those partition functions are determined, we would like to see if they enable us to read off the Higgsing tree pattern. One plausible way is to look at degenerations among various Weyl-invariant Jacobi modular forms.

Meanwhile, their partition functions can also be studied from the two dimensional world-sheet, which have interesting chiral algebra structures [51] and can possibly be related to the super-conformal index and BPS spectrum of four dimensional SCFTs. It would be worthwhile to clarify their relations.

Last but not least, to define an elliptic CY manifold we need to choose a two dimensional base, either compact or non-compact. In [56], we obtain our strongest results for the non-compact bases, where the dynamics in the gravity sector is frozen. If instead we choose a compact base, its partition function captures the BPS content of black holes. Clearly, the more we can compute, the better we can understand those microscopic states and hence, the black hole entropy. However, since here the indices of g_s for the partition functions of different base degrees are mostly positive, we need new methods to compute it. It would also be interesting to see to what extent we could use GW invariants at low genera to determine the partition functions.

Appendix A

Modular Forms

In this appendix, we give a quick introduction to the theory of modular forms. Modular forms appear in a lot of places in modern mathematics: arithmetics, geometry, combinatorics, etc. They also arise quite naturally in modern physics, typically as partition functions of systems with certain symmetries. In section A.1, we discuss modular form of a single variable, also known as the elliptic modular form, then in section A.2 modular form with additional elliptic parameters, i.e., Jacobi modular form. Finally we will talk about Jacobi modular forms with extra Weyl group symmetries in section A.3, used in chapter 7.

For introductions to elliptic modular forms, one can read chapter 7 of the classical book [165], on which the section A.1 is based, or the first chapter of [35]. The standard reference for the section A.2 is [60]. For the section A.3 on Weyl-invariant Jacobi forms, many details can be found in [27].

A.1 Elliptic modular forms

Definition 2. *Suppose k is an integer. A function $f : \mathbb{H} \to \mathbf{C}$ is called a modular form of weight k for full modular group $SL(2,\mathbb{Z})$ if f is holomorphic on $H \cup \{\infty\}$ and satisfies the following equation*

$$f(\frac{a\tau+b}{c\tau+d}) = (cz+d)^k f(\tau), \text{ for any } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{A.1.1}$$

In particular, if we choose the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we find that $f(\tau+1) = f(\tau)$. Thus introducing $q = \exp(2\pi \mathrm{i} \tau)$, f can be expanded as a power series at $\tau = i\infty$

$$f(\tau) = \sum_{n\geq 0} a_n q^n \tag{A.1.2}$$

The condition $n \geq 0$ is guaranteed by the holomorphicity at the infinity. If furthermore $a_0 = 0$, we say that f is a *cusp form*. Notice that in this expansion, the full $SL(2,\mathbb{Z})$ symmetry is no longer manifest.

naively, possible examples of modular form should be related to summation over integer points of a lattice, which are reshuffled under $SL(2,\mathbb{Z})$ transformation. Indeed, it's not difficult to write down examples of this sort, also known as the Eisenstein series,

Definition 3. *For $k > 2$ an even integer, the level k Eisenstein series is defined as*

$$G_k(\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m+n\tau)^k}, \tag{A.1.3}$$

The condition $k > 2$ is to ensure absolute convergence[1] of the infinite sum. It's straightforward to verify that G_k is a modular form of weight k. The construction for odd k doesn't give anything new since in that case G_k is always zero. But still, this already gives us infinitely many examples. Notice that we can always rescale G_k by an overall factor, and we make use of this freedom to define $E_k = \frac{1}{\zeta(k)} G_k$ such that the constant term is one. Expand E_k at $\tau = \mathrm{i}\infty$ and obtain

$$E_k = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n, \tag{A.1.4}$$

where B_k is the kth Bernoulli number and $\sigma_i(n)$ denotes the sum of the ith powers of the positive divisors of n. We also list the first few examples:

$$\begin{aligned} E_4(\tau) &= 1 + 240q + 2160q^2 + \cdots, \\ E_6(\tau) &= 1 - 504q + 16632q^2 + \cdots, \\ E_8(\tau) &= 1 + 480q + 61920q^2 + \cdots. \end{aligned} \tag{A.1.5}$$

Moreover, we can write down a cusp form by canceling the constant term,

$$\Delta(\tau) = \frac{1}{1728}(E_4(\tau)^3 - E_6(\tau)^2) = q - 24q^2 + \cdots. \tag{A.1.6}$$

Since the product of two modular forms is also a modular form with weight the sum of separate weights, the space of modular forms forms a ring. Tis in fact greatly simplifies the structure, and the first striking result is that the ring of modular forms is generated by only two elements E_4 and E_6. cusp modular forms of weight k and the space of holomorphic modular forms of weight k. Clearly $M_k^c \subset M_k$.

Theorem 2. *1) When k is odd, $k < 0$ and $k = 2$, $M_k = 0$.*

[1]This is needed to ensure that we can rearrange the order of summation.

2) When $k = 0, 4, 6, 8, 10$, M_k is one dimensional with basis $1, E_2, E_4, E_6, E_4^2, E_4E_6$ respectively, while $M_k^c = 0$.

3) $M_k = M_k^c \oplus \mathbb{C}E_k$.

4) Multiplication by Δ is an isomorphism from M_{k-6} to M_k^c.

Proof. [165]. □

Corollary 1. *M_* is generated by E_4 and E_6.*

It turns out that we also need functions that are not exactly modular. The first one is the famous *Dedekind eta-function*,

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{i=1}^{\infty} (1 - q^i) \,. \tag{A.1.7}$$

The factor $q^{\frac{1}{24}}$ may seem strange, but actually it nicely reflects the presence of zero-point energy of physical systems. The Dedekind η function has the transformation rule,

$$\eta(\tau + 1) = \exp(\frac{\mathrm{i}\pi}{12})\,\eta(\tau), \;\; \eta(-\frac{1}{\tau}) = \frac{\sqrt{\tau}}{\sqrt{\mathrm{i}}}\,\eta(\tau). \tag{A.1.8}$$

Nevertheless, η^{24} is a cusp modular form of weight 12, using the fact that T and S matrices generate the whole $SL(2,\mathbb{Z})$ group. Since M_{12}^c is one dimensional, by comparing the first factor we immediately conclude that

$$\Delta(\tau) = \eta(\tau)^{24} \,. \tag{A.1.9}$$

Another function that is close to being modular is the so-called qausi modular form E_2. It can be motivated from the Eisenstein series. Although in the original definition 3 k must be greater than two to ensure absolute convergence, the Taylor series expansion (*A*.1.4) can be extended formally to $k = 2$ with finite radius of convergence. I.e.,

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n = 1 - 24q - 72q^2 + \cdots \tag{A.1.10}$$

However, we are no longer guaranteed to have modular invariance, since the naive definition 3 is not absolutely convergent. In fact, by noticing the fact that

$$\frac{1}{2\pi \mathrm{i}} \frac{d}{dz} \log \Delta(z) = E_2(z) \,, \tag{A.1.11}$$

which can be checked explicitly, together with (A.1.9) we can show

$$E_2(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^2 E_2(\tau) - \frac{6\mathrm{i}}{\pi}c(c\tau+d)\ . \tag{A.1.12}$$

In fact, we can enlarge our ring M_* to be the ring of quasi-modular forms $\tilde{M}_*$, by relaxing the condition (A.1.1) to be

$$f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k \sum_{j=0}^{s} f_j(\tau)(\frac{c}{cz+d})^j \tag{A.1.13}$$

for a set of holomorphic functions $f_j(\tau)$ and integers k and s. E_2 corresponds to the simplest case with $k = 2$ and $s = 1$. Similar to the ring M_*, the $\tilde{M}_*$ is also finitely generated. Actually, it can be shown [35] that for the full modular group $SL(2,\mathbb{Z})$ $\tilde{M}_*$ is generated from M_* just by adding E_2.

On the other hand, we can alternatively modify E_2 to be modular which can be verified by explicit computation,

$$\hat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi \mathrm{Im}(\tau)}\ , \tag{A.1.14}$$

at the expense that it's no longer holomorphic in τ.

In short, if we choose E_2, we keep holomorphicity but we lose modularity; If we choose $\hat{E}_2$ instead, we are able to preserve modularity but lose holomorphicity. This fact is intimately related to the holomorphic anomalies introduced in chapter 3 and greatly simplifies the computations in chapter 4.

A.2 Jacobi modular forms

A natural generalization of elliptic modular forms is Jacobi modular forms.

Definition 4. *A Jacobi modular form is a function $\phi : \mathbb{H}\times\mathbb{C} \to \mathbb{C}$ that depends on a modular parameter $\tau \in \mathbb{H}$ and an elliptic parameter $z \in \mathbb{C}$. It transforms under the action of $SL(2,\mathbb{Z})$ on $\mathbb{H}\times\mathbb{C}$ as*

$$\tau \mapsto \tau_\gamma = \frac{a\tau+b}{c\tau+d}, \quad z \mapsto z_\gamma = \frac{z}{c\tau+d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\ , \tag{A.2.1}$$

as

$$\phi\left(\tau_\gamma, z_\gamma\right) = (c\tau+d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}} \phi(\tau,z)\,, \tag{A.2.2}$$

$$\phi(\tau, z+\lambda\tau+\mu) = e^{-2\pi \mathrm{i} m(\lambda^2\tau+2\lambda z)}\phi(\tau,z) \quad \forall \lambda,\mu \in \mathbb{Z}\,, \tag{A.2.3}$$

(A.2.2) is known as the *modular* transformation which is a generalization of modular transform, while the second one (A.2.3) is known as the *elliptic* transform.

We can immediately extract some information from the definition. For example, in (A.2.2), if we choose $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$, we see that

$$\phi(\tau,-z) = (-1)^k \phi(\tau,z)\,, \tag{A.2.4}$$

meaning that the parity in z is correlated with its weight. Moreover, if we choose $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\lambda = 0, \mu = 1$ in equations (A.2.2) and (A.2.3) respectively, we see that the Jacobi form is invariant under the shift $\tau \to \tau + 1$ and $z \to z+1$, hence it enjoys a double Fourier expansion

$$\phi(\tau,z) = \sum_{n,r} c(n,r) q^n y^r, \qquad \text{where} \;\; q = e^{2\pi \mathrm{i}\tau}, \;\; y = e^{2\pi \mathrm{i} z}\;. \tag{A.2.5}$$

It can be shown that $c(n,r)$ only depends on r and an $SL(2,\mathbb{Z})$ invariant combination $4nm - r^2$, i.e., $c(n,r) = C(4nm - r^2, r)$. We can further define three subrings of Jacobi modular forms: holomorphic Jacobi forms $J^h_{*,*}$ satisfy the constraint $c(n,r) = 0$ unless $4nm \geq r^2$, cusp forms $J^c_{*,*}$ satisfy $c(n,r) = 0$ unless $4nm > r^2$ and weak Jacobi forms $J^w_{*,*}$ satisfy $c(n,r) = 0$ unless $n \geq 0$. Clearly we have $J^c_{*,*} \subset J^h_{*,*} \subset J^w_{*,*}$. In chapter 5, we will choose the biggest subring $J^w_{*,*}$ as our ansatz.

An important theorem whose proof can be found in [60] shows that $J^w_{*,*}$ of integer index is freely generated over the ring of elliptic modular forms by two generators $\phi_{-2,1}(\tau,z)$ and $\phi_{0,1}(\tau,z)$. Using the notation,

$$A(\tau,z) = \phi_{-2,1}(\tau,z) \quad \text{and} \quad B(\tau,z) = \phi_{0,1}(\tau,z)\;, \tag{A.2.6}$$

the above statement can be written concisely as

$$J^{\mathrm{w}}_{k,m} = \bigoplus_{j=0}^{m} M_{k+2j} A^j B^{m-j}\;. \tag{A.2.7}$$

As a remark, notice that for $J^w_{*,*}$ we can have non trivial elements with negative weights, but since both generators have positive index, there is no weak Jacobi modular form with negative index. This fact will play a very important role in the chapter 6.

We will define A and B in terms of the famous Jacobi theta functions. For a and b $\in \{0, 1/2\}$, we have

$$\Theta \begin{bmatrix} a \\ b \end{bmatrix} (\tau,z) = \sum_{n\in\mathbb{Z}} e^{\pi \mathrm{i}(n+a)^2\tau + 2\pi \mathrm{i} z(n+a) + 2\pi \mathrm{i} bn}\;. \tag{A.2.8}$$

The four theta functions in our conventions are $\theta_1 = \mathrm{i}\Theta\left[\begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2} \end{smallmatrix}\right]$, $\theta_2 = \Theta\left[\begin{smallmatrix} \frac{1}{2} \\ 0 \end{smallmatrix}\right]$, $\theta_3 = \Theta\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ and $\theta_4 = \Theta\left[\begin{smallmatrix} 0 \\ \frac{1}{2} \end{smallmatrix}\right]$.

In terms of $\theta_i(\tau, z)$, we can write down our two generators

$$\begin{aligned} A(\tau,z) &= -\frac{\theta_1(\tau,z)^2}{\eta^6(\tau)}\,, \\ B(\tau,z) &= 4\Big(\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2} + \frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2} + \frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\Big). \end{aligned} \tag{A.2.9}$$

The following Taylor expansions of A and B at $z = 0$ are useful in the chapter 6,

$$\begin{aligned} A(\tau,z) &= -(2\pi z)^2 + \frac{E_2}{12}(2\pi z)^4 + \frac{-5E_2^2+E_4}{1440}(2\pi z)^6 + \frac{35E_2^3 - 21E_2E_4 + 4E_6}{362880}(2\pi z)^8 \\ &\quad + \mathcal{O}(z^{10})\,, \\ B(\tau,z) &= 12 - E_2(2\pi z)^2 + \frac{E_2^2+E_4}{24}(2\pi z)^4 + \frac{-5E_2^3 - 15E_2E_4 + 8E_6}{4320}(2\pi z)^6 + \mathcal{O}(z^8)\,. \end{aligned} \tag{A.2.10}$$

Note that the coefficients of this expansion take values in the ring of quasi-modular forms $\mathbb{C}[E_2, E_4, E_6]$, which is guaranteed by the weakness condition: $c(n, r) = 0$ unless $n \geq 0$.

Up to now, we only considered Jacobi forms of one elliptic variable. It's possible to generalize them to Jacobi forms of many elliptic variables, e.g., by taking products. Formally, We fix an integral lattice L of rank $n > 0$, equipped with a positive definite inner product $(,)$. L is required to be even under this inner product, that is to say, for any $l \in L, (l, l) \in 2\mathbb{Z}$. Then we have

Definition 5. *A Jacobi modular form for the lattice L is a function $\phi : \mathbb{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}$ that depends on a modular parameter $\tau \in \mathbb{H}$ and $\boldsymbol{z} \in L \otimes \mathbb{C} \simeq \mathbb{C}^n$. It transforms under the action of $SL(2,\mathbb{Z})$ on $\mathbb{H} \times \mathbb{C}^n$ as*

$$\tau \mapsto \tau_\gamma = \frac{a\tau+b}{c\tau+d}, \quad \boldsymbol{z} \mapsto \boldsymbol{z}_\gamma = \frac{\boldsymbol{z}}{c\tau+d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})\,, \tag{A.2.11}$$

as

$$\phi\left(\tau_\gamma, \boldsymbol{z}_\gamma\right) = (c\tau+d)^k e^{\frac{2\pi i m c(\boldsymbol{z},\boldsymbol{z})}{c\tau+d}}\phi(\tau,\boldsymbol{z})\,, \tag{A.2.12}$$

$$\phi(\tau, \boldsymbol{z} + \boldsymbol{\lambda}\tau + \boldsymbol{\mu}) = e^{-2\pi i m\Big((\boldsymbol{\lambda},\boldsymbol{\lambda})\tau + 2(\boldsymbol{\lambda},\boldsymbol{z})\Big)}\phi(\tau,\boldsymbol{z}) \quad \forall \boldsymbol{\lambda}, \boldsymbol{\mu} \in L^*\,, \tag{A.2.13}$$

where in (A.2.13) L^* stands for the dual lattice, whose elements are defined to have integer inner product with all the elements in L. Note that comparing with definition 4, we basically just replace z by the $\boldsymbol{z}$. To name a small difference, the index m which is very important in definition 4 actually only plays a minor role here, since we can always rescale the inner product $(,)$ to be $m(,)$. However, we choose to preserve the index m because below we will use a fixed inner product on the lattice.

A.3 Weyl-invariant Jacobi forms

Lattices naturally occur in the study of Lie algebras. In fact, given a Lie algebra $\mathfrak{g}$, on the dual of complexified Cartan sub-algebra $\mathfrak{h}^*$ we have a bilinear form induced by the (suitably normalized) Killing form on $\mathfrak{h}^*$. Moreover, it's well known that on the root lattice there is a discrete group of automorphisms called the Weyl group, generated by reflections of simple roots. Therefore, we can impose the Weyl group invariance upon definition 5

$$\phi(\tau, \sigma(\boldsymbol{z})) = \phi(\tau, \boldsymbol{z}), \qquad \forall \sigma \in W. \tag{A.3.1}$$

Before proceeding further, let's first introduce our notation. For a given Lie algebra $\mathfrak{g}$, the bigraded ring $J_{*,*}(\mathfrak{g}) = \oplus_{w,n} J_{w,n}(\mathfrak{g})$ denotes Jacobi modular forms on $\mathfrak{h}_{\mathbb{C}}$ satisfying equations (A.2.12), (A.2.13) and (A.3.1). It is a polynomial ring over M_* which is the elliptic modular forms generated by E_4 and E_6. It was shown in [179] that if $\mathfrak{g}$ is a simple Lie algebra other than $\mathfrak{e}_8$, $J_{*,*}(\mathfrak{g})$ is freely generated by

$$\varphi_0, \varphi_1, \ldots, \varphi_r\,, \tag{A.3.2}$$

whose weights and indices are given respectively by

$$(-d_i, a_i^\vee)\ . \tag{A.3.3}$$

$d_0 = 0, a_0^\vee = 1$, while for $i = 1, \ldots, r$, d_i are the exponents of the Casimirs of $\mathfrak{g}$, and $a_i^\vee$ dual Coxeter numbers. We call the generators of the ring $J_{*,*}(\mathfrak{g})$ the fundamental Jacobi forms. Their explicit forms were constructed in Bertola's thesis [27] for $\mathfrak{g} = \mathfrak{a}_n, \mathfrak{b}_n, \mathfrak{g}_2, \mathfrak{c}_3, \mathfrak{d}_4$, and in [161] for $\mathfrak{g} = \mathfrak{e}_6, \mathfrak{e}_7$; For the case of $\mathfrak{e}_8$, fundamental Jacobi forms A_i and B_i were proposed in [162], whose explicit forms can be found in A.3.3. It was proved in [175] that $J_{*,*}(\mathfrak{e}_8)$ is properly contained inside the polynomial algebra of A_i and B_i over the rational field $\mathbb{C}(E_4, E_6)$.

As a special case, we have actually already encountered the $J_{*,*}(\mathfrak{a}_2)$. Remember that the root lattice of $\mathfrak{a}_1$ has rank one, generated by $\alpha_1^\vee = e_1 - e_2$, hence we can identify $z\alpha_1^\vee$ with z. Being invariant under the Weyl group just means an even function in the z invariable. Comparing with definition 4, we readily see that $J_{*,*}(\mathfrak{a}_1)$ is the same as $J^{\mathrm{w}}_{k,m}$ introduced before.

A.3.1 $J_{*,*}(\mathfrak{a}_2)$

The next example, which is also used in chapter 7, is $J_{*,*}(\mathfrak{a}_2)$. Its complexified Cartan subalgebra is two-dimensional, and a vector $\boldsymbol{z}$ can be parametrized as

$$\boldsymbol{z} = \sum_{j=1}^{3} u_j e_j = \sum_{j=1}^{2} x_j \alpha_j^\vee = \sum_{j=1}^{2} m_j w_j\ . \tag{A.3.4}$$

$\{e_j\}$ is the standard basis of $\mathbb{C}^3$, in which $\mathfrak{h}_{\mathbb{C}}$ can be embedded as a hyperplane. The parameters u_j should satisfy

$$u_1 + u_2 + u_3 = 0 \ . \tag{A.3.5}$$

α_j are the simple roots and ω_j are the fundamental weights. They can be taken to be [33]

$$\alpha_1 = e_1 - e_2 \ , \quad \alpha_2 = e_2 - e_3 \ , \tag{A.3.6}$$

and

$$w_1 = \frac{1}{3}(2e_1 - e_2 - e_3) \ , \quad w_2 = \frac{1}{3}(e_1 + e_2 - 2e_3) \ . \tag{A.3.7}$$

Accordingly, the different parametrizations are related by

$$\begin{cases} u_1 = x_1 & = \frac{2}{3}m_1 + \frac{1}{3}m_2 \ , \\ u_2 = -x_1 + x_2 & = -\frac{1}{3}m_1 + \frac{1}{3}m_2 \ , \\ u_3 = -x_2 & = -\frac{1}{3}m_1 - \frac{2}{3}m_2 \ . \end{cases} \tag{A.3.8}$$

According to (A.3.2), $J_{*,*}(\mathfrak{a}_2)$ is generated by forms

$$\varphi_3 \in J_{-3,1}(\mathfrak{a}_2) \ , \quad \varphi_2 \in J_{-2,1}(\mathfrak{a}_2) \ , \quad \varphi_0 \in J_{0,1}(\mathfrak{a}_2) \tag{A.3.9}$$

The explicit constructions in [27] are as follows: defining

$$\mathfrak{d}_x = \frac{1}{2\pi}\frac{\partial}{\partial x} \ , \tag{A.3.10}$$

the ring generators then read

$$\begin{aligned} \varphi_3 &= -\mathrm{i}\,\eta(\tau)^{-9}\prod_{j=1}^{3}\theta_1(\tau,u_j)|_{u_*\to x_*} \ , \\ \varphi_2 &= \Big(\sum_{j=1}^{3}\frac{\mathfrak{d}_{u_j}\theta_1(\tau,u_j)}{\theta_1(\tau,u_j)}\Big)\cdot\varphi_3|_{u_*\to x_*} \ , \\ \varphi_0 &= \Big(-\mathfrak{d}_\tau - \frac{E_2(\tau)}{4} + \frac{1}{3}(\mathfrak{d}_{x_1}^2 + \mathfrak{d}_{x_2}^2 + \mathfrak{d}_{x_1}\mathfrak{d}_{x_2})\Big)\circ\varphi_2 \ , \end{aligned} \tag{A.3.11}$$

where $u_* \to x_*$ means a change of parametrization according to (A.3.8).

$J^{\hat{D}}_{*,*}(\mathfrak{a}_2)$

In fact, as discussed in chapter 7, we are also interested in a subring inside $J_{*,*}(\mathfrak{a}_2)$, which contains Weyl invariant Jacobi forms invariant further under the affine Dynkin diagram automorphism group $D(\hat{\mathfrak{a}}_2)$. A set of generators is conjectured in [48] to be

$$\tilde{\phi}_0 \in J^{\hat{D}}_{0,3}(\mathfrak{a}_2) \ , \quad \tilde{\phi}_2 \in J^{\hat{D}}_{-2,3}(\mathfrak{a}_2) \ , \quad \tilde{\phi}_6 \in J^{\hat{D}}_{-6,6}(\mathfrak{a}_2) \ . \tag{A.3.12}$$

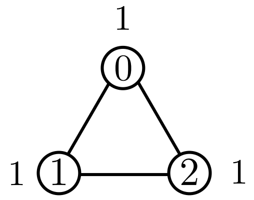


Figure A.1: Affine $\mathfrak{a}_2$ Dynkin diagram. The number outside of a circle is its comark.

To express these generators explicitly, we first write down Weyl invariant Jacobi forms that are invariant under the Dynkin diagram automorphism group $D(\mathfrak{a}_2)$. We denote them as $J^D_{*,*}(\mathfrak{a}_2)$. They are conjectured to have a basis,

$$\phi_0 \in J^D_{0,3}(\mathfrak{a}_2)\ , \quad \phi_2 \in J^D_{-2,3}(\mathfrak{a}_2)\ , \quad \phi_6 \in J^D_{-6,6}(\mathfrak{a}_2)\,, \tag{A.3.13}$$

where

$$\begin{aligned} \phi_6 &= -\,\eta^{-18}\prod_{j=1}^{3}\theta_1^2(m_j)\Big|_{m_3=-m_1-m_2}\,, \\ \phi_2 &= -8\mathrm{i}\,\eta^{-9}\prod_{j=1}^{3}\theta_1(m_j)\Big(\sum_{k=1}^{3}\frac{\mathfrak{d}_{m_k}\theta_1(m_k)}{\theta_1(m_k)}\Big)\Big|_{m_3=-m_1-m_2}\,, \\ \phi_0 &= \frac{3}{4}\Big(-\mathfrak{d}_\tau - \frac{E_2}{4} + \frac{1}{3}(\mathfrak{d}^2_{m_1} + \mathfrak{d}^2_{m_2} - \mathfrak{d}_{m_1}\mathfrak{d}_{m_2})\Big)\circ\phi_2\ . \end{aligned} \tag{A.3.14}$$

In terms of generators of $J_{*,*}(\mathfrak{a}_2)$ and elliptic modular forms, they look as follows

$$\begin{aligned} \phi_0 &= 6\varphi_0^3 + \frac{E_4\varphi_0\varphi_2^2}{8} - \frac{E_6\varphi_2^3}{72} - \frac{E_6\varphi_0\varphi_3^2}{16} + \frac{E_4^2\varphi_2\varphi_3^2}{192}\ , \\ \phi_2 &= 24\varphi_0^2\varphi_2 - E_4\varphi_0\varphi_3^2 - \frac{E_4\varphi_2^3}{6} + \frac{E_6\varphi_2\varphi_3^2}{12}\ , \\ \phi_6 &= 4\varphi_0^3\varphi_2^3 - 27\varphi_0^4\varphi_3^2 + \frac{5}{8}E_4\varphi_0^2\varphi_2^2\varphi_3^2 + \frac{E_6\varphi_0^2\varphi_3^4}{16} - \frac{E_4\varphi_0\varphi_2^5}{12} - \frac{E_6\varphi_0\varphi_2^3\varphi_3^2}{24} - \frac{E_4^2\varphi_0\varphi_2\varphi_3^4}{96} \\ &\quad + \frac{E_6\varphi_2^6}{216} + \frac{E_4^2\varphi_2^4\varphi_3^2}{2304} + \frac{E_4E_6\varphi_2^2\varphi_3^4}{2304} - \frac{E_6^2\varphi_3^6}{27648} + \frac{E_4^3\varphi_3^6}{27648}\ . \end{aligned} \tag{A.3.15}$$

They are normalized so that in the power series expansion in Q_0, Q_1, Q_2 with

$$q = Q_0Q_1Q_2 \tag{A.3.16}$$

the leading term has coefficient 1.

The generators of $J^D(\mathfrak{g})$ and generators of $J^{\widehat{D}}(\mathfrak{g})$ only differ by some prefactors,

$$\begin{aligned} J^D_{k,m}(\mathfrak{g}) &\longrightarrow \quad J^{\widehat{D}}_{k,m}(\mathfrak{g})\,, \\ \phi_{k,m} &\longmapsto \quad \tilde{\phi}_{k,m} = \Big(Q_1Q_2\Big)^{\frac{m}{2}}\phi_{k,m}\ . \end{aligned} \tag{A.3.17}$$

A.3.2 $J_{*,*}(\mathfrak{g}_2)$

The complexified Cartan subalgebra of $\mathfrak{g}_2$ can be realized as the hyperplane of $\mathbb{C}^3$ satisfying $z_1 + z_2 + z_3 = 0$. Its root lattice is the same as that of $\mathfrak{a}_2$, while the root diagram of $\mathfrak{g}_2$

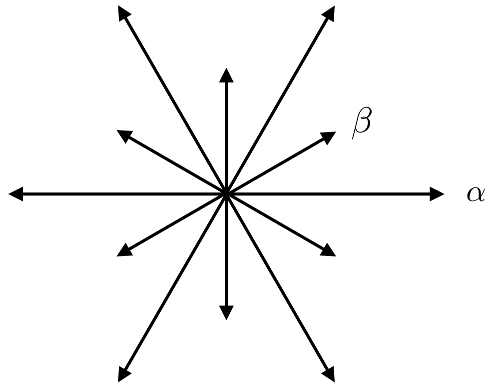


Figure A.2: Root system of $\mathfrak{g}_2$.

looks like Figure A.2. Intuitively, $\mathfrak{g}_2$ has two copies of the roots of $\mathfrak{a}_2$ but with different lengths. The Weyl goup $W(\mathfrak{g}_2)$ is the dihedral group D_6 of order 12. In fact, it can be shown that $W(\mathfrak{g}_2))/W(\mathfrak{a}_2) \simeq \mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by the involution:

$$i : (z_1, z_2, z_3) \to (-z_3, -z_2, -z_1)\,. \tag{A.3.18}$$

Therefore, by analyzing the generators for $J_{*,*}(\mathfrak{a}_2)$ under this involution, [27] proposed the following generators for $J_{*,*}(\mathfrak{g}_2)$:

$$\psi_0 = \phi_0 \in J_{0,1}(\mathfrak{g}_2); \qquad \psi_2 = \phi_2 \in J_{-2,1}(\mathfrak{g}_2); \qquad \psi_6 = \frac{1}{2}\phi_3^2 \in J_{-6,2}(\mathfrak{g}_2)\,. \tag{A.3.19}$$

A.3.3 $J_{*,*}(\mathfrak{e}_8)$

Finally, we list the generators of E_8 Weyl invariant Jacobi forms conjectured in [160]:

$$\begin{aligned}
A_1 &= \Theta(\tau, \mathbf{m}) = \frac{1}{2}\sum_{k=1}^{4}\prod_{j=1}^{8}\theta_4(\tau, m_j)\ , \quad A_4 = \Theta(\tau, 2\mathbf{m})\ , \\
A_n &= \frac{n^3}{n^3+1}\Big(\Theta(n\tau, n\mathbf{m}) + \frac{1}{n^4}\sum_{k=0}^{n-1}\Theta(\tfrac{\tau+k}{n}, \mathbf{m})\Big)\ , \quad n = 2, 3, 5\ , \\
B_2 &= \frac{8}{15}\Big((\theta_3^4+\theta_4^4)\Theta(2\tau, 2\mathbf{m}) + \frac{1}{2^4}(-\theta_2^4-\theta_3^4)\Theta(\tfrac{\tau}{2}, \mathbf{m}) + \frac{1}{2^4}(\theta_2^4-\theta_4^4)\Theta(\tfrac{\tau+1}{2}, \mathbf{m})\Big)\ , \\
B_3 &= \frac{81}{80}\Big(h(\tau)^2\Theta(3\tau, 3\mathbf{m}) - \frac{1}{3^5}\sum_{k=0}^{2}h(\tfrac{\tau+k}{3})^2\Theta(\tfrac{\tau+k}{3}, \mathbf{m})\Big)\ , \\
B_4 &= \frac{16}{15}\Big(\theta_4(2\tau)^4\Theta(4\tau, 4\mathbf{m}) - \frac{1}{2^4}\theta_4(2\tau)^4\Theta(\tau+\tfrac{1}{2}, 2\mathbf{m}) - \frac{1}{4^5}\sum_{k=0}^{3}\theta_2(\tfrac{\tau+k}{2})^4\Theta(\tfrac{\tau+k}{4}, \mathbf{m})\Big), \\
B_6 &= \frac{9}{10}\Big(h(\tau)^2\Theta(6\tau, 6\mathbf{m}) + \frac{h(\tau)^2}{2^4}\sum_{k=0}^{1}\Theta(\tfrac{3\tau+3k}{2}, 3\mathbf{m}) - \frac{1}{3^5}\sum_{k=0}^{2}h(\tfrac{\tau+k}{3})^2\Theta(\tfrac{2\tau+2k}{3}, 2\mathbf{m}). \\
&\quad . - \frac{1}{3\cdot 6^4}\sum_{k=0}^{5}h(\tfrac{\tau+k}{3})^2\Theta(\tfrac{\tau+k}{6}, \mathbf{m})\Big)\ ,
\end{aligned}$$

where we have set $\mathbf{m} = \sum_{i=1}^{8} m_i e_i$ and

$$h(\tau) = \theta_2(2\tau)\theta_2(6\tau) + \theta_3(2\tau)\theta_3(6\tau) \ . \tag{A.3.20}$$

A_n and B_n have index n and weight 4 and 6 respectively.

Appendix B

Toric Geometries

In this appendix, we review briefly the basic results in toric geometry, which plays an indispensable role in topological string theory. Mathematical oriented readers can consult [65, 46] for more details. [45] also contain lots of useful information. The first three sections B.1, B.2 and B.3 of this appendix are mainly based on chapter 7 of [103]. For the last section B.4, [145] contains a more detailed discussion.

Definition 6. *An r-dimensional toric variety X is a complex algebraic variety containing an algebraic torus $T = \mathbb{C}^r$ as a dense open set, together with an action of T on X whose restriction to $T \subset X$ is the multiplication.*

The nomenclature is explained by the definition. Since tori can be constructed through the quotient of Euclidean space by lattices, toric varieties are also naturally related to lattices. Let's fix N to be a rank r lattice, and set $N_\mathbb{R} = N \otimes \mathbb{R}$. We also introduce the dual lattice $M = \mathrm{Hom}(T, \mathbb{C}^*) \simeq \mathrm{Hom}(N, \mathbb{Z})$, and its underlying vector space $M_\mathbb{R} = M \otimes \mathbb{R}$. The action of M on N will be denoted as $\langle , \rangle : M \times N \to \mathbb{Z}$.

In the mathematical literature, there are typically two ways to define a toric variety X through the lattice. The first one makes use of N while the second one needs M. Since they are both useful in the main text, let's introduce both of them.

B.1 Fans

The first construction starts from (strongly convex rational) polyhedron cones in N.

Definition 7. *A (strongly convex rational) polyhedron cone $\sigma \subset N_\mathbb{R}$ is a set*

$$\sigma = \{a_1\nu_1 + a_2\nu_2 + \cdots + a_k\nu_k | a_i \geq 0\} \tag{B.1.1}$$

generated by a finite set of vectors $\nu_i \in N$ such that $\sigma \cap (-\sigma) = 0$.

A collection Σ of polyhedron cones is known as a fan if it satisfies the following conditions:

$$1.\textit{Each face of a polyhedron cone is also a polyhedron cone in } \Sigma, \tag{B.1.2}$$

$$2.\textit{The intersection of two polyhedron cones in } \Sigma \textit{ is a face of each.} \tag{B.1.3}$$

Given a fan Σ, let $\Sigma(1)$ be the set of one dimensional rays of Σ, denoted as $\{\nu_\rho\}_{\rho=1}^n$, where ν_ρ generate the intersection of the ray ρ with lattice N. The toric variety X_Σ is constructed as a quotient of a subset of $\mathbb{C}^n$ as follows.

We associate a coordinate x_ρ for each ν_ρ. Let $\mathscr{S}$ denote a subset of $\Sigma(1)$ that do not form a cone in Σ. Correspondingly, we single out a subspace $V(\mathscr{S})$ in $\mathbb{C}^n$ that is defined by setting $x_\rho = 0$ for $\rho \in \mathscr{S}$. If we denote $Z(\mathscr{S})$ by the union of all possible $V(\mathscr{S})$, then X_Σ is the quotient of $\mathbb{C}^n - Z(\mathscr{S})$ by a group G, and the algebraic torus can be identified as a dense open set $(\mathbb{C}^*)^n/G$.

The group G can be written down explicitly in terms of coordinates. If the coordinate of ν_ρ is $(\nu_{i1}, \cdots, \nu_{ir})$ inside N_R, first we consider the map ϕ:

$$\phi : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^r,\ (x_1, \cdots, x_n) \to (\prod_{i=1}^n x_i^{\nu_{i1}}, \cdots, \prod_{i=1}^n x_i^{\nu_{ir}}) \tag{B.1.4}$$

Then the group G is defined as the kernel of ϕ.

One of the useful features of the fan is an explicit correspondence between polyhedron cones and T-invariant subvarieties. Suppose $\sigma \in \Sigma$ is a cone generated by edges $\rho_1, \cdots, \rho_k$, then we can associate to it a codimension k subvariety

$$Z_\sigma = \{x \in X_\Sigma\ |x_{\rho_1} = \cdots = x_{\rho_k} = 0\}, \tag{B.1.5}$$

where x_i is assigned to ρ_i as before. For example, each edge ρ corresponds to a T-invariant divisor D_ρ in X_σ. If they constitutes a two dimensional cone in Σ, this means that their intersection gives a codimension two T-invariant subvariety, etc. (Notice $\sigma \mapsto Z_\sigma$ reverses the order of inclusion)

In particular, if we consider all effective (complex) one-cycles of X, they form a cone known as the Mori cone. Given the fan Σ, it is possible to find the generators of the Mori cone explicitly based on the above correspondence.

Theorem 3. *The Mori cone of a toric variety X is generated by curves corresponding to all $(r-1)$ dimensional cones.*

Proof. [65]. □

In practice, we choose to represent the data of Σ as two matrices $P_{ij}|Q_{kl}$, where $(P_{i1}, \cdots, P_{ir})$ is the coordinate of the edge ρ_i, while Q_{kl} encodes relations among the edges: $\sum_{i=1}^r Q_{il}\rho_i = 0$ for all l. Actually, it can be shown that $(Q_{1l}, \cdots, Q_{nl})$ represents a

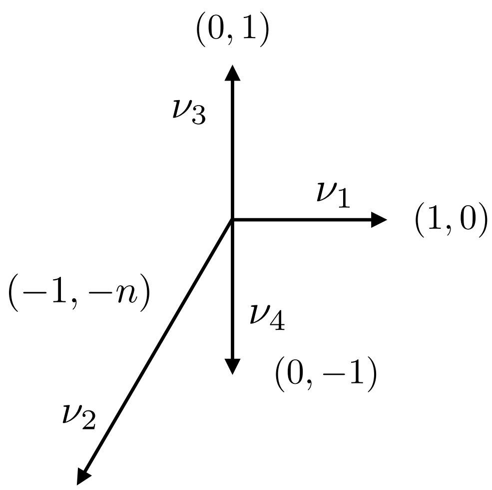


Figure B.1: Hirzebruch surface $\mathbb{F}_n$.

one cycle C_l, and Q_{kl} is nothing but the intersection number of C_l with the divisor D_k. Since we are free to choose a set of basis for one-cycles, we shall always choose $\{C_l\}$ to be generators of the Mori cone whenever it is possible. In the main text, we sometimes also use the notation $\ell_i^{(j)}$ for Q_{ij}.

After all these abstract definitions, it's better to look at some concrete example. Below, let's construct the Hirzebruch surface $\mathbb{F}_n$, which plays an important role from chapter 5 to chapter 7, from its fan.

Example 5. *We consider the compact toric variety associated with the fan Σ with edges $\Sigma(1) = \{(1,0), (-1,-n), (0,1), (0,-1)\}$, shown in Figure B.1.*

Clearly $\{\nu_1, \nu_2\}$ and $\{\nu_3, \nu_4\}$ don't span a cone in Σ, and any set of edges that doesn't span a cone must contain at least one of these sets. Therefore, we see that $Z(\Sigma) = \{x_1 = x_2 = 0\} \cup \{x_3 = x_4 = 0\}$. Moreover, the group G which is the kernel of the map $\phi : (\mathbb{C}^*)^4 \to (\mathbb{C}^*)^2$ is given by

$$(x_1, x_2, x_3, x_4) \to (t_1 t_2^{-1}, t_2^{-n} t_3 t_4^{-1}). \tag{B.1.6}$$

As a result, G is actually the following map

$$(t_1, t_2, t_3, t_4) \to (t_1, t_1, t_1^n t_2, t_2). \tag{B.1.7}$$

First of all, according to our rules (B.1.5), four edges ν_i correspond to four T-invariant divisors D_i. Since v_1, v_2 do not span a cone, D_1 and D_2 do not intersect. Similarly for D_3 and D_4. However, all other possible pairings of D_i with D_j $(i \neq j)$ intersect, simply because the corresponding edges constitute two-dimensional cones in Σ.

From the fan it's easy to see that $\mathbb{F}_n$ is a $\mathbb{CP}^1$ bundle over $\mathbb{CP}^1$. Consider the map $\mathbb{F}_n \to \mathbb{CP}^1$ by $(t_1, t_2, t_3, t_4) \to (t_1, t_2)$. From (B.1.7) we see that this map is well-defined and the fiber is also a $\mathbb{CP}^1$. The fiber over $(1,0)$ and $(0,1)$ are D_1 and D_2 respectively.

If $n = 0$, we can define another projection map by $(t_1, t_2, t_3, t_4) \to (t_3, t_4)$ with the base also a $\mathbb{CP}^1$, hence we have $\mathbb{F}_0 \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$.

Furthermore, by noticing that we have two relations $1 \cdot \nu_3 + 1 \cdot \nu_4 = 0$ *and* $1 \cdot \nu_1 + 1 \cdot \nu_2 - n \cdot \nu_4 = 0$ *, we can write down the matrix* $P|Q$ *from the general recipe outlined in the previous paragraph,*

$$\left[\begin{array}{rr|rr} 1 & 0 & 0 & 1 \\ -1 & -n & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & -n \end{array}\right] \tag{B.1.8}$$

From the matrix Q*, we can obtain the intersection numbers between divisors and one-cycles. Since We already showed that both* D_1 *and* D_2 *are fibers, so* $D_1 \sim D_2$ *or* $D_3 \sim D_4$ *in the Picard group. Let's denote the divisors corresponding to* D_3 *and* D_4 *by* H *and* E *respectively. From the intersection numbers of* H *and* E *with one-cycles, we see that* $H = E + nF$*. Since* n *is non negative, it's easy to see that we should choose* H *and* F *as generators of the Mori cone.*

On the other hand, since F_n *is two-dimensional, one-cycles are themselves divisors. Again the intersection numbers determine them. The first column vector is* f*, since* $F^2 = 0$ *and* $f \cdot H = D_1 \cdot D_2 = F \cdot E = D_1 \cdot D_4 = 1$*. Similarly, the second column vector is* E*, since* $F \cdot E = 1$*,* $E \cdot H = D_3 \cdot D_4 = 0$ *and* $E \cdot E = E \cdot (H - nF) = -n$*.*

B.2 Polytopes

Next we switch our gear and discuss another description of toric variety in terms of polytopes. Polytopes live in $M_{\mathbb{R}}$, which is the dual space of $N_{\mathbb{R}}$.

Definition 8. *An integral polytope in* $M_{\mathbb{R}}$ *is the convex hull of a finite set of points of* M*.*

Now let's spell out the details of the construction. A Polytope encodes the data of projective embedding of X into a projective space. Consider an r-dimensional polytope $\Delta \subset M_{\mathbb{R}}$. The points of $\Delta \cap M_{\mathbb{R}}$ are denoted as $m_0, \cdots, m_k$. Recall that $M = \mathrm{Hom}(T, \mathbb{C}^*)$, hence m_i can be regarded as nowhere vanishing holomorphic functions on T. These functions give us a map

$$f : T \to \mathbb{CP}^k, \qquad f(t) = (m_0(t), \cdots, m_k(t))\,. \tag{B.2.1}$$

Since m_i are nowhere vanishing, f is well-defined and in fact gives an embedding. The toric variety $\mathbb{CP}^k_\Delta$ is defined as the closure of $f(t)$ in $\mathbb{CP}^k$. The algebraic torus is clearly the image of $f(t)$. Note that the toric variety structure does not depend on the ordering of m_i.

On the other hand, we can rewrite (B.2.1) as $y_i = m_i(t)$, where $(y_0, \cdots, y_k)$ are homogeneous coordinates on $\mathbb{CP}^k$. Suppose we have a linear relation among m_i: $\sum a_i m_i = 0$. Then $\mathbb{CP}^k_\Delta$ is contained in the following subvariety:

$$\prod_{a_i > 0} y_i^{a_i} = \prod_{a_i < 0} y_i^{-a_i}\,. \tag{B.2.2}$$

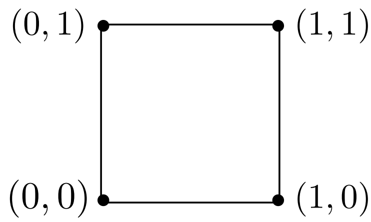


Figure B.2: The polytope of $\mathbb{F}_0$.

Conversely, we can use all such equations (B.2.2) to define our $\mathbb{CP}^k_\Delta$ as a projective subvariety.

It's straightforward to recover the fan from a given Δ. First of all, for each face F of Δ, we define the cone

$$\sigma_F = \{v \in N_\mathbb{R} | \langle m, \nu\rangle \leq \langle m', \nu\rangle \text{ for all } m \in F \text{ and } m' \in \Delta\} \tag{B.2.3}$$

The collection of all possible F forms a fan, known as the normal fan Σ_Δ.

Theorem 4.

$$X_{\Sigma_\Delta} \simeq \mathbb{CP}^k_\Delta\,. \tag{B.2.4}$$

Proof. [65]. □

Now let's come back to our old friend $\mathbb{F}_n$. For simplicity, we only show how to describe $F_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$ from its polytope.

Example 6. *The polytope of* $\mathbb{CP}^1 \times \mathbb{CP}^1$ *is Figure B.2. The integral points of* $\Delta \cap M_\mathbb{R}$ *can be easily found to be*

$$\{m\}_{i=0}^4 = \{(0,0), (1,0), (0,1), (1,1)\}. \tag{B.2.5}$$

This defines a projective embedding into $\mathbb{CP}^3$ *as follows,*

$$f : (t_1, t_2) = (1, t_1, t_2, t_1 t_2). \tag{B.2.6}$$

The closure of its image gives us the embedding of $\mathbb{CP}^1 \times \mathbb{CP}^1$*,*

$$f : ([x_0, x_1], [y_0, y_1]) = ([x_0 y_0, x_0 y_1, x_1 y_0, x_1 y_1]) \tag{B.2.7}$$

Moreover, if we take its normal fan, constituting normal vectors for all the faces, we get back the Figure B.1 after setting $n = 0$.

Next, let's discuss the inverse problem: how to recover the polytope Δ from the toric variety X? Because we always assume X is projective, i.e., can be embedded in a projective space $\mathbb{CP}^k$, there always exists a very ample divisor $\mathcal{O}_X(1)$ on X. To construct Δ, we need to fix an isomorphism between $\mathcal{O}_X(1)$ and $\mathcal{O}(D)$, where D is a T-invariant divisor. Different choices will result in a translation in $M_\mathbb{R}$, so there is no essential difference. We also need to assume that the action of T can be extended to an action on $\mathbb{CP}^k$, which is just by coordinate multiplications.

Holomorphic sections of divisor D can be identified with meromorphic functions f on X satisfying $(f) + D \geq 0$, with (f) being the divisor defined by f. Therefore, if we restrict each coordinate function x_i on $\mathbb{CP}^k$ to X, it can be regarded as a meromorphic function f_i on X. The assumption that T is an action on $\mathbb{CP}^k$ implies that its further restriction on $T \in X$ is a character. We denote all such characters by lattice points $\{m_i\}$, and their convex hull is the Δ that we are looking for.

Example 7. *Let's consider again our favorite example $\mathbb{F}_0$. It can be represented as $[(x_0, x_1, x_2, x_3)\setminus\{(0, 0, x_2, x_3) \cup (x_0, x_1, 0, 0)\}]/\sim$, where $\sim$ means the equivalence relation*

$$(x_0, x_1, x_2, x_3) \sim (\lambda_1 x_0, \lambda_1 x_1, x_2, x_3),\ (x_0, x_1, x_2, x_3) \sim (x_0, x_1, \lambda_2 x_2, \lambda_2 x_3), \tag{B.2.8}$$

for all $\lambda_{1,2} \in \mathbb{C}^*$.

We choose to identify $\mathcal{O}_{\mathbb{F}_0}(1)$ with $\mathcal{O}(D_0 + D_2)$. A basis of $\Gamma(\mathcal{O}(D_0 + D_2))$ is given by four homogeneous monomials of $(1, 1)$ bidegree. From our choice, they correspond to meromorphic functions on $\mathbb{F}_0$,

$$V = \left\{ \frac{s}{x_0 x_2} : s \in \{x_0x_2, x_1x_2, x_0x_3, x_2x_3\} \right\}. \tag{B.2.9}$$

Then it's easy to see we indeed recover Figure B.2 or (B.2.5).

B.3 Blow-ups

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Singularities appear naturally when studying algebraic varieties, and how to resolve them is an important subject in algebraic geometry. A big theorem proved by Heisuke Hironaka [99] says that over a complex field, all the singularities can be resolved by performing sufficiently many times of blow-ups. If we restrict to toric varieties, the blow-up can be visualized quite easily. Therefore, in this section we will discuss pictures rather than definitions. For rigorous definitions and discussions, we refer readers to [90].

At this point we already know that there are two ways to construct a toric variety. Thanks to theorem 4, if we understand one case, then we can readily infer the other one. Here, we choose to study the fan in more detail.

First of all, it's helpful to know what types of singularities could possibly arise. In terms of fans, we have the following characterizations for different levels of being singular:

Theorem 5. *Given a fan Σ and its corresponding toric variety X_Σ,*

- *If each cone σ in Σ is generated by a $\mathbb{Z}-$basis for the intersection of σ with N, then X_Σ is actually smooth.*
- *If the generating vectors $\{\nu_i\}$ each cone σ in Σ forms a basis for the vector space they span, then Σ is called simplicial and X_Σ in general is an orbifold.*
- *If the generating vectors $\{\nu_i\}$ of one cone σ in Σ do not form a basis for the vector space, X_Σ has more severe singularities.*

Proof. [65]. □

It's well-known that blow-ups create new cycles in the resolved geometry. Because of the correspondence (B.1.5), it's natural to expect that resolution amounts to adding new cones. That is indeed the case. We first introduce the notion of a subdivision:

Definition 9. *A fan σ' is a subdivision of σ' if*

- $\Sigma(1) \in \Sigma'(1)$,
- *each cone of Σ' is contained in some cone of Σ.*

Suppose Σ' with generators $\Sigma'(1) = \{\nu_1, \cdots, \nu_\alpha\}$ subdivides σ with generators $\Sigma(1) = \{\nu_1, \cdots, \nu_\beta\}$. Then it can be checked that there is a well-defined projection map $\pi : \mathbb{C}^m - Z(\Sigma') \longmapsto \mathbb{C}^n - Z(\Sigma)$ compatible with the group action, hence π descend to a map from $X_{\Sigma'}$ to X_Σ. It is birational because it induces an isomorphism on the torus which is a dense open set.

Intuitively, blow-ups means "adding new edges" and subdividing a fan. Base on theorem 4, we can check that it is the same as "cutting the corner" of the corresponding polytope.

In practice, if we want to blow up a T-invariant smooth point $p \in X_\sigma$, we first identify the corresponding r-dimensional cone $\sigma \in \Sigma$. If the primitive generators of σ are $\{\mu_1, \cdots, \nu_k\}$, then all we need to do is to add a new edge

$$\nu_{k+1} = \nu_1 + \cdots \nu_k \,, \tag{B.3.1}$$

and properly subdivide σ. These new cones together with other cones in Σ form the new fan Σ'. This procedure can be easily generalized to lower dimensional cones, related to blow-ups along higher dimensional subvarieties.

Furthermore, we can blow up a singular point to resolve singularities. Now comes a important remark. For complex threefolds, there is a very important phenomenon which doesn't exist for lower dimensional varieties, known as the flop. In particular, this means that there can be different ways to resolve a singularity. Abstract as it sounds, it is actually very concrete in terms of toric geometry. A prototype example is the resolution of conifold singularity.

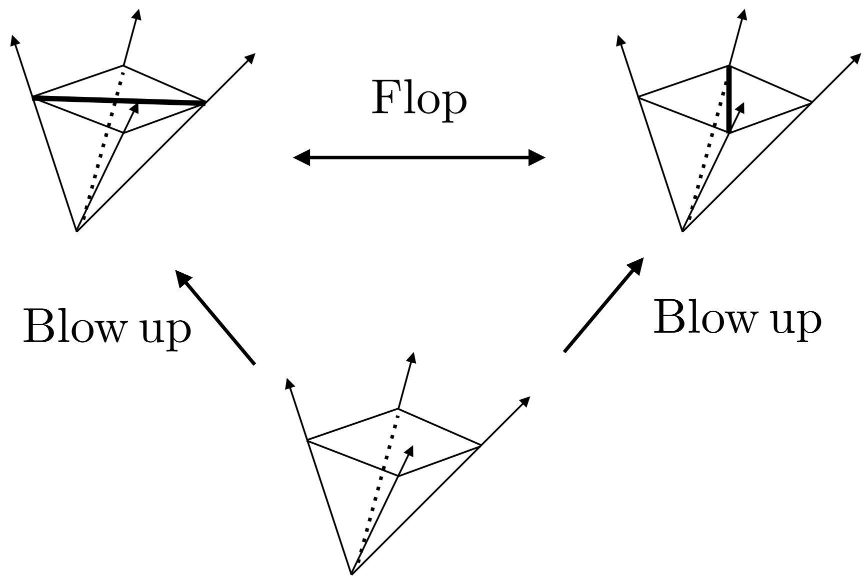

Figure B.3: Resolutions of conifold singularity and flop.

Example 8. *Consider the fan* Σ *with cone generated by* $(1,0,0),(0,1,0),(1,1,-1)$ *and the faces they form. The toric variety is singular at one point corresponding to the three dimensional cone formed by* $(1,0,0),(0,1,0),(1,1,-1)$. *It is not an orbifold singularity since the four vectors do not constitute a set of basis. This codimension three singularity is the famous conifold point.*

Singularity can be resolved by blow-ups. However, here they are two ways to perform the blow-up, as shown in Figure B.3.

We did not add any new edge to the fan, so there's no exceptional divisor, i.e., $\Sigma'(1) = \Sigma(1)$. *Nevertheless, we added new two dimensional cone (spanned by* $(1,0,0),(0,1,0)$ *in the first case and by* $(0,0,1),(1,1,-1)$ *in the second case), so there's a new exceptional curve in the resolved geometry. This phenomenon is known as the flop.*

Since the flop can change the one-cycles, it can shuffle the GV invariants. If we want to compare results in the topological string with those in the gauge theory, we need to select the corresponding CY geometry among all possible flops.

B.4 Toric Diagrams

In the physics literature, there is another very common way to present specifically a three-dimensional toric non compact CY variety, known as the toric diagram. Roughly speaking, a toric diagram starts from local patches which are $\mathbb{C}^3$ and encodes how to glue them together globally. Below let's see how this works.

To begin with, it's essential to first consider the simplest case $\mathbb{C}^3$. To make the torus fibration manifest, we will rather view it as a $T^2 \times \mathbb{R}$ fibration over $\mathbb{R}^3$, and we will only show the degeneration pattern.

More precisely, let z_i be complex coordinates on $\mathbb{C}^3$, $i = 1, 2, 3$. The canonical symplectic form $\omega = \sum_i dz_i \wedge d\bar{z}_i$ turns it into symplectic manifold. We introduce three commuting Hamiltonian functions,

$$\begin{aligned} r_\alpha(z) &= |z_1|^2 - |z_3|^2 \,, \\ r_\beta(z) &= |z_2|^2 - |z_3|^2 \,, \\ r_\gamma(z) &= \mathrm{Im}(z_1 z_2 z_3) \,. \end{aligned} \tag{B.4.1}$$

The Hamiltonians generate three vector fields on $\mathbb{C}^3$ via the standard procedure

$$\partial_\nu z_i = \{r_\nu, z_i\}_\omega, \quad \nu = \alpha, \beta, \gamma. \tag{B.4.2}$$

where $\{,\}_\omega$ is the Poisson bracket. The integral curves give us exactly the fibration structure mentioned earlier. The base $\mathbb{R}^3$ parameterizes the values of (B.4.1), while over each point p we have the fiber $T^2 \times \mathbb{R}$. The elliptic part of the fiber can be easily seen from the integrated form of the flow (B.4.2),

$$e^{i\alpha r_\alpha + i\beta r_\beta} : \ (z_1, z_2, z_3) \ \rightarrow \ (e^{i\alpha} z_1, e^{i\beta} z_2, e^{-i(\alpha+\beta)} z_3). \tag{B.4.3}$$

We denote the cycle generated by r_α the $(0,1)$ cycle and the cycle generated by r_β the $(1,0)$ cycle.

Notice that the $(0,1)$ cycle degenerates along the subspace $z_1 = z_3 = 0$, which in terms of (B.4.1) is a subspace of $\mathbb{R}^3$ given by $r_\alpha = r_\gamma = 0, r_\beta \geq 0$. Similarly, along $z_2 = z_3 = 0$ the $(1,0)$-cycle degenerates over the subspace $r_\beta = r_\gamma = 0$ and $r_\alpha \geq 0$ of $\mathbb{R}^3$. Finally, the $(1,1)$ cycle is degenerate along $r_\alpha - r_\beta = 0 = r_\gamma$ and $r_\alpha \leq 0$.

The toric diagram of $\mathbb{C}^3$ encodes the degeneration loci of T^2 in the $\mathbb{R}^3$ base. Namely, we consider a planar graph by taking $r_\gamma = 0$ and drawing the lines in the $r_\alpha - r_\beta$ plane. From the above discussion, the degeneration loci are straight lines described by the equation $pr_\alpha + qr_\beta = \mathrm{const}$. Over this line the $(-q, p)$ cycle of the T^2 degenerates. Therefore we correlate the degenerating cycles unambiguously with the lines in the graph (up to $(q,p) \to (-q,-p)$). This yields the graph in Figure B.4.

Taking the symmetries of $\mathbb{C}^3$ into account, the vectors in the toric diagram can always be assumed to satisfy

$$\sum_i v_i = 0. \tag{B.4.4}$$

Aside from that, there is still a residual $SL(2,\mathbb{Z})$ symmetry acting on the torus. In terms of one cycles, this just means the freedom to choose a set of basis in $H^1(T^2, \mathbb{R})$. This means that different toric diagrams can correspond to the same toric geometry.

After discussing the local structure, now let's see how to patch them together. Suppose we are given a toric Calabi-Yau threefold as a symplectic quotient of $\mathbb{C}^{N+3}$,

$$\mu_A = \sum_{j=1}^{N+3} Q_A^j |z_j|^2 = t_A, \qquad A = 1, \cdots, N. \tag{B.4.5}$$

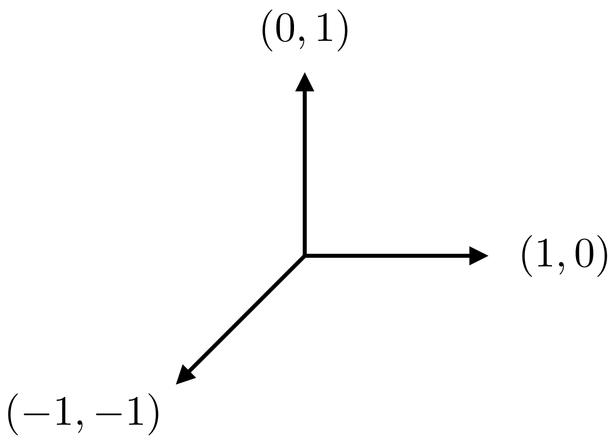

Figure B.4: Toric diagram of $\mathbb{C}^3$.

We first find a decomposition of coordinates $\{z_j\}_{j=1}^{N+3}$ into triplets $U_a = (z_{i_a}, z_{j_a}, z_{k_a})$ that correspond to the decomposition of X into $\mathbb{C}^3$ patches. We pick one of the patches and associate to it two Hamiltonians r_α, r_β, just as we did for $\mathbb{C}^3$ before. These two coordinates will be global coordinates in the base $\mathbb{R}^3$, therefore they will generate a globally defined T^2 fiber. The third coordinate in the base is $r_\gamma = \mathrm{Im}(\prod_{j=1}^{N+3} z_j)$, which is manifestly gauge invariant and moreover, patch by patch, can be identified with the coordinate used in the $\mathbb{C}^3$ example above. The (B.4.5) can then be used to find the action of $r_{\alpha,\beta}$ on the other patches.

Example 9. *The example that we look at is $K \to \mathbb{F}_0$, which is the total space of canonical line bundle K over $\mathbb{F}_0$. It can be defined as the symplectic quotient inside $\mathbb{C}^5$*

$$\begin{aligned} |z_1|^2 + |z_2|^2 - 2|z_0|^2 &= t_1 \,, \\ |z_3|^2 + |z_4|^2 - 2|z_0|^2 &= t_2 \,. \end{aligned} \tag{B.4.6}$$

The two $U(1)$ actions on the z_i are

$$(z_0, z_1, z_2, z_3, z_4) \to (e^{-2i\alpha - 2i\beta} z_0, e^{i\alpha} z_1, e^{i\alpha} z_2, e^{i\beta} z_3, e^{i\beta} z_4). \tag{B.4.7}$$

Notice that $z_{1,2,3,4}$ describe the basis $\mathbb{F}_0$, while z_0 parameterizes the direction of the fiber.

This geometry can be glued together in terms of four local patches U_i defined by $z_i \neq 0$, for $i = 1, 2, 3, 4$, since they cannot all be zero simultaneously. they all look like $\mathbb{C}^3$, because for example, for $z_1 \neq 0$, we can "solve" for z_1 and z_3 in terms of the other three unconstrained coordinates which then parameterize $\mathbb{C}^3$: $U_3 = (z_0, z_2, z_4)$.

Let us now construct the local toric diagram. In the $U_1 = (z_0, z_2, z_4)$ patch we take as our Hamiltonians

$$\begin{aligned} r_\alpha =&\ |z_4|^2 - |z_0|^2, \\ r_\beta =&\ |z_2|^2 - |z_0|^2. \end{aligned}$$

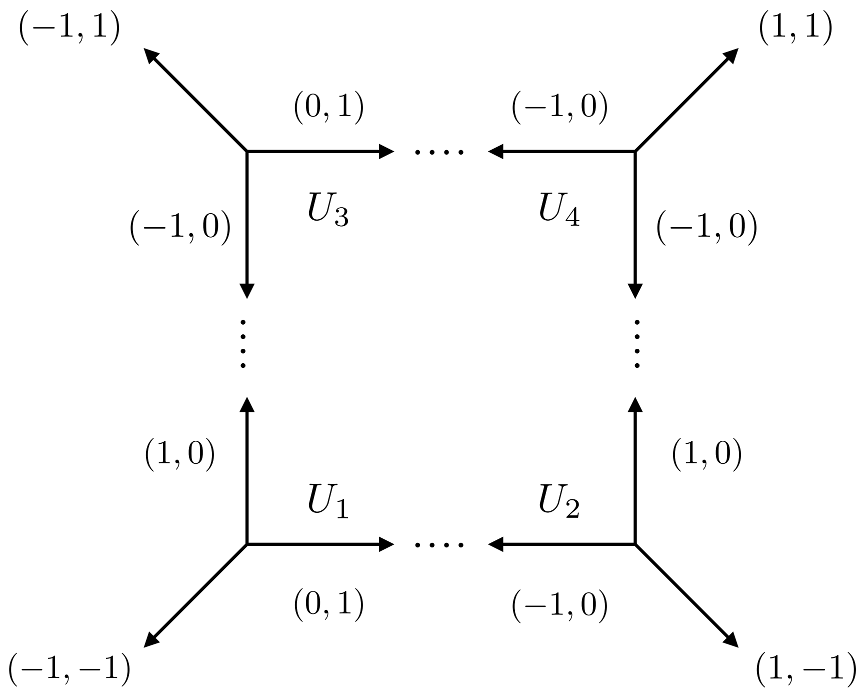

Figure B.5: Toric diagram of local $\mathbb{F}_0$.

The graph of the degenerate fibers in the $r_\alpha - r_\beta$ *plane is the same as in the* $\mathbb{C}^3$ *example Figure B.4. The third direction in the base,* r_γ *is now given by the gauge invariant product* $r_\gamma = \mathrm{Im}(z_0 z_1 z_2 z_3 z_4)$.

The same two Hamiltonians $r_{\alpha,\beta}$ *generate the action in the* $U_2 = (z_0, z_1, z_4)$ *patch, and we can use the constraint (B.4.6) to rewrite them. Since both* z_0 *and* z_4 *are coordinates of this patch* r_α *does not change. However,* r_β *must be changed because* z_2 *is not a good coordinate. Using the first equation in (B.4.6), we can rewrite* r_β *as*

$$r_\beta = t_1 + |z_0|^2 - |z_1|^2, \tag{B.4.8}$$

Consequently, the action in U_2 *becomes*

$$e^{(i\alpha r_\alpha + i\beta r_\beta)} : (z_0, z_1, z_4) \to (e^{i(-\alpha+\beta)} z_0, e^{-i\beta} z_1, e^{i\alpha} z_4). \tag{B.4.9}$$

We see from the above that the fibers degenerate over three lines. Two of them is obvious, which are just the (1,0) and (0,-1) cycles. The final one is $r_\alpha + r_\beta = t$*, corresponding to* $z_2 = z_3 = 0$*, and where a* $(1, -1)$ *cycle degenerates. To glue* U_1 *and* U_2 *patches, we need to identify the leg* $(0, 1)$ *in* U_1 *and* $(0, -1)$ *in* U_2*. The gluing of other patches is similar, and we end up with the graph for* $K \to \mathbb{F}_0$ *shown in Figure B.5.*

Appendix C

Computations in One-Instanton Sector

C.1 Instanton solution

We would like to solve the instanton profile from its equations of motion

$$\mathrm{i}\dot{x} - \sin y = 0 \; , \tag{C.1.1a}$$

$$\mathrm{i}\dot{y} + \sin x = 0 \; . \tag{C.1.1b}$$

We take the derivative w.r.t. time on (C.1.1a) and multiply it with $\dot{x}$, and after using (C.1.1b) to remove all appearance of y, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(\sqrt{1+\dot{x}^2} \pm \cos x) = 0 \; , \tag{C.1.2}$$

where $\pm$ comes from converting $\cos y$ to $\sin y$, and the above equation integrates to the identity

$$E(\beta) = \sqrt{1+\dot{x}^2} \pm \cos x \; . \tag{C.1.3}$$

We interpret the integration constant $E(\beta)$ to be the conserved energy of the saddle point configuration. Indeed, when $\dot{x}$ is small, the r.h.s. of (C.1.3) becomes

$$\frac{1}{2}\dot{x}^2 + 1 \pm \cos x \tag{C.1.4}$$

which resembles the conserved energy of a saddle point configuration in non-relativistic QM where $1 \pm \cos x$ is the inverted potential. For the 1-instanton configuration $x_1(t)$, the energy $E(\beta)$ reaches the maximum value in the limit $\beta \to \infty$, and it corresponds to the oscillation between two neighboring highest points of the inverted potential. In (C.1.3), we have $E(\infty) = 2$ regardless of the sign in the inverted potential, so we simply take $+$ without loss of generality

$$\sqrt{1+\dot{x_1}^2} + \cos x_1 = 2 \; . \tag{C.1.5}$$

Solving (C.1.5), we find the following profile of 1-instanton

$$x_1(t) = 2\cos^{-1}\left(-\frac{\sqrt{2}\tanh(t-t_0)}{\sqrt{1+\tanh^2(t-t_0)}}\right) , \tag{C.1.6}$$

as well as

$$y_1(t) = \cos^{-1}(2-\cos(x_1(t))) = \cos^{-1}\left(1+\frac{2}{\cosh 2(t-t_0)}\right) . \tag{C.1.7}$$

Using the conservation law

$$\cos y_1 + \cos x_1 = 2 , \tag{C.1.8}$$

we find that the action is given by

$$\begin{aligned} A =& \int_{-\infty}^{\infty} \mathrm{d}t(-\cos x_1 - \cos y_1 + 2 - \mathrm{i}\dot{x}_1 y_1) \\ =& -\mathrm{i}\int_0^{2\pi} y_1(x_1)\mathrm{d}x_1 = 2\int_0^{\pi}\cosh^{-1}(2-\cos x)\mathrm{d}x \\ =& 2\int_0^{\pi}\log\left(2-\cos x+\sqrt{(3-\cos x)(1-\cos x)}\right)\mathrm{d}x \\ =& 4\int_0^{\pi}\log\left(\sin\frac{x}{2}+\sqrt{1+\sin^2\frac{x}{2}}\right)\mathrm{d}x \\ =& 8\int_0^1\frac{\mathrm{d}t}{1-t^2}\log(t+\sqrt{1+t^2}) = 8\mathrm{C} . \end{aligned} \tag{C.1.9}$$

In the last line we performed the change of variables $t = \sin x/2$, and used one of the definitions of the Catalan's constant

$$\mathrm{C} = \int_0^1 \frac{\sinh^{-1} t}{\sqrt{1-t^2}}\mathrm{d}t . \tag{C.1.10}$$

C.2 The moduli-space metric

We want to find the moduli-space metric of the one instanton. We can do this by adding a factor $\lambda\delta\tilde{x}^2$ to the action (4.2.25) before integration, so as to lift the zero mode. Upon modifying (4.2.25) by adding such a term, we can do the Gaussian integral and simply get

$$Z_1^\lambda = \frac{1}{\sqrt{\tilde{\mathsf{O}}+\lambda}} . \tag{C.2.1}$$

Now let us write the small deviation around the instanton solution as

$$\delta\tilde{x} \approx \left(\partial_{t_0}x_1\Big|_{t_0=0}/\sqrt{\cos y_1}\right)t_0 + \delta\tilde{x}^\perp = -\dot{x}_1 t_0/\sqrt{\cos y_1} + \delta\tilde{x}^\perp , \tag{C.2.2}$$

where $\delta\tilde{x}$ is orthogonal to $\cos y_1\dot{x}_1$. The first term is a small deviation from the instanton solution in the direction of the zero mode, and t_0 specifies a shift of its position in time.

In fact t_0 is precisely the coordinate we want to isolate, and over which we will integrate exactly, producing a factor of β. Recall that our goal is to find a way to write the path integral in (4.2.25), as

$$Z_1 = \int \mathrm{d}t_0 \; \frac{\mu}{\sqrt{\det' \tilde{\mathsf{O}}}} \,, \tag{C.2.3}$$

where the prime indicates that the zero-mode has been excluded from the determinant. The μ above is the measure of the zero-mode moduli t_0 (also referred to as moduli space metric), which is what we wish to find.

To find it we will add the term $\lambda \delta\tilde{x}^2$ into the action as before, and integrate over t_0. We should get (C.2.1), up to a constant, which will precisely correspond to μ^{-1}. To do this, let us plug in the expression (C.2.2) for $\delta\tilde{x}$ into the path integral (4.2.25). It only amounts to adding the term $\lambda\delta\tilde{x}^2$ into the action, since the zero mode is annihilated by $\tilde{\mathsf{O}}$. Then it is easy to see that the action contains the term

$$e^{-\frac{\lambda N^2}{2\phi} t_0^2} \,. \tag{C.2.4}$$

where N is given by (4.2.27). If we now integrate over t_0 and $\delta\tilde{x}^{\perp}$ we produce a term

$$\sqrt{\frac{2\pi\phi}{\lambda N^2}} \frac{\mu}{\sqrt{\det'(\tilde{\mathsf{O}} + \lambda)}} \,, \tag{C.2.5}$$

where the prime on the determinant means we have excluded the zero mode of the $\tilde{\mathsf{O}}$ operator. The λ in the denominator however combines with the primed determinant to give the complete determinant

$$\sqrt{\frac{2\pi\phi}{N^2}} \frac{\mu}{\sqrt{\det(\tilde{\mathsf{O}} + \lambda)}} \,. \tag{C.2.6}$$

Comparing with (C.2.1), we can read off the measure to be

$$\mu = \sqrt{\frac{N^2}{2\pi\phi}} \,. \tag{C.2.7}$$

C.3 The one-instanton determinant

In this appendix, we will compute the determinant of the one-instanton fluctuation operator using the Gel'fand-Yaglom theorem, explained for instance in [43, 58, 169, 140]. Consider an ordinary differential operator O, with a canonical second derivative term $\mathsf{O} = -\partial_t^2 + \dots$. We wish to compute the determinant of the operator. For that purpose we consider the space of functions on which the operator acts to be defined on an interval $t \in [-\beta/2, \beta/2]$

with the Dirichlet boundary conditions[1] for the eigenfunctions $\phi(t)$, i.e.

$$\phi(-\beta/2) = \phi(\beta/2) = 0 \ . \tag{C.3.1}$$

Then the Gel'fand-Yaglom theorem states that the determinant of the operator O is

$$\det \mathsf{O} \propto \Psi(\beta/2) \ , \tag{C.3.2}$$

where $\Psi(t)$ is a zero mode of O, i.e.

$$\mathsf{O} \circ \Psi(t) = 0 \tag{C.3.3}$$

satisfying a *different* boundary condition

$$\Psi(-\beta/2) = 0 \ , \quad \dot{\Psi}(-\beta/2) = 1 \ . \tag{C.3.4}$$

The proportionality identity can be made precise by regularizing the operator determinant with that of a simple operator, for instance, the harmonic oscillator

$$\frac{\det \mathsf{O}}{\det \mathsf{O}_0} = \frac{\Psi(\beta/2)}{\Psi_0(\beta/2)} \ , \tag{C.3.5}$$

where $\Psi_0(t)$ is the zero mode of the harmonic oscillator $\mathsf{O} = -\partial_t^2 + 1$ with the boundary condition Eqs. (C.3.4), and it is simply

$$\Psi_0(t) = \sinh(t + \beta/2) \ . \tag{C.3.6}$$

To treat $\det' \mathsf{O}$ with zero mode removed, we can use the relation

$$\det' \mathsf{O} = \lim_{\lambda\to 0} \frac{\mathrm{d}}{\mathrm{d}\lambda} \det \mathsf{O}_\lambda \ , \tag{C.3.7}$$

with

$$\mathsf{O}_\lambda := \mathsf{O} + \lambda \ . \tag{C.3.8}$$

Therefore we need to compute the zero mode of O_λ satisfying the boundary condition Eqs. (C.3.4) up to order λ.

Now we could take the operator O to simply be the fluctuation operator $\tilde{\mathsf{O}}$ given by (4.2.26). However notice that we have

$$\det(\tilde{\mathsf{O}} + \lambda) = \det[f(t)\tilde{\mathsf{O}}\frac{1}{f(t)} + \lambda] \ , \tag{C.3.9}$$

[1] More appropriate boundary conditions for computing path-integral determinants would be periodic boundary conditions, as Euclidean time is periodic. However in the limit of large Euclidean time-expanse – the limit relevant for the ground state properties of the system – the boundary conditions do not matter. Since the formulas are simpler when Dirichlet boundary conditions are used. But everything can be generalized to periodic boundary conditions if so desired. Indeed if one wished to study the excited spectrum of the theory, one would need to do precisely this.

where $f(t)$ is an arbitrary, nonsingular function with no zeros. By taking the derivative with respect to λ and setting $\lambda = 0$ we get

$$\det{}'(\tilde{\mathsf{O}}) = \det{}'[f(t)\tilde{\mathsf{O}}\frac{1}{f(t)}] \ . \tag{C.3.10}$$

If we take $f(t) = \sqrt{\cos y_1(t)}$ we can define the operator

$$\mathsf{O} = \sqrt{\cos y_1(t)}\tilde{\mathsf{O}}\frac{1}{\sqrt{\cos y_1(t)}} = \cos y_1 \left(-\partial_t \frac{1}{\cos y_1(t)}\partial_t + \cos x_1(t)\right) \tag{C.3.11}$$

so that we will compute $\det{}'(O)$ instead of $\det(O)$.

In order to compute it we first have to consider the determinant of $\det \mathsf{O}_\lambda$, where $\mathsf{O}_\lambda = \mathsf{O} + \lambda$, at least for small λ. We already know that O has a zero mode given by $\dot{x}_1$. To use Gel'fand-Yaglom theorem we look for a solution

$$\mathsf{O}_\lambda \Psi_\lambda = 0 \ , \tag{C.3.12}$$

where Ψ_λ satisfies Eqs. (C.3.4). Now assuming λ is small we can write

$$\Psi_\lambda = \Psi^{(0)} + \lambda\Psi^{(1)}(t) + \mathcal{O}(\lambda^2) \ , \tag{C.3.13}$$

where

$$\mathsf{O}\Psi^{(0)} = 0 \tag{C.3.14}$$

and

$$\mathsf{O}\Psi^{(1)} = -\Psi^{(0)} \ . \tag{C.3.15}$$

The first of these equations reduces to

$$\mathsf{O}\Psi^{(0)} = \cos y_1(t) \left(-\partial_t \frac{1}{\cos y_1(t)}\partial_t + \cos x_1(t)\right) \Psi^{(0)} = 0 \ . \tag{C.3.16}$$

This is a second order ODE, and we already know that one solution is

$$\psi_1(t) = \dot{x}_1(t) \ , \tag{C.3.17}$$

although it does not satisfy the boundary condition Eqs. (C.3.4). In order to find a second independent solution, we notice that the operator O can be factorized in the following way. We introduce operators

$$\mathsf{Q} = \frac{1}{\cos y_1}\partial_t - \mathsf{i}\frac{\sin x_1}{\sin y_1} \ , \quad \mathsf{Q}^\dagger = \frac{1}{\cos y_1}\partial_t + \mathsf{i}\frac{\sin x_1}{\sin y_1} \ . \tag{C.3.18}$$

They satisfy

$$\mathsf{Q}^\dagger\mathsf{Q} = -\frac{1}{\cos^2 y_1}\mathsf{O} \ , \quad \mathsf{Q}\mathsf{Q}^\dagger = -\frac{1}{\cos^2 y_1}\mathsf{O} + \frac{2}{\cos y_1}\left(\frac{\cos x_1}{\cos y_1} + \frac{\sin^2 x_1}{\sin^2 y_1}\right) \ . \tag{C.3.19}$$

We want to find the most general homogeneous solution to the equation $\mathsf{O}\psi = 0$. This is the same as finding such a solution for the operator $\mathsf{Q}^\dagger\mathsf{Q}$. We observe that $\mathsf{Q}^\dagger$ annihilates $1/\dot{x}_1$. If one can find ψ_2 such that $\mathsf{Q}\psi_2 = 1/\dot{x}_1$, then one concludes immediately from Eqs. (C.3.19) that ψ_2 is another solution to Eq. (C.3.16). Indeed by making an appropriate ansatz we find

$$\psi_2(t) = \dot{x}_1(t)\int^t \mathrm{d}t' \frac{\cos y_1(t')}{\dot{x}_1^2(t')} \ . \tag{C.3.20}$$

Furthermore since the Wronskian is not identically vanishing

$$W_{21}(t) := \psi_2(t)\partial_t\psi_1(t) - \psi_1(t)\partial_t\psi_2(t) = -\cos y_1(t) \ , \tag{C.3.21}$$

the two solutions are linearly independent. From $\psi_2(t)$ we can construct the solution to Eq. (C.3.16) satisfying the boundary condition Eqs. (C.3.4)

$$\Psi^{(0)}(t) = \frac{\dot{x}_1(-\beta/2)}{\cos y_1(-\beta/2)}\dot{x}_1(t)\int_{-\beta/2}^t \mathrm{d}t' \frac{\cos y_1(t')}{\dot{x}_1^2(t')} \ . \tag{C.3.22}$$

Let us proceed to the next order in λ, namely Eq. (C.3.15),

$$\left(\partial_t^2 - \frac{\dot{y}_1\sin y_1}{\cos y_1}\partial_t - \cos y_1\cos x_1\right)\Psi^{(1)} = -\mathsf{O}\circ\Psi^{(1)} = \Psi^{(0)} \ , \tag{C.3.23}$$

and $\Psi^{(1)}(t)$ satisfies the boundary condition

$$\Psi^{(1)}(-\beta/2) = 0 \ , \quad \dot{\Psi}^{(1)}(-\beta/2) = 0 \ . \tag{C.3.24}$$

One way to solve Eq. (C.3.23) is to first find the modified Green's function $G(t,t')$ satisfying

$$\mathsf{O}G(t,t') = \cos y_1\delta(t-t') \ , \tag{C.3.25}$$

so that $\Psi^{(1)}$ is given by

$$\Psi^{(1)}(t) = \int_{-\beta/2}^{\beta/2}\mathrm{d}t' G(t,t')\Psi^{(0)}(t')\frac{1}{\cos y_1} \ . \tag{C.3.26}$$

We claim that the Green's function is given by

$$G(t,t') = \begin{cases} -\psi_1(t)\psi_2(t') + \psi_2(t)\psi_1(t') \ , & t > t' \ , \\ 0 \ , & t \le t' \ . \end{cases} \tag{C.3.27}$$

Indeed, when both $t < t'$ and $t > t'$, Eq. (C.3.25) is trivially satisfied since $\psi_1(t), \psi_2(t)$ are annihilated by O. In the neighborhood of $t \to t'$, let us plug Eq. (C.3.27) into Eq. (C.3.25), integrate both sides from $t = t' - \epsilon$ to $t = t' + \epsilon$ and take the limit $\epsilon \to 0$. The r.h.s. is simply $\cos y_1(t')$, while the l.h.s. is given by

$$\partial_t G(t,t')\Big|_{t=t'^+} - \partial_t G(t,t')\Big|_{t=t'^-} = -W_{21}(t') = \cos y_1(t') \ , \tag{C.3.28}$$

where we have used Eq. (C.3.21). Therefore Eq. (C.3.27) is the correct modified Green's function. We can now write down $\Psi^{(1)}(t)$

$$\Psi^{(1)}(t) = \int_{-\beta/2}^{t} \mathrm{d}t' \Psi^{(0)}(t') \frac{1}{\cos y_1(t')} \left(\psi_1(t')\psi_2(t) - \psi_2(t')\psi_1(t)\right) \ . \tag{C.3.29}$$

This function indeed satisfies the boundary condition Eqs. (C.3.24).

Now we are ready to compute the operator determinant using the Gel'fand-Yaglom theorem. Combining Eqs. (C.3.5),(C.3.6),(C.3.7) and (C.3.29), we have

$$\frac{\det' \mathsf{O}}{\det \mathsf{O}_0} = \frac{\det' \tilde{\mathsf{O}}}{\det \mathsf{O}_0} = \frac{\dot{x}_1(-\beta/2)\dot{x}_1(\beta/2)}{\sinh\beta \cos y_1(-\beta/2)} \int_{-\beta/2}^{\beta/2} \mathrm{d}t \, \frac{\dot{x}_1^2(t)}{\cos y_1} \int_{-\beta/2}^{t} \mathrm{d}t' \frac{\cos y_1(t')}{\dot{x}_1^2(t')} \int_{t}^{\beta/2} \mathrm{d}t'' \frac{\cos y_1(t'')}{\dot{x}_1^2(t'')} \, . \tag{C.3.30}$$

Appendix D

Principal Parts of Z_3 and Z_4 for the Massless E-string

D.1 Base degree 3

For simplicity, we only consider the massless limit of the E-string at base degree three. The generalization to the massive case is straightforward but cumbersome. The decomposition (6.1.18) of $\mathcal{Z}_3$ is given by

$$\mathcal{Z}_3 = \mathcal{F}_3 + \mathcal{Z}_1\mathcal{Z}_2 - \frac{1}{3}\mathcal{Z}_1^3\,, \tag{D.1.1}$$

where

$$\mathcal{F}_3 = \mathfrak{f}_3(\tau, z) + \frac{1}{3}\left(\frac{\eta^{36}(\tau)}{\eta^{12}(3\tau)}\right)\mathcal{Z}_1(3\tau, 3z)\,. \tag{D.1.2}$$

We can read off the poles of $\mathcal{Z}_3$ from the expression (5.2.5) for the denominator specialized to $b_2(B) = 1$, $k = 3$:

$$z = \frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2}, \frac{m\tau+n}{3}\,, \tag{D.1.3}$$

where $m, n \in \{0, 1, 2\}$.

In order to determine the negative index Laurent data, assuming that we already know the expressions for $\mathcal{Z}_1$ and $\mathcal{Z}_2$, the only input we need is the genus zero free energy at base-wrapping degree 3.

Following the general strategy outlined in subsection 6.3.2, we determine the principal parts of $\mathcal{Z}_3$ as follows:

$z = 0$ At the origin, all three terms in $\mathcal{Z}_3$ contribute, yielding

$$\begin{aligned} D_{0,6}\,\mathcal{Z}_3 &= -\frac{(2\pi \mathsf{i})^{-6}}{6}E_4^3\,, \\ D_{0,4}\,\mathcal{Z}_3 &= \frac{(2\pi \mathsf{i})^{-4}}{12}(E_2E_4^3+E_4^2E_6)\,, \\ D_{0,2}\,\mathcal{Z}_3 &= -\frac{(2\pi \mathsf{i})^{-2}}{155520}(3240E_2^2E_4^3+2359E_4^4+6480E_2E_4^2E_6+2645E_4E_6^2)\,. \end{aligned} \tag{D.1.4}$$

$z = \frac{1}{3}, \frac{2}{3}$ The second order poles at $z = \frac{1}{3}$, $z = \frac{1}{2}$ and $z = \frac{2}{3}$ appear due to the multi-covering contribution of $\mathcal{Z}_1$ in $\mathcal{F}_3$. Therefore, the Laurent coefficients can be simply read off as

$$D_{\frac{1}{3},2}\,\mathcal{Z}_3 = D_{\frac{2}{3},2}\,\mathcal{Z}_3 = -\frac{(2\pi \mathsf{i})^{-2}}{27}\frac{\eta^{36}(\tau)}{\eta^{12}(3\tau)}E_4(3\tau)\,. \tag{D.1.5}$$

$z = \frac{1}{2}, \frac{\tau}{2}, \frac{\tau+1}{2}$ These poles are due to $\mathcal{F}_3$ and $\mathcal{Z}_1\mathcal{Z}_2$. Relating them via modularity to the Laurent coefficients at the real poles, we obtain

$$\begin{aligned} D_{\frac{1}{2},2}\,\mathcal{Z}_3 &= \frac{(2\pi \mathsf{i})^{-2}}{8}\left(\frac{\eta^{24}(\tau)}{\eta^{12}(2\tau)}\right)\frac{E_4(\tau)E_4(2\tau)}{\phi_{-2,1}(\tau,\frac{1}{2})}\,, \\ D_{\frac{\tau}{2},2}\,\mathcal{Z}_3 &= -\frac{(2\pi \mathsf{i})^{-2}}{2}q^{5/4}\left(\frac{\eta^{24}(\tau)}{\eta^{12}(\frac{\tau}{2})}\right)\frac{E_4(\tau)E_4(\frac{\tau}{2})}{\phi_{-2,1}(\tau,\frac{\tau}{2})}\,, \\ D_{\frac{\tau}{2},1}\,\mathcal{Z}_3 &= -\frac{1}{2\pi \mathsf{i}}q^{5/4}\left(\frac{\eta^{24}(\tau)}{\eta^{12}(\frac{\tau}{2})}\right)\frac{[3\phi_{-2,1}(\tau,\frac{\tau}{2})-\partial_z\phi_{-2,1}(\tau,\frac{\tau}{2})]E_4(\tau)E_4(\frac{\tau}{2})}{\phi_{-2,1}(\tau,\frac{\tau}{2})^2}\,, \\ D_{\frac{\tau+1}{2},2}\,\mathcal{Z}_3 &= -\mathsf{i}\frac{(2\pi \mathsf{i})^{-2}}{2}q^{5/4}\left(\frac{\eta^{24}(\tau)}{\eta^{12}(\frac{\tau+1}{2})}\right)\frac{E_4(\tau)E_4(\frac{\tau+1}{2})}{\phi_{-2,1}(\tau,\frac{\tau+1}{2})}\,, \\ D_{\frac{\tau+1}{2},1}\,\mathcal{Z}_3 &= -\frac{1}{2\pi}q^{5/4}\left(\frac{\eta^{24}(\tau)}{\eta^{12}(\frac{\tau+1}{2})}\right)\frac{[3\phi_{-2,1}(\tau,\frac{\tau+1}{2})-\partial_z\phi_{-2,1}(\tau,\frac{\tau+1}{2})]E_4(\tau)E_4(\frac{\tau+1}{2})}{\phi_{-2,1}(\tau,\frac{\tau+1}{2})^2}\,. \end{aligned} \tag{D.1.6}$$

$z = \frac{m\tau+n}{3}, (m \neq 0)$ The remaining six poles are due to $\mathfrak{f}_3$ only. We obtain

$$\begin{aligned} D_{\frac{\tau}{3},2}\,\mathcal{Z}_3 &= \frac{1}{q^2}D_{\frac{2\tau}{3},2}\,\mathcal{Z}_3 = -\frac{(2\pi \mathsf{i})^{-2}}{3}q^{2/3}\left(\frac{\eta^{36}(\tau)}{\eta^{12}(\frac{\tau}{3})}\right)E_4(\tfrac{\tau}{3})\,, \\ D_{\frac{\tau}{3},1}\,\mathcal{Z}_3 &= \frac{1}{2q^2}D_{\frac{2\tau}{3},1}\,\mathcal{Z}_3 = -4\frac{(2\pi \mathsf{i})^{-1}}{3}q^{2/3}\left(\frac{\eta^{36}(\tau)}{\eta^{12}(\frac{\tau}{3})}\right)E_4(\tfrac{\tau}{3})\,, \\ D_{\frac{\tau+1}{3},2}\,\mathcal{Z}_3 &= \frac{1}{q^2}D_{\frac{2\tau+2}{3},2}\,\mathcal{Z}_3 = -\frac{(2\pi \mathsf{i})^{-2}e^{\pi i/3}}{3}q^{2/3}\left(\frac{\eta^{36}(\tau)}{\eta^{12}(\frac{\tau+1}{3})}\right)E_4(\tfrac{\tau+1}{3})\,, \\ D_{\frac{\tau+1}{3},1}\,\mathcal{Z}_3 &= \frac{1}{2q^2}D_{\frac{2\tau+2}{3},1}\,\mathcal{Z}_3 = -4\frac{(2\pi \mathsf{i})^{-1}e^{\pi i/3}}{3}q^{2/3}\left(\frac{\eta^{36}(\tau)}{\eta^{12}(\frac{\tau+1}{3})}\right)E_4(\tfrac{\tau+1}{3})\,, \\ D_{\frac{\tau+2}{3},2}\,\mathcal{Z}_3 &= \frac{1}{q^2}D_{\frac{2\tau+1}{3},2}\,\mathcal{Z}_3 = -\frac{(2\pi \mathsf{i})^{-2}e^{2\pi i/3}}{3}q^{2/3}\left(\frac{\eta^{36}(\tau)}{\eta^{12}(\frac{\tau+2}{3})}\right)E_4(\tfrac{\tau+2}{3})\,, \\ D_{\frac{\tau+2}{3},1}\,\mathcal{Z}_3 &= \frac{1}{2q^2}D_{\frac{2\tau+1}{3},1}\,\mathcal{Z}_3 = -4\frac{(2\pi \mathsf{i})^{-1}e^{2\pi i/3}}{3}q^{2/3}\left(\frac{\eta^{36}(\tau)}{\eta^{12}(\frac{\tau+2}{3})}\right)E_4(\tfrac{\tau+2}{3})\,. \end{aligned} \tag{D.1.7}$$

We have checked these results by using them to compute the full Jacobi form $\mathcal{Z}_3$, following the discussion of subsection 6.2.2, and checking against the results in [79].

D.2 Base degree 4

Here we shall explain the complications that arise when we consider higher orders in the base degree. Let's take $\mathcal{Z}_4$ as an example. At base degree four, we have the decomposition,

$$\mathcal{Z}_4 = \mathcal{F}_4 + \mathcal{Z}_1\mathcal{Z}_3 + \frac{1}{2}\mathcal{Z}_2^2 - \mathcal{Z}_1^2\mathcal{Z}_2 + \frac{1}{4}\mathcal{Z}_1^4. \tag{D.2.1}$$

where

$$\mathcal{F}_4 = \mathfrak{f}_4(\tau,z) + \frac{1}{2}\left(\frac{\eta^{48}(\tau)}{\eta^{24}(2\tau)}\right)\mathcal{Z}_2(2\tau,2z) - \frac{1}{4}\left(\frac{\eta^{48}(\tau)}{\eta^{24}(2\tau)}\right)\mathcal{Z}_1(2\tau,2z)^2. \tag{D.2.2}$$

Among all the poles of $\mathcal{Z}_4$, the most intricate one is the point $z = \frac{\tau}{2}$, Again we use the strategy detailed in chapter 6. First assume we have two expansions:

$$\mathcal{Z}_4(\tau,z) = \sum_{n=-4}^{\infty} c_{4,n}(\tau)(z-\frac{\tau}{2})^n = \sum_{n=-4}^{\infty} b_{4,n}(\tau)(z-\frac{1}{2})^n. \tag{D.2.3}$$

Making use of the S-transform, we have the following relation,

$$\sum_{n=-4}^{\infty} c_{4,n}(\tau)(z-\frac{\tau}{2})^n = \tau^{24}\boldsymbol{e}[\frac{10\,z^2}{\tau}]\sum_i\sum_{n=-4}^{\infty}\frac{b_{4,n,i}(-\frac{1}{\tau})}{\tau^n}(z-\frac{\tau}{2})^n \tag{D.2.4}$$

In this case, there are five terms in $m_{\boldsymbol{k},i}$ which contribute: $\frac{1}{2}\mathcal{Z}_2(2\tau,2z)$, $-\frac{1}{4}\mathcal{Z}_1(2\tau,2z)^2$, $\frac{1}{2}\mathcal{Z}_2^2$, $\mathcal{Z}_1\mathcal{Z}_3$ and $-\mathcal{Z}_1^2\mathcal{Z}_2$, with the first three having fourth order poles at $z = \frac{1}{2}$ while the last two of only second order. This is the simplest example to demonstrate a very important fact: even though naively in $c_{4,n}(\tau)$ terms proportional to τ will appear, they will actually cancel with each other so that the final result has τ dependence only in q, as they should.

Furthermore, since $\frac{1}{\tau}$ is always accompanied by z^2, the claimed cancellation will only occur at the second and first order pole and only the first three terms are relevant. The result of the Laurent expansion of each term, after applying the general method in chapter 6, is as follows ($\cdots$ represents irrelevant terms)

$$\mathcal{Z}_2(\tau,z)^2 = \boldsymbol{e}[\frac{-6z^2}{\tau}]\left(\frac{(2\pi\mathsf{i})^{-2}}{2}(\frac{\eta^{24}(\tau)}{\eta^{12}(\frac{\tau}{2})})q^{3/4}E_4(\tfrac{\tau}{2})\right)^2(\frac{1}{(z-\frac{\tau}{2})^2}+3\frac{2\pi i}{z-\frac{\tau}{2}}+\cdots)^2, \tag{D.2.5}$$

$$\left(\frac{\eta^{48}(\tau)}{\eta^{24}(2\tau)}\right)\mathcal{Z}_2(2\tau,2z) = \boldsymbol{e}[\frac{-6(z-\tau/2)^2}{\tau}]\left(\frac{(2\pi\mathsf{i})^{-4}}{2}\frac{\eta^{48}(\tau)}{\eta^{24}(\frac{\tau}{2})}E_4(\frac{\tau}{2})^2\right)(\frac{1}{(z-\frac{\tau}{2})^4}+\cdots), \tag{D.2.6}$$

$$\left(\frac{\eta^{48}(\tau)}{\eta^{24}(2\tau)}\right)\mathcal{Z}_1(2\tau,2z)^2 = \boldsymbol{e}[\frac{-4(z-\tau/2)^2}{\tau}]\left((2\pi\mathsf{i})^{-2}(\frac{\eta^{24}(\tau)}{\eta^{12}(\frac{\tau}{2})})E_4(\tfrac{\tau}{2})\right)^2(\frac{1}{(z-\frac{\tau}{2})^2}+\cdots)^2. \tag{D.2.7}$$

Plug them back to Eqs. (D.2.1), (D.2.2) and (D.2.4), we can write down the expansion of $\mathcal{Z}_4$ at $z=\frac{\tau}{2}$,

$$\mathcal{Z}_4 \supset \frac{1}{2}\mathcal{Z}_2(\tau,z)^2 + \frac{1}{2}\left(\frac{\eta^{48}(\tau)}{\eta^{24}(2\tau)}\right)\mathcal{Z}_2(2\tau,2z) - \frac{1}{4}\left(\frac{\eta^{48}(\tau)}{\eta^{24}(2\tau)}\right)\mathcal{Z}_1(2\tau,2z)^2, \tag{D.2.8}$$

where ($\cdots$ represents irrelevant terms)

$$\mathcal{Z}_2(\tau,z)^2 \supset \left(\frac{(2\pi\mathsf{i})^{-2}q^{1/2}}{4}\frac{\eta^{48}(\tau)}{\eta^{24}(\frac{\tau}{2})}E_4^2(\tfrac{\tau}{2})\right)(\frac{4/\tau}{(z-\frac{\tau}{2})^2}+\frac{40(2\pi\mathsf{i})/\tau}{z-\frac{\tau}{2}}+\cdots), \tag{D.2.9}$$

$$\left(\frac{\eta^{48}(\tau)}{\eta^{24}(2\tau)}\right)\mathcal{Z}_2(2\tau,2z) \supset \left(\frac{(2\pi\mathsf{i})^{-2}q^{1/2}}{2}\frac{\eta^{48}(\tau)}{\eta^{24}(\frac{\tau}{2})}E_4^2(\frac{\tau}{2})\right)(\frac{4/\tau}{(z-\frac{\tau}{2})^2}+\frac{40(2\pi\mathsf{i})/\tau}{z-\frac{\tau}{2}}+\cdots), \tag{D.2.10}$$

$$\left(\frac{\eta^{48}(\tau)}{\eta^{24}(2\tau)}\right)\mathcal{Z}_1(2\tau,2z)^2 \supset \left((2\pi\mathsf{i})^{-2}q^{1/2}\frac{\eta^{48}(\tau)}{\eta^{24}(\frac{\tau}{2})}E_4^2(\tfrac{\tau}{2})\right)(\frac{6/\tau}{(z-\frac{\tau}{2})^2}+\frac{60(2\pi\mathsf{i})/\tau}{z-\frac{\tau}{2}}+\cdots). \tag{D.2.11}$$

Indeed, the sum of the above three terms in $\mathcal{Z}_4$ vanishes as claimed.

Finally, similar to the base degree three case, We have determined all the principal parts and use them to reconstruct the full Jacobi form $\mathcal{Z}_4$, following the discussion of subsection 6.2.2, and checking against the results in the literature.

Appendix E

Some Enumerative Invariants

E.1 Genus zero GW invariants for massless E strings

In this subsection, we list some genus zero GW invariants for massless E strings. Note that they are in general rational numbers rather than just integers. Meanwhile notice that the column of fiber degree 0 hints at the denominator structure (3.1.13).

b/e	0	1	2	3	4
1	1	252	5130	54760	419895
2	$\frac{1}{8}$	0	$-\frac{18441}{2}$	-673760	$-\frac{82133595}{4}$
3	$\frac{1}{27}$	0	0	$\frac{2545912}{3}$	115243155
4	$\frac{1}{64}$	0	0	0	$-\frac{1828258569}{16}$

(E.1.1)

E.2 GV invariants for $\mathfrak{a}_2$ and $\mathfrak{g}_2$ models

In this subsection, we list some GV invariants of $\mathfrak{a}_2$ and $\mathfrak{g}_2$ models. Similarly, Refined GV invariants can also be obtained using the methods outlined below.

In the literature, there are many different strategies to obtain the partition functions and extract those invariants. Here we would like to mention two possible ways. One way is to look at the field theory side. We can work out the precise gauge theory dynamics living on the six dimensional space-time, then use localization technique to determine the partition functions, which is the strategy adopted in e.g., [126, 112, 125, 124]. The other way is to explore modularity. We write down the modular ansatz and find sufficiently many boundary conditions to determine the unknowns. For example, in [48], it was found that the $\mathfrak{a}_2$ models for base degree up to 3 can be completely solved by imposing precise vanishing conditions. Since the precise vanishing conditions for the $\mathfrak{g}_2$ model haven't been worked out yet, we obtain the corresponding GV invariants based on field theoretic computations [124]. To extract them, in particular we set the flavor mass to zero.

E.2.1 $\mathfrak{a}_2$ model

Base degree 1

Fiber degree 0, genus 0

m_2/m_1	0	1	2	3	4	5
0	1	3	5	7	9	11
1	3	4	8	12	16	20
2	5	8	9	15	21	27
3	7	12	15	16	24	32
4	9	16	21	24	25	35
5	11	20	27	32	35	36

(E.2.1)

Fiber degree 1, genus 1

m_2/m_1	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	2	6	10	14	18
2	0	6	8	16	24	32
3	0	10	16	18	30	42
4	0	14	24	30	32	48
5	0	18	32	42	48	50

(E.2.2)

Fiber degree 2, genus 2

m_2/m_1	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	3	9	15	21
3	0	0	9	12	24	36
4	0	0	15	24	27	45
5	0	0	21	36	45	48

(E.2.3)

Base degree 2

Fiber degree 0, genus 0

m_2/m_1	0	1	2	3	4	5
0	0	0	-6	-32	-110	-288
1	0	0	-10	-70	-270	-770
2	-6	-10	-32	-126	-456	-1330
3	-32	-70	-126	-300	-784	-2052
4	-110	-270	-456	-784	-1584	-3360
5	-288	-770	-1330	-2052	-3360	-6076

(E.2.4)

Fiber degree 1, genus 1

m_2/m_1	0	1	2	3	4	5
0	0	0	0	-16	-144	-704
1	0	0	-8	-132	-936	-4308
2	0	-8	-36	-272	-1860	-8964
3	-16	-132	-272	-776	-3192	-131920
4	-144	-936	-1860	-3192	-7436	-22384
5	-704	-4308	-8964	-131920	-22384	-45936

(E.2.5)

Fiber degree 2, genus 2

m_2/m_1	0	1	2	3	4	5
0	0	0	0	0	-30	-348
1	0	0	0	-24	-430	-3600
2	0	0	-24	-224	-2250	-15660
3	0	-24	-224	-880	-4816	-30092
4	-30	-430	-2250	-4816	-13050	-51354
5	-348	-3600	-15660	-30092	-51354	-117168

(E.2.6)

E.2.2 $\mathfrak{g}_2$ model

Base degree 1

Fiber degree 0, genus 0

m_2/m_1	0	1	2	3	4	5
0	1	3	5	7	9	11
1	0	3	4	8	12	16
2	0	0	5	8	9	15
3	0	0	0	7	12	15
4	0	0	0	0	9	16
5	0	0	0	0	0	11

(E.2.7)

Fiber degree 1, genus 1

m_2/m_1	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	-2	-6	-10	-14
2	0	0	0	-6	-8	-16
3	0	0	0	0	-10	-16
4	0	0	0	0	0	-14
5	0	0	0	0	0	0

(E.2.8)

Fiber degree 2, genus 2

m_2/m_1	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	0	3	9
3	0	0	0	0	0	9
4	0	0	0	0	0	0
5	0	0	0	0	0	0

(E.2.9)

Base degree 2

Fiber degree 0, genus 0

m_2/m_1	0	1	2	3	4	5
0	0	0	-6	-32	-110	-288
1	0	0	0	-10	-70	-270
2	0	0	-6	-10	-32	-126
3	0	0	0	-32	-70	-126
4	0	0	0	0	-110	-270
5	0	0	0	0	0	-288

(E.2.10)

Fiber degree 1, genus 1

m_2/m_1	0	1	2	3	4	5
0	0	0	0	16	144	704
1	0	0	0	8	132	936
2	0	0	0	8	36	272
3	0	0	0	16	132	272
4	0	0	0	0	144	936
5	0	0	0	0	0	704

(E.2.11)

Fiber degree 2, genus 2

m_2/m_1	0	1	2	3	4	5
0	0	0	0	0	-30	-348
1	0	0	0	0	-24	-430
2	0	0	0	0	-24	-224
3	0	0	0	0	-24	-224
4	0	0	0	0	-30	-430
5	0	0	0	0	0	-348

(E.2.12)

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RÉSUMÉ

Cette thèse porte sur diverses applications de la théorie des cordes topologiques basée sur différents types de variétés de Calabi-Yau (CY). Le premier type considéré est la variété torique CY, qui est intimement liée aux problèmes spectraux des différents opérateurs. L'exemple particulier considéré dans la thèse ressemble beaucoup au modèle de Harper-Hofstadter en physique de la matière condensée. Nous étudions d'abord les secteurs non perturbatifs dans ce modèle et proposons une nouvelle façon de les calculer en utilisant la théorie topologique des cordes. Dans la deuxième partie de la thèse, nous considérons les fonctions de partition sur des variétés de CY elliptiquement fibrées. Celles-ci présentent un comportement modulaire intéressant. Nous montrons que pour les géométries qui ne conduisent pas à des symétries de jauge non abéliennes, les fonctions de partition des cordes topologiques peuvent être reconstruites avec seulement les invariants de Gromov-Witten du genre zéro. Finalement, nous discutons des travaux en cours concernant la relation entre les fonctions de partitionnement des cordes topologiques sur les soi-disant arbres de Higgsing dans la théorie de F.

MOTS CLÉS

Théorie de cordes topologiques, invariant de Gromov-Witten, symétrie miroir, résurgence, genre elliptique.

ABSTRACT

This thesis focuses on various applications of topological string theory based on different types of Calabi-Yau (CY) manifolds. The first type considered is the toric CY manifold, which is intimately related to spectral problems of difference operators. The particular example considered in the thesis closely resembles the Harper-Hofstadter model in condensed matter physics. We first study the non-perturbative sectors in this model, and then propose a new way to compute them using topological string theory. In the second part of the thesis, we consider partition functions on elliptically fibered CY manifolds. These exhibit interesting modular behavior. We show that for geometries which do not lead to non-abelian gauge symmetries, the topological string partition functions can be reconstructed based solely on genus zero Gromov-Witten invariants. Finally, we discuss ongoing work regarding the relation of the topological string partition functions on the so-called Higgsing trees in F-theory.

KEYWORDS

Topological string theory, Gromov-Witten invariant, mirror symmetry, resurgence, elliptic genus.