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Topologies on finite groups

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It has been shown by D. Stephen that the number N of open sets in a non-discrete topology on a finite set with n elements is not greater than $3 \times 2^{n-2}$. We show that for admissable topologies on a finite group $N \leqq 2^{n/r}$, where r is the least order of its non-trivial normal subgroups. This is clearly a sharper bound.

Define semi-topological group and topological group as in [2]. (We do not assume that these topological spaces are Hausdorff.)

THEOREM 1. *Let X be a non-discrete finite semi-topological group of order n (> 1). Then the number N of open sets satisfies $N \leqq 2^{n/r}$, where r is the least order of the non-trivial normal subgroups of X.*

Proof. Let 0_x be the intersection of all open sets containing $x \in X$. Because there are only a finite number of open sets 0_x is open. We show that 0_e (where e is the identity of X) is a normal subgroup.

Since, for each $x \in X$, $x0_e$ and $x^{-1}0_x$ are open (Theorem 1, Chapter II of [2]) and contain x and e respectively, $0_x \subseteq x\, 0_e$ and $0_e \subseteq x^{-1}\, 0_x$ which imply

$$0_x = x\, 0_e \,. \tag{1}$$

Similarly

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(2) $$0_x = 0_e x \ .$$

Clearly if $y \in 0_e$, $0_e \supseteq 0_y = y0_e$. (In fact since 0_e and $y0_e$ have the same finite number of elements $0_e = y0_e$.) Thus

(3) $$0_e^{\,2} \subseteq 0_e \ .$$

Since X is a finite group and 0_e is non-empty, (1), (2) and (3) imply 0_e is a normal subgroup.

The 0_x are the cosets of 0_e. Indeed $\{0_e\}$ is an open basis at e for the topology. Thus the number of distinct 0_x is $\frac{n}{m}$, where m is the order of 0_e. (Since the topology was assumed to be non-discrete m is strictly greater than one.) The proof is completed by noting that the number of open sets is $2^{n/m} \leqq 2^{n/r}$. (See [3]).

The following example shows that the bound in the above theorem is the best possible (under the conditions of the theorem).

EXAMPLE. Let X be the additive group of integers mod n and $0_e = \left\{0, \frac{n}{p}, \frac{2n}{p}, \ldots, \frac{(p-1)n}{p}\right\}$, where p is the smallest prime dividing n. By Theorem 4.5 of [1], the topology on X which has $\{0_e\}$ as an open basis at 0 makes X a topological group. It is obvious that the number of open sets is $2^{n/p}$.

COROLLARY 2. *Every finite semi-topological group is a topological group.*

Proof. By Theorem 1 0_e is a normal subgroup and $\{0_e\}$ is an open basis for the topology. The result now follows from Theorem 4.5 of [1].

For completeness we include the following remark.

REMARK. It is easily shown that if there is a topology on a group X (not necessarily finite) such that the mapping $(x,y) \to x^{-1}y$ is continuous in the first variable then $(x,y) \to xy$ is continuous in each variable separately and $x \to x^{-1}$ is continuous. If X is also finite then by Corollary 2, X is a topological group.

References

[1] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis*, Vol. 1, (Academic Press, New York, 1963).

[2] Taqdir Husain, *Introduction to topological groups*, (W.B. Saunders Company, Philadelphia and London, 1966).

[3] D. Stephen, "Topologies on finite sets", *Amer. Math. Monthly* 75 (1968), 739-741.

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