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Topologies on the real line

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Abstract
We prove that if a topology on the real line endows it with a topological group structure (additive) for which the interval $(0, +\infty)$ is an open set, so this topology is stronger than the usual topology. As a consequence we obtain characterizations of the usual topology as group topology and as ring topology. We also proved that if a topology on the real line is compatible with its usual lattice structure and is T_1, so this topology is stronger than the usual topology, and as a consequence we obtain a characterization of the usual topology as lattice topology.

Keywords Usual topology on the real line · Topological group · Topological ring · Topological lattice

Mathematics Subject Classification 54D05 · 54F65 · 54H12 · 54H13

1 Introduction

On many occasions the field $\mathbb{R}$ of real numbers appears as a subspace (subgroup, subring, sublattice, ...) of a topological space (group, ring, lattice, ...), and in such cases it is natural to ask whether there is any relationship between the induced topology in $\mathbb{R}$ and the usual topology of $\mathbb{R}$. To try to answer the previous question, it can be very useful to know in some way the different topologies that exist on $\mathbb{R}$. In the paper [1], its authors study "how many" topologies there are on $\mathbb{R}$ that endow it with a topological group structure ($\mathbb{R}$ with its sum), and they obtain that there are many (perhaps too many). They prove that there are even an infinite number of

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non-homeomorphic connected compact Hausdorff group topologies on $\mathbb{R}$. Almost forty years before, Halmos showed in [4] that $\mathbb{R}$ may be endowed with a compact group topology. In this brief work we prove that the number of different ring topologies on $\mathbb{R}$ is significantly less than that of group topologies. For example, there is no a topology on $\mathbb{R}$ with which it is a compact Hausdorff topological ring.

In 1936, Ward characterised in [5] the real numbers topologically among the metric spaces. He showed that every metric space that is separable, connected and locally connected, and in which each point is a strong cut point (its complementary has exactly two components) is homeomorphic to $\mathbb{R}$. In 1970, Franklin and Krishnarao proved in [2] that this characterization remains valid for regular spaces: every regular space that is separable, connected and locally connected, and in which each point is a strong cut point is homeomorphic to $\mathbb{R}$. And, in 1971, they proved in [3] that a separable, connected, locally compact Hausdorff space in which each point is a strong cut point is homeomorphic to the real line.

At the end of [1], its authors give a characterization of the usual topology of $\mathbb{R}$ as a topological group. Similarly, at the end of Sect. 2 of this work we give a characterization of the usual topology of $\mathbb{R}$ as a ring topology.

In Sect. 3 we consider topologies on $\mathbb{R}$ for which "supremum" and "infimum" operations are continuous (lattice topologies). We prove that if one of these topologies is T_1, then that topology is stronger than the usual topology of $\mathbb{R}$, and as consequence we obtain a characterization of the usual topology of $\mathbb{R}$ as a lattice topology.

2 Group topologies and ring topologies

Definitions 1 By a *topological group* we shall mean a (additive) group G endowed with a topology such that the group sum $G \times G \xrightarrow{+} G$ and the involutive map $G \to G$, $x \mapsto -x$, are continuous.

By a *topological ring* we shall mean a commutative ring A endowed with a topology such that the ring sum $A \times A \xrightarrow{+} A$ and the ring multiplication $A \times A \xrightarrow{\cdot} A$ are continuous.

Let A be a topological ring, in which case $(A, +)$ is also a topological group. We will denote by C_0 the connected component of 0 in A.

Remark For unitary topological rings, it is common to require that the map $a \mapsto a^{-1}$ is continuous on the set of invertible elements. In the definition that we have given in this work we do not demand the continuity of such application because we will not use it.

Lemma 2 *Let A be a topological ring. The connected component C_0 of 0 is an ideal.*

Proof On the one hand, for every $a \in A$, the map $A \xrightarrow{a+} A$, that sends b to $a + b$, is an homeomorphism, so that the connected component of a is $a + C_0$. So, if $a, b \in C_0$, then $a + C_0 = C_0$ and $a + b \in C_0$. On the other hand, let us consider $a \in A$ and $b \in C_0$. The continuity of the map $A \xrightarrow{a\cdot} A$ implies that $a \cdot C_0$ is a connected set containing 0, so $a \cdot C_0 \subseteq C_0$ and $a \cdot b \in C_0$. □

Corollary 3 *If a topological ring is a field, then it is connected or totally disconnected.*

Proof Let A be a topological ring that is a field, in which case the only ideals of A are 0 and A. If $C_0 = 0$, then the connected component of any element $a \in A$ is $a + C_0 = \{a\}$. Obviously, if $C_0 = A$, then A is connected. □

Corollary 4 *If a topological ring is a field and it is not connected, then it is Hausdorff.*

Proof Let A be a topological ring that is a field. If A is not connected, then it is totally disconnected, and since the connected components are always closed subsets, every one point subset is closed. It is well-known that if the topological group $(A, +)$ is a T_1-space, then it is Hausdorff. □

Corollary 5 *If a topological ring is a field, and is not connected and locally connected, then it is endowed with its discrete topology.*

Notation Given an element $x \in \mathbb{R}$, $x^+ := \max\{x, 0\}$ is its *positive part*.

Lemma 6 *The map $x \mapsto x^+$ is continuous at $x = 0$ for any topology on $\mathbb{R}$.*

Proof If U is an open subset containing 0, then $0 \in U \subseteq (x^+)^{-1}(U)$. □

Lemma 7 *Let τ be a topology on the real line such that the map $\mathbb{R} \to \mathbb{R}$, $x \mapsto -x$, is continuous (and hence is homeomorphism). We have:*

(i) *If the set $(0, +\infty)$ is open, then the map $x \mapsto x^+$ is continuous.*
(ii) *If the topology τ is Hausdorff and the map $x \mapsto x^+$ is continuous, then the set $(0, +\infty)$ is open.*

Proof (i) If the set $(0 + \infty)$ is open, then also is open the set $(-\infty, 0)$. As the map $x \mapsto x^+$ is continuous at $x = 0$ (Lemma 6), it is enough to prove that this map is continuous on $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, +\infty)$, which is clear: on the open set $(-\infty, 0)$ the map is constant, and on the open set $(0, +\infty)$ the map is the identity.

(ii) When the topology τ is Hausdorff the set $\{0\}$ is closed. If in addition the map $x \mapsto x^+$ is continuous, then the set $(x^+)^{-1}\big(\{0\}\big) = (-\infty, 0]$ is closed, and hence $(0 + \infty)$ is an open set. □

Lemma 8 *Let τ be a topology on the real line such that $(\mathbb{R}, +)$ is a topological group. The following conditions are equivalent:*

(i) *the set $(0, +\infty)$ is open;*
(ii) *the topology τ is stronger than the usual topology of $\mathbb{R}$.*

Proof Let us suppose that the set $(0, +\infty)$ is open. Since the translation maps and the map $x \mapsto -x$ are homeomorphisms, we have: given $a, b \in \mathbb{R}$, the sets $(-\infty, b)$ and $(a, +\infty)$ are open, and therefore $(a, b) = (-\infty, b) \cap (a, +\infty)$ is also open when $a < b$. □

Corollary 9 *Let τ be a topology on the real line such that $(\mathbb{R}, +)$ is a topological group. If τ is Hausdorff, then the following conditions are equivalent:*

(i) *the map* $x \mapsto x^+$ *is continuous;*
(ii) *the topology* τ *is stronger than the usual topology of* $\mathbb{R}$.

Proof It follows from Lemma 7 and Lemma 8. □

According to [1, Theorem 6], there are exactly $\aleph_0$ non-isomorphic compact Hausdorff groups topologies on $\mathbb{R}$. We have:

Lemma 10 *Let* τ *be a topology on the real line such that* $(\mathbb{R}, +)$ *is a topological group. If the map* $x \mapsto x^2$ *is continuous, then the topology* τ *is not compact Hausdorff.*

Proof Let us suppose that τ is a compact Hausdorff topology on the real line such that $(\mathbb{R}, +)$ is a topological group. If the map $x \mapsto x^2$ is continuous, then its image $[0, +\infty)$ is a compact subset of $\mathbb{R}$, and hence $[0, +\infty)$ is closed (because τ is Hausdorff). Applying Lemma 8 we obtain that the topology τ is stronger than the usual topology of $\mathbb{R}$, and as consequence the usual topology is also compact, which is false. □

Corollary 11 *There is no compact Hausdorff topology on the real line such that* $\mathbb{R}$ *is a topological ring.*

Definition 12 A subset X of $\mathbb{R}$ is said to be *semi-bounded* when it is bounded below or above for the usual order of $\mathbb{R}$, i.e., when there is $a \in \mathbb{R}$ such that $X \subseteq (-\infty, a]$ or $X \subseteq [a, \infty)$.

Theorem 13 *Let* τ *be a topology on the real line such that* $\mathbb{R}$ *is a topological ring. The following conditions are equivalent:*

(i) *there exists a non empty open subset in* τ *which is semi-bounded;*
(ii) *the topology* τ *is stronger than the usual topology of* $\mathbb{R}$.

Proof Clearly, it is only necessary to prove that (i) implies (ii). The map $\mathbb{R} \to \mathbb{R}$, $x \mapsto -x$, is an homeomorphism that transforms bounded below sets in bounded above sets and conversely, so it is enough to consider the bounded below case. Moreover, since the translation maps are also homeomorphisms, we can suppose that there exists a non empty open subset U such that $U \subseteq [a, +\infty)$ for some $a > 0$. By Lemma 8, it is enough to prove that the set $(0, +\infty)$ is open. Suppose, on the contrary, that there exists $\lambda > 0$ such that $\lambda \in \overline{(-\infty, 0]}$ (=the closure of the set $(-\infty, 0]$). For any $\alpha > 0$, the homeomorphism $\mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{\alpha}{\lambda}x$, sends λ to α and transforms $(-\infty, 0]$ in itself, so $\alpha \in \overline{(-\infty, 0]}$. Then, $\overline{(-\infty, 0]} = \mathbb{R}$. But, it is not possible because we have supposed that $U \cap (-\infty, 0] = \emptyset$, so we conclude that $(0, +\infty)$ is an open set in τ. □

Part (i) of the following result, easy to prove, can be obtained as a particular case of [1, Theorem 11], whose proof is quite elaborate.

Proposition 14 *Let* τ *be a locally connected topology on the real line such that* $(\mathbb{R}, +)$ *is a topological group and the set* $(0, +\infty)$ *is open. We have:*

(i) *if* τ *is connected, then* τ *is the usual topology of* $\mathbb{R}$*;*
(ii) *if* τ *is not connected, then* τ *is the discrete topology of* $\mathbb{R}$.

Proof Let us denote by $\mathbb{R}_u$ the real line endowed with its usual topology. As the identity map $(\mathbb{R}, \tau) \to \mathbb{R}_u$ is continuous (Lemma 8), if X is a non empty connected subset for τ, then X is also connected in $\mathbb{R}_u$, and therefore X is an interval. Hence, every point of $\mathbb{R}$ has a basis of τ-neighborhood formed by intervals. Recall that if $\mathcal{U}$ is a basis of τ-neighborhood of 0, then, for each $\alpha \in \mathbb{R}$, $\alpha + \mathcal{U} := \{\alpha + U : U \in \mathcal{U}\}$ is a basis of τ-neighborhood of α. Moreover, if an interval I is a τ-neighborhood of 0, then $-I$ also is so. We distinguish two cases.

First case: for each $b > 0$, the interval $[0, b)$ is not a τ-neighborhood of 0. Then, the intervals of type (a, b) with $a < 0 < b$ are a basis of τ-neighborhood of 0, and therefore τ is the usual topology of $\mathbb{R}$.

Second case: There exists $b > 0$ such that $[0, b)$ is a τ-neighborhood of 0. Then $(-b, 0]$ is a τ-neighborhood of 0 and therefore $\{0\} = (-b, 0] \cap [0, b)$ is as well. Then the set $\{0\}$ is open for τ, and τ is the discrete topology of $\mathbb{R}$. □

For a ring topology on $\mathbb{R}$, in the above proof the first case is the connected case, and the second is the totally disconnected case (see Corollary 3 and Corollary 5). Hence, apply Theorem 13 we obtain the following characterization:

Theorem 15 *Let τ be a topology on the real line such that $\mathbb{R}$ is a topological ring. If τ is connected, locally connected, and has a non empty semi-bounded open subset, then τ is the usual topology of $\mathbb{R}$.*

3 Lattice topologies

Definitions 16 A *lattice* is a non empty set X endowed with an order relationship "$\leq$" for which every non empty finite subset has supremum and infimum. As usual, the supremum and the infimum of a subset $\{x_1, \ldots, x_n\}$ of X will be denoted by $x_1 \vee \cdots \vee x_n$ and $x_1 \wedge \cdots \wedge x_n$, respectively.

By a *topological lattice* we shall mean a lattice X endowed with a topology for which the maps $X \times X \xrightarrow{\vee} X$ and $X \times X \xrightarrow{\wedge} X$ are continuous.

Examples 17 (a) The real line $\mathbb{R}$ with its usual order is a lattice. It is well known that the usual topology on $\mathbb{R}$ is a "lattice topology", i.e., $\mathbb{R}$ with the usual topology is a topological lattice. Also, it is clear that the discrete topology of $\mathbb{R}$ is a lattice topology. Note that the above two topologies are locally connected and Hausdorff. The usual topology is connected, but the discrete topology is not.

(b) Let us consider on $\mathbb{R}$ the topology τ whose open sets are $\mathbb{R}$, the empty set, and the collection of intervals $\{(a, +\infty) : a \in \mathbb{R}\}$. It is easy to see that τ is a lattice topology.

Lemma 18 *Let τ be a lattice topology on $\mathbb{R}$. If $(\mathbb{R}, \tau)$ is a T_1-space (i.e., every point of $\mathbb{R}$ is τ-closed), then τ is stronger than the usual topology of $\mathbb{R}$. In particular, τ is Hausdorff.*

Proof Given $b \in \mathbb{R}$, let us consider the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) := x \wedge b$. As the set $\{b\}$ is τ-closed and the map f is continuous, we obtain that the set $[b, +\infty) = f^{-1}(b)$ is τ-closed, and therefore the interval $(-\infty, b)$ is τ-open. Similarly it is proved that for each $a \in \mathbb{R}$, the interval $(a, +\infty)$ is τ-open. So, given $a, b \in \mathbb{R}$ with $a < b$, the interval $(a, b) = (a, +\infty) \cap (-\infty, b)$ is also τ-open. □

Theorem 19 *Let τ be a lattice topology on $\mathbb{R}$. If $(\mathbb{R}, \tau)$ is a locally connected and connected T_1-space, then τ is the usual topology of $\mathbb{R}$.*

Proof Let us denote by $\mathbb{R}_u$ the real line endowed with its usual topology. As the identity map $(\mathbb{R}, \tau) \to \mathbb{R}_u$ is continuous (Lemma 18), if X is a non empty connected subset for τ, then X is also connected in $\mathbb{R}_u$, and therefore X is an interval. Hence, every point of $\mathbb{R}$ has a basis of τ-neighborhood formed by intervals.

Let us suppose that τ is not the usual topology. Then there are a point $a \in \mathbb{R}$ and an interval I of $\mathbb{R}$, such that I is a τ-neighborhood of a and I is not neighborhood of a for the usual topology. Under these conditions, one of the following equalities must necessarily occur: (i) $I = \{a\}$; (ii) $I = [a, +\infty)$; (iii) $I = (-\infty, a]$; (iv) $I = [a, b)$ with $a < b$; (v) $I = (b, a]$ with $b < a$. It is easy to see that in any of the five previous cases we obtain that there is a τ-neighborhood of a in $\mathbb{R}$ that is open, closed and different from $\mathbb{R}$, which can not be because τ is connected. Therefore τ is the usual topology of $\mathbb{R}$. □

Example 20 The lattice topology τ of Example 17 (b) is locally connected and connected, but $(\mathbb{R}, \tau)$ is not a T_1-space.

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