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TOPOLOGY OF CERTAIN SUBMANIFOLDS IN THE EUCLIDEAN SPHERE

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ABSTRACT. Using the nonexistence theorem for stable harmonic maps, we study the fundamental group of certain submanifolds in the Euclidean sphere.

1. Introduction. In [3] R. Schoen and S. T. Yau made the first attempt to study the geometry of manifolds by using harmonic maps. They proved that if M is a complete noncompact stable immersed hypersurface in a manifold of nonnegative curvature and D is a compact domain in M with smooth simply connected boundary, then there is no nontrivial homomorphism from $\pi_1(D)$ into the fundamental group of a compact manifold with nonpositive curvature.

In this paper we consider certain submanifolds in the Euclidean sphere. First of all we generalize the *nonexistence theorem* in our previous paper [5] as follows:

Let M be a compact n-dimensional immersed submanifold with second fundamental form B and mean curvature H in the Euclidean sphere. When $n > 2 + \tilde{B}$ there is no nonconstant stable harmonic map from M to any Riemannian manifold N, where

$$\tilde{B} = \left(\sum_{i,j=1}^{n} \left(2\langle B_{e_k,e_i} B_{e_k,e_j}\rangle - \langle H, B_{e_i,e_j}\rangle\right)^2 \right)^{1/2}.$$

According to the J. Simons' theorem [4], when M as above is minimal, it cannot be stable.

Using the above nonexistence theorem and Eells-Sampson's theorem [1], we find a topological restriction similar to that in Schoen-Yau's theorem. The result is the following:

Let M be a compact n-dimensional submanifold with second fundamental form B and mean curvature H in the Euclidean sphere. When $n > 2 + \tilde{B}$ there is no nontrivial homomorphism from the fundamental group $\pi_1(M)$ into the fundamental group of a compact manifold with nonpositive curvature.

2. Preliminary notation. We refer the basic notion about harmonic maps to the paper [2]. Our purpose in this section is to sketch the immersed submanifolds.

Let M be a compact n-dimensional Riemannian manifold in $\overline{M}$, which is an m-dimensional Riemannian manifold. Set $m = n + p$, where p is the codimension

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of M in $\overline{M}$. We shall use the following ranges of indices throughout this paper:

$$1 \leq A, B, C, \ldots \leq m = n + p,$$
$$1 \leq i, j, k, l \leq n,$$
$$n + 1 \leq \alpha, \beta, \gamma, \ldots \leq n + p.$$

Let TM and NM denote the tangent bundle and the normal bundle of M, respectively, such that for any $x \in M \subset \overline{M}$ we have an orthogonal splitting

$$T_x(\overline{M}) = T_x(M) \oplus N_x(M).$$

With respect to this splitting we decompose any vector $X \in T_x(\overline{M})$ as $X = (X)^T \oplus (X)^N$. M inherits the Riemannian connection from one $\overline{\nabla}$ of $\overline{M}$ as follows: let $\tilde{X}$ and $\tilde{Y}$ be vector fields on M. Then for $X = \tilde{X}(x)$

$$\nabla_X \tilde{Y} = \left(\overline{\nabla}_X \tilde{Y}\right)^T, \tag{2.1}$$

which is the unique Riemannian connection induced by the metric inherited from $\overline{M}$.

The second fundamental form of M in $\overline{M}$ is a section of $\mathrm{Hom}(TM \otimes TM, NM)$, defined as follows: for any $X, Y \in T_xM$

$$B_{X,Y} = \left(\overline{\nabla}_X \tilde{Y}\right)^N \tag{2.2}$$

where $\tilde{Y}$ is an extension of Y to a local tangent vector field on M. At each point $x \in M$, B_x represents a symmetric bilinear map of T_xM into N_xM. Thus we can define

$$H_x = \mathrm{trace}(B_x) \tag{2.3}$$

for each $x \in M$. H is called a *mean curvature vector field.*

Sometimes it is convenient to consider B in adjoint form. For $\nu \in N_xM$ we define $A^\nu: T_xM \to T_xM$ with

$$A^\nu(X) = -\left(\overline{\nabla}_X \tilde{\nu}\right)^T \tag{2.4}$$

for $X \in T_xM$, where $\tilde{\nu}$ is any local extension of ν to a normal vector field. It is easy to check the relation

$$\langle A^\nu(X), Y\rangle = \langle B_{X,Y}, \nu\rangle. \tag{2.5}$$

We define the squared length of B at each $x \in M$ in the usual way as

$$\|B\|^2 = \sum_{i,j=1}^{n} \|B_{e_i,e_j}\|^2 \tag{2.6}$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal basis of M.

Let R and $\overline{R}$ be the curvature tensors of M and $\overline{M}$, respectively. For any $X, Y, Z, W \in T_xM$ we have Gauss' formula for submanifolds:

$$\langle R_{X,Y}Z, W\rangle = \langle \overline{R}_{X,Y}Z, W\rangle - \langle B_{X,W}, B_{Y,Z}\rangle + \langle B_{X,Z}, B_{Y,W}\rangle, \tag{2.7}$$

which we shall have occasion to use below. We adopt the sign convention of [2] about the curvature.

Let us consider a submanifold in the Euclidean sphere $M \subset S^m \subset \mathbf{R}^{m+1}$. Let θ denote the $(n + 1)$-dimensional vector space of vector fields on S^m by

$$\theta = \left\{\operatorname{grad} f\big|_{S^m} : f \text{ is linear on } \mathbf{R}^{m+1}\right\}.$$

For any $V \in \theta$ there is a unique decomposition $V|_M = V^T + V^N$. We denote $\theta^T = \{V^T : V \in \theta\}$.

It is easy to check the following relations:

$$\nabla_X V^N = -B_{X,V^T}, \tag{2.8}$$

and

$$\nabla_X V^T = A^{V^N}(X) - fX. \tag{2.9}$$

3. The proof of the results. Let $M \subset S^m$ be a compact immersed submanifold. We consider any harmonic map ϕ: $M \to \overline{M}$, where the image manifold is any Riemannian manifold. By means of this map ϕ we obtain an induced vector bundle $\phi^{-1}TM$ over M which inherits a Riemannian connection $\tilde{\nabla}$ from the canonical connection in $\overline{M}$. Choose a local orthonormal basis $\{e_i\}$, such that $(\nabla_{e_i} e_j)_x = 0$ at a given point $x \in M$. For $V^T \in \theta^T$, taking the cross section $\phi_* V^T$, we have the following lemmas:

LEMMA 3.1.

$$\tilde{\nabla}_{e_i}\phi_*\left(\nabla_{e_i} V^T\right) = \langle(\nabla_{e_i} A)^{V^N}(e_i), e_j\rangle\phi_* e_j - \langle B_{e_i,e_j}, B_{e_i,V^T}\rangle\phi_* e_j$$
$$+\langle B_{e_i,e_j}, V^N\rangle\tilde{\nabla}_{e_i}\phi_* e_j - \phi_* V^T. \tag{3.1}$$

PROOF. Using (2.5), (2.8) and (2.9), we have

$$\begin{aligned}
\tilde{\nabla}_{e_i}\phi_*\left(\nabla_{e_i} V^T\right) &= \tilde{\nabla}_{e_i}\phi_*\left(A^{V^N}(e_i) - fe_i\right)\\
&= \tilde{\nabla}_{e_i}\phi_*\langle A^{V^N}(e_i), e_j\rangle e_j - \langle V^T, e_i\rangle\phi_* e_i\\
&= \tilde{\nabla}_{e_i}\langle A^{V^N}(e_i), e_j\rangle\phi_* e_j - \phi_* V^T\\
&= \langle\nabla_{e_i} A^{V^N}(e_i), e_j\rangle\phi_* e_j + \langle A^{V^N}(e_i), e_j\rangle\tilde{\nabla}_{e_i}\phi_* e_j - \phi_* V^T\\
&= \langle(\nabla_{e_i} A)^{V^N}(e_i), e_j\rangle\phi_* e_j + \langle A^{\nabla_{e_i} V^N}(e_i), e_j\rangle\phi_* e_j\\
&\quad +\langle B_{e_i,e_j}, V^N\rangle\tilde{\nabla}_{e_i}\phi_* e_j - \phi_* V^T\\
&= \langle(\nabla_{e_i} A)^{V^N}(e_i), e_j\rangle\phi_* e_j - \langle B_{e_i,e_j}, B_{e_i,V^T}\rangle\phi_* e_j\\
&\quad +\langle B_{e_i,e_j}, V^N\rangle\tilde{\nabla}_{e_i}\phi_* e_j - \phi_* V^T. \quad \text{Q.E.D.}
\end{aligned}$$

LEMMA 3.2.

$$\varphi_*(\nabla^*\nabla V^T) = \langle(\nabla_{e_i} A)^{V^N}(e_i), e_j\rangle\phi_* e_j - \langle B_{e_i,V^T}, B_{e_i,e_j}\rangle\phi_* e_j - \phi_* V^T. \tag{3.2}$$

PROOF. Using (2.5), (2.8) and (2.9), we obtain

$$\begin{aligned}
\phi_*(\nabla^*\nabla V^T) &= \phi_*\left(\nabla_{e_i}\nabla_{e_i} V^T\right) = \phi_*\nabla_{e_i}\left(A^{V^N}(e_i) - fe_i\right)\\
&= \phi_*\left((\nabla_{e_i} A)^{V^N}(e_i) + A^{\nabla_{e_i} V^N}(e_i) - V^T\right)\\
&= \phi_*\left(\langle(\nabla_{e_i} A)^{V^N}(e_i), e_j\rangle e_j - \langle B_{e_i,V^T}, B_{e_i,e_j}\rangle e_j - V^T\right)\\
&= \langle(\nabla_{e_i} A)^{V^N}(e_i), e_j\rangle\phi_* e_j - \langle B_{e_i,V^T}, B_{e_i,e_j}\rangle\phi_* e_j - \phi_* V^T.
\end{aligned}$$

LEMMA 3.3.

$$
\begin{aligned}
-\tilde{\nabla}^*\tilde{\nabla}\phi_* V^T = R^{\overline{M}}\left(\phi_* e_i, \phi_* V^T\right)\phi_* e_i + (2-n)\phi_* V^T \\
-\langle(\nabla_{e_i}A)^{V^N}(e_i), e_j\rangle\phi_* e_j - \langle H, B_{V^T,e_j}\rangle\phi_* e_j \\
-2\langle B_{e_i,e_j}, V^N\rangle\tilde{\nabla}_{e_i}\phi_* e_j + 2\langle B_{e_i,e_j}, B_{e_i,V^T}\rangle\phi_* e_j.
\end{aligned} \tag{3.3}
$$

PROOF. In our case $M \subset S^m$ by using Gauss formula (2.7), we have

$$
\begin{aligned}
\langle R_{XY}Z, W\rangle = \langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle \\
-\langle B_{X,W}, B_{Y,Z}\rangle + \langle B_{X,Z}, B_{Y,W}\rangle.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\langle \mathrm{Ric}\, Y, W\rangle &= \langle R_{e_iY}e_i, W\rangle \\
&= (n-1)\langle Y, W\rangle - \langle B_{e_i,W}, B_{e_i,Y}\rangle + \langle H, B_{Y,W}\rangle.
\end{aligned}
$$

Namely

$$
\mathrm{Ric}\, Y = (n-1)Y - \langle B_{e_i,Y}, B_{e_i,e_j}\rangle e_j + \langle H, B_{Y,e_j}\rangle e_j. \tag{3.4}
$$

By Weitzenböck's formula and (3.4)

$$
\begin{aligned}
-(\tilde{\nabla}^*\tilde{\nabla}\, d\phi)V^T &= R^{\overline{M}}\left(\phi_* e_i, \phi_* V^T\right)\phi_* e_i - \phi_*(\mathrm{Ric}\, V^T) \\
&= R^{\overline{M}}\left(\phi_* e_i, \phi_* V^T\right)\phi_* e_i - (n-1)\phi_* V^T \\
&\quad + \langle B_{e_i,V^T}, B_{e_i,e_j}\rangle\phi_* e_j - \langle H, B_{V^T,e_j}\rangle\phi_* e_j.
\end{aligned} \tag{3.5}
$$

Thus from (3.1), (3.2) and (3.5), we have

$$
\begin{aligned}
-\tilde{\nabla}^*\tilde{\nabla}\phi_* V^T &= -\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}\, d\phi(V^T) \\
&= -\tilde{\nabla}_{e_i}\left((\tilde{\nabla}_{e_i}\, d\phi)V^T + d\phi(\nabla_{e_i}V^T)\right) \\
&= -(\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}\, d\phi)V^T - 2(\tilde{\nabla}_{e_i}\, d\phi)\nabla_{e_i}V^T - d\phi\left(\nabla_{e_i}\nabla_{e_i}V^T\right) \\
&= -(\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}\, d\phi)V^T - 2\tilde{\nabla}_{e_i}\phi_*\left(\nabla_{e_i}V^T\right) + \phi_*\left(\nabla_{e_i}\nabla_{e_i}V^T\right) \\
&= R^{\overline{M}}\left(\phi_* e_i, \phi_* V^T\right)\phi_* e_i - (n-1)\phi_* V^T \\
&\quad + \langle B_{e_i,V^T}, B_{e_i,e_j}\rangle\phi_* e_j - \langle H, B_{V^T,e_j}\rangle\phi_* e_j \\
&\quad - 2\langle(\nabla_{e_i}A)^{V^N}(e_i), e_j\rangle\phi_* e_j + 2\langle B_{e_i,e_j}, B_{e_i,V^T}\rangle\phi_* e_j \\
&\quad - 2\langle B_{e_i,e_j}, V^N\rangle\tilde{\nabla}_{e_i}\phi_* e_j + 2\phi_* V^T \\
&\quad + \langle(\nabla_{e_i}A)^{V^N}(e_i), e_j\rangle\phi_* e_j - \langle B_{e_i,V^T}, B_{e_i,e_j}\rangle\phi_* e_j - \phi_* V^T \\
&= R^{\overline{M}}\left(\phi_* e_i, \phi_* V^T\right)\phi_* e_i + (2-n)\phi_* V^T - \langle(\nabla_{e_i}A)^{V^N}(e_i), e_j\rangle\phi_* e_j \\
&\quad - \langle H, B_{V^T,e_j}\rangle\phi_* e_j - 2\langle B_{e_i,e_j}, V^N\rangle\tilde{\nabla}_{e_i}\phi_* e_j + 2\langle B_{e_i,e_j}, B_{e_i,V^T}\rangle\phi_* e_j.
\end{aligned}
$$

Q.E.D.

Using the second variation formula for harmonic maps, we have

$$
\begin{aligned}
i &= I\left(\phi_* V^T, \phi_* V^T\right) \\
&= \int_M \langle -\tilde{\nabla}^* \tilde{\nabla} \phi_* V^T - R^{\overline{M}}\left(\phi_* e_i, \phi_* V^T\right)\phi_* e_i, \phi_* V^T \rangle * 1 \\
&= \int_M \Big\{ (2-n) \|\phi_* V^T\|^2 \\
&\qquad - \left[\langle (\nabla_{e_i} A)^{V^N}(e_i), e_j \rangle + 2\langle B_{e_i,e_j}, B_{e_i,V^T} \rangle - \langle H, B_{V^T,e_j} \rangle \right] \\
&\qquad\qquad \cdot \langle \phi_* e_j, \phi_* V^T \rangle - 2\langle B_{e_i,e_j}, V^N \rangle \langle \tilde{\nabla}_{e_i} \phi_* e_j, \phi_* V^T \rangle \Big\} * 1. \quad (3.6)
\end{aligned}
$$

Now we choose an orthonormal basis $\{x, e_i, \nu_\alpha\}$ for $\mathbf{R}^{N+1}$, where e_i are (parallel to) tangent vectors to M at the point $x \in M$. This basis determines an orthonormal basis $\{X, E_i, F_\alpha\}$ for θ and a corresponding basis $\{X^T, E_i^T, F_\alpha^T\}$ for θ^T such that $X(x) = 0$, $E_i(x) = e_i$ and $F_\alpha(x) = \nu_\alpha$, namely $E_i^T(x) = e_i$, $E_i^N(x) = 0$, $F_\alpha^T(x) = 0$ and $F_\alpha^N(x) = \nu_\alpha$. Hence

$$
\text{trace } i = (2-n)E(\phi) + \int_M \left[2\langle B_{e_k,e_j}, B_{e_k,e_i} \rangle - \langle H, B_{e_i,e_j} \rangle \right] \langle \phi_* e_i, \phi_* e_j \rangle * 1 \tag{3.7}
$$

where $E(\phi)$ is the energy integral of the harmonic map ϕ.

We have the following lemma whose proof is not difficult; we leave it to the readers.

LEMMA 3.4. *If A and B are symmetric matrices and B is semipositive definite, then trace $AB \leqslant$ (trace $A^2)^{1/2}$trace B.*

Therefore (3.7) becomes

$$
\text{trace } i \leqslant (2-n)E(\phi) + \int_M \tilde{B} \langle \phi_* e_k, \phi_* e_k \rangle * 1, \tag{3.8}
$$

where

$$
\tilde{B} = \left(\sum_{i,j=1}^{n} \left(2\langle B_{e_k,e_i}, B_{e_k,e_j} \rangle - \langle H, B_{e_i,e_j} \rangle \right)^2 \right)^{1/2}.
$$

Thus we obtain the following:

THEOREM 1. *Let M be an n-dimensional compact submanifold with second fundamental form B and mean curvature H in the Euclidean sphere S^m. When $n > 2 + \tilde{B}$ there is no nonconstant stable harmonic map from M to any Riemannian manifold, where*

$$
\tilde{B} = \left(\sum_{i,j=1}^{n} \left(2\langle B_{e_k,e_i}, B_{e_k,e_j} \rangle - \langle H, B_{e_i,e_j} \rangle \right)^2 \right)^{1/2}.
$$

REMARK. If $B = O$, M is a sphere S^n with the usual totally geodesic imbedding, this theorem becomes Theorem 3.1 of our previous paper [5].

Using the above Theorem 1, we obtain a certain topological restriction on M.

THEOREM 2. *Let M be an n-dimensional compact submanifold with second fundamental form B and mean curvature H in the Euclidean sphere S^m and let $\bar{M}$ be a compact Riemannian manifold with nonpositive sectional curvature. If $n > 2 + \tilde{B}$, then there is no nontrivial homomorphism from the fundamental group $\pi_1(M)$ into $\pi_1(\bar{M})$, where*

$$\tilde{B} = \left(\sum_{i,j=1}^{n} \left(2\langle B_{e_k,e_i}, B_{e_k,e_j}\rangle - \langle H, B_{e_i,e_j}\rangle\right)^2 \right)^{1/2}.$$

PROOF. Let h: $\pi_1(M) \to \pi_1(\bar{M})$ be a homomorphism. Since M is compact and $\bar{M}$ is $K(\pi, 1)$, there exists a smooth map f: $M \to \bar{M}$, such that its induced map f_* between the fundamental groups is h. By Eells-Sampson's theorem **[1]** there exists a harmonic map ϕ which is homotopic to f and has minimum energy in its homotopy class. It follows both that $\phi_* = h$ and that ϕ is a stable harmonic map. But Theorem 1 tells us ϕ is constant so that h is trivial. Q.E.D.

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