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Topological Superstring in $D=2$ and Topological Supergravity

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We examined the quantization of a topological sigma model in $2D$ superspace with a $2D$ flat target manifold. It is shown that the topological supergravity is obtained as the quantum theory by taking the superdiffeomorphism invariance into account. We also discuss the global (twisted) supersymmetries realized in the quantum theory.

Various topological field theories have been studied intensively since Witten proposed the topological Yang-Mills theories.[1] Among many topological models $2D$ topological gravity theories[2),3)] and topological string theories[4)] have been attracting much attention since it was conjectured that those models are equivalent to some large N matrix models.[3),4)] Now it seems to be a very interesting problem to characterize the non-critical string theories from the point of view of topological field theories.[5)]

Recently the Nambu-Goto type string in $D=2$ space-time has been studied as a $2D$ topological field theory, and the $2D$ topological gravity turns out to be obtained through a certain quantization procedure.[6)] When the dimension of space-time is 2, there are no physical degree of freedom of the string. Therefore it is expected that the Nambu-Goto action in $D=2$

$$S=-\int d^2\sigma\sqrt{-\det(\partial_\mu X^a\partial_\nu X_a)}\,, \tag{1}$$

where $a=0,\ 1$ is the space-time index and $\mu=0,\ 1$ is the world-sheet index, may be regarded as a topological field theory. Actually we may be allowed to rewrite (1) as

$$S=-\frac{1}{2}\int d^2\sigma\epsilon^{\mu\nu}\epsilon_{ab}\partial_\mu X^a\partial_\nu X^b \tag{2}$$

using the speciality of $D=2$. This gives a topological model, since the lagrangian is a total derivative. If we suppose the $2D$ space-time manifold to be Euclidean and S^2, then the corresponding action will be given by

$$S=-\frac{1}{2}\int d^2\sigma\epsilon^{\mu\nu}J_{ab}(X)\partial_\mu X^a\partial_\nu X^b\,, \tag{3}$$

where $J_{ab}(X)$ is the complex structure on the S^2. Actually this is a simple case of the topological sigma model.[7)] This action is nothing but the wrapping number of the string world-sheet to S^2, which will play a role similar to the Pontryagin index in the topological Yang-Mills theories.[1)] However this argument may sound to be wrong to some readers, because the Nambu-Goto action (1) is usually supposed to give the area of the world-sheet not a topological number. It, therefore, should be noted that we have assumed the positive definiteness of det $\partial_\mu X^a$ in rewriting (1) into (2). This assumption seems to be natural, if we regard det $\partial_\mu X^a$ as the zweibein $e_\mu{}^a$ on the world-sheet. Otherwise the models based on the actions (1) and (2) give us different physical states, which means (1) and (2) are not equivalent systems. In another point of view those models can be canonically transformed into each other, however this transformation is found to be singular for some configurations of the string.[6] Anyway we would like to start with the action (2) and consider its supersymmetric extension,*)

$$S=\frac{1}{4}\int d^2\sigma d^2\theta \epsilon^{AB}\epsilon_{ab}D_A(\Phi^a D_B\Phi^b)\,, \tag{4}$$

where $D=D_1=\partial/\partial\theta+i\theta(\partial_0-\partial_1)$ and $\bar{D}=D_2=\partial/\partial\bar{\theta}+i\bar{\theta}(\partial_0+\partial_1)$ are the supercovariant derivatives, Φ^a, which are superfields decomposed as

$$\Phi^a(\sigma,\theta)=X^a+\theta\psi^a+\bar{\theta}\bar{\psi}^a \tag{5}$$

and ϵ^{AB} is an off-diagonal tensor defined by $\epsilon^{12}=\epsilon^{21}=1$. Here ψ^a and $\bar{\psi}^a$ denote components of a real (Majorana) spinor. If we write down the lagrangian in terms of the component fields, then

$$\begin{aligned} L=&\partial_0 X^-\partial_1 X^+-\partial_1 X^-\partial_0 X^+ \\ &+\frac{i}{2}\psi^-(\partial_0+\partial_1)\psi^+-\frac{i}{2}\psi^+(\partial_0+\partial_1)\psi^- \\ &-\frac{i}{2}\bar{\psi}^-(\partial_0-\partial_1)\bar{\psi}^+ +\frac{i}{2}\bar{\psi}^+(\partial_0-\partial_1)\bar{\psi}^-\,. \end{aligned} \tag{6}$$

As is easily seen, this lagrangian is also a total derivative.

Now we shall consider the quantization of this system. In Ref. 6) the bosonic action was quantized by means of the Batalin-Fradkin-Vilkovisky (BFV) Hamiltonian formulation.[9] Here we also apply the BFV quantization to our model, since the usual BRST quantization may be rather awkward for our purpose.

The primary constraints are immediately read off from (6) as

$$\begin{aligned} &\hat{\phi}_\pm=P_\pm\pm\partial_1 X^\mp\approx 0\,, \\ &\hat{\varphi}_\pm=\pi_\pm\mp\frac{i}{2}\psi^\mp\approx 0\,, \\ &\hat{\bar{\varphi}}_\pm=\bar{\pi}_\pm\pm\frac{i}{2}\bar{\psi}^\mp\approx 0\,, \end{aligned} \tag{7}$$

where $P_\pm$, $\pi_\pm$ and $\bar{\pi}_\pm$ are the canonical conjugate variables to $X^\pm$, $\psi^\pm$ and $\bar{\psi}^\pm$ respectively. All these constraints are commuting, i.e., 1st class. However let us reduce the degree of freedom of the fermionic phase space by the proper gauge fixing so that we obtain the usual Poisson brackets between the fermionic variables,

*) Here we are interested in the world-sheet supersymmetry. The Green-Schwarz superstring in $D=2$, which are found to be topological, has been also investigated.[8]

$$\{\psi^+(\sigma), \psi^-(\sigma')\} = -i\delta(\sigma-\sigma'),$$
$$\{\bar\psi^+(\sigma), \bar\psi^-(\sigma')\} = -i\delta(\sigma-\sigma'). \tag{8}$$

The gauge fixing conditions may be given by

$$\chi_- = \pi_- - \frac{i}{2}\psi^+ \approx 0,$$
$$\bar\chi_+ = \bar\pi_+ - \frac{i}{2}\bar\psi^- \approx 0. \tag{9}$$

Now the remaining 1st class constraints (we shall call them topological constraints) are found to be

$$\hat\phi_\pm = P_\pm \pm \partial_1 X^\mp \approx 0,$$
$$\hat{\bar\varphi}_+ = i\bar\psi^- \approx 0,$$
$$\hat\varphi_- = i\psi^+ \approx 0. \tag{10}$$

Hereafter we shall abbreviate bar of $\hat{\bar\varphi}_+$ and write it as $\hat\varphi_+$ for the simplicity.

As is well known the superstring action (Neveu-Schwarz-Ramond formulation) is invariant under the superdiffeomorphism. In the $D=2$ case the corresponding constraints may be expressed by

$$\phi_+ = \frac{1}{2}(P_+ + \partial_1 X^-)(P_- + \partial_1 X^+) + \frac{i}{2}\bar\psi^+\partial_1\bar\psi^- + \frac{i}{2}\bar\psi^-\partial_1\bar\psi^+,$$
$$\phi_- = \frac{1}{2}(P_+ - \partial_1 X^-)(P_- - \partial_1 X^+) - \frac{i}{2}\psi^+\partial_1\psi^- - \frac{i}{2}\psi^-\partial_1\psi^+,$$
$$\varphi_+ = -\frac{i}{2}[\bar\psi^+(P_+ + \partial_1 X^-) + \bar\psi^-(P_- + \partial_1 X^+)],$$
$$\varphi_- = -\frac{i}{2}[\psi^+(P_+ - \partial_1 X^-) + \psi^-(P_- - \partial_1 X^+)], \tag{11}$$

where $\varphi_\pm$ are fermionic constraints for the supersymmetry transformation. Now we suppose that our model is the constrained system characterized by both the topological constraints and the superdiffeomorphism constraints. Then these constraints are reducible reflecting that the gauge transformations generated by them are dependent. The reducibility relation between the constraints are read off as

$$\phi_+ = \hat\phi_+\tilde\phi_- - \frac{1}{2}\partial_1\hat{\bar\varphi}_+\bar\psi^+ + \frac{1}{2}\hat{\bar\varphi}_+\partial_1\bar\psi^+,$$
$$\phi_- = \hat\phi_-\tilde\phi_+ + \frac{1}{2}\partial_1\hat\varphi_+\psi^+ - \frac{1}{2}\hat\varphi_+\partial_1\psi^+,$$
$$\varphi_+ = -\frac{i}{2}\hat\phi_+\bar\psi^+ - \hat{\bar\varphi}_+\tilde\phi_-,$$
$$\varphi_- = -\frac{i}{2}\hat\phi_-\psi^- - \hat\varphi_-\tilde\phi_+, \tag{12}$$

where we introduced

$$\tilde{\phi}_{\mp}=\frac{1}{2}(P_{\mp}\pm\partial_1 X^{\pm}) \tag{13}$$

for convenience. Such reducibility of the constraints seems to be fairly crucial in the topological field theories such as topological (super)gravity.

We are now in a position to perform the BFV quantization. First we enlarge the phase space introducing ghost multiplet for each constraint as follows,

$$\begin{aligned}
&(\hat{c}^{\pm}, \hat{\bar{p}}_{\pm}), (\hat{p}^{\pm}, \hat{\bar{c}}_{\pm}), (\hat{N}^{\pm}, \hat{B}_{\pm}) \quad \text{for} \quad \hat{\phi}_{\pm}\,,\\
&(\hat{\gamma}^{\pm}, \hat{\bar{\pi}}_{\pm}), (\hat{\pi}^{\pm}, \hat{\bar{\gamma}}_{\pm}), (\hat{\chi}^{\pm}, \hat{\omega}_{\pm}) \quad \text{for} \quad \hat{\varphi}_{\pm}\,,\\
&(c^{\pm}, \bar{p}_{\pm}), (p^{\pm}, \bar{c}_{\pm}), (N^{\pm}, B_{\pm}) \quad \text{for} \quad \phi_{\pm}\,,\\
&(\gamma^{\pm}, \bar{\pi}_{\pm}), (\pi^{\pm}, \bar{\gamma}_{\pm}), (\chi^{\pm}, \omega_{\pm}) \quad \text{for} \quad \varphi_{\pm}\,.
\end{aligned} \tag{14}$$

However the constraints are reducible as explained before. Therefore we have to impose additional constraints between the antighosts according to the general procedure to handle the reducible constraints.[9] They may be given by

$$\begin{aligned}
\phi'_{+}&=\bar{p}_{+}-\hat{\bar{p}}_{+}\tilde{\phi}_{-}+\frac{1}{2}\partial_1\hat{\bar{\pi}}_{+}\bar{\psi}^{+}-\frac{1}{2}\hat{\bar{\pi}}_{+}\partial_1\bar{\psi}^{+}\,,\\
\phi'_{-}&=\bar{p}_{-}-\hat{\bar{p}}_{-}\tilde{\phi}_{+}-\frac{1}{2}\partial_1\hat{\bar{\pi}}_{-}\psi^{-}+\frac{1}{2}\hat{\bar{\pi}}_{-}\partial_1\psi^{-}\,,\\
\varphi'_{+}&=\bar{\pi}_{+}+\frac{i}{2}\hat{\bar{p}}_{+}\bar{\psi}^{+}+\hat{\bar{\pi}}_{+}\tilde{\phi}_{-}\,,\\
\varphi'_{-}&=\bar{\pi}_{-}+\frac{i}{2}\hat{\bar{p}}_{-}\psi^{-}+\hat{\bar{\pi}}_{-}\tilde{\phi}_{+}\,,
\end{aligned} \tag{15}$$

where it is noted that the ghost number of these constraints are -1. Corresponding to the new constraints we shall introduce multiplets of so-called ghosts for ghosts as

$$\begin{aligned}
&(c'^{\pm}, \bar{p}'_{\pm}), (p'^{\pm}, \bar{c}'_{\pm}), (N'^{\pm}, B'_{\pm}) \quad \text{for} \quad \phi'_{\pm}\,,\\
&(\gamma'^{\pm}, \bar{\pi}'_{\pm}), (\pi'^{\pm}, \bar{\gamma}'_{\pm}), (\chi'^{\pm}, \omega'_{\pm}) \quad \text{for} \quad \varphi'_{\pm}\,.
\end{aligned} \tag{16}$$

Here it should be noted that the ghost number of $c'^{\pm}$ and $\gamma'^{\pm}$ are 2, $\bar{p}'_{\pm}$ and $\bar{\pi}'_{\pm}$ are -2, and also $(c'^{\pm}, \bar{p}'_{\pm})$ are bosonic and $(\gamma'^{\pm}, \bar{\pi}'_{\pm})$ are fermionic ghost variables.

Using these constraints and the ghost variables one can construct the nilpotent BRST charge as

$$\begin{aligned}
Q_B&=Q_B{}^{(+)}+Q_B{}^{(-)}\,,\\
Q_B{}^{(+)}&=\int d\sigma\Big\{c^{+}\phi_{+}+\hat{c}^{+}\hat{\phi}_{+}+c'^{+}\Big(\bar{p}_{+}-\hat{\bar{p}}_{+}\tilde{\phi}_{-}+\frac{1}{2}\partial_1\hat{\bar{\pi}}_{+}\bar{\psi}^{+}-\frac{1}{2}\hat{\bar{\pi}}_{+}\partial_1\bar{\psi}^{+}\Big)\\
&\quad+\gamma^{+}\varphi_{+}+\hat{\gamma}^{+}\hat{\varphi}_{+}+\gamma'^{+}\Big(\bar{\pi}_{+}+\frac{i}{2}\hat{\bar{p}}_{+}\bar{\psi}^{+}+\hat{\bar{\pi}}_{+}\tilde{\phi}_{-}\Big)+c^{+}p_{+}\partial_1 c^{+}-\frac{i}{2}\gamma^{+2}p_{+}
\end{aligned}$$

$$+c^+\Big(\frac{3}{2}\bar{\pi}_+\partial_1\gamma^+ +\frac{1}{2}\partial_1\bar{\pi}_+\gamma^+ +\hat{\bar{p}}_+\partial_1\hat{c}^+ +\frac{1}{2}\hat{\bar{\pi}}_+\partial_1\hat{\gamma}^+ -\frac{1}{2}\partial_1\hat{\bar{\pi}}_+\hat{\gamma}^+$$

$$+2\bar{p}'_+\partial_1 c'^+ +\partial_1\bar{p}'_+ c'^+ +\frac{3}{2}\bar{\pi}'_+\partial_1\gamma'^+ +\frac{1}{2}\partial_1\bar{\pi}'_+\gamma'^+\Big)$$

$$+\gamma^+\Big(\partial_1\hat{c}^+\hat{\bar{\pi}}_+ +\frac{i}{2}\hat{\gamma}^+\hat{\bar{p}}_+ +\frac{3}{2}\partial_1 c'^+\bar{\pi}'_+ +c'^+\partial_1\bar{\pi}'_+ +i\gamma'^+\bar{p}'_+\Big)\Big\}\,. \tag{17}$$

$Q_B{}^{(-)}$ is given by an expression similar to $Q_B{}^{(+)}$. This BRST charge is determined by the structure of the constraint algebra and totally independent of the choice of gauge.

We may also construct the symmetry generators corresponding to the constraints, which we started with, by taking the Poisson brackets of the BRST charge and the antighosts,

$$\begin{aligned}
&\hat{L}_\pm(\sigma)\equiv\{Q_B,\ \hat{\bar{p}}_\pm(\sigma)\}\,, \qquad \hat{G}_\pm(\sigma)\equiv\{Q_B,\ \hat{\bar{\pi}}_\pm(\sigma)\}\,,\\
&L_\pm(\sigma)\equiv\{Q_B,\ \bar{p}_\pm(\sigma)\}\,, \qquad G_\pm(\sigma)\equiv\{Q_B,\ \bar{\pi}_\pm(\sigma)\}\,,\\
&L'_\pm(\sigma)\equiv\{Q_B,\ \bar{p}'_\pm(\sigma)\}\,, \qquad G'_\pm(\sigma)\equiv\{Q_B,\ \bar{\pi}'_\pm(\sigma)\}\,.
\end{aligned} \tag{18}$$

They are found to be as follows:

$$\hat{L}_+=\hat{\phi}_+-\partial_1(c^+\hat{\bar{p}}_+)-\partial_1(\gamma^+\hat{\bar{\pi}}_+)\,,$$

$$\hat{G}_+=\hat{\varphi}_+-\frac{1}{2}\partial_1 c^+\hat{\bar{\pi}}_+-c^+\partial_1\hat{\bar{\pi}}_+ +\frac{i}{2}\hat{\bar{p}}_+\gamma^+\,,$$

$$L_+=\phi_+ +\hat{p}_+\partial_1\hat{c}^+ +\frac{1}{2}\hat{\pi}_+\partial_1\hat{\gamma}^+ -\frac{1}{2}\partial_1\hat{\pi}_+\hat{\gamma}^+$$

$$+2\bar{p}_+\partial_1 c^+ +\partial_1\bar{p}_+ c^+ +\frac{3}{2}\bar{\pi}_+\partial_1\gamma^+ +\frac{1}{2}\partial_1\bar{\pi}_+\gamma^+$$

$$+2\bar{p}'_+\partial_1 c'^+ +\partial_1\bar{p}'_+ c^+ +\frac{3}{2}\bar{\pi}'_+\partial_1\gamma'^+ +\frac{1}{2}\partial_1\bar{\pi}'_+\gamma'^+$$

$$\equiv L_+{}^M+L_+{}^{\mathrm{Gh}}\,,$$

$$G_+=\varphi_+ +\partial_1\hat{c}^+\hat{\pi}_+ +\frac{i}{2}\hat{\gamma}^+\hat{p}_+$$

$$-\frac{3}{2}\partial_1 c^+\bar{\pi}_+ -c^+\partial_1\bar{\pi}_+ -i\bar{p}_+\gamma^+ +\frac{3}{2}\partial_1 c'^+\bar{\pi}'_+ +c'^+\partial_1\bar{\pi}'_+ +i\bar{p}'_+\gamma'^+$$

$$\equiv G_+{}^M+G_+{}^{\mathrm{Gh}}\,,$$

$$L'_+=\bar{p}_+-\hat{\bar{p}}_+\tilde{\phi}_- +\frac{1}{2}\partial_1\hat{\bar{\pi}}_+\bar{\psi}^+ -\frac{1}{2}\hat{\bar{\pi}}_+\partial_1\bar{\psi}^+$$

$$-2p'_+\partial_1 c^+ -\partial_1 p'_+ c^+ -\frac{3}{2}\pi'_+\partial_1\gamma^+ -\frac{1}{2}\partial_1\pi'_+\gamma^+$$

$$\equiv\bar{p}_+ +L'^M_+ +L'^{\mathrm{Gh}}_+\,,$$

$$G'_+ = \bar{\pi}_+ + \bar{\hat{\pi}}_+ \hat{\phi}_- + \frac{i}{2}\hat{\bar{p}}_+ \bar{\psi}^+ + \frac{3}{2}\bar{\pi}'_+ \partial_1 c^+ + \partial_1 \bar{\pi}'_+ c^+ + i\bar{p}'_+ \gamma^+$$

$$\equiv \bar{\pi}_+ + G'^M_+ + G'^{\mathrm{Gh}}_+ , \tag{19}$$

where the superscript M denotes the part composed of the string variables and the hat ghosts, and the superscript Gh denotes other ghost variables. We may write down the BRST charge neatly in terms of these generators as

$$Q_B^{(+)} = \int d\sigma \Big\{ \hat{c}^+ \hat{\phi}_+ + \hat{\gamma}^+ \hat{\varphi}_+ + c'^+ \bar{p}_+ + \gamma'^+ \bar{\pi}_+$$

$$+ c^+\Big(L_+{}^M + \frac{1}{2}L_+{}^{\mathrm{Gh}}\Big) + c'^+\Big(L_+{}'^M + \frac{1}{2}L'^{\mathrm{Gh}}_+\Big)$$

$$+ \gamma^+\Big(G_+{}^M + \frac{1}{2}G_+{}^{\mathrm{Gh}}\Big) + \gamma'^+\Big(G_+{}'^M + \frac{1}{2}G'^{\mathrm{Gh}}_+\Big)\Big\} . \tag{20}$$

The $(-)$ sector of the generators and of the BRST charge are expressed by forms similar to (19) and (20).

One of the interesting gauge choices is the superconformal gauge which is familiar in the superstring theories. The original Hamiltonian is vanishing, therefore the gauge fixed Hamiltonian is given merely by

$$H \equiv \{Q_B, \Psi\} , \tag{21}$$

where Ψ is a so-called gauge fermion.

The gauge fermion corresponding to the superconformal gauge is

$$\Psi = \bar{c}_\pm (N^\pm - 1) + \hat{\bar{c}}_\pm \hat{N}^\pm + \bar{c}'_\pm N'^\pm + \bar{\gamma}_\pm \chi^\pm + \hat{\bar{\gamma}}_\pm \hat{\chi}^\pm + \bar{\gamma}'_\pm \chi'^\pm$$

$$+ \bar{p}_\pm N^\pm + \hat{\bar{p}}_\pm \hat{N}^\pm + \bar{p}'_\pm N'^\pm + \bar{\pi}_\pm \chi^\pm + \hat{\bar{\pi}}_\pm \hat{\chi}^\pm + \bar{\pi}'_\pm \chi'^\pm . \tag{22}$$

After performing Legendre transfomation and solving non-dynamical equations of motion such as $\bar{c}_\pm + \bar{p}_\pm = 0$, $\bar{\gamma}_\pm + \bar{\pi}_\pm = 0$, etc., we will obtain the following lagrangian:

$$L = \partial_+ X^+ \partial_- X^- + i\bar{\psi}^+ \partial_- \bar{\psi}^- + i\psi^+ \partial_+ \psi^-$$

$$- \hat{\bar{c}}_+ \partial_- \hat{c}^+ - \hat{\bar{c}}_- \partial_+ \hat{c}^- - \hat{\bar{\gamma}}_+ \partial_- \hat{c}^+ - \hat{\bar{\gamma}}_- \partial_+ \hat{\gamma}^-$$

$$- \bar{c}_+ \partial_- c^+ - \bar{c}_- \partial_+ c^- - \bar{\gamma}_+ \partial_- \gamma^+ - \bar{\gamma}_- \partial_+ \gamma^-$$

$$- \bar{c}'_+ \partial_- c'^+ - \bar{c}'_- \partial_+ c'^- - \bar{\gamma}'_+ \partial_- \gamma'^+ - \bar{\gamma}'_- \partial_+ \gamma'^- , \tag{23}$$

where $\partial_\pm$ denote $\partial_0 \pm \partial_1$. Since this is a conformally invariant free theory, it may be convenient to reformulate this on a complex plane. The ghosts also denote $(\hat{c}, \hat{b}_z)$, $(\hat{\gamma}_\theta, \hat{\beta}_\theta)$, (c^z, b_{zz}), (c'^z, b'_{zz}), $(\gamma^\theta, \beta_{z\theta})$ and $(\gamma'^\theta, \beta'_{z\theta})$ instead of $(\hat{c}, -i\hat{\bar{c}}/2\pi)$, $(c, -i\bar{c}/2\pi)$, $(c', -i\bar{c}'/2\pi)$, $(\hat{\gamma}, -i\hat{\bar{\gamma}}/2\pi)$, $(\gamma, -i\bar{\gamma}/2\pi)$ and $(\gamma', -i\bar{\gamma}'/2\pi)$. Then the action is rewritten as

$$S = \frac{1}{2\pi}\int d^2z \{\partial X^+ \bar{\partial} X^- + i\bar{\psi}^+ \bar{\partial}\bar{\psi}^- + i\psi^+ \partial\psi^-$$

$$+ \hat{b}^+ \bar{\partial}\hat{c}^+ + \hat{b}^- \partial\hat{c}^- + \hat{\beta}^+ \bar{\partial}\hat{\gamma}^+ + \hat{\beta}^- \partial\hat{\gamma}^-$$

$$+b^{+}\bar{\partial}c^{+}+b^{-}\partial c^{-}+\beta^{+}\bar{\partial}\gamma^{+}+\beta^{-}\partial\gamma^{-}$$
$$+b'^{+}\bar{\partial}c'^{+}+b'^{-}\partial c'^{-}+\beta'^{+}\bar{\partial}\gamma'^{+}+\beta'^{-}\partial\gamma'^{-}\}\,, \tag{24}$$

where ∂ and $\bar{\partial}$ mean $\partial/\partial z$ and $\partial/\partial\bar{z}$. Thus we obtain eventually the conformal field theory which is regarded as the topological superstring coupled to the topological supergravity.[10),11),*)] The generators which are relevant to characterize this conformal field theory turn out to be

$$\hat{L}_{z}=-2i\partial X^{-}+\partial(c\hat{b})+\partial(\gamma\hat{\beta})\,,$$
$$\hat{G}_{\theta}=2i\psi^{-}+\frac{1}{2}\partial c\hat{\beta}+c\partial\hat{\beta}+\hat{b}\gamma\,,$$
$$L^{M}_{zz}=-\partial X^{+}\partial X^{-}-\frac{1}{2}\psi^{+}\partial\psi^{-}+\frac{1}{2}\partial\psi^{+}\psi^{-}-\hat{b}\partial\hat{c}-\frac{1}{2}\hat{\beta}\partial\hat{\gamma}+\frac{1}{2}\partial\hat{\beta}\hat{\gamma}\,,$$
$$G^{M}_{z\theta}=-\psi^{-}\partial X^{+}-\psi^{+}\partial X^{-}-\partial\hat{c}\hat{\beta}+\hat{\gamma}\hat{b}\,,$$
$$L'^{M}_{zz}=-\hat{b}\partial X^{+}-\frac{1}{2}\partial\hat{\beta}\psi^{+}+\frac{1}{2}\hat{\beta}\partial\psi^{+}\,,$$
$$G'^{M}_{z\theta}=\hat{\beta}\partial X^{+}+2\hat{b}\psi^{+}\,,$$
$$L^{\mathrm{Gh}}_{zz}=-2b\partial c-\partial bc-\frac{3}{2}\beta\partial\gamma-\frac{1}{2}\partial\beta\gamma-2b'\partial c'-\partial b'c'-\frac{3}{2}\beta'\partial\gamma'-\frac{1}{2}\partial\beta'\gamma'\,,$$
$$G^{\mathrm{Gh}}_{z\theta}=-2b\gamma+\frac{3}{2}\partial c\beta+c\partial\beta+2b'\gamma'-\frac{3}{2}\partial c'\beta'-c'\partial\beta'\,,$$
$$L'^{\mathrm{Gh}}_{zz}=2b'\partial c+\partial b'c+\frac{3}{2}\beta'\partial\gamma+\frac{1}{2}\partial\beta'\gamma\,,$$
$$G'^{\mathrm{Gh}}_{z\theta}=2b'\gamma-\frac{3}{2}\beta'\partial c-\partial\beta'c\,, \tag{25}$$

where we have used the equations of motion.

One may also consider the following gauge conditions instead of the ordinary superconformal gauge,

$$N_{\pm}=1\,,\quad \tilde{\phi}_{\pm}=0\,,\qquad N'_{\pm}=0\,,$$
$$\chi_{\pm}=0\,,\quad \bar{\psi}^{+}=\psi^{-}=0\,,\quad \chi'_{\pm}=0\,. \tag{26}$$

Then we will find the part of the matter and hat ghosts to disappear and the quantum system to be purely the topological supergravity.

It is known generally that such topological field theories are ruled by so-called twisted $N=2$ supersymmetry.[12)] So we shall examine what kinds of supersymmetries are realized in this model. Actually we may find out that a twisted $N=2$ superconformal algebras (SCA's) and also an ordinary $N=2$ SCA's in both of the string sector and the gravity sector separately.

*) In Ref. 11) the Liouville sector is given in addition to the ghost sector. However only the ghost sector is essential for the moduli space of the topological supergravity and we may eliminate the Liouville sector by a proper gauge fixing. Our model does not produce the Liouville terms, because the gravity freedom does not exist.

In the twisted $N=2$ SCA the conformal spins of the two supercurrents are modified to be 1 and 2 due to twisting of the energy momentum T_{zz}. These fermionic currents, which are denoted by G_z and $\bar{G}_{zz}$, may be given in terms of a part of the BRST charge and L'_{zz}. Indeed the following sets of generators are found to satisfy the twisted SCA's:

$$
(\mathrm{TG})\begin{cases}
T_{zz}=-2b\partial c-\partial bc-2b'\partial c'-\partial b'c'-\frac{3}{2}\beta\partial\gamma-\frac{1}{2}\partial\beta\gamma-\frac{3}{2}\beta'\partial\gamma'-\frac{1}{2}\partial\beta'\gamma' , \\
\tilde{J}_z=-bc-2b'c'-\frac{1}{2}\beta\gamma-\frac{3}{2}\beta'\gamma' , \\
G_z=-2bc'-2\beta\gamma' , \\
\bar{G}_{zz}=2b'\partial c+\partial b'c+\frac{3}{2}\beta'\partial\gamma+\frac{1}{2}\partial\beta'\gamma
\end{cases}
\tag{27}
$$

and

$$
(\mathrm{TS})\begin{cases}
T_{zz}=-\partial X^+\partial X^- -\hat{b}\partial\hat{c}-\frac{1}{2}\psi^+\partial\psi^-+\frac{1}{2}\partial\psi^+\psi^- -\frac{1}{2}\hat{\beta}\partial\hat{\gamma}+\frac{1}{2}\partial\hat{\beta}\hat{\gamma} , \\
\tilde{J}_z=-\hat{b}\hat{c}-\frac{1}{2}\psi^+\psi^- -\frac{1}{2}\hat{\beta}\hat{\gamma} , \\
G_z=2\hat{c}\partial X^- -2\hat{\gamma}\psi^+ , \\
\bar{G}_{zz}=-\hat{b}\partial X^+ -\frac{1}{2}\partial\hat{\beta}\psi^+ +\frac{1}{2}\hat{\beta}\partial\psi^+ .
\end{cases}
\tag{28}
$$

The central charge of the algebra (TG) is calculated to be $c^{(\mathrm{TG})}=c^{(b,c,b',c')}+c^{(\beta,\gamma,\beta',\gamma')}=-9+6=-3$, for the algebra (TS) $c^{(\mathrm{TS})}=c^{(X,\hat{b},\hat{c})}+c^{(\psi,\hat{\beta},\hat{\gamma})}=3+0=3$. Here it should be noticed that the $U(1)$ currents differ from the ghost number currents generally.

On the other hand, we can find $N=2$ SCA's by decomposing the $N=1$ supercurrents $G^M_{z\theta}$ or $G^{\mathrm{Gh}}_{z\theta}$ given in (25) into $G_{z\theta}$ and $\bar{G}_{z\theta}$ properly. The generators will be as follows,

$$
(\mathrm{SG})\begin{cases}
T_{zz}=-2b\partial c-\partial bc-\frac{3}{2}\beta\partial\gamma-\frac{1}{2}\partial\beta\gamma-2b'\partial c'-\partial b'c'-\frac{3}{2}\beta'\partial\gamma'-\frac{1}{2}\partial\beta'\gamma' , \\
J_z=-2bc-3\beta\gamma-2b'c'-3\beta'\gamma' , \\
G_{z\theta}=-2b\gamma+2b'\gamma' , \\
\bar{G}_{z\theta}=3\beta\partial c+2\partial\beta c-3\beta'\partial c'-2\partial\beta'c
\end{cases}
\tag{29}
$$

and

$$
(\mathrm{SS})\begin{cases}
T_{zz}=-\partial X^+\partial X^- -\frac{1}{2}\psi^+\partial\psi^-+\frac{1}{2}\partial\psi^+\psi^- -\hat{b}\partial\hat{c}-\frac{1}{2}\hat{\beta}\partial\hat{\gamma}+\frac{1}{2}\partial\hat{\beta}\hat{\gamma} , \\
J_z=-\psi^+\psi^- +\hat{\beta}\hat{\gamma} , \\
G_{z\theta}=-\psi^-\partial X^+ -\hat{\beta}\partial\hat{c} , \\
\bar{G}_{z\theta}=-2\psi^+\partial X^- +2\hat{\gamma}\hat{b} .
\end{cases}
\tag{30}
$$

Both of the central charges of these algebra are vanishing as is expected; $c^{(\mathrm{SG})}$

$=c^{(b,c,\beta,\gamma)}+c^{(b',c',\beta',\gamma')}=-15+15=0$ and $c^{(SS)}=c^{(X,\psi)}+c^{(\tilde{b},\tilde{c},\tilde{\beta},\tilde{\gamma})}=3-3=0$. The (b, c, β, γ) part of the algebra (SG) has been already known as the hidden $N=2$ SCA in the $N=1$ superghost system.[13)]

Thus we obtain two spin 3/2 supercurrents, a spin 1 supercurrent and a spin 2 supercurrent in each sector. Therefore it may be possible for these systems to have the enlarged symmetry, i.e., the twisted $N=4$ SCA's.[14),11)] In order to have above mentioned supercurrents the $N=4$ SCA to be twisted has to contain an $SU(2)\times SU(2)$ current algebra. However, the commutation relations (or O.P.E.) between the supercurrents given in (27)~(30) show us that some of such $SU(2)\times SU(2)$ currents as well as spin 1/2 fermionic currents are missing. Therefore the twisted $N=4$ symmetry does not seem to realize in this system.

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